# The Curious Events Leading to the Theory of Shock Waves

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*Natura non facit saltus.*
Lucretius Caro (98 BC-55 BC).

*I will tell you these stories, not in the fashion of those textbook writers who manufacture historical notices so as to bear out their own views of how science ought have developed, but instead as they really did occur.*
Clifford Truesdell (1919-2000).

## Abstract

We review the history of the development of the modern theory of shock waves. Several attempts at an early-theory quickly collapsed for lack of foundations in mathematics and thermodynamics. It is not until the works of Rankine and later Hugoniot that a full theory is established. Rankine is the first to show that within the shock a non-adiabatic process must occur. Hugoniot showed that in the absence of viscosity and heat conduction conservation of energy implies conservation of entropy in smooth regions and a jump in entropy across a shock. Even after the theory is fully developed, old notions continue to pervade the literature well into the early part of the 20th Century.

## Beginnings

The period between Poisson's 1808 paper on the theory of sound and Hugoniot's fairly complete 1887 exposition of the theory of shock waves is a period characterized by many insecurities brought about by weak foundations in mathematics and thermodynamics. In the early 1800's few British scientists had read the works of Johann and Daniel Bernoulli, d'Alembert and Euler [3]. Truesdell summarizes the prevailing current in England thus: "The mathematics taught in Cambridge in the early nineteenth century was so antiquated that experiment and mathematical theory had turned their backs upon each other. In order to set up a mathematical framework general enough to cover the phenomena of tides and waves and resistance and deformation and heat flow and attraction and magnetism, the young British mathematicians had to turn, finally, straight to what had been until then the enemy camp: the French Academy, where the mantle of the Basel school, inherited from Euler by Lagrange, had been passed on to Laplace, Legendre, Fourier, Poisson and Cauchy" [33]. However, to be fair the truth is that the mathematical apparatus needed to effectively deal with discontinuous functions[1] did not exist anywhere and the long established attitude among British scientists to ignore the work of scientists in the

[1] Our understanding of the meaning of a function has it roots in the acrimonious debate between d'Alembert and Euler over the solution of a vibrating string [17]. Euler's view employed the notion of "improper" functions which allowed for the representation of discontinuities consistent with the physical observations of D. Bernoulli [18]. The dispute declined with the passing of both protagonists in 1783. The issue of the regularity of solution did not resurface until it was forced on mathematicians by Riemann and other physicists dealing with discontinuous waves in the latter part of the nineteenth century. The first steps towards a theory of generalized solutions to hyperbolic partial differential equations were taken only at the beginning of the twentieth century [16]. The theory reached maturity in the works of Sobolev [28], 1934 and Schwartz [26], 1950.

Continent was equally reciprocated by their peers in the Continent. In addition there was a generally held belief that nature would not tolerate a discontinuity and this, compounded by a lack of appreciation for the singular nature of the inviscid equations, cast a fog of confusion over the problem.

The main events that follow unfold as a tale of two cities, in Cambridge and Paris. In order to appreciate the events in Paris leading to Hugoniot's paper of 1887, we have to start by considering the contributions of Gaspard Monge (1746-1818). Monge's work is unique in that he approached the solution of partial differential equations by means of geometrical constructions. In 1773, he presented his approach to the French academy [19], in addition to other works on the calculus of variation, infinitesimal geometry, the theory of partial differential equations and combinatorics. His work on solutions to first-order partial differential equations established the foundations for the method of characteristics which would be later expanded by Earnshaw, Riemann and Hadamard. But beyond his technical contributions, Monge is also important to this story for his role in the creation in 1794 of what would become one of the most prestigious education centers in the world, the École Polytechnique of Paris. The political turmoil of the French revolution from 1790 to 1793, culminating in the Reign of Terror, resulted in the closing of all institution of higher learning and left the republic without a much needed supply of civilian and military engineers. The École was created to meet this demand. As Dickens observed about Paris in 1794: "it was the best of times, it was the worst of times, it was the age of wisdom, it was the age of foolishness …" Four hundred students, the best the country had to offer, were enrolled the first year for a three-year curriculum in "revolutionary courses" in mathematics and chemistry [8]. Over the years, the school would count among its faculty and students Lagrange, Poisson, Fourier, Duhamel, Cauchy, Carnot, Biot, Fresnel, Hugoniot, Navier, Saint-Venant, Sturm, Liouville, Hadamard and Poincaré, to list but a few.

Siméon-Denis Poisson (1781-1840) entered the École at age seventeen. There he was trained by Laplace[2] and Lagrange who quickly recognized his mathematical talents. As a student Poisson had troubles with Monge's descriptive geometry which was a requirement for students going into public service, but because he was interested in a career in pure science he was able to avoid taking the course. Soon after completing his studies, Poisson was appointed repetiteur[3] at the École. Appointment to full professor was a difficult proposition, but in a lucky break a vacancy was created 1802 by Napoleon when he sent Fourier to Grenoble in south-east France to be the prefect of Isére. With the support of Laplace, Poisson took the position in 1806. A year later, Poisson delivered his lecture "on the theory of sound", which appeared the following year (1808) in the École's journal [20]. The opening paragraph begins by giving credit to Lagrange and continues with: "However, at the time of their publication, very little was known about the use of partial differential equations on which the solution for these types of problems depend. There was disagreement on the use of discontinuous functions which are nevertheless

[2] Laplace considered second only to Lagrange as a mathematical savant was not a professor at the École Polytechnique, but an examiner. As an examiner he traveled to cities throughout France administering public, individual, oral exams to aspiring students. Laplace's influence on the school was considerable.

