Hard and Soft Constraints in Reliability-Based Design Optimization

L. G. Crespo
National Institute of Aerospace, Hampton, VA, USA

D. P. Giesy
NASA Langley Research Center, Hampton, VA, USA

S. P. Kenny
NASA Langley Research Center, Hampton, VA, USA

ABSTRACT: This paper proposes a framework for the analysis and design optimization of models subject to parametric uncertainty where design requirements in the form of inequality constraints are present. Emphasis is given to uncertainty models prescribed by norm bounded perturbations from a nominal parameter value and by sets of componentwise bounded uncertain variables. These models, which often arise in engineering problems, allow for a sharp mathematical manipulation. Constraints can be implemented in the hard sense, i.e., constraints must be satisfied for all parameter realizations in the uncertainty model, and in the soft sense, i.e., constraints can be violated by some realizations of the uncertain parameter. In regard to hard constraints, this methodology allows (i) to determine if a hard constraint can be satisfied for a given uncertainty model and constraint structure, (ii) to generate conclusive, formally verifiable reliability assessments that allow for unprejudiced comparisons of competing design alternatives and (iii) to identify the critical combination of uncertain parameters leading to constraint violations. In regard to soft constraints, the methodology allows the designer (i) to use probabilistic uncertainty models, (ii) to calculate upper bounds to the probability of constraint violation, and (iii) to efficiently estimate failure probabilities via a hybrid method. This method integrates the upper bounds, for which closed form expressions are derived, along with conditional sampling. In addition, an ℓ_∞ formulation for the efficient manipulation of hyper-rectangular sets is also proposed.

1 INTRODUCTION

Design under uncertainty arises in numerous disciplines including engineering, economics, finance and management. Achieving balance between robustness and performance is one of the fundamental challenges faced by scientists and engineers.

Literature in probabilistic control (Crespo and Kenny 2005), stochastic programming (Kall and Wallace 1994) and stochastic approximations (Ermoliev 1983) provide several mathematical tools for Reliability-Based Design Optimization (RBDO). The algorithms at disposal can be classified according to the way they integrate inequality constraints. *Hard constraints* (Darlington et al. 1999) are those that must be satisfied for all possible realizations of the uncertain parameter. Strategies to solve the resulting semi-infinite optimization problem usually require nested searches for the identification of the worst-parameter-realization (Darlington et al. 1999). On the other hand, *Soft constraints* (Acevedo and Pistikopoulos 1998; Samsatli et al. 1998; Rustem and Nguyen 1998; Royset et al. 2001; Crespo and Kenny 2005) are those that are not required to be satisfied for all possible realizations of the uncertain parameter. They result from slack design requirements, overly large uncertainty sets or limiting design structures. Chance-constrained programming (Kall and Wallace 1994), sampling-based techniques, asymptotic approximations (Rackwitz 2001) and penalty-based optimization (Rustem and Nguyen 1998; Kall and Wallace 1994) are some of the strategies most commonly used to tackle this problem.

This paper proposes strategies for robustness analysis and RBDO of problems subject to hard and/or soft constraints. A unifying mathematical framework is first introduced.

2 FRAMEWORK

Definition 1 (Uncertainty Model). *If $\boldsymbol{p} \in \mathbb{R}^{\dim(\boldsymbol{p})}$ is an uncertain parameter, its* uncertainty model *is given by the* support $\Delta_{\boldsymbol{p}} \subset \mathbb{R}^{\dim(\boldsymbol{p})}$*, and a* designated point $\bar{\boldsymbol{p}} \in \Delta_{\boldsymbol{p}}$*.*

The uncertainty model is specified by the analyst/designer. The intent is that the support set $\Delta_{\boldsymbol{p}}$ be chosen so that the actual value of the parameter $\boldsymbol{p}$ lies somewhere within it. A *realization* of the uncertain parameter is a value of the parameter selected from the support set. The designated point, whose selection is subjective, can be interpreted as the realization that best represents $\boldsymbol{p}$.

Consider now the situation that a system depends on the uncertain parameter $\boldsymbol{p} \in \mathbb{R}^{\dim(\boldsymbol{p})}$ and the design variable $\boldsymbol{d} \in \mathbb{R}^{\dim(\boldsymbol{d})}$. Suppose that $\boldsymbol{g} : \mathbb{R}^{\dim(\boldsymbol{p})} \times \mathbb{R}^{\dim(\boldsymbol{d})} \longrightarrow \mathbb{R}^{\dim(\boldsymbol{g})}$ is a set of constraint functions on the system (with the convention that positive values represent constraint violations). If these are considered hard constraints, the system corresponding to given values of $\boldsymbol{d}$ and $\Delta_{\boldsymbol{p}}$ will be judged acceptable if $\boldsymbol{g}(\boldsymbol{p},\boldsymbol{d}) \leq \boldsymbol{0}\ \forall \boldsymbol{p} \in \Delta_{\boldsymbol{p}}$.

For a given vector $\boldsymbol{g}$ of constraints and a given design $\boldsymbol{d}$, denote the failure (or constraint violation) set by $\mathcal{F}(\boldsymbol{d},\boldsymbol{g}) = \bigcup_{i=1}^{\dim(\boldsymbol{g})} \mathcal{F}_i(\boldsymbol{d},\boldsymbol{g})$ where $\mathcal{F}_i(\boldsymbol{d},\boldsymbol{g}) = \{\boldsymbol{p} : \boldsymbol{g}_i(\boldsymbol{p},\boldsymbol{d}) > 0\}$ is the set of $\boldsymbol{p}$-values where constraint number i is violated. In a context where $\boldsymbol{g}$ and $\boldsymbol{d}$ are understood, $\mathcal{F}(\boldsymbol{d},\boldsymbol{g})$ and $\mathcal{F}_i(\boldsymbol{d},\boldsymbol{g})$ will be represented simply as $\mathcal{F}$ and $\mathcal{F}_i$. Using the symbol ∂ to represent the topological boundary of a set, if the constraint functions are continuous, each $\boldsymbol{p} \in \partial\mathcal{F}_i$ satisfies $\boldsymbol{g}_i(\boldsymbol{p},\boldsymbol{d}) = 0$ and each $\boldsymbol{p} \in \partial\mathcal{F}$ satisfies $\boldsymbol{g}_j(\boldsymbol{p},\boldsymbol{d}) = 0$ for some j [1].

The manifold $\partial\mathcal{F}$ commonly referred as the *limit state surface*, separates the parameter space into two regions: one where all the constraints are satisfied and one where they are not. For a fixed design $\boldsymbol{d}$, the failure region $\mathcal{F}$ either overlaps the support set $\Delta_{\boldsymbol{p}}$, case in which the design $\boldsymbol{d}$ is called *non-robust*; or $\mathcal{F}$ and $\Delta_{\boldsymbol{p}}$ are disjoint, case in which the design $\boldsymbol{d}$ is called *robust*. In the latter case, the degree of robustness can be quantified by measuring the separation between the two sets.

