

# Dynamic theory of relativistic electrons stochastic heating by whistler mode waves with application to the Earth magnetosphere

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**Abstract.**

In the Hamiltonian approach an electron motion in a coherent packet of the whistler mode waves propagating along the direction of an ambient magnetic field is studied. The physical processes by which these particles are accelerated to high energy are established. Equations governing a particle motion were transformed into a closed pair of nonlinear difference equations. The solutions of these equations have shown there exists the energetic threshold below that the electron motion is regular, and when the initial energy is above the threshold an electron moves stochastically. Particle energy spectra and pitch angle electron scattering are described by the Fokker-Planck-Kolmogorov equations. Calculating the stochastic diffusion of electrons due to a spectrum of whistler modes is presented. The parametric dependence of the diffusion coefficients on the plasma particle density, magnitude of wave field, and the strength of magnetic field is studied. It is shown that significant pitch angle diffusion occurs for the Earth radiation belt electrons with energies from a few keV up to a few MeV.

## 1. Introduction

It is suggested that whistler mode waves are responsible for electron acceleration caused by the interaction of these waves and Earth's radiation belt electrons. As a result, electrons may be accelerated up to relativistic energies (*Baker et al.*, [1986]). Thus *Horne et al.*, [1998] have identified potential whistler wave modes that are capable of resonating with electrons over the important energy range from 100 keV to a few MeV in different regions of Earth's magnetosphere. The basic concept of energy diffusion of relativistic electrons resulting from resonant interaction with whistlers in the magnetosphere has been discussed, for example, by *Walker* [1993], *Reeves* [1996] and *Summers et al.*, [1998]. Using the Chirikov overlap criterion (*Chirikov*, [1979]), *Karimabadi et al.*, [1990] considered the conditions at which all resonance state break up. They showed that the stochasticity occurs when the amplitude of the obliquely propagating wave is large enough. Electron precipitation caused by chaotic motion due to coupling the bounce motion of non-relativistic electrons with a large-amplitude whistler wave has been studied by *Faith et al.*, [1997].

The goal of work is to describe the high-energy electron motion in a coherent packet of whistler modes. Stochastic dynamics of charged particles in the field of a wave packet is one of the fundamental problem in the theory of plasma physics (*Lichtenberg et al.*, [1983]; *Zaslavsky et al.*, [1991]). Chaotic dynamics of relativistic particles in the spectrum of waves is of particular interest. Thus the stochastic dynamics of relativistic electrons in the time-like wave packet has been discussed by *Chernikov et al.*, [1989]. *Krotov et al.*, [1998] have been shown that the stochastic heating of relativistic particles by the Langmuir waves in space plasmas can be regarded as a possible mechanism for the formation of the

energy spectrum of cosmic rays. They also studied the evolution of distribution function caused by the stochasticity. *Nagornykh et al.*, [2002] developed the relativistic theory for the stochastic motion of electrons in the presence of obliquely propagating electrostatic wave. In that case, the ambient magnetic field plays an important role in randomizing the phase of particle with respect to the wave phase. Stochastic motion of relativistic electrons in the whistler wave packet with application of the results to electron heating in the Jovian magnetosphere was studied by *Khazanov et al.*, [2007].

The purpose of the present paper is to investigate the dynamics of energetic electrons that are confined by the Earth magnetic field and undergo bounded motion about the equatorial plane. We propose mapping equation which describe the particle motion transiting through a wave packet in a periodic system iteratively. These equation exhibit chaotic motion when the particle energy exceeds a certain threshold value at given magnitude of the wave field.

The paper is organized as follows. In section two we derive the canonical equation of motion in terms of the action-angle variables, and discuss physical processes underlying the results. In particular, it is shown that nonlinear nature of resonance wave-particle interaction leads to essentially different representation of the whistler wave field (the so-called space-like and time-like wave packet representations). In section three, it is shown that the stochastic motion of relativistic electrons can be described by a closed set of nonlinear difference equations. The solutions of these equations are obtained both analytically and numerically. In section four we derive equations that describe the dynamics of resonant non-relativistic electrons. Solutions of these equations reveal the existence of an energetic threshold below which the particle can't gain net energy from the wave

field. When the initial velocity is above the threshold, the electron moves stochastically and eventually gains a net energy. The Fokker-Planck-Kolmogorov equation is derived in section five, and the effects of radial drift and pitch angles scattering associated with particle stochastic heating are studied. Application of our results to high-energy electrons observed in Earth's radiation belts is described in section six. In section seven we give the conclusions of our studies.

## 2. Basic equations

Let us consider a relativistic particle of charge  $|e|$  and mass  $m$  in the wave packet of extraordinary electromagnetic waves propagating along an external uniform magnetic field of strength  $B$ . The Hamiltonian corresponding to the problem is

$$H(\mathbf{r}, \mathbf{p}; t) = \sqrt{m^2 + (\mathbf{p} + \mathbf{A})^2}, \quad (1)$$

and the canonical equations of motion are

$$\dot{\mathbf{p}} = [\mathbf{p}, H], \quad \dot{\mathbf{r}} = [\mathbf{r}, H], \quad (2)$$

where  $\mathbf{p}$  is the particle momentum,  $\mathbf{r}$  is the position vector,  $\mathbf{A} = \mathbf{A}^w + \mathbf{A}^{ext}$  the vector potential, the superscripts  $w$  and  $ext$  denote both the wave and external fields, and  $[ , ]$  stand for the Poisson brackets.

We have employed here and throughout this paper, the frame of reference in which the speed of light  $c = 1$  and charge  $|e| = 1$ .

We denote by  $R$  the set of all real numbers. Then  $\mathbf{p} \in R^3, \mathbf{r} \in R^3$ , and the smooth manifold  $M = R^6$  will be a canonical space of this dynamic system, and  $R^6 = R^3 \times R^3$  is a direct product space.

In order to write down equations of the particle motion one must specify a coordinate system. We have chosen a Cartesian spatial coordinates system whose  $z$  axis is directed along the external magnetic field, the plane perpendicular to this direction is spanned by the orthogonal coordinates  $x$  and  $y$ .