[3] A repetiteur is a professor's aid who explains the lectures to the students.

fundamental for representing the status of the air at the origin of the motion: thankfully, these difficulties have been removed with the progress made in the analysis, whilst those which persist relate to the nature of the problem". Without further reference to discontinuous functions, the paper proceeds to prove several general theorems for the solution of partial differential equations governing the propagation of sound waves. In section §24 dealing with disturbances of finite amplitude he introduces the governing equation for the velocity potential $\varphi$ and particle velocity $d\varphi/dx$:

$$\frac{d\varphi}{dt} + a\frac{d\varphi}{dx} + \frac{1}{2}\frac{d^2\varphi}{dx^2} = 0,$$

here $a$ is the speed of sound which is assumed constant. Poisson finds the exact solution for a traveling wave in one direction in the form,

$$\frac{d\varphi}{dx} = f\left(x - at - \frac{d\varphi}{dt}t\right), \tag{1}$$

where $f$ is an arbitrary function. Poisson's other major contribution to the theory of sound was his derivation in 1823 of the gas law for sound waves with infinitesimal amplitudes, $p \propto \rho^\gamma$, later called *Poisson isentrope*[4] [21].

Twenty years younger than Poisson, George Airy (1801-1892) graduated top of his class at Trinity College, Cambridge in 1823. That year, Airy was awarded the first Smith prize, given for proficiency in mathematics and natural philosophy. Among Airy's examiners for the Smith's prize were Robert Woodhouse and Thomas Torton, both former Lucasian Chair holders. Only three years after graduating from Trinity College, Airy was elected to the Lucasian Chair. The Chair paid Airy very little[5] and thus in 1828, less than two years after being appointed Lucasian Chair, Airy took a much higher paying job by replacing Woodhouse as Plumian Professor of astronomy at Cambridge and Director of the Cambridge Observatory. In 1835, Airy moved to Greenwich to become the Astronomer Royal of the Royal Observatory at Greenwich. In his position as Astronomer Royal, Airy published a long article on tides and waves [1] in volume 3 of *Encyclopaedia Metropolitana*. In the article he makes the following reference to Poisson and Cauchy: "…[he] does not comprehend those special cases which have been treated at so great length by Poisson…and Cauchy… With respect to these we may express here an opinion, borrowed from others writers, but in which we join, that as regards their physical results these elaborate treatises are entirely uninteresting; although they rank among the leading works of the present century in regard to the improvement of pure mathematics". Airy's remark captures the views of his colleagues at Cambridge towards the works of

[4] Although the isentropic relation is credited to Poisson's 1823 paper, its discovery belongs to Laplace who developed it in a short note [15] published in 1816, in the same journal as Poisson's 1823 paper. Laplace wrote: "The real speed of sound equals the product of the speed according to the Newtonian formula by the square root of the ratio of the specific heat[s] …" That is, $a^2 = \gamma p / \rho$.

[5] The Lucasian Chair paid Airy £99 per year compared to £500 per year as Plumian Professor.

Poisson, Cauchy and other leading scientists in the Continent. In the *Metropolitana* article, Airy studies waves of finite amplitudes in water canals and makes the observation that the crests tend to gain upon the hollows so that the fore slopes become steeper and steeper. The significance of Airy's observation is understood by James Challis (1803-1882), a colleague of Airy, who is best known for his role in the British failure to discover Neptune [27]. In 1845 Challis was director of the Cambridge Observatory, the same position previously held by Airy. That year, John Adams, a young mathematical prodigy, approached both Challis and Airy with calculations he had made based on irregularities on the orbit of Uranus predicting the position of a new planet. His calculations were ignored until late in June of 1846 when calculations by Joseph Le Verrier, a French mathematician, became known in England. After seeing Le Verrier's prediction, Airy suggested that Challis should conduct a search for the planet, but by this time it was too late. The discovery of Neptune was snatched from the British by the Berlin Observatory on September 23, 1846. Challis record in fluid dynamics was not impressive either [3], however in 1848 he published an article in the Philosophical Magazine [2] based on Airy's observation about the behavior of waves of finite amplitude in the *Metropolitana*. In this article Challis writes that if we consider a sinusoidal motion,

$$w = m \sin \frac{2\pi}{\lambda}(z - (a + w)t),$$

an apparent contradiction occurs. The contradiction comes about because when the velocity $w$ vanishes at some $t = t_1$ we have $z = at_1 + n\lambda/2$ and since at the same time $t = t_1$ $w$ reaches its maximum value ($\pm m$) when $z = at_1 + \ n\lambda/2 + mt_1 - \lambda/4$, and since $m$, $t_1$ and $\lambda$ are arbitrary, we may have $mt_1 - \lambda/4 = 0$, in which case the maximum and the points of no velocity occur at the same point.