Selecting the support set usually involves some engineering judgment. One reasonable choice might be to confine each component of $\boldsymbol{p}$ to a bounded interval. This leads to the choice of $\Delta_{\boldsymbol{p}}$ as a hyper-rectangle. A natural choice for the designated point $\bar{\boldsymbol{p}}$ is the geometric center of $\Delta_{\boldsymbol{p}}$. If $\boldsymbol{m}$ is the vector of half-lengths of the sides of the hyper-rectangle, the rectangle is represented by the notation $\mathcal{R}_{\boldsymbol{p}}(\bar{\boldsymbol{p}},\boldsymbol{m})$, and defined by $\mathcal{R}_{\boldsymbol{p}}(\bar{\boldsymbol{p}},\boldsymbol{m}) = \{\boldsymbol{p} : \boldsymbol{p}_i \in [\bar{\boldsymbol{p}}_i - \boldsymbol{m}_i, \bar{\boldsymbol{p}}_i + \boldsymbol{m}_i], 1 \leq i \leq \dim(\boldsymbol{p})\}$. If vector inequalities ($\boldsymbol{a} \leq \boldsymbol{b}$) and vector absolute value ($|\boldsymbol{a}|$) are understood to hold componentwise, this hyper-rectangle is given by $\{\boldsymbol{p} : |\boldsymbol{p} - \bar{\boldsymbol{p}}| \leq \boldsymbol{m}\}$ where $\boldsymbol{m} > 0$.

Another reasonable choice for the support set is a hyper-sphere. The hyper-sphere of radius R centered at $\bar{\boldsymbol{p}}$ will be denoted as $\mathcal{S}_{\boldsymbol{p}}(\bar{\boldsymbol{p}},R)$.

Definition 2 (Homothetic Sets). *Two sets A and B are* homothetic *with respect to the* homothetic center *x at a* similitude ratio *of α if $B = \{b : b = \alpha(a-x) + x, a \in A\}$. Such a B is completely determined by A, x, and α. We use the notation $B = \mathcal{H}(A,x,\alpha)$ to express this relationship.*

$B = \mathcal{H}(A,x,\alpha)$ means that B can be created from A by forming every vector from the fixed point x to each point of A; stretching or shrinking it by a factor of α; and, with the roots of the stretched or shrunk vectors all fixed at x, collecting all the new endpoints to form the set B.

For purposes of this paper, two uncertainty models will be called *proportional* if they have the same designated point and they are homothetic with respect to that designated point as homothetic center. This means that one of the two support sets can be formed from the other by expansion or contraction by some positive factor, which is the similitude ratio, about the common designated point. For instance, the hyper-rectangles $\mathcal{R}_{\boldsymbol{p}}(\bar{\boldsymbol{p}},\boldsymbol{m})$ and $\mathcal{R}_{\boldsymbol{p}}(\bar{\boldsymbol{p}},\alpha\boldsymbol{m})$ are proportional sets for $\alpha > 0$.

The notions of *Critical Parameter Value* (CPV) and *Parametric Safety Margin* (PSM) are now introduced. For simplicity sake, the presentation of the material will concentrate on the case where the designated point is in the non-failure region. Intuitively, one imagines that a set proportional to the support set of the uncertainty model is being expanded homothetically with respect to its designated point until its boundary just touches the failure region. The point where the expanding set touches the infeasible region is the CPV. The PSM is a metric that quantifies the size of the set proportional to the support set that has the CPV on its surface. The mathematical background for these notions is presented next.

Definition 3 (Critical Similitude Ratio). *Let $\boldsymbol{d}$ be a fixed design and let $\Delta_{\boldsymbol{p}}$ and $\bar{\boldsymbol{p}}$ be the support set and designated point of the uncertainty model of $\boldsymbol{p}$. Suppose that $\boldsymbol{g} : \mathbb{R}^{\dim(\boldsymbol{p})} \times \mathbb{R}^{\dim(\boldsymbol{d})} \longrightarrow \mathbb{R}^{\dim(\boldsymbol{g})}$ is a set of constraint functions. The set of expansion/contraction factors which produce homothets of $\Delta_{\boldsymbol{p}}$ with respect to $\bar{\boldsymbol{p}}$ which contain constraint violation points is denoted*

$$\tilde{A}(\bar{\boldsymbol{p}},\Delta_{\boldsymbol{p}},\boldsymbol{d},\boldsymbol{g}) \triangleq \left\{\alpha \geq 0 : \mathcal{H}_{\boldsymbol{p}}(\Delta_{\boldsymbol{p}},\bar{\boldsymbol{p}},\alpha) \bigcap \mathcal{F} \neq \emptyset\right\}.$$

[1] The converse is not universally true. We will assume in this paper that the boundary of $\mathcal{F}_i$ is exactly described by the points where that constraint function assumes the value of zero; i.e., $\partial\mathcal{F}_i = \{\boldsymbol{p} : \boldsymbol{g}_i(\boldsymbol{p},\boldsymbol{d}) = 0\}$. We believe that in a realistic engineering setting, the loss of generality in making this assumption is unimportant.

The greatest lower bound of $\tilde{A}(\bar{\boldsymbol{p}}, \Delta_{\boldsymbol{p}}, \boldsymbol{d}, \boldsymbol{g})$*, given by*

$$\tilde{\alpha}(\bar{\boldsymbol{p}}, \Delta_{\boldsymbol{p}}, \boldsymbol{d}, \boldsymbol{g}) \triangleq \inf\left(\tilde{A}(\bar{\boldsymbol{p}}, \Delta_{\boldsymbol{p}}, \boldsymbol{d}, \boldsymbol{g})\right) \tag{1}$$

will be called the Critical Similitude Ratio.

Arguments that can be understood from context will be omitted.

Definition 4 (Critical Parameter Value). *Let* $\boldsymbol{d}$ *be a given design and let* $\Delta_{\boldsymbol{p}}$*,* $\bar{\boldsymbol{p}}$*, and* $\boldsymbol{g}$ *define the uncertainty model and the set of inequality constraints. Any* $\tilde{\boldsymbol{p}} \in \mathcal{H}_{\boldsymbol{p}}(\Delta_{\boldsymbol{p}}, \bar{\boldsymbol{p}}, \tilde{\alpha})$ *on a limit state function, i.e.* $\boldsymbol{g}_i(\tilde{\boldsymbol{p}}, \boldsymbol{d}) = 0$ *for some* i*, will be called the* Critical Parameter Value.

Some properties of the CPV are provided in reference (Crespo, Giesy, and Kenny 2006). Whenever $\Delta_{\boldsymbol{p}}$ is compact and the $\boldsymbol{g}$ are continuous functions, the existence of a CPV is guaranteed. Note also that the CPV might not be a realization of the uncertain parameter, i.e., $\tilde{\boldsymbol{p}}$ might not belong to $\Delta_{\boldsymbol{p}}$.