Making use of the connecting relations,

$$\mathbf{B} = \text{rot}\mathbf{A}, \quad \mathbf{E} = -\partial\mathbf{A}/\partial t, \quad (3)$$

we have in the coordinate representation

$$\mathbf{r} = (x, y, z), \quad \mathbf{B}^{ext} = (0, 0, B), \quad (4)$$

$$\mathbf{A}^w = \left( \sum_k A_k \sin \varphi, \sum_k A_k \cos \varphi, 0 \right),$$

$$\varphi = zk - t\omega_k, \quad (5)$$

$$\mathbf{A}^{ext} = (-By, Bx, 0)/2. \quad (6)$$

Here the expression for  $\mathbf{A}^{ext}$  is written in the axial gauge,  $A_k$  is the amplitude of mode in the wave packet,  $k$  is the wave number, and  $\omega_k$  is the dispersion equation.

The dispersion relation for electron branch of the whistler mode waves in the cold magnetoplasma is written as

$$k^2/\omega^2 = 1 + \omega_p^2/[\omega(\omega_B - \omega)], \quad (7)$$

where  $\omega_B$  and  $\omega_p$  are the gyrofrequency and electron plasma frequency, respectively.

This equation in a long-wavelength approximation ( $\omega_B\omega/\omega_p^2 \ll 1$ ) reduces to

$$v_{ph}^2 = \omega(\omega_B - \omega)/\omega_p^2, \quad v_{ph}^2 \ll 1. \quad (8)$$

We now take into account axial symmetry of the non-perturbative problem, and introduce the new variables, an action( $I$ ), and an angle ( $\theta$ ), by a canonical transformation

$$(x, p_x; y, p_y) \rightarrow (\theta, I) :$$

$$x = r \cos \theta, \quad p_x = -(mr\omega_B/2) \sin \theta,$$

$$y = r \sin \theta, \quad p_y = (mr\omega_B/2) \cos \theta; \quad (9)$$

$$r = \sqrt{2m\omega_B I / m\omega_B}, \quad \omega_B = B/m, \quad (10)$$

where  $r$  is the gyroradius.

The Hamiltonian (1) in this representation becomes

$$H(z, p_z; \theta, I; t) = H_0(p, I) + \sqrt{2m\omega_B I} H_0^{-1} \cdot \sum_k A_k \cos(zk + \theta - t\omega_k), \quad (11)$$

$$H_0(p, I) = \sqrt{m^2 + p^2 + 2m\omega_B I}. \quad (12)$$

Here we have assumed that the ratio  $\mu = A/m \ll 1$  is the small parameter of the problem, and retain in (11) only the leading terms.

Associated with (11) the equations of motion are

$$\dot{p} = [p, H] = \sqrt{2m\omega_B I} H_0^{-1} \cdot \sum_k k A_k \sin \psi, \quad (13)$$

$$\dot{I} = [I, H] = \sqrt{2m\omega_B I} H_0^{-1} \cdot \sum_k A_k \sin \psi, \quad (14)$$

$$\dot{z} = [z, H] = p H_0^{-1}, \quad \dot{\theta} = [\theta, H] = \omega_B m H_0^{-1}. \quad (15)$$

In (15) we omit the terms of the order of  $\mu^2$  and introduce the definition for the phase

$$\psi \stackrel{def}{=} zk + \theta - \omega_k t. \quad (16)$$

Now the structure of wave packet,  $A^w(t, z) = \sum_k A_k \exp[i(zk + \theta - t\omega_k)]$ , is to be specialized.

The most frequently used representations of wave packet are the so-called time- and space-like representations *Zaslavsky et al.*, [1991], *Zaslavsky*, [1998]. Such wave packet may be excited in a plasma due to the intrinsic instabilities. In this particular physical situation,

we make following simplifying assumptions regarding the structure of the packet,

$$k = k_0 + n\Delta k, \quad \omega = \omega_0 + n\Delta\omega, \quad \Delta k/k_0 \ll 1, \quad \Delta\omega/\omega_0 \ll 1, \quad n \in Z, \quad (17)$$

where  $k_0, \omega_0$  are the characteristic wave number and frequency,  $\Delta\omega(\Delta k)$  is the group dispersion in wave spectrum, so that

$$\Delta\omega = v_{gr}\Delta k, \quad v_{gr} = d\omega/dk, \quad (18)$$

$v_{gr}$  is the group velocity,

$$\Delta k = 2\pi/L, \quad \Delta\omega = 2\pi/T, \quad (19)$$

$L, T$  are the length- and the time- scales of the problem,  $Z$  denotes the set of all integers. Then we suppose that the characteristic spectral amplitude  $A_0$  is a slowly varying function on  $t$  and  $z$  such that

$$\dot{A}_0/\omega A_0 \simeq 1/\omega T (\ll 1), \quad \nabla A_0/k A_0 \simeq 1/k L (\ll 1), \quad (20)$$

and write down the wave packet in the form

$$A^w(t, z) = A_0 \exp(i\psi_0) \sum_{n \in Z} A_k \exp[i(n\Delta k z + n\Delta\omega t)], \quad (21)$$

$$\psi_0 = k_0 z + (\omega_B m H_0^{-1} - \omega_0)t. \quad (22)$$

Define the parameter,  $\eta$  namely, the ratio of the particle velocity along an ambient magnetic field to the group velocity,

$$\eta = v_z/v_{gr}. \quad (23)$$

In the limit,  $\Delta k \rightarrow 0$ ,  $\eta \rightarrow 0$ , expression (21) can be transformed into

$$A^w(t, z) = A_0 \exp(i\psi_0) \sum_{n \in Z} \delta(t/T - n) \quad (24)$$



There is the time-like representation (TLR) of a wide wave packet. The Poisson sum formula

$$\sum_{n \in \mathbb{Z}} \exp(in\Delta\omega t) = \sum_{n \in \mathbb{Z}} \delta(t/T - n) \quad (25)$$

has employed in 24. Here  $\delta(\cdot)$  and the Dirac  $\delta$  function. In another limit  $\Delta\omega \rightarrow 0$ ,  $\eta \rightarrow \infty$ , we can easily show that the wave field takes the form of the space-like (SL) wave packet

$$A^w(t, z) = A_0 \exp(i\psi_0) \sum_{n \in \mathbb{Z}} \delta(z/L - n). \quad (26)$$

This packet represents a periodic sequence of impulses with characteristic spatial period  $L = 2\pi/\Delta k$ .