George Stokes (1819-1903) graduated from Pembroke College, Cambridge, in 1841. Like Airy, he was first of his class, was a recipient of the Smith's prize and was appointed to the Lucasian Chair. While attending Pembroke College, Stokes had already crossed paths with Challis and had quarreled with his views on fluid mechanics on several occasions [4]. Thus it appears that Stokes seized the opportunity to embarrass Challis with a comment on the following issue of the Philosophical Magazine entitled "On a Difficulty in the Theory of Sound" [29]. Unlike Challis and many others of his British contemporaries, Stokes had both read and understood the works of Fourier, Cauchy and Poisson. Stokes begins his article by writing down Poisson's exact solution to the traveling wave problem, Eq. (1). He follows with an illustration that shows how a sinusoidal solution satisfying Eq. (1) would change in time. He then describes the motion as follows: "It is evident that in the neighborhood of the points *a*, *c* [the compression side] the curve becomes more and more steep as *t* increases, while in the neighborhood of the points *o*, *b*, *z* [the expansion side] its inclination becomes more and more gentle." He continues by finding the rate of change of the tangent to the velocity curve,

$$\frac{\frac{dw}{dz}}{1+\frac{dw}{dz}t}, \tag{A}$$

and remarks that:

"At those points of the original curve at which the tangent is horizontal, $dw/dz = 0$, and therefore the tangent will constantly remain horizontal at the corresponding points of the altered curve. For the points for which $dw/dz$ is positive, the denominator of the expression (A) increases with $t$, and therefore the inclination of the curve continually decreases. But when $dw/dz$ is negative, the denominator of (A) decreases as $t$ increases, so that the curve becomes steeper and steeper. At last, for sufficiently large values of $t$, the denominator of (A) becomes infinite[6] for some value of $z$. Now the very formation of the differential equations of motion with which we start, tacitly supposes that we have to deal with finite and continuous functions; and therefore in the case under consideration we must not, without limitation, push our results beyond the least value of $t$ which renders (A) infinite. This value is evidently the reciprocal, taken positively, of the greatest negative value of $dw/dz$; $w$ here, as in the whole of this paragraph, denoting the velocity when $t = 0$." After finding the breakdown-time for Challis' problem ($t = \lambda/2\pi m$), Stokes explains that Challis' paradox occurs because he considered larger values of times. Stokes continues with:

"Of course, after the instant at which the expression (A) becomes infinite, some motion or other will go on, and we might wish to know what the nature of that motion was. Perhaps the most natural supposition to make for trial is, that a surface of discontinuity is formed, in passing across which there is an abrupt change of density and velocity. The existence of such a surface will presently be shown to be possible, on the two suppositions that the pressure is equal in all directions about the same point, and that it varies as the density…even on the supposition of the existence of a surface of discontinuity, it is not possible to satisfy all the conditions of the problem by means of a single function of the form $f\{z-(a+w)t\}$."

Stokes then proceeds to find the jump conditions for mass and momentum:

$$\rho w - \rho' w' = (\rho - \rho')\gamma, \tag{2}$$

$$(\rho w - \rho' w')\gamma - \left(\rho w^2 - \rho' w'^2\right) = a^2(\rho - \rho'), \tag{3}$$

[6] Stokes should have said that the denominator of (A) becomes zero. Equation A represents the new tangent of inclination, when $1+\frac{dw}{dz}t \to 0$ the tangent $\to \infty$.

where $\gamma$ is the speed of the discontinuity. We immediately recognize Eq. (2) as the conservation of mass across a shock wave. However, Eq. (3) does not look familiar. Conceptually Eq. (3) expresses a correct balance of momentum across a shock wave. The reason it does not look familiar is that Stokes has replaced the right hand side term which corresponds to the pressure difference across the shock with the density difference using the prevalent Newtonian theory of sound: Boyle's law and constant speed of sound (isothermal flow). Stokes then writes: "The equations (2), (3) being satisfied, it appears that the discontinuous motion is dynamically possible. This result, however, is so strange, that it may be well to consider more in detail the simplest possible case of such a motion". After considering in detail the motion of a shock moving with a constant velocity $\gamma$ and finding no contradictions Stokes writes: "The strange results at which I have arrived appear to be fairly deducible from the two hypotheses already mentioned. It does not follow that the discontinuous motion considered can ever take place in nature[7], for we have all along been reasoning on an ideal elastic fluid which does not exist in nature." He then discusses the effects of heat addition and friction, concluding that: "It appears, then, almost certain that the internal friction would effectively prevent the formation of a surface of discontinuity, and even render the motion continuous again if it were for an instant discontinuous". The following year, Stokes was appointed Lucasian Professor of Mathematics at Cambridge. He retained this chair until his death in 1903. Stokes' note on Challis' paradox goes unnoticed for many years.