Formal definitions of the PSM for hyper-spherical and hyper-rectangular supports are provided in Section 3, as are expressions for the calculation of these PSMs. In general, the PSM is a measure of the robustness of the design $\boldsymbol{d}$ to uncertainty in $\boldsymbol{p}$ that will assume non-negative values in the *Feasible Design Space* (FDS). The FDS is the set of designs satisfying all the constraints at the designated point. The PSM is proportional to the degree of resilience of $\boldsymbol{d}$ to uncertainty in $\boldsymbol{p}$. If the PSM assumes the value of zero, there is no resilience since at least one of the constraints is active for $\bar{\boldsymbol{p}}$, i.e. there exists an infinitesimally small perturbation from $\bar{\boldsymbol{p}}$ leading to a constraint violation. Negative PSM values are attained by designs outside the FDS. On the other hand, the PSM is infinite when $\boldsymbol{g}$ is insensitive to $\boldsymbol{p}$. In general, the PSM is uniquely specified by the design point $\boldsymbol{d}$, the support set $\Delta_{\boldsymbol{p}}$, the designated point $\bar{\boldsymbol{p}}$, the constraint functions $\boldsymbol{g}$, and the critical similitude ratio $\tilde{\alpha}$.

Definition 5 (Robust Design Space). *The* Robust Design Space *(RDS) is the set of designs* $\mathbf{d}$ *for which* $\boldsymbol{g}(\boldsymbol{p}, \boldsymbol{d}) \leq \mathbf{0}$ *for all realizations of* $\mathbf{p}$*. Each member of this set is called a* robust design.

In this paper, *robustness* refers to the ability of a given design to satisfy the design requirements prescribed in $\boldsymbol{g} \leq \mathbf{0}$ given an uncertain model for $\boldsymbol{p}$. In principle, such requirements can be enforced by either hard and soft constraints. Developments for the former type are presented next.

3 HARD CONSTRAINTS

Problem Statement: evaluate whether the design $\boldsymbol{d}$ is robust for the support set $\Delta_{\boldsymbol{p}}$ and the hard constraint $\boldsymbol{g}(\boldsymbol{p}, \boldsymbol{d}) \leq \mathbf{0}$.

In general, the support set can have any shape and size. We assume hereafter that $\Delta_{\boldsymbol{p}}$ is a compact set with a designated point $\bar{\boldsymbol{p}}$. A general robustness test is available in (Crespo et al. 2006). Support sets with hyper-spherical and hyper-rectangular support geometries are studied next.

3.1 *Hyper-Spheres in* $\boldsymbol{p}$*-space*

Hyper-spherical supports result from uncertainty models where norm bounded perturbations from the nominal parameter value $\bar{\boldsymbol{p}}$ are allowed. This implies that $\bar{\boldsymbol{p}}$ is the geometric center of the hyper-sphere. Problems with this class of support sets are the simplest since the CPV is calculated by solving a minimum norm problem in $\boldsymbol{p}$-space. The CPV for $\Delta_{\boldsymbol{p}} = \mathcal{S}_{\boldsymbol{p}}(\bar{\boldsymbol{p}}, R)$ is given by

$$\tilde{\boldsymbol{p}} = \underset{\boldsymbol{p}}{\operatorname{argmin}} \left\{ \|\boldsymbol{p} - \bar{\boldsymbol{p}}\| : \prod_{k=1}^{dim(\boldsymbol{g})} \boldsymbol{g}_k(\boldsymbol{p}, \boldsymbol{d}) = 0 \right\}. \tag{2}$$

Definition 6 (Spherical PSM). *The* Spherical Parametric Safety Margin *corresponding to the design* $\mathbf{d}$ *for* $\Delta_{\mathbf{p}} = \mathcal{S}_{\mathbf{p}}(\bar{\mathbf{p}}, R)$ *is defined as*

$$\rho_{\mathcal{S}}(\bar{\boldsymbol{p}}, \tilde{\boldsymbol{p}}, \boldsymbol{d}) \triangleq \|\tilde{\boldsymbol{p}} - \bar{\boldsymbol{p}}\|, \tag{3}$$

where $\tilde{\mathbf{p}}$ *is the corresponding CPV and* $\bar{\mathbf{p}}$*, the designated point of the uncertainty domain, satisfies* $\boldsymbol{g}(\bar{\boldsymbol{p}}, \boldsymbol{d}) \leq \mathbf{0}$*. Note that* $\rho_{\mathcal{S}} = \tilde{\alpha} R$*.*

Lemma 1 (P-Test). *Let* $\boldsymbol{g} \leq \mathbf{0}$ *describe a set of inequality constraints and* $\Delta_{\mathbf{p}} = \mathcal{S}_{\mathbf{p}}(\bar{\boldsymbol{p}}, R)$ *be the support of the uncertain parameter* $\mathbf{p}$*. The design* $\mathbf{d}$ *is robust if and only if* $\rho_{\mathcal{S}} \geq R$*.*

The exact boundary of the RDS corresponding to a hyper-spherical support set is confined by an iso-spherical-PSM manifold, i.e. surface in the design space prescribed by $\rho_{\mathcal{S}} = c$ where c is a constant.

3.2 *Hyper-Rectangles in* $\boldsymbol{p}$*-space*

When the components of the uncertain parameter are independently bounded the support set is a hyper-rectangle. Note that this geometry allows for the manipulation of uncertain parameters having different fidelities. If $\bar{\boldsymbol{p}}$ is the geometric center of the set, such a hyper-rectangle is $\mathcal{R}_{\boldsymbol{p}}(\bar{\boldsymbol{p}}, \boldsymbol{m})$. A robustness test for this geometry is introduced subsequently. The mathematical background for this is presented next.

Definition 7 (Q-Transformation). *Let* $\Delta_{\boldsymbol{p}} = \mathcal{R}_{\boldsymbol{p}}(\bar{\boldsymbol{p}}, \boldsymbol{m})$ *be the support of the uncertain parameter* $\mathbf{p}$*. The* Q-Transformation*, denoted as* $\boldsymbol{q} = Q(\boldsymbol{p}, \bar{\boldsymbol{p}}, \boldsymbol{m})$*, and given by*

$$Q(\boldsymbol{p}, \bar{\boldsymbol{p}}, \boldsymbol{m}) \triangleq \frac{\max\{|\boldsymbol{k}|\}\boldsymbol{k}}{\|\boldsymbol{k}\|}, \tag{4}$$

where
$$\boldsymbol{k} = \mathrm{diag}\{\boldsymbol{m}\}^{-1}(\boldsymbol{p}-\bar{\boldsymbol{p}}),$$
transforms $\Delta_{\boldsymbol{p}}$ into a unit hyper-sphere in $\boldsymbol{q}$-space. The inverse transformation, $\boldsymbol{p} = Q^{-1}(\boldsymbol{q}, \bar{\boldsymbol{p}}, \boldsymbol{m})$, is given by

$$Q^{-1}(\boldsymbol{q}, \bar{\boldsymbol{p}}, \boldsymbol{m}) = \bar{\boldsymbol{p}} + \mathrm{diag}\{\boldsymbol{m}\}\frac{\|\boldsymbol{q}\|\boldsymbol{q}}{\max\{|\boldsymbol{q}|\}}. \qquad (5)$$

The Q-Transformation maps hyper-rectangles proportional to $\mathcal{R}_{\boldsymbol{p}}(\bar{\boldsymbol{p}}, \boldsymbol{m})$ into hyper-spheres proportional to $\mathcal{S}_{\boldsymbol{q}}(\boldsymbol{0}, 1)$. The connectivity of the uncertainty set is preserved by the transformation. Notice however, that differentiable functions on $\boldsymbol{p}$-space are transformed into functions on $\boldsymbol{q}$-space which can have derivative discontinuities at points corresponding to $\boldsymbol{p}$-space points where the faces of the homothets meet. The notation $Q(\boldsymbol{p})$ will be used when the arguments $\bar{\boldsymbol{p}}$ and $\boldsymbol{m}$ are understood by context.