Note that both representation are often used. Thus it is established *Zaslavsky et al.*, [1991], *Zaslavsky*, [1998]. That the TLR is available for the problem if the condition  $\eta^2 \ll 1$  holds. The TLR of the electric field of the electrostatic waves was used by *Chernikov et al.*, [1989] to derive the relativistic generalization of the standard map. On the other hand, the SLR has been utilized in *Klimov et al.*, [1995] to describe the stochastic motion of relativistic particle in the electrostatic field of Langmuir waves, whose group velocity is small as known.

### 3. Particle dynamics in a space-like wave packet

We specify first the wave spectrum of a packet. Let us assume that the wave packet is given by (26). Thus, we will consider a relativistic electron motion in the space-like packet (SLP) of the whistler mode waves. In this approach dropping the subscript "0" we write down the equations of motion (14) and (15) in the form:

$$\dot{p} = kA\sqrt{2m\omega_B}IH_0^{-1} \sin \psi \sum_{n \in \mathbb{Z}} \delta(\zeta - n), \quad (27)$$

$$\dot{I} = \sqrt{2m\omega_B I} H_0^{-1} A \sin \psi \sum_{n \in Z} \delta(\zeta - n), \quad (28)$$

$$\dot{z} = p H_0^{-1}, \quad \dot{\theta} = \omega_B m H_0^{-1}; \quad (29)$$

$$H_0(p, I) = \sqrt{m^2 + p^2 + 2m\omega_B I}, \quad (30)$$

$$\psi = \omega(p, I) = k p H_0^{-1} + \omega_B m H_0^{-1} - \omega. \quad (31)$$

Here  $A, \omega$  and  $k$  are the magnitude, frequency, and wave number of the fundamental (characteristic) mode,  $\zeta = (z/L)(\text{mod } 1)$ ,  $L$  is the characteristic spacescale,  $\delta_n \equiv \delta(\zeta - n)$ ,  $\delta(\cdot)$  is the Dirac delta function, and  $Z$  denotes the set of all integers.

The evident symmetries of rotation,  $\psi \rightarrow \psi \exp(i\psi_0)$ ,  $\psi_0$  is an arbitrary constant, and a translation,  $\{p, I\}(\zeta) \rightarrow \{p, I\}(\zeta + 1)$ , allow us to represent dynamics of the system as certain iterative process by identification of the planes  $n\zeta$  and  $(n + 1)\zeta$ . The group symmetry is realized first as the invariant of motion,

$$p - kI = \text{inv}. \quad (32)$$

Introducing the new variables,

$$\varepsilon_z = p/m, \quad \varepsilon_t = \sqrt{2m\omega_B I}/m, \quad \varepsilon = E/m, \quad (33)$$

where  $E = \sqrt{m^2 + p^2 + 2m\omega_B I}$  is the particle energy, and the parameter,  $\alpha$ ,

$$\alpha = 2v_{ph}\omega_B/\omega, \quad (34)$$

we write down expression (32) as

$$\varepsilon_z - \alpha^{-1}\varepsilon_t^2 = 0, \quad \alpha = (2\omega_B/\omega)v_{ph}, \quad (35)$$

where the constant of integration has chosen equals zero. In view of (35) the number of dimensions reduces to two. Let the variables  $z$  and  $p$  will be the represented pair.

Now the explicit form of iteration system is to be found. Denote by  $\tilde{d}p, \tilde{d}\zeta, \tilde{d}t$  the co-ordinate basis of 1-form on an extended phase space. Then, using the further result  $L\tilde{d}\zeta = pH_0^{-1}\tilde{d}t$ , we represent equations (27, 31) as

$$\tilde{d}\varepsilon_z - \varepsilon_z^{-1}\varepsilon_t N(A/m) \sin \psi \sum_{z \in Z} \delta_n \tilde{d}\zeta = 0, \quad (36)$$

$$\tilde{d}\psi - (\omega - \omega_B \varepsilon^{-1} - k\varepsilon_z \varepsilon^{-1}) L(\varepsilon/\varepsilon_z) \tilde{d}\zeta = 0, \quad (37)$$

where  $\delta_n \equiv \delta(\zeta - n)$ . Making use of the invariant of motion (35), and integrating one by one resulting equations, we obtain the closed set of nonlinear difference equations

$$\begin{aligned} u_{n+1} &= u_n + 3/2 \cdot Q \sin \psi_n, \\ \psi_{n+1} &= \psi_n + N \left( 1 + 1/2 \cdot \alpha u_{n+1}^{-2/3} \right. \\ &\quad \left. - u_{n+1}^{-2/3} v_{ph} \sqrt{1 + u_{n+1}^{4/3} + \alpha u_{n+1}^{2/3}} \right) \pmod{2\pi}, \end{aligned} \quad (38)$$

expressed in terms of the variables  $u, \psi$ , where  $u_{n+1}$  and  $u_n$  are respectively the values of the normed momentum at times  $(n+1)\zeta$  and  $n\zeta$ , and  $\psi_{n+1} - \psi_n$  is the phase shift acquired by the particle. Another deriving the above equations was given by *Khazanov et al.*, [2007]. We have employed here the following notations:

$$\begin{aligned} u &= \varepsilon_z^{3/2}, \quad Q = \sqrt{\alpha} \cdot Nw, \\ N &= [kL], \quad w = A/m. \end{aligned} \quad (39)$$