Lord Rayleigh (1842-1919) entered Cambridge as a student in 1861. There he was a student of the mathematician E. J. Routh[8]. As an undergraduate, Rayleigh attended the lectures of Stokes and was inspired by Stokes' approach which combined experimental and theoretical methods. Rayleigh's first volume of *The Theory of Sound* was published the same year of the following letter to Stokes. The letter is in response to a conversation Rayleigh had with Stokes concerning Challis' paradox [33]:

*4 Carlton Gardens, S. W.*
*June 2/77*

*Dear Prof. Stokes,*

*In consequence of our conversation the other evening I have been looking at you paper "On a difficulty in the theory of sound", Phil. Mag. Nov. 1848. The latter half of the paper appears to me be liable to an objection, as to which (if you have time to look at the matter) I should be glad to hear your opinion.*

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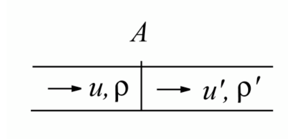
[7] Stokes is echoing the popular adage originally due to the Latin philosopher Lucretius Caro: *Natura non facit saltus* (nature makes no jumps). Leibniz expressed the same thought in his *Nouveaux essays sur l'entendement humain* (1705): "C'est une de mes grandes maximes et des plus vérifiées, que la nature ne fait jamais des sauts", and Darwin quotes it seven times in his *Origin of Species* (1859).

[8] Routh is best known for the Routh-Hurwitz theorem which can be used to establish if a polynomial is stable. Here stable is in the sense that the roots lie to the right of the imaginary axis.




*By impressing a suitable velocity on all the fluid the surface of separation at A may be reduced to rest. When this is done, let the velocities and densities on the two sides be* $u, \rho, u', \rho'$. *Then by continuity*

$$u\rho = u'\rho'.$$

*The momentum leaving a slice including A in unit time =* $\rho u \cdot u'$, *momentum entering =* $\rho u^2$.

*Thus*[9] $p - p' = a^2(\rho - \rho') = \rho u(u' - u)$.

*From these two equations*

$$u = a\sqrt{\frac{\rho'}{\rho}}, \quad u' = a\sqrt{\frac{\rho}{\rho'}}.$$

*This, I think, is your argument, and you infer that the motion is possible. But the energy condition imposes on u and* $u'$ *a different relation, viz.*

$$u'^2 - u^2 = 2a^2 \log\frac{\rho}{\rho'},$$

*so that energy is lost or gained at the surface of separation A.*

*It would appear therefore that on the hypotheses made, no discontinuous change is possible.*

*I have put the matter very shortly, but I dare say what I have said will be intelligible to you.*

In order to follow Rayleigh's argument, consider the energy balance across the shock:

$$\rho'\left(\tfrac{1}{2}u'^2 + e'\right)\tilde{u}' - \rho\left(\tfrac{1}{2}u^2 + e\right)\tilde{u} = pu - p'u',$$

where $\tilde{u}$ is the velocity relative to the shock wave and $e$ is the internal energy. A change in internal energy is given by

$$de = TdS + \frac{p}{\rho}\frac{d\rho}{\rho},$$

[9] The second equal sign was replaced by a minus sign in [33].

where $T$ is the temperature and $S$ is the entropy. If, as assumed by Rayleigh, the shock is stationary and the flow is reversible and it obeys Boyle's law, then

$$u'^2 - u^2 = 2(e - e') = 2\int_{\rho'}^{\rho} a^2 \frac{d\rho}{\rho} = 2a^2 \log\frac{\rho}{\rho'}.$$

Stokes replies to Rayleigh [33]:

*Cambridge,*
*5th June, 1877.*

*Dear Lord Rayleigh,*

*Thank you for pointing out the objections to the queer kind of motion I contemplated in the paper you refer to. Sir W. Thomson[10] pointed the same out to me many years ago, and I should have mentioned it if I had had occasion to write anything bearing on the subject, or if, with out that, my paper had attracted attention. It seemed, however, hardly worth while to write a criticism on a passage in a paper which was buried among other scientific antiquities.*

*P.S. You will observe I wrote somewhat doubtfully about the possibility of the queer motion.*

It is apparent that Stokes doesn't have the determination or confidence in his position to defend the convincing case he had presented in 1848.

Years latter, in a letter to W. Thomson dated October 15, 1880 [35], Stokes tells Thomson that he is reviewing his paper "On a Difficulty in the Theory of Sound" for inclusion in his collected works. Stokes reminds Thomson that both he and Rayleigh had pointed out years earlier that his analysis violated the principle of conservation of energy. Then Stokes argues: "The conservation of energy gived another relation, which can be satisfied, so that it appears that such a motion is possible." Two weeks later, on November 1 [35], Stokes changes his mind and writes to Thomson: "On further reflection I see that I was wrong, and that a surface of discontinuity in crossing which the density of the gas changes abruptly is impossible." In this letter, Stokes gives no details for this conclusion. The details come two days later [35] when he writes: "I mentioned to you I think in a letter that I had found that my surface of discontinuity was bosh. In fact, the equation of energy applied to a slice infinitely near the surface of discontinuity leads to

$$\int_{\rho'}^{\rho} \frac{p}{\rho^2} d\rho = \tfrac{1}{2}\left(w'^2 - w^2\right),$$

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[10] W. Thomson (1824-1907) was named Lord Kelvin in 1866.

where $\rho, \rho'$ are the densities, and $w, w'$ the velocities on the two sides of the surface of discontinuity. But this equation is absurd, as violating the second law of motion. In this way the existence of a surface of discontinuity is proved to be impossible." On December 4, Stokes again writes to Thomson [35]: "I have cut out the part of the paper which related to the formation of a surface of discontinuity…The equation I sent you was wrong, as I omitted the considerations of the work of the pressures at the two ends of the elementary portion…" And thus, in the volume of his collected works of 1883 [31], Stokes adds the following footnote to his 1848 paper at the point where he said that such a surface was possible: "Not so: see substituted paragraph at the end", and removes the entire section dealing with the description of the discontinuity. After claiming that he had made a mistake by considering only conservation of mass and momentum, he says: "It was however pointed out to me by Sir William Thomson, and afterwards by Lord Rayleigh, that the discontinuous motion supposed above involves a violation of the principle of the conservation of energy"[11].