The Q-Transformation allows identification of the corresponding CPV by solving the following minimum norm problem in $\boldsymbol{q}$-space

$$\tilde{\boldsymbol{p}} = \underset{\boldsymbol{p}}{\mathrm{argmin}}\left\{\| \, Q(\boldsymbol{p})\| : \prod_{k=1}^{dim(\boldsymbol{g})} \boldsymbol{g}_k(\boldsymbol{p}, \boldsymbol{d}) = 0\right\}. \qquad (6)$$

Definition 8 (Rectangular PSM). *The* Rectangular Parametric Safety Margin *corresponding to the design $\boldsymbol{d}$ for $\Delta_{\boldsymbol{p}} = \mathcal{R}_{\boldsymbol{p}}(\bar{\boldsymbol{p}}, \boldsymbol{m})$, is defined as*

$$\rho_{\mathcal{R}}(\bar{\boldsymbol{p}}, \tilde{\boldsymbol{p}}, \boldsymbol{m}, \boldsymbol{d}) \stackrel{\Delta}{=} \|Q(\tilde{\boldsymbol{p}})\|\|\boldsymbol{m}\|, \qquad (7)$$

where $\boldsymbol{g}(\bar{\boldsymbol{p}}, \boldsymbol{d}) \leq \boldsymbol{0}$.

$\mathcal{H}_{\boldsymbol{p}}(\mathcal{R}, \bar{\boldsymbol{p}}, \tilde{\alpha})$ is tightly bounded by an hyper-sphere of radius equal to $\rho_{\mathcal{R}}$ centered at $\bar{\boldsymbol{p}}$.

Lemma 2 (Q-Test). *Let $\boldsymbol{g} \leq \boldsymbol{0}$ describe a set of inequality constraints, and $\Delta_{\boldsymbol{p}} = \mathcal{R}_{\boldsymbol{p}}(\bar{\boldsymbol{p}}, \boldsymbol{m})$ be the support of the uncertain parameter. The design $\boldsymbol{d}$ is robust if and only if $\rho_{\mathcal{R}} \geq \|\boldsymbol{m}\|$.*

The CPVs in Equations 2 and 6 are $\boldsymbol{p}$ values, not necessarily realizations in $\Delta_{\boldsymbol{p}}$, on the intersection between the surface of the homothetic set and the failure region. As before, the exact boundary of the RDS for hyper-rectangular support sets is prescribed by an iso-rectangular-PSM manifold. Relevant aspects related to the existence of the RDS are considered in (Crespo, Giesy, and Kenny 2006).

3.2.1 *Infinity Norm Formulation*

An alternate way to search for the CPV for hyper-rectangular supports is presented here. This formulation allows us to circumvent the problems caused by discontinuities in the gradient of the Q transformation.

Recall that the infinity norm in a finite dimensional space is defined as $\|\boldsymbol{x}\|^{\infty} = \sup_i\{|\boldsymbol{x}_i|\}$. Let us define the $\boldsymbol{m}$-scaled infinity norm as $\|\boldsymbol{x}\|^{\infty}_{\boldsymbol{m}} = \sup_i\{|\boldsymbol{x}_i|/m_i\}$. A distance between the vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ can be defined as $\|\boldsymbol{x}-\boldsymbol{y}\|^{\infty}_{\boldsymbol{m}}$. Using this distance, the unit ball centered at $\bar{\boldsymbol{p}}$ is just $\mathcal{R}(\bar{\boldsymbol{p}}, \boldsymbol{m})$.

The problem of finding, for a fixed design $\boldsymbol{d}$, the CPV for the vector $\boldsymbol{g}(\boldsymbol{p}, \boldsymbol{d})$ of constraint functions and the uncertainty model with designated point $\bar{\boldsymbol{p}}$ becomes the problem of finding a vector $\tilde{\boldsymbol{p}}$ of minimal distance in this $\boldsymbol{m}$-scaled infinity norm from $\bar{\boldsymbol{p}}$ such that $\boldsymbol{p}$ touches the failure set. This can be expressed in the form of a constrained optimization problem

$$\tilde{\boldsymbol{p}} = \underset{\boldsymbol{p}}{\mathrm{argmin}}\{\|\boldsymbol{p}-\bar{\boldsymbol{p}}\|^{\infty}_{\boldsymbol{m}} : \boldsymbol{p} \in \partial\mathcal{F}\}$$

This optimization problem is restated as $\tilde{\boldsymbol{p}} = \tilde{\boldsymbol{p}}_i$ where

$$i = \underset{j}{\mathrm{argmin}}\left\{\|\tilde{\boldsymbol{p}}_j - \bar{\boldsymbol{p}}\|^{\infty}_{\boldsymbol{m}}\right\}, \qquad (8)$$

$$\tilde{\boldsymbol{p}}_j = \underset{\boldsymbol{p}}{\mathrm{argmin}}\{\|\boldsymbol{p}-\bar{\boldsymbol{p}}\|^{\infty}_{\boldsymbol{m}} : \boldsymbol{p} \in \partial\mathcal{F}_j\}. \qquad (9)$$

That is, the CPV problem is solved for each individual constraint function, and the answer is selected which is closest to the designated point in the $\boldsymbol{m}$-scaled infinity norm. It should be pointed out that, in the preceding discussion, if the $\boldsymbol{m}$-scaled infinity norm is replaced by the standard Euclidean norm, what results is a formulation of the CPV calculation for spherical uncertainty domains.