Here  $N$  is the characteristic number of modes in the SL packet. The quantity  $w$  has a clear physical meaning:  $w$  is the dimensionless representation of the ratio of the work of the wave field at one wavelength, to the particle rest energy. Regarding the relationship

(3) between fields  $A^w$  and  $B^w$ , it is convenient to represent an expression for  $w$  in the form

$$w = \alpha b/2, \quad b = B^w/B. \quad (40)$$

In the relativistic limit, when the inequality  $\varepsilon^2 \gg 1$  is valid, the set of equations (38) goes over into the map:

$$\begin{aligned} g^n : \quad & u_{n+1} = u_n + 3/2 \cdot Q \sin \psi_n, \\ & \psi_{n+1} = \psi_n + (\pi^{5/3}/Q)u_{n+1}^{-2/3} \pmod{2\pi}, \end{aligned} \quad (41)$$

written in the perception

$$u \rightarrow \pi u/u_b, \quad Q \rightarrow \pi Q/u_b, \quad u_b = \left(\alpha^{3/2} N^2 w/2\right)^{3/5}. \quad (42)$$

The map  $g^n$ , or rather the family of maps depending upon the parameter, becomes suitable for sequential analysis. So hereafter we deal with high-frequency heating of relativistic particles, expressed in terms of the discrete group  $g^n$ , that acts on the smooth manifold.

Denote by

$$J = \frac{\partial(u_{n+1}, \psi_{n+1})}{\partial(u_n, \psi_n)} \quad (43)$$

the Jacobi matrix of map (41). It is important to note that the Jacobian of (43) is equal to one, therefore,  $g^n$  has a structure of the differentiable area-preserving map. It stands to reason that  $u$  and  $\psi$  are the canonical pair of variables.

We now study the behavior of this dynamic system by computational analysis. We have numerically integrated equations (38) and (41) for several different values of  $Q$  from  $10^{-3}$  to 0.1. Figure 1 shows some our results computed for these equations after  $(10^6 - 10^4)$  iterations for one trajectory in the  $(u, \psi)$ -phase space. The initial conditions were chosen in a random fashion and correspond to the region of small values of  $(u, \psi)$ .

*Figure 1 here*

Next we describe the structure of the point set. Let us consider a pair  $(M, g^n)$ , where  $M$  is a smooth manifold and  $g^n$  is the differentiable area-preserving map (41). From (41) follows that a pair  $(M, g^n)$  is invariable under the inversion of a point with respect to a circle  $\psi(\text{mod}2\pi) \in S$ , and the reflection of a point on a  $O\psi$  - axis. On that basis, we define the following equivalence rules:

$$(-\pi, u) \sim (\pi, u), \quad (\psi, -\pi) \sim (\psi, \pi), \quad (44)$$

where  $\sim$  stands for the equivalence sign.

The identification of these points is indicated by the arrows in Figure 1.

The equivalence rules (44) allow us to represent phase space of the system as the space having the topology of a torus,  $T^2 = S \times S$ ,  $\psi(\text{mod}2\pi) \in S, u \in S$  (Kosniowski, [1980]). Now we discuss local topology of the manifold considering a Jacobi matrix given by (43). Denote through  $\lambda_1$  and  $\lambda_2$  the eigenvalues of the matrix  $J$ . Recalling that local structure is determined by neighbors of fixed points, we have from (43) and (39)

$$\det J = \lambda_1 \cdot \lambda_2 = 1, \quad (45)$$

$$\text{tr} J = \lambda_1 + \lambda_2 = 2 + (\pi^5 u^{-5})^{1/3}, \quad (46)$$

where  $\det J$  and  $\text{tr} J$  denote the determinant and the trace of this matrix, respectively.

As pointed out in *Arnold et al.*, [1968] the condition

$$|\text{tr} J| = 3 \quad (47)$$

corresponds to a topological modification of a phase space, and its validity implies that the manifold have topology of a hyperbolic torus. Thereupon from (47) subject to (46) we find

$$\lambda_1 = \frac{3 + \sqrt{5}}{2}, \quad \lambda_2 = \frac{3 - \sqrt{5}}{2}. \quad (48)$$

Since  $\lambda_1$  and  $\lambda_2$  are the dynamic characteristics such that  $\lambda_2 > 0$ ,  $\lambda_1 > 1$ , and the ratio  $\lambda_1/\lambda_2$  is an irrational number, therefore the map  $(\psi_n, u_n) = g^n(\psi_0, u_0)$ ,  $(\psi_0, u_0)$  is an initial point of phase curve, forms a stochastic phase flow with the mean rate of a loss of information,  $K$ ,

$$K = \ln \lambda_1, \quad (49)$$

$K$  is the Kolmogorov entropy.

Present conditions (46) and (47) ensure that the relation

$$|u_b| = \pi \quad (50)$$

take place, and it determines the upper bound of  $\{u\}$ .

Like that the set  $\{u, \psi\}$  is a compact, whose structural stability is determined by the fractal dimension,  $d_f$

$$d_f = 1 - \ln \lambda_1 / \ln \lambda_2, \quad d_f = 2. \quad (51)$$

We call any compact a probabilistic fractal if its topological dimension is less than  $d_f$  and  $K > 0$  Mandelbrot, [1982]. In our case both these conditions are valid.

As seen in Figure 1 for a given values of  $Q$  the stochastic region extends to values of  $u$  predicted by above equation (50). We know, that  $u$  depends on the parameters of this problem as given by (39). By that condition (50) determines in itself an equivalence class in the  $Q$ -parametric space.

Considering (39,42) and (35), from (50) the following equation results

$$\varepsilon_b = \alpha \left( \frac{1}{4} b N^2 \right)^{2/5}, \quad (52)$$

which determines the upper value of the energy spectrum for the SLR case. Note the dependence  $\varepsilon_b$  on the driving field  $b$  is weak enough. This agrees to the numerical solution. Finally, it should be noted that the phase flow,  $(M, g^n)$  is structural stable and typical because  $d_f$  and  $K$  be an invariants, and  $\sup\{\varepsilon\}$  smoothly depends on the driving amplitude  $b$ , which is the controlling parameter of the system. Phase flows with such properties are said to be the stochastic (strange) attractors. Certainly the flow  $(M, g^n)$  is such attractor as was to be proved.