The difficulties that these prominent scientists were having originated from, as Truesdell has succinctly put it, "the insufficiency of thermodynamics as it was then (and often still now is) understood" [33]. Through much of the first half of the nineteenth century, particularly in Great Britain, the Newtonian theory of sound, based on Boyle's law, $p \propto \rho$, constant speed of sound and isothermal conditions was accepted, even though it clearly contradicted experimental observations. For a reenactment of the painful birth of thermodynamics in the nineteenth century read Truesdell's play in five acts [32].

The first step in improving the thermodynamics was taken by William Rankine (1820-1872). Rankine attended Edinburgh University for two years and left without a degree to practice engineering. Around 1848, Rankine started developing theories on the behavior of matter, particularly a theory of heat. In 1851, before taking the chair of civil engineering and mechanics at the University of Glasgow, Rankine writes: "Now the velocity with which a disturbance of density is propagated is proportional to the square root, not of the total pressure divided by the total density, but of the variation of pressure divided by the variation of density…"[12] [22].

Samuel Earnshaw (1805-1888) studied at St. John's College, Cambridge and later became a cleric and tutor of mathematics and physics. In 1860 Earnshaw submitted for publication to the Philosophical Transactions of the Royal Society a paper on the theory of sound of finite amplitudes [6]. Stokes, in his capacity of secretary of the Royal Society, asked Thomson, in a letter dated April 28, 1859 [34], to review the paper. Thomson's review stretches through seven letters to Stokes, from May 11, 1859 to June 20, 1860 [34]. Thomson is decidedly against publication. In the paper, Earnshaw develops a simple wave solution in one direction for gases satisfying an arbitrary relation between pressure and density. Earnshaw works with the Lagrangian formulation of the

[11] What is so odd here is that by 1880 both Thomson and Stokes were familiar with Rankine's paper of 1870, yet failed to understand its significance in relation to Stokes' 1848 paper.
[12] One step forward, two steps back, see footnote 4.

equations to find the relation between the velocity and the density for $p \propto \rho^{\gamma}$. Like others before him, he observes that the differential equations might not have a unique solution. He remarks: "I have defined a bore to be a tendency to discontinuity of pressure; and it has been shown that as a wave progresses such a tendency necessarily arises. As, however, discontinuity of pressure is a physical impossibility, it is certain Nature has a way of avoiding its actual occurrence". Thomson has trouble with Earnshaw's Lagrangean formulation and feels that his "aerial bore" is a rehash of Stokes' paper on "A Difficulty in the Theory of Sound". In his last letter to Stokes on this subject Thomson writes [34]: "On the whole I think if called on to vote, it would be against the publication… On speaking to Rankine I found *the* idea he had taken from Earnshaw's paper,… was superposition of transm$^{n}$ vel. On wind vel.: & he thought it good. This however is of course fully expressed in Poisson's solution."

Bernhard Riemann (1826-1866) received his Ph.D. from the University of Göttingen in 1851. After Dirichlet died in 1857, vacating the chair previously held by Gauss, Riemann became a full professor. Most of Riemann's papers were in pure mathematics and differential geometry and they have been extremely important to theoretical physics. Unique among his contributions is his more applied paper on the propagation of sound waves of finite amplitude published in 1860 [25]. The paper is very easy to read with notation very similar to that used today. Early in the paper, Riemann introduces what we know today as Riemann variables which he denotes *r* and *s*. For an isentropic gas he writes the governing equations as

$$\sqrt{\varphi'(\rho)} + u = \frac{k+1}{2} r + \frac{k-3}{2} s,$$
$$\sqrt{\varphi'(\rho)} - u = \frac{k-3}{2} r + \frac{k+1}{2} s,$$

where *k* is the ratio of specific heats and $\varphi' = dp/d\rho = a^2$. Shortly after introducing *r* and *s* Riemann describes how a compression wave would necessarily steepen leading to multiple vales of $\rho$ at one point. Then he says: "Now since in reality this cannot occur, then a circumstance would have to occur where this law will be invalid..., and from this moment on a discontinuity occurs… so that a larger value of $\rho$ will directly follow a smaller one… The compression waves [Verdichtungswellen], that is, the portions of the wave where the density decreases in the direction of propagation, will accordingly become increasingly more narrow as it progresses, and finally go over into compression shocks [Verdichtungsstösse]". He derives the jumps in mass and momentum for an isentropic (reversible) flow and establishes that the speed of the shock wave, $d\xi/dt$, is bounded by

$$u_1 + \sqrt{\varphi'(\rho_1)} > \frac{d\xi}{dt} > u_2 + \sqrt{\varphi'(\rho_2)}\,.$$

Then he discusses what we know today as the Riemann problem, i.e. the wave patterns corresponding to various initial conditions with jumps in *u* and $\rho$ at $x = 0$. Riemann, like

Stokes before him, failed to understand the true nature of the shock layer. The problem is one of physics, not mathematics, and its solution must wait for a better understanding of thermodynamics.