Assuming that $\boldsymbol{g}(\bar{\boldsymbol{p}}, \boldsymbol{d}) < 0$ and $\partial\mathcal{F}_i = \{\boldsymbol{p} : \boldsymbol{g}_i(\boldsymbol{p}, \boldsymbol{d}) = 0\}$, the problem in Equation (9) can be formulated as

$$\tilde{\boldsymbol{p}}_j = \underset{\boldsymbol{p}}{\mathrm{argmin}}\{\|\boldsymbol{p}-\bar{\boldsymbol{p}}\|^{\infty}_{\boldsymbol{m}} : \boldsymbol{g}_j(\boldsymbol{p}, \boldsymbol{d}) = 0\}.$$

Rewriting this using the definition of the $\boldsymbol{m}$-scaled infinity norm gives

$$\tilde{\boldsymbol{p}}_j = \underset{\boldsymbol{p}}{\mathrm{argmin}}\left\{\max_{1\leq i\leq \dim(\boldsymbol{p})} \frac{|p_i - \bar{p}_i|}{m_i} : \boldsymbol{g}_j(\boldsymbol{p}, \boldsymbol{d}) = 0\right\}.$$

The "max" can be eliminated and the objective function made differentiable by introducing the similitude ratio α defined before

$$\langle\tilde{\boldsymbol{p}}_j, \tilde{\alpha}_j\rangle = \underset{\boldsymbol{p},\alpha}{\mathrm{argmin}}\{\alpha : \boldsymbol{g}_j(\boldsymbol{p}, \boldsymbol{d}) = 0, |p_i - \bar{p}_i| \leq \alpha m_i\},$$

where $1 \leq i \leq \dim(\boldsymbol{p})$. The non-differentiabilities in the objective functions can be eliminated and the problem turned into an "inequality constraint only" optimization problem which is more "optimizer friendly" by changing the constraint on $\boldsymbol{g}_j$ from $=$ to $\geq$ (assuming that the constraint functions $\boldsymbol{g}$ are continuous) since the optimum must occur on $\partial\mathcal{F}$.

4 SOFT CONSTRAINTS

Up to this point, only hard constraints have been considered. In practice, there might be cases in which the design architecture, i.e., the form in which $\boldsymbol{g}$ depends on $\boldsymbol{d}$, or the size of the support set, lead to an empty RDS. In such cases, soft constraints allow for the search of designs that minimize the severity of the constraint violation. Deterministic (Rustem and Nguyen 1998) and probabilistic (Royset, Kiureghian, and Polak 2001; Crespo and Kenny 2005) approaches can be used to tackle problems where partial feasibility is satisfactory. Non-probabilistic uncertainty models have been used thus far. This has been the case since the enforcement of hard constraint makes the probabilistic information inconsequential. In what follows, probabilistic uncertainty models are used. A probabilistic uncertainty model is fully prescribed by a joint probability density function of $\boldsymbol{p}$, denoted by $f_{\boldsymbol{p}}(\boldsymbol{p})$, supported in $\Delta_{\boldsymbol{p}}$.

Strategies for the efficient solution of RBDO problems that involve the estimation of the failure probability $P[\mathcal{F}(\boldsymbol{d},\boldsymbol{g})]$ are studied in this section. Notice that if the probability of constraint violation can be eliminated, the tools for hard constraints must be used instead. The implementation of soft constraints must be preceded by determining that the RDS is an empty set. Failure to recognize this will lead to ineffective numerical implementations.

By definition, the homothetic sets corresponding to the critical similitude ratio are fully contained in the non-failure region. If a probabilistic uncertainty model for $\boldsymbol{p}$ is available, the probability of being within such a set can be quantified. Obviously, this probability depends, among several other factors, on the geometry of the support set. In order to maximize the probability of being within such a homothet, expansions/contractions in the standard normal space $\boldsymbol{u}$ are desirable. The reader can refer to (Rackwitz 2001) for a review on the transformation from $\boldsymbol{p}$-space to $\boldsymbol{u}$-space. We will use $\boldsymbol{u} = U(\boldsymbol{p})$ to refer to the probability preserving transformation. Most of the concepts introduced earlier can be naturally extended to this setting with one notable exception. Since the transformation of the support set $\Delta_{\boldsymbol{p}}$ covers the entire $\boldsymbol{u}$-space, the concept of proportionality between the homothets and the support set is no longer attainable. This gives us the freedom to select dilating sets of arbitrary geometry in $\boldsymbol{u}$-space. As before, we will concentrate on hyper-spherical and hyper-rectangular geometries. We will denote with $\bar{\boldsymbol{u}}$ the designated point in standard normal space. Analogous to the spherical and rectangular PSMs concepts used for hard constraints, are the spherical- and rectangular-Reliability Indices (RI) for soft constraints.

4.1 *Hyper-Spheres in $\boldsymbol{u}$-space*

The CPV for this type of dilating sets is calculated by solving a minimum norm problem in $\boldsymbol{u}$-space. In particular, the CPV is given by

$$\tilde{\boldsymbol{u}} = \underset{\boldsymbol{u}}{\operatorname{argmin}} \left\{ \|\boldsymbol{u} - \bar{\boldsymbol{u}}\| : \prod_{i=1}^{dim(\boldsymbol{g})} \boldsymbol{g}_i(U^{-1}(\boldsymbol{u}), \boldsymbol{d}) = 0 \right\}.$$

Definition 9 (Spherical RI). *The reliability index corresponding to the design $\boldsymbol{d}$ for dilating spheres centered at $\bar{\boldsymbol{u}}$, called the* Spherical Reliability Index, *is defined as*

$$\beta_{\mathcal{S}}(\bar{\boldsymbol{u}}, \tilde{\boldsymbol{u}}, \boldsymbol{d}) \triangleq \|\tilde{\boldsymbol{u}} - \bar{\boldsymbol{u}}\|, \tag{10}$$

where $\boldsymbol{g}(\bar{\boldsymbol{u}}, \boldsymbol{d}) \leq \boldsymbol{0}$.

If $\bar{\boldsymbol{u}} = \boldsymbol{0}$, $\beta_{\mathcal{S}}$ is the conventional reliability index and the CPV is the *Most Probable Point*. With this information at hand, a bound to the failure probability can be derived.

Theorem 1 (Bound to $P[\mathcal{F}]$: Hyper-Spheres). *The number $\psi_{\mathcal{S}}$, given by*

$$\psi_{\mathcal{S}} = 1 - \operatorname{erf}(\gamma) + \sqrt{\frac{2}{\pi}} e^{-\gamma^2} f_1\left(\dim(\boldsymbol{u}), \beta_{\mathcal{S}}\right), \tag{11}$$

$$f_1(n, r) \triangleq \left(\prod_{j=1}^{\frac{n-3}{2}} h(j) \right)^{-1} \sum_{k=1}^{\frac{n-1}{2}} \left[r^{2k-1} \prod_{i=1}^{\frac{n-1}{2}-k} h(i) \right],$$

when $\dim(\boldsymbol{u})$ *is an odd number, and by*

$$\psi_{\mathcal{S}} = e^{-\gamma^2} f_2\left(\dim(\boldsymbol{u}), \beta_{\mathcal{S}}\right), \tag{12}$$

$$f_2(n, r) \triangleq \left(\prod_{j=1}^{\frac{n}{2}-1} h(j) \right)^{-1} \sum_{k=1}^{n/2} \left[r^{2(k-1)} \prod_{i=1}^{\frac{n}{2}-k} h(i) \right],$$

when $\dim(\boldsymbol{u})$ *is an even number, is an* upper bound *to the failure probability $P[\mathcal{F}]$ corresponding to a hyper-spherical set with designated point $\bar{\boldsymbol{u}} = \boldsymbol{0}$ and spherical reliability index $\beta_{\mathcal{S}}$. In these expressions, $h(i) = n - 2i$ and $\gamma = \beta_{\mathcal{S}}/\sqrt{2}$.*