At last, the achieved fractal measure  $d_f = 2$  infers that points of phase curve evenly fill all obtainable phase space, or in other words, all states of our dynamic system are equivalent. This issue is supported by simulation in Figure 2 on which is represented a joint probability density,  $\rho(\psi, u)$ . The following algorithm was used for computing  $\rho$ . We partition all the phase space on the identical cells with the mesh size  $\Delta\psi\Delta u = (2\pi/40) \times (2\pi/30)$ . Like that  $\rho(\psi, u)$  is proportional to the number of phase points in the element of phase space,  $\Delta\psi\Delta u$ . Figure 2 reveals that diffusion in the phase angle is very fast. Significant phase angle diffusion occurs on timescales order of tens of  $T$ , where  $T$  is the step of one iteration. On these timescales, the change in  $u$  is smaller (the numerical calculations indicate the characteristic time for establishing the uniform distribution in  $u$  is proportional to  $T/Q^2$  at  $Q \ll 1$ ), therefore, the variable  $u$  is a slow varying coordinate on the strange attractor. Statistical aspect of the problem will be studied in more detail in Sec.5.

*Figure 2 here*

#### 4. Non-relativistic electron motion in a time-like wave packet

We now discussed the dynamics of non-relativistic electrons in the wave packet of the whistler mode. In the case, the condition  $v_z/v_{gr} < 1$  is valid, therefore by means of (24)

the wave field of packet may be approximated as

$$A^w(z, t) = A_0 \cos(k_0 z - \omega_B \varepsilon^{-1} t - \omega_0 t) \sum_n \delta(\tau - n). \quad (53)$$

Here  $\tau = t/T$ ,  $T$  is the timescale of the problem.

Dropping the subscript "0" in (53) and substituting this expression in (10) we write down the Hamiltonian as

$$H(z, p; \theta, I; t) = H_0(p, I) + \sqrt{2\omega_B I/m} A \cos \psi \sum_{n \in \mathbb{Z}} \delta(\tau - n), \quad (54)$$

$$H_0 = p^2/(2m) + \omega_B I, \quad mv_t^2/2 = \omega_B I, \quad (55)$$

$$\psi = zk + \theta - t\omega. \quad (56)$$

The equations of motion associated with (54) are

$$\dot{p} = k\sqrt{2\omega_B I/m} A \sin \psi \sum \delta(\tau - n), \quad (57)$$

$$\dot{I} = \sqrt{2\omega_B I/m} A \sin \psi \sum \delta(\tau - n) \quad (58)$$

$$\dot{z} = v_z = p/m, \quad \dot{\theta} = \omega_B. \quad (59)$$

In (59) we retain only the leading terms on condition that  $A/mv_t$  is a small parameter of the problem.

Since the whistler wave frequency  $\omega$  is smaller than the electron cyclotron frequency  $\omega_B$ , electrons in general must move in the opposite direction as waves in order to match the resonance condition

$$kv_z + \omega_B - \omega = 0. \quad (60)$$



In view of this fact and also taking into account the relation  $\omega_B I = mv_t^2/2$ , we write the invariant of Hamiltonian flow (58, 59) as

$$v_t^2 = \alpha(v_r - |v_z|), \quad \alpha = 2(\omega_B/\omega)v_{ph} \quad (61)$$

where  $v_r = v_{ph}(\omega_B - \omega)/\omega$  is the resonance speed of an electron.

To understand the physical picture of the stochastic motion we use the overlap criterion *Chirikov*, [1979]. In the broad wave spectrum case given by (25), the condition (60) describes the family of the resonance states. First varying this equation we evaluate the interval between an adjacent resonance states in the velocity space as  $k\Delta v_z \simeq \Delta\omega = T^{-1}$ , and

$$\Delta v_z \simeq v_{ph}/N, \quad N = [\omega T], \quad (62)$$

where  $N$  is the characteristic number of modes in the TL-packet.

Then it avails oneself of equation (57) to estimate the width,  $\delta v_z$ , of resonance state in the velocity space,

$$\delta v \simeq \alpha b N v_t / 2 v_{ph}, \quad (63)$$

where we have again used the notation  $w = A/m = \alpha b/2$ ,  $b = B^w/B$ .

Now from the overlap criterion,  $\delta v \geq \Delta v$ , and the two above expressions the following relation results

$$v_t \geq v_c = v_{ph}\omega/\omega_B N^2 b. \quad (64)$$

Note the term in the RHS of equation (57) corresponds to the Lorentz force caused by spiralling an electron in the magnetic field of wave. Consequently we conclude from above the equation that the stochasticity occurs when the Lorentz force acting of electron in the axial direction exceeds some defined value.

Now we may be able to estimate the value of wave field at which an electron motion becomes stochastic. We substitute the value of  $v_c$  from (64) in (61) at  $v_z = 0$  to obtain

$$b \geq b_c = (v_{ph}/2v_r)^{1/2}(\omega/\omega_B)^{3/2}N^{-2}. \quad (65)$$

Now we turn to the equations of motion. We use (61) and the expression  $mv_t^2/2 = \omega_B I$  once more to represent the equations of motion in the form

$$\dot{v}_t = \frac{1}{2}\omega_B \alpha b \sin \psi \sum_{n \in \mathbb{Z}} \delta(\tau - n), \quad (66)$$

$$\dot{\psi} = \omega(v_t, \omega) = \omega_B - \omega - kv_r + (k/\alpha)v_t^2. \quad (67)$$

By integrating these equations can be transformed into a map,  $G^n$ ,

$$G^n : \begin{cases} u_{n+1} = u_n + Q \sin \psi_n, & \{u \in U \subset R \mid 0 < u \leq 1\}, \\ \psi_{n+1} = \psi_n + su_{n+1}^2 \pmod{2\pi}, \end{cases} \quad (68)$$

where subscript  $n$  refers to values taken at time  $t = nT$ , and the new variable,  $u$ , has introduced by the relation

$$u = v_t/v_0, \quad v_0 = \sqrt{\alpha v_{gr}}. \quad (69)$$

Here  $v_0$  is the all perpendicular (to  $B$ ) speed of electron, and the new parameters are

$$Q = \frac{1}{2}N\alpha b\omega_B/v_0\omega, \quad N = [\omega T], \quad s = Nv_0^2/\alpha v_{ph}. \quad (70)$$

We have numerically integrated equations (68) for values of  $Q$  from 0.0005 to 0.01. Our results are shown in Figure 3.