**Fig. 0.** William Rankine.

**Resolution**

Rankine makes his main contribution in his 1870 paper on the thermodynamic theory of waves [23] published in the Philosophical Transactions of the Royal Society of London. Previous papers by Earnshaw and Riemann shared some similarities; not so with Rankine's paper which was more focused on thermodynamics[13]. He begins with: "The object of the present investigation is to determine the relations which must exist between the laws of the elasticity of any substance, whether gaseous, liquid, or solid, and those of the wave-like propagation of a finite longitudinal disturbance in that substance." Here "elasticity" is the eighteenth-century term for what we now call pressure. Later he writes: "It is to be observed, in the first place, that no substance yet known fulfills the condition expressed by the equation $\frac{dp}{ds} = -m^2 = \text{constant}$, [14] between finite limits of disturbance, at a constant temperature, nor in a permanency of type may be possible in a wave of longitudinal disturbance, there must be both change of temperature and conduction of heat during the disturbance". Therefore, Rankine by explaining that the shock transition is a non-adiabatic process, where the particles exchange heat with each other, but no heat is received from the outside, resolved the objections that had been raised by Rayleigh and others concerning the conservation of energy. He goes on to find, for a perfect gas, the jump conditions for a shock wave moving with speed *a* into an undisturbed medium with pressure and specific volume defined respectively by *P* and *S*. He writes:

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[13] In papers written in 1850 and 1851, Rankine developed a theory of thermodynamics which included an entropy function [32], but it will take another 15 years for R. Clausius to coin the term and fully develop the concept.

[14] Rankine denotes by *s* the "bulkiness" $= 1/\rho$, and by *m* the "mass velocity" $= \rho\tilde{u}$, where $\tilde{u}$ is the velocity relative to the shock wave. In its discrete form, we call this expression Prandtl's relation: $[p]/[s] = -m^2$.

$$m^2 = \frac{1}{S}\left\{(\gamma+1)\frac{p}{2} + (\gamma-1)\frac{P}{2}\right\},$$

$$a^2 = S\left\{(\gamma+1)\frac{p}{2} + (\gamma-1)\frac{P}{2}\right\},$$

$$u = (p-P)\sqrt{\frac{S}{(\gamma+1)\frac{p}{2} + (\gamma-1)\frac{P}{2}}}.$$

Thomson in a letter to Stokes dated March 7, 1870 [35] writes: "I have read Rankine's paper with great interest. The simple elementary method by which he investigates the condition for sustained uniformity of type is in my opinion very valuable. It ought as soon as it is published to be introduced into every elementary book henceforth written on the subject."

Pierre-Henri Hugoniot (1851-1887) entered the École Polytechnique in 1870. That summer, France declared war on Germany[15] and patriotic feelings ran high among the students, see Fig. 0. In 1872 Hugoniot entered the marine artillery service. The artillery service would turn Hugoniot's attention to research on the flight characteristics of projectiles. In 1879 he was appointed professor of mechanics and ballistics at the Lorient Artillery School and three years later he became assistant director of the Central


**Fig. 0** Hugoniot with classmates from the École Polytechnique, 1870. Hugoniot is second from left, front row, see insert. Photo courtesy of Professor Jean-Francois Gouyet, École Polytechnique.

[15] The Franco-Prussian war lasted from July 19, 1870 to May 10, 1871.

Laboratory of Marine Artillery. He returned to the École as an auxiliary assistant in mechanics in 1884.

Here, in the course of a few months, he completed his memoir "On the Propagation of Motion in Bodies" which he submitted for publication on October of 1885. The publication was delayed because as the editor explains: "…The author, carried off before his time, was unable to make the necessary changes and additions to his original text…" The memoir appears in two parts. The first is published in 1887 [11], it consists of three chapters. The first begins with an exposition of the theory of characteristic curves for partial differential equations of which Hugoniot says: "The theories set out herein are not entirely new; however, they are currently being expounded in the works of Monge and Ampère and have not, to my knowledge, been brought together to form a body of policy". In the second chapter he sets down the equations of motion for a perfect gas and in the third chapter he discusses the motion in gases in the absence of discontinuities. It is the second memoir, published in 1889 [12], that is most interesting. Chapter four covers the motion of a non-conducting fluid in the absence of external forces, friction and viscosity. Here Hugoniot analysis is very similar to that of Earnshaw. Finally, chapter five "examine[s] the phenomena which occur when discontinuities are introduced into the motion". It is in this last chapter that Hugoniot writes the famous *Hugoniot-equation* relating the internal energy to the kinetic energy. It appears in §150, not in the usually quoted form,

$$e' - e = \tfrac{1}{2}(p + p')(v' - v), \tag{4}$$

but as

$$\frac{p + p_1}{2} = \frac{p_1 - p}{m - 1}\frac{1}{z_1 - z} + \frac{p_1 z_1 - pz}{m - 1}\frac{1}{z_1 - z}, \tag{5}$$

where $m = \gamma$, the ratio of specific heats, $v = z + 1$, and $e = pv/(\gamma - 1)$. Of course, (4) follows easily from (5), but the elegant way in which (4) connects the three thermodynamic variables, $p, v$ and $e$, is lost in (5). Equation (4) states that the increase in internal energy across a shock is due to the work done by the mean pressure in compressing the flow by an amount $v' - v$. Fig. 0 shows the three *p-v* relations used by various authors for $\gamma = 1.4$. Boyle's law and Poisson's isentrope are constitutive relations, while the Hugoniot curve establishes what states are possible across a shock wave.