4.2 *Hyper-Rectangles in $\boldsymbol{u}$-space*

The developments that follow are an extension of the ideas presented in Section 3.2. The CPV resulting from dilating a hyper-rectangle centered at $\bar{\boldsymbol{u}}$ having the half-length vector $\boldsymbol{m}$ is given by

$$\tilde{\boldsymbol{u}} = \underset{\boldsymbol{u}}{\operatorname{argmin}} \left\{ \|\boldsymbol{u} - \bar{\boldsymbol{u}}\|_{\boldsymbol{m}}^{\infty} : \prod_{i}^{\dim(\boldsymbol{g})} \boldsymbol{g}_i(U^{-1}(\boldsymbol{u}), \boldsymbol{d}) = 0 \right\}.$$

An equivalent formulation based on the Q-Transformation requires $Q(\boldsymbol{u}, \bar{\boldsymbol{u}}, \boldsymbol{m})$ as the cost function. In this context, $\boldsymbol{q} = Q(\boldsymbol{u}, \bar{\boldsymbol{u}}, \boldsymbol{m})$ maps hyper-rectangles in $\boldsymbol{u}$-space proportional to $\mathcal{R}_{\boldsymbol{u}}(\bar{\boldsymbol{u}}, \boldsymbol{m})$ into hyper-spheres in $\boldsymbol{q}$-space proportional to $\mathcal{S}_{\boldsymbol{q}}(\boldsymbol{0}, 1)$. For the reminder of this section, $Q(\boldsymbol{u})$ refers to $Q(\boldsymbol{u}, \bar{\boldsymbol{u}}, \boldsymbol{m})$.

Definition 10 (Rectangular RI). *The reliability index corresponding to the design $\boldsymbol{d}$ for a dilating hyper-rectangle with designated point $\bar{\boldsymbol{u}}$ and the half-lengths vector $\boldsymbol{m}$, called the* Rectangular Reliability Index, *is defined as*

$$\beta_{\mathcal{R}}(\bar{\boldsymbol{u}}, \tilde{\boldsymbol{u}}, \boldsymbol{d}) \triangleq \left\| Q^{-1}\left(\frac{\|\tilde{\boldsymbol{q}}\| \, \boldsymbol{1}}{\sqrt{\dim(\boldsymbol{u})}} \right) - \bar{\boldsymbol{u}} \right\|, \quad (13)$$

where $\boldsymbol{g}(\bar{\boldsymbol{u}}, \boldsymbol{d}) \leq \boldsymbol{0}$ and $\tilde{\boldsymbol{q}} = Q(\tilde{\boldsymbol{u}})$.

Theorem 2 (Bound to $P[\mathcal{F}]$: Hyper-Rectangles). *The number $\psi_{\mathcal{R}}$, given by*

$$\psi_{\mathcal{R}} = 1 - \prod_{i=1}^{\dim(\boldsymbol{u})} \left[1 + \Phi(\bar{\boldsymbol{u}}_i - \sigma_i) - \Phi(\bar{\boldsymbol{u}}_i + \sigma_i)\right] \quad (14)$$

is an upper bound *to the failure probability $P[\mathcal{F}]$ for a hyper-rectangular set with designated point $\bar{\boldsymbol{u}}$ and a rectangular reliability index $\beta_{\mathcal{R}}$. In this expression, Φ is the cumulative distribution function of a standard normal variable and $\sigma_i = \beta_{\mathcal{R}} \boldsymbol{m}_i / \|\boldsymbol{m}\|$.*

Note that as long as the search for the CPV converges to the global minimum, the upper bounds above are exact and do not require of confidence intervals. A method for approximating failure probabilities that uses both the failure bounds and sampling is presented next.

4.3 *Hybrid Method (HM)*

Let ψ be the probability bound associated to the homothetic set $\mathcal{H}_{\boldsymbol{u}}(\tilde{\alpha})$. In this context

$$P[\mathcal{F}] = \psi P\left[\boldsymbol{g}(U^{-1}(\boldsymbol{u}), \boldsymbol{d}) > \boldsymbol{0} | \mathcal{H}^c_{\boldsymbol{u}}(\tilde{\alpha})\right]. \quad (15)$$

The conditional probability will be estimated via sampling. Equation 15 requires the generation of samples outside $\mathcal{H}_{\boldsymbol{u}}(\tilde{\alpha})$. Since this practice avoids sampling the non-failure region of the $\boldsymbol{u}$-space inside the homothet, the accuracy of the approximation is proportional to the reliability index. If n_s is the number of samples, an approximation to Equation 15 is

$$P[\mathcal{F}] = \psi \frac{\sum_{i=1}^{n_s} \omega_i \mathcal{I}\left[\boldsymbol{g}(U^{-1}(\boldsymbol{u}_i), \boldsymbol{d}) > \boldsymbol{0}\right]}{\sum_{i=1}^{n_s} \omega_i}, \quad (16)$$

where $\mathcal{I}$, the indicator function, is equal to one if its argument is true and zero if it is false, $\omega_i > 0$ is a weighting factor and $\boldsymbol{u}_i \in \mathcal{H}^c_{\boldsymbol{u}}(\tilde{\alpha})$ is a conditional sample.

A rejection-based algorithm for the generation of these samples is presented next. This algorithm generates Monte Carlo samples in standard normal space and rejects those within the homothet. Its implementation is as follows. Instantiate the counter i to one.

1. Generate a sample of the standard normal space, namely $\hat{\boldsymbol{u}}$.
2. *Hyper-spheres*: regard this sample acceptable if $\|\hat{\boldsymbol{u}} - \bar{\boldsymbol{u}}\| > \beta_{\mathcal{S}}$. *Hyper-rectangles*: regard the sample acceptable if $\|Q(\hat{\boldsymbol{u}})\| > \|Q(\beta_{\mathcal{R}})\|$.
3. If $\hat{\boldsymbol{u}}$ is acceptable, make $\boldsymbol{u}_i = \hat{\boldsymbol{u}}$ and increase the counter i by one.
4. If $i < n_s + 1$ go to Step 1 and repeat.

An approximation to $P[\mathcal{F}]$ results from using the samples $\boldsymbol{u}_i$ and assuming $\omega_i = 0$ for all i, in Equation 16. The number of non-acceptable samples generated is proportional to the size of $\mathcal{H}_{\boldsymbol{u}}(\tilde{\alpha})$. Therefore, the efficiency of this algorithm diminishes with the value of the RI, hence with the value of $P[\mathcal{F}]$. An alternative method, well-suited for cases in which $P[\mathcal{F}]$ is small, was developed but omitted here due to space limitations.