*Figure 3  
here*

The overall picture of the phase space is quite different for  $u < u_c$  and  $u \geq u_c$ . In the first case, the motion is regular. The figures indicate the existence of a threshold for the initial particle velocity above which the trajectory becomes chaotic and the extent of the stochastic region increases linearly with  $Q$ .

Thus the Jacobian of the matrix of  $Q^n$ ,  $\det J = 1$ , therefore  $G^n$  is the measure-preserving map, and  $\psi, u$  are the symplectic pair on the smooth manifold,  $M : U \times S$ . Thereby we apply the condition (50) to (68), to find the expression

$$u_c = \inf\{u\} = (2Qs)^{-1}, \quad (71)$$

that well determines the lower bound  $\inf\{u\}$  of the stochastic set.

Substituting (69) and (70) in (71) we get

$$v_c = v_{ph}\omega/\omega_B N^2 b, \quad (72)$$

which is in good agreement with the numerical solutions and results of qualitative analysis.

So we resume that nonlinear electron acceleration by a wave packet of the whistler mode waves is always a stochastic process.

## 5. Diffusion evolution

Particle dynamics in random electromagnetic fields is known to be described by the quasilinear theory (QLT) (*Sagdeev et al.*, [1969]). The QLT approach in particular, has been employed to deal with diffusion of electrons in the turbulent field of whistler waves packets (*Kennel et al.*, [1966]), (*Summers et al.*, [2004]). In coherent electromagnetic fields, on the other hand, the particle dynamics are not described by these theories. The nature of diffusion in this case is described by stochastic dynamics of particles, when the motion along stochastic trajectories gives rise to the so-called deterministic diffusion (*Lichtenberg et al.*, [1983]). General mathematical and physical aspects of this problem have been discussed, for instance, in reviews (*Arnold et al.*, [1968], *Zaslavsky et al.*, [1991]).

The purpose of the present paper is to investigate electron dynamics in a coherent packets of whistler mode waves. We have employed here the method in which dynamics

of the phase variables on stochastic attractor is included in a Fokker-Planck-Kolmogorov (FPK) equation (*Zaslavsky et al.*, [1991]).

In the present section, we shall exploit the canonical Hamiltonian structure that has been developed above. Thus, the drift kinetic equation follows at once since the canonical structure equations of motion (28, 29) have been established. The distribution function (probability density)  $f(u; t)$  obeys the Fokker-Planck-Kolmogorov (FPK) equation

$$\frac{\partial f(u; t)}{\partial t} = \frac{1}{2} \frac{\partial}{\partial u} D \frac{\partial f}{\partial u}, \quad (73)$$

which holds if  $Q \ll 1$  (*Zaslavsky et al.*, [1991]).

Here  $D$  is the conventional diffusion coefficient in phase space,

$$D = \langle (u_{n+1} - u_n)^2 \rangle T^{-1}, \quad (74)$$

in which  $(u_{n+1} - u_n)$  is substituted from (41), and  $\langle \cdot \rangle$  denotes the phase average,  $T$  is the timescale of mapping (41). The function  $f(u, t)$  belongs to the space of all differentiable functions supported in  $[-\pi, \pi]$ , and the functional

$$\int_{-\pi}^{\pi} f(u, t) du = 1 \quad (75)$$

is the condition of normalization.

First, by means of (41) we calculate using (74) the diffusion coefficient

$$D = 9Q^2/8T. \quad (76)$$

Then, making use of the result (50) proved above along with (73) and (76), we evaluate the characteristic time for redistribution  $u$  over the spectrum

$$t_d \simeq u_b^2/D = T(4\pi/3Q)^2. \quad (77)$$

A objective of this study is to determine the time-independent distribution function and the rate of heating. This requires a solution of the FPK equation together with a normalization (75). Thus a solution of the FPK equation along with the boundary condition  $f(-\pi) = f(\pi)$  in the limiting case  $t \geq t_d$  may be given in the form of the uniform distribution

$$f(u) = (2\pi)^{-1}. \quad (78)$$

Now, it becomes relevant to determine how the system evolves in time at  $t \leq t_d$ . We exploit the FPK equation with  $f(u)$  and its derivative  $\partial f / \partial u$  vanishing at the boundary. We introduce the moment  $\langle u^2 \rangle = \int_{-\pi}^{\pi} du u^2 f(u)$ , multiply equation (73) by  $u^2$ , and integrate the resulting equation over  $u$  to obtain

$$d \langle u^2 \rangle / dt = D. \quad (79)$$

We know, that the variables  $u$  and  $\varepsilon$  are related by (42). This allows us to attach all possible states of  $\varepsilon$  a probabilistic measure, namely, the probability density,  $f(\varepsilon, t)$ , which is associated with  $f(u, t)$  via the measure-preserving point transformation

$$f(\varepsilon, t) = f(u, t)(du/d\varepsilon). \quad (80)$$

Representing our results we start with the SLR case. In the case the variables  $u$  and  $\varepsilon$  are associated by relation (42). Then equation (39) along with the normalization

$$\int_0^{\varepsilon_b} f(\varepsilon) d\varepsilon = 1$$

allow us to derive the time-independent distribution

$$f(\varepsilon)d\varepsilon = \varepsilon_b^{-1}d\varepsilon, \quad \{\varepsilon \in R_+ | 1 \leq \varepsilon \leq \varepsilon_b\}, \quad (81)$$