Later, in §155, Hugoniot explains that in the absence of viscosity and heat conduction the conservation of energy implies that $p/\rho^m = \text{constant}$, but that across a shock this relation is no longer valid and is replaced by

$$p_1 = p\frac{(m+1)\frac{\rho_1}{\rho} - (m-1)}{(m+1) - (m-1)\frac{\rho_1}{\rho}}.$$

It is Hadamard's *Lectures on the Propagation of Waves* [9] that brings Hugoniot's work to the attention of the Cambridge community. In the preface to his *Lectures*, Hadamard explains that Chapters I through IV were prepared during the years 1898 to 1899, but the publication was delayed. He also acknowledges his friend Pierre M. Duhem[16] for pointing out the theory of Hugoniot: "Dans le cas des gaz, on est, au contraire, conduit à la théorie d'Hugoniot, sur laquelle l'attention a été attire depuis quelques années, grace aux *leçons d'Hydrodynamique, Elasticitè et Acoustique* de M. Duhem". However, Duhem's lectures [5] deal primarily with Hugoniot's treatment of waves of small amplitude, see Fig. 0, not discontinuities. Hadamard writes the *Hugoniot-equation* in Chapter IV, § 209, of his *Lectures*:

$$\frac{(p_1 + p_2)(\omega_2 - \omega_1)}{2} = \frac{1}{m-1}(p_1\omega_1 - p_2\omega_2),$$

here $\omega = v$, and he attributes it to Hugoniot: "Telle est la relation qu'Hugoniot a substituée à (66) pour exprimer que la condensation ou dilatation brusque se fait sans absorption ni dégagement de chaleur. On lui donne actuellement le nom de *loi adiabatique dynamique,* la relation (66), qui convient aux changements lents, étant designée sous le nom de *loi adiabatique statique".* The *adiabatique statique*, equation (66), that Hadamard mentions is, of course, Poisson's isentrope.

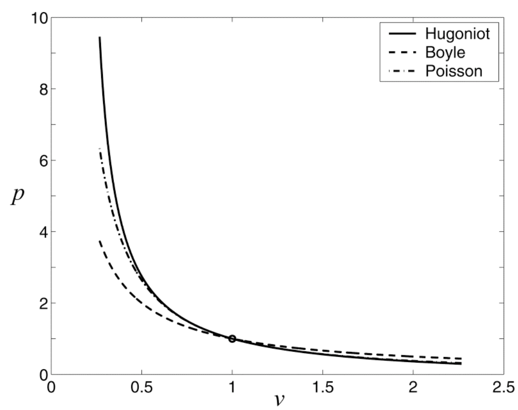


**Fig. 0.** *p-v* relations.

## Acceptance

[16] When Jacques Hadamard (1865-1963) entered the École Normale Supérieure in 1885, Pierre M. Duhem (1861-1916) was a third year student there and the two became close friends. Duhem is well known for his work on themodynamics and history and philosophy of science.

By 1910, all the principal players, Stokes, Earnshaw, Riemann, Rankine and Hugoniot, had passed away. Thus the review article by Rayleigh, "Aerial Plane Waves of Finite Amplitude" [24], is intended for a new generation of scientists. Rayleigh divides the review into two main parts: "Waves of Finite Amplitude without Dissipation" and "Permanent Regime under the Influence of Dissipative Forces". The first part, aside from a review of the work of Earnshaw and Riemann, is a rehash of his letter to Stokes of 1877. Once again he states: "… I fail to understand how a loss of energy can be admitted in a motion which is supposed to be subject to the isothermal or adiabatic laws, in which no dissipative action is contemplated". In the second part of the paper, he reviews Rankine's 1870 paper calling it "very remarkable…although there are one or two serious deficiencies, not to say errors…" He also reviews Hugoniot's 1887-1889 memoirs, thus: "The most original part of Hugoniot's work has been supposed to be his treatment of discontinuous waves involving a sudden change of pressure, with respect to which he formulated a law often called after his name by French writers. But a little examination reveals that this law is *precisely the same* as that given 15 years earlier by Rankine, a fact which is the more surprising inasmuch as the two authors start from quite different points of view". Of the Hugoniot-curve he says: "…however valid [it] may be, its fulfillment does not secure that the wave so defined is possible. As a matter of fact, a whole class of such waves is certainly impossible, and I would maintain, further, that a wave of the kind is never possible under the conditions, laid down by Hugoniot, of no viscosity or heat-conduction". Rayleigh makes two small contributions in the article. He shows that the increase in $\int dQ/\theta$ (the entropy) across the shock, for weak shocks, is of the order of the 3$^{rd}$ power of the pressure jump, and he estimates that the shock wave thickness for air under ordinary conditions is of the order of $\frac{1}{3}\times10^{-5}$ cm.