5 EXAMPLE

A two-dimensional problem in $\boldsymbol{d}$ and $\boldsymbol{p}$ has been selected to allow for visualization. The same example is considered in (Crespo, Giesy, and Kenny 2006), where additional aspects of the methodology are exercised. The constraint set is $\boldsymbol{g} = [3\boldsymbol{d}_2 - 4\boldsymbol{p}_1^2 - 4\boldsymbol{d}_1\boldsymbol{p}_2 \sin(\boldsymbol{p}_2\boldsymbol{d}_1 - \boldsymbol{p}_1^2), -\sin(\boldsymbol{p}_1^2\boldsymbol{p}_2 - \sin(2\boldsymbol{p}_1 - 2)) - \boldsymbol{d}_1\boldsymbol{d}_2\boldsymbol{p}_1 + \boldsymbol{p}_2, \boldsymbol{d}_1 + \boldsymbol{p}_1^2\boldsymbol{d}_2 - 4\boldsymbol{p}_2^2\boldsymbol{p}_1 - 4\sin(2\boldsymbol{p}_1 - 2\boldsymbol{p}_2), 2(\boldsymbol{p}_1 + \boldsymbol{p}_2)\sin(\boldsymbol{p}_1^2 - \boldsymbol{d}_2) - 2\boldsymbol{p}_1\boldsymbol{p}_2(\boldsymbol{d}_2 + 2\boldsymbol{p}_1^2 - 2) + \boldsymbol{d}_1 - 6\boldsymbol{p}_1]^T$. The designated point is $\bar{\boldsymbol{p}} = [1, 1]^T$. The boundary of the FDS will be shown in subsequent figures as a solid line. In this example, soft constraints are considered. In what follows, $\bar{\boldsymbol{u}} = \boldsymbol{0}$ and $f_{\boldsymbol{p}}(\boldsymbol{p}) = f_{\boldsymbol{p}1}(\boldsymbol{p1}) f_{\boldsymbol{p}2}(\boldsymbol{p2})$, where $f_{\boldsymbol{p}1}(\boldsymbol{p1}) = N(1, 0.05)$ and $f_{\boldsymbol{p}2}(\boldsymbol{p2}) = N(1, 0.2)$. Since the support of this joint probability density function is not a subset of largest homothetic set found in (Crespo, Giesy, and Kenny 2006), the usage of soft constraints is justified. This implies that the RDS is empty and that $P[\mathcal{F}]$ is non-zero for all designs. The distribution of $-\log(P[\mathcal{F}])$ for $n_s = 10000$ samples, generated via Monte Carlo Sampling (MCS), is shown in Figure 1. Figures showing $-\log(P[\mathcal{F}])$ are used to illustrate the behavior of the approximation when the failure probabilities are small. The "hole" at the interior of the FDS results from using a finite number of samples. In the vicinity of this region, the approximation to $P[\mathcal{F}]$ is inaccurate and very noisy. For a given number of samples, sampling-based RBDO algorithms are not only unable to discriminate among designs attaining small failure probabilities, but more importantly, wrongly identify zero-failure probability solutions. Besides, reliability assessments resulting from sampling lead to piece-wise constant functions at every design point no matter what n is. Spherical (not shown) and

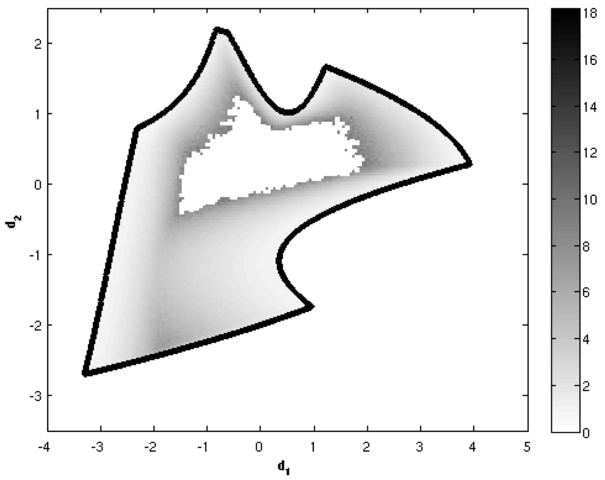

Figure 1: $-\log(P[\mathcal{F}])$ via MCS for $n_s = 10000$.

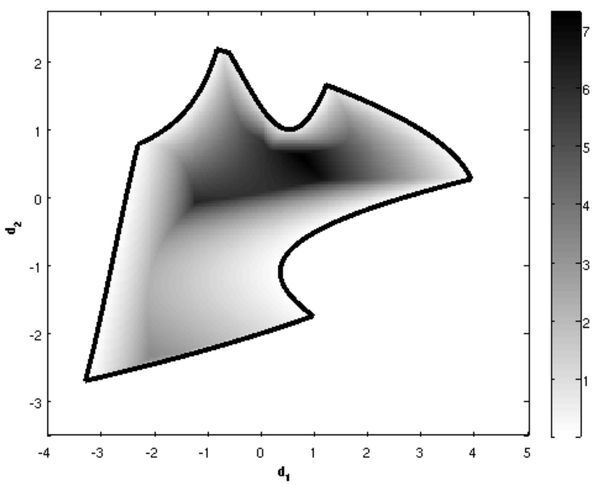

Figure 2: Rectangular RIs for $\boldsymbol{m} = [1, 1]^T$.

rectangular (Figure 2) RIs in the FDS were calculated. Noticeable differences between the two functions are apparent. Locally, either the spherical-RI or the rectangular-RI can be the largest. Overall, the maximum rectangular-RI is greater than the maximum spherical-RI. Failure probabilities bounds for circular and squared homothets are shown in Figures 3 and 4. Note that in both cases the bounds are smooth functions. Since (i) the calculation of the bound not only requires considerably fewer function evaluations that MCS (about 50 times less for this example), and (ii) the bound is continuous everywhere in the FDS, substantial advantages would result from using the bound ψ as an approximation to $P[\mathcal{F}]$. Locally, either $\psi_{\mathcal{S}}$ or $\psi_{\mathcal{R}}$ can be the smallest bound. The First Order Reliability Method (FORM) and Second Order Reliability Method generate approximations that also overcome some of the deficiencies of sampling. The HM

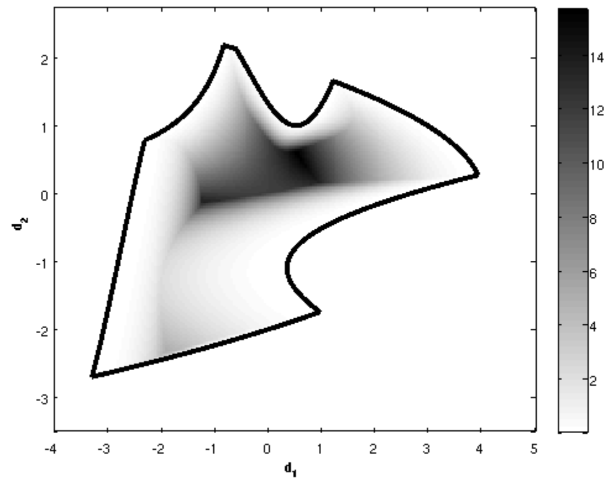

Figure 3: $-\log(\psi_{\mathcal{S}})$ for hyper-spheres.