and the FPK equation describing the evolution of an energetic spectrum in the allowed range of particle energies

$$\frac{\partial}{\partial t} f(\varepsilon, t) = \frac{\partial}{\partial \varepsilon} D(\varepsilon) \frac{\partial}{\partial \varepsilon} f(\varepsilon, t), \quad (82)$$

$$D(\varepsilon) = \frac{\alpha^3 N^2 b^2}{16 \varepsilon T}. \quad (83)$$

One may observe that there is an explicit dependence of  $D$  on  $\varepsilon^{-1}$ . Consequently the heating rate of electron decreases with its energy as  $\varepsilon^{-2}$ , namely,

$$\dot{\varepsilon} = \frac{\alpha^3 N^2 b^2}{48 T \varepsilon^2}, \quad 1 \leq \varepsilon \leq \varepsilon_b. \quad (84)$$

The restriction on  $\varepsilon$  in (84) is needed because the energetic spectrum is bounded above by  $\varepsilon_b$ .

Then from (82), (83) and (73) we find the characteristic time to establish the distribution (81)

$$t_d = 16 T \left( N / 64 b^2 \right)^{2/5}. \quad (85)$$

Now via (81) we evaluate the means

$$\langle \varepsilon \rangle = (1/2) \varepsilon_b, \quad \langle \varepsilon^2 \rangle = (1/3) \varepsilon_b^2, \quad (86)$$

and the relative standard deviation,

$$\frac{\sqrt{\langle \varepsilon^2 \rangle - \langle \varepsilon \rangle^2}}{\sqrt{\langle \varepsilon^2 \rangle}} = 0.5. \quad (87)$$

The distribution function  $f(u, t)$  in the non-relativistic case also obeys the FPK equation as a consequence of the fact that the equations of motion, expressed in coordinates  $(u, \psi)$  reflect the underlying canonical Hamiltonian structure of the equations (61). By reason of (71), we write the normalization as

$$\int_0^1 f(u) du = 1. \quad (88)$$

Then by means of (70) along with (71) and (70), we calculate by formulas (74) and (77) the diffusion coefficient

$$D = Q^2/2T = \alpha^2 N^2 b^2 \omega_B^2 / 8T v_0^2 \omega^2, \quad (89)$$

and the characteristic time for redistribution  $u$  over the spectrum

$$t_d = 4T/Q^2 = 16T v_0^2 \omega^2 / N^2 \alpha^2 b^2 \omega_B^2. \quad (90)$$

Next using the normalization for  $f(u, t)$  in the FPK equation, it follows when  $t > t_d$  the distribution function can be given by

$$f(u) du = (1 - u_c)^{-1} du. \quad (91)$$

This means that the random variable  $u$  is evenly distributed on  $[u_c, 1]$ .

Now from the transformation  $f(u) du = f(v) dv$  and the relation  $u = v_t/v_b$  we obtain

$$f(v_t) = v_b^{-1}. \quad (92)$$

Taking account of (88), by (92) we derive the quadratic mean of  $v_t$

$$\begin{aligned} \langle v_t^2 \rangle &= \int_0^{v_b} f(v_t) v_t^2 dv_t = v_b^2/3 \\ &= 4/3 \cdot v_{ph}^2 (\omega_B - \omega)/\omega, \end{aligned} \quad (93)$$

the mean value of  $|v_z|$  and the mean velocity

$$\langle |v_z| \rangle = 2/3 \cdot v_{ph} (\omega_B - \omega)/\omega_B, \quad (94)$$

$$\langle v_z \rangle = 0. \quad (95)$$

Then we define by

$$T_t = \langle m v_t^2 / 2 \rangle, \quad T_z = \langle m v_z^2 / 2 \rangle, \quad (96)$$

to find the ratio

$$T_t/T_z = 5\alpha/8v_{gr} = 5/8 \cdot \omega_B^2/\omega(\omega_B - \omega), \quad (97)$$

describing the anisotropy of distribution (61) in the  $(v_z, v_t)$  phase space. Anisotropic distribution function (61) with its derivative

$$\frac{dv_t}{dv_z} = -\frac{1}{2}\sqrt{\alpha/(v_{gr} - |v_z|)} \operatorname{sgn} v_z, \quad (98)$$

that tends to infinity as  $|v_z| \rightarrow v_{gr}$ , describe the so-called pancake distribution in the  $(v_z, v_t)$  phase space.

Again it is useful to evaluate the energy distribution function,  $f(\varepsilon)$ , for the domain of non-relativistic energies. Considering (61), we note that the particle energy  $E$  is a specified function of  $v_t$ . Thus the measure-preserving transformation  $f(\varepsilon) d\varepsilon = f(v_t) dv_t$  determines this problem completely, subject to appropriate boundary conditions. To that end, we have to find the lower ( $\varepsilon_c$ ) and upper ( $\varepsilon_0$ ) boundaries of an energetic spectrum. At first, from (61) we evaluate threshold value of particle speed

$$v_{th} = |v_z|_c = v_{gr} - v_c^2/\alpha. \quad (99)$$

Now it is clear that  $\varepsilon_c$  and  $\varepsilon_0$  are given by

$$\varepsilon_c = 1/2 \cdot \left[ (v_{gr} - v_c^2/\alpha)^2 + v_c^2 \right], \quad (100)$$

$$\varepsilon_0 = v_b^2/2 = \alpha v_{gr}/2. \quad (101)$$

The threshold energy  $\varepsilon_c$  is a function of the magnitude of wave field,  $b$ , and, in view of (71)  $\varepsilon_c$  hinges on  $b$  as follows:

$$\varepsilon_c \rightarrow \infty \text{ as } b^{-4}, \text{ as } b \rightarrow 0, \quad (102)$$

$$(\varepsilon_c - v_{gr}^2/2) \rightarrow 0 \text{ as } b^{-2}, \text{ as } b \rightarrow \infty. \quad (103)$$