We can conclude that our understanding of shock waves was hampered by three factors: first, a lack of understanding of what is an admissible solution to a partial differential equation; second, the incomplete knowledge of thermodynamics at the times; and third, as is evident in Rayleigh's paper, the lack of understanding that the shock wave manifested itself in the inviscid equations as a singular limit[17] of the viscous, heat conducting, Navier-Stokes equations. As a postscript, consider how Lamb perpetuated the folly.

Horace Lamb (1849-1934) was student of Stokes and Maxwell at Cambridge. Lamb was a prolific writer who authored many books in fluid mechanics, mathematics, and classical physics. His texts were used in British universities for many years. In his Presidential address to the British Association in 1904, he provided the following insight into his writings: "It is ... essential that from time to time someone should come forward to sort out and arrange the accumulated material, rejecting what has proved unimportant, and welding the rest into a connected system". His acclaimed[18] book "Hydrodynamics"

[17] The conceptual leap needed is to see the inviscid shock jumps as the outer limits of the viscous shock layer as viscosity vanishes. Curiously, Stokes made a significant contribution to asymptotic theory with what we call today Stokes phenomenon [30], see [7] for an overview.

[18] "The leading treatise on classical hydrodynamics", *Mathematical Gazette*; "Difficult to find a writer on any mathematical topic with equal clearness and lucidity", *Philosophical Magazine*; "…it has become

[14], based on his brief 250 page "Treatise on the Mathematical Theory of the Motion of Fluids" of 1879, was first published in 1895 and was then revised and expanded until the current 700 page 6th edition of 1932. In it Lamb discusses the conditions for a discontinuous wave in §284. Lamb mentions the works of Rankine [23] and Hugoniot, as described by Hadamard [9], but he sides with the Stokes-Rayleigh way of thinking: "These results are [the jump conditions for mass and momentum], however, open to the criticism that in actual fluids the equation of energy cannot be satisfied consistently with (1) [mass conservation] and (2) [momentum conservation]". Of Hugoniot's result he says in a footnote: "…the argument given in the text [referring to Hadamard's book [9]] is inverted. The possibility of a wave of discontinuity being *assumed*, it is pointed out that the equation of energy will be satisfied if we equate expression (10) [ $\frac{1}{2}(p_1 + p_0)(u_0 - u_1)$ ] to the increment of intrinsic energy. On this ground the formula

$$\frac{1}{2}(p_1 + p_0)(v_0 - v_1) = \frac{1}{\gamma - 1}(p_1 v_1 - p_0 v_0)$$

is propounded, as governing the transition from one state to the other… But no physical evidence is adduced in support of the proposed law".

Today, the Cambridge legacy continues to resonate with Hawking, the current Lucasian Chair holder, who writes: "It seems to be a good principle that the prediction of a singularity by a physical theory indicates that the theory has broken down, i.e. it no longer provides a correct description of observations" [10].

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the foundation on which nearly all subsequent workers in hydrodynamics have built. The long-continued supremacy of this book in a field where much development has been taking place is very remarkable, and is evidence of the complete mastery which its author retained over his subject throughout his life", *G. I. Taylor's eulogy to Lamb, Nature, 1934*.

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# Chapitre IX.

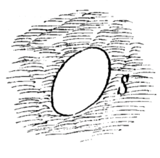
## Propagation d'un petit Mouvement dans un autre.

## Méthode d'Hugoniot.

### §.1. Quelques définitions.

Imaginons un fluide qui, jusqu'à l'instant $t = t_0$,

Fig. 21.




est soumis à certaines conditions aux limites parfaitement déterminées. C'est, par exemple, un fluide indéfini assujetti à s'appuyer sur la partie extérieure d'une surface immobile $S$ (fig 21).

Ce fluide est en mouvement. Ce mouvement est représenté par les équations analytiques

$$(1)\ldots\ldots\ldots\ldots \begin{cases} u = U(x, y, z, t) \\ v = V(x, y, z, t) \\ w = W(x, y, z, t) \\ \pi = P(x, y, z, t) \\ \rho = R(x, y, z, t) \end{cases}$$

obtenues en intégrant les équations aux dérivées partielles du mouvement et en tenant compte des conditions aux limites qui sont imposées au fluide jusqu'à l'instant $t = t_0$

Imaginons maintenant qu'à partir de l'instant $t = t_0$ les conditions aux limites auxquelles le fluide était assujetti soient remplacées par des conditions analytiquement différentes. Par exemple, la surface $S$ sur laquelle ce fluide doit s'appuyer, au lieu de demeurer immobile, se met en mouvement suivant une certaine loi.

Pour les valeurs de $t$ supérieures à $t_0$, le mouvement du fluide ne pourra plus, en général, être représenté par les équations (1) où les cinq fonctions $U, V, W, P, R$, sont cinq fonctions analytiques

**Fig. 0.** Opening page to Chapter IX of Duhem's lectures on hydrodynamics. Here he presents Hugoniot's Method for the treatment of waves of small amplitude.

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