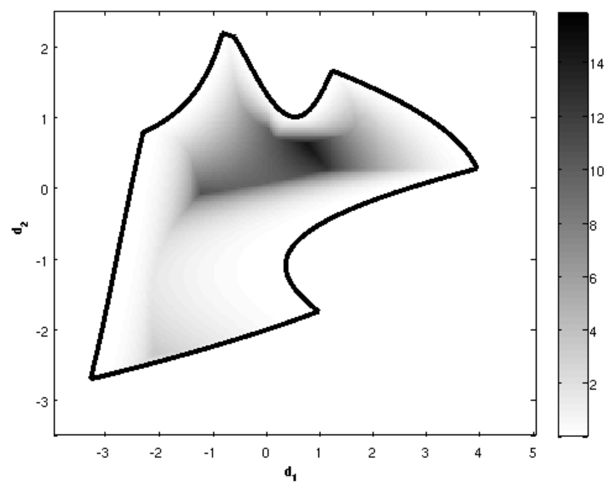

Figure 4: $-\log(\psi_{\mathcal{R}})$ for hyper-rectangles.

is used next. Figures 5 and 6 show $-\log(P[\mathcal{F}])$ for circular and squared homothets respectively. In both cases, $n_s = 2000$ samples are used. The approximation not only inherits the nice features of the bound, such as its smoothness, but it is also is more accurate and efficient than MCS. Overall, less than one fourth of the function evaluations used to generate the MCS results were used by the HM. As expected, the HM approximation compares well to the FORM approximation when $P[\mathcal{F}]$ is small. On the other hand, the HM outperforms the FORM approximation when the failure probabilities are not small. For instance, while

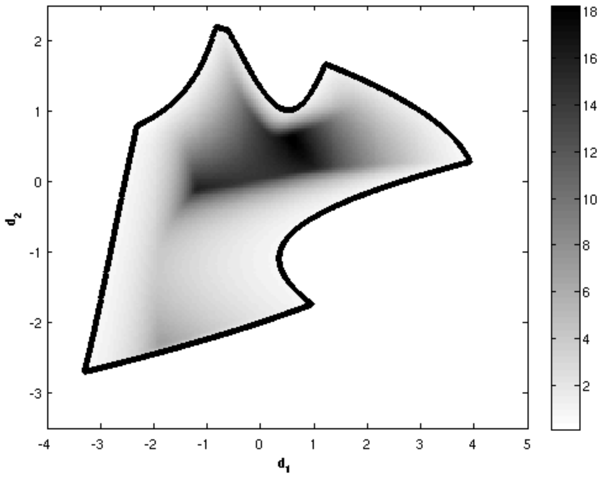

Figure 5: $-\log(P[\mathcal{F}])$ for circular homothets via HM.

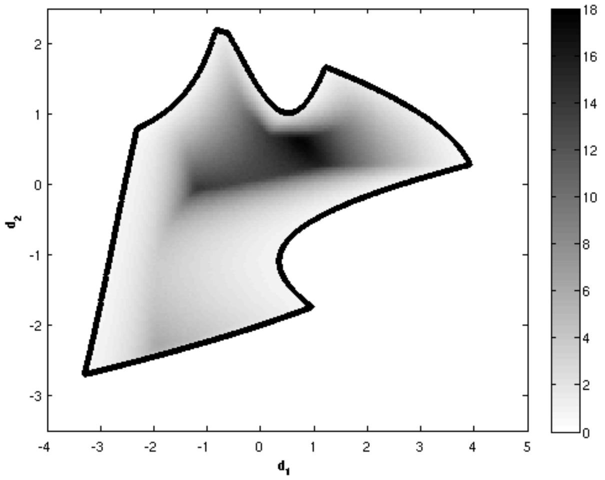

Figure 6: $-\log(P[\mathcal{F}])$ for squared homothets via HM.

the HM leads to $P[\mathcal{F}] = 0.623$ for $\boldsymbol{d} = [-2.2, 0.73]^T$, FORM leads to $P[\mathcal{F}] = 0.379$. The corresponding MCS approximation, that for this case can be used as a yardstick, is $P[\mathcal{F}] = 0.614$. Therefore, the HM compares very well to MCS for not-small failure probabilities values and to FORM for small values.

6 CONCLUSIONS

Strategies for analyzing and implementing hard and soft constraints in a reliability-based setting are proposed. Emphasis is given to uncertainty models prescribed by hyper-spheres and hyper-rectangles. While the former geometry can be used to deal with uncertain parameters of comparable fidelity, the second one allows for the manipulation of parameters with different fidelity. The mathematical framework developed allows the determination of whether it is feasible or not to satisfy hard constraints. Strategies for implementing hard constraints into design optimization schemes are proposed. Probabilistic uncertainty models are used in the implementation of soft constraints. The methodology permits determination of closed-form expressions for upper bounds to the failure probability. A hybrid method for the approximation of failure probabilities, based on the upper bounds and conditional sampling, is proposed. Numerical experiments show that this method results in substantial improvements in accuracy and efficiency as compared to alternative methods.

REFERENCES

Acevedo, J. and E. N. Pistikopoulos (1998). Stochastic optimization based algorithms for process synthesis under uncertainty. *Comp. Chem. Engng 22*(1), 647–671.

Crespo, L. G., D. P. Giesy, and S. P. Kenny (2006, September). Strict feasibility in optimization under uncertainty. In *11th AIAA/ISSMO Multidisciplinary Analysis and Optimization Conference*, Portsmouth, VA USA. AIAA-2006-7035.

Crespo, L. G. and S. P. Kenny (2005). Reliability-based control design for uncertain systems. *AIAA Journal of Guidance, Control, and Dynamics 28*(4).

Darlington, J., C. Pantelides, B. Rustem, and B. Tanyi (1999). An algorithm for constrained nonlinear optimization under uncertainty. *Automatica 35*, 217–228.

Ermoliev, Y. (1983). Stochastic quasigradient methods and their applications to systems optimization. *Stochastics 9*(1), 1–36.

Kall, P. and S. Wallace (1994). *Stochastic Programming*. New York: Wiley.

Rackwitz, R. (2001). Reliability analysis, a review and some perspectives. *Structural Safety 23*, 365–395.

Royset, J., A. D. Kiureghian, and E. Polak (2001). Reliability-based optimal structural design by the decoupling approach. *Reliability Engineering and System Safety 73*, 213–221.

Rustem, B. and Q. Nguyen (1998). An algorithm for the inequality constrained minimax problem. *SIAM Journal of Optimization 8*(1), 265–283.

Samsatli, N., L. Papageorgiou, and N. Shah (1998). Robustness metrics for dynamic optimization models under parametric uncertainty. *Process Systems Engineering 44*(9), 1993–2006.