The corresponding solution for  $f(\varepsilon)$  provided that

$$\int_{\varepsilon_c}^{\varepsilon_0} f(\varepsilon) d\varepsilon = 1$$

can be expressed in the form

$$f(\varepsilon) d\varepsilon = 1/2 \cdot [(\varepsilon_0 - \varepsilon_c)(\varepsilon - \varepsilon_c)]^{-1/2} d\varepsilon, \quad \varepsilon_0 > \varepsilon > \varepsilon_c. \quad (104)$$

This solution relates directly to the behavior of the system near the order-chaos bifurcation transition. The function  $f(\varepsilon)$  undergoes a sudden change at  $\varepsilon = \varepsilon_c$ , and in the region of regular motion ( $\varepsilon < \varepsilon_c, \varepsilon = \text{const}$ ) has the form of the Dirac  $\delta$ -function,  $f(\varepsilon) = \delta(\varepsilon - \text{const})$ .

Thus  $f(\varepsilon)$  describes the density of states in an energy space, the mean particle energy and relative standard deviation can be calculated by

$$\int_{\varepsilon_c}^{\varepsilon_0} \varepsilon f(\varepsilon) d\varepsilon = 1/3(\varepsilon_0 - \varepsilon_c) + \varepsilon_c, \quad (105)$$

$$\frac{\sqrt{\langle \varepsilon^2 \rangle - \langle \varepsilon \rangle^2}}{\sqrt{\langle \varepsilon^2 \rangle}} = 0.4. \quad (106)$$

The latter indicates the high level of fluctuations in the energetic spectrum of an electron.

Evolution of  $a$  means at  $t < t_d$  is governed by (79), where  $D$  is now given by (90).

Equation (79) in the explicit form may be written as

$$\frac{d}{dt} \langle v_t^2 \rangle = \alpha^2 b^2 N^2 \omega_B^2 / 8T\omega^2. \quad (107)$$

Then we have respect to the invariant of motion along with (107), and define by  $\mu_a = v_t^2/2B$  the first adiabatic invariant to find the rate of a change of  $\mu_a$

$$\dot{\mu}_a = \alpha^2 b^2 N^2 \omega_B^2 / 16TB\omega^2, \quad (108)$$

and the heating rate

$$\dot{\varepsilon} = D_\varepsilon, D_\varepsilon = \alpha^2 b^2 N^2 \omega_B^2 / 16 T \omega^2. \quad (109)$$

This result is nontrivial, because chaotic motion in the  $(u, \psi)$  phase space leads to important and easily observable macroscopic effects such as the stochastic heating of plasma particles.

Now we describe the effects associated with stochastic heating of high-energy particles. First, we discuss the pitch angle distribution in a range of non-relativistic energies.

Denote by  $\chi_p$  the pitch angle, and let  $\chi$  be its complementary angle. Then we define

$$\tan \chi = v_z / v_t, \quad \chi \in (-\pi/2, \pi/2), \quad (110)$$

and use equations (61) to represent  $\tan \chi$  as a function of  $u (u = v_t / v_b)$ , namely,

$$\tan \chi = (v_{gr} / \alpha)^{1/2} \cdot (1 - u^2) / u. \quad (111)$$

Then via (91) and the measure-preserving point transformation  $f(u) du = f(\chi) d\chi$ , we find the angle distribution

$$\begin{aligned} f(\chi) = & (\alpha / v_{gr})^{1/2} \left( u^2 / (1 + u^2) \right. \\ & \left. + (v_{gr} / \alpha) (1 - u^2)^2 / (1 + u^2) \right). \end{aligned} \quad (112)$$

Equation (112) determines the angle distribution function in a parametric form. Nonlinear transformation (111) allows us to express (112) as a function of a proper variable  $\chi$ . Note that  $f(\chi)$  is invariable under transformation

$$u \rightarrow -u, \quad \chi \rightarrow -\chi; \quad f(\chi) = f(-\chi), \quad (113)$$

therefore  $f(\chi)$  is a symmetric function on  $(-\pi/2, \pi/2)$ . However, the explicit expression for  $f(\chi)$  has a very complicated form. The following asymptotic formulas are valid:

$$f(\chi) \approx 1/2 \cdot (\alpha/v_{gr})^{1/2} \left(1 - (\alpha/v_{gr})^{1/2} \tan \chi\right) \quad \text{as } \chi \rightarrow 0, \quad (114)$$

$$f(\chi) \approx (v_{gr}/\alpha)^{1/2} (1 + \tan^{-1} \chi), \quad \text{as } \chi \rightarrow \pm\pi/2, \quad (115)$$

$$f(\chi) \approx (v_{gr}/\alpha)^{1/2} (1 + 2(\varepsilon/\varepsilon_c - 1)) \quad \text{as } \varepsilon \downarrow \varepsilon_c, \chi \rightarrow \pm\pi/2, \quad (116)$$

$$f(\chi) \approx 1/2 \cdot (v_{gr}/\alpha)^{-1/2} \left(\sqrt{\varepsilon/\varepsilon_c} - 1\right) \quad \text{as } \varepsilon \uparrow \varepsilon_0, \chi \rightarrow 0. \quad (117)$$

where  $\varepsilon_c$  and  $\varepsilon_0$  are given by (100) and (101).

Making use of these equations, we conclude that the function  $f(\chi)$  governing the distribution of angles is a convex symmetric function of  $\chi$ , having the maximum at  $\chi = 0$  ( $\chi_p = \pi/2$ ). The result indicates also that particles are equally likely to be scattered in the direction with or against the wave. To this may be added that the given function is an increasing function of the particle energy on  $(\varepsilon_c, \varepsilon_0)$ , and it hinges asymptotically on  $\varepsilon$  as  $(\varepsilon - \varepsilon_c)$  near the energetic threshold, and as  $\sqrt{\varepsilon/\varepsilon_c} - 1$  at  $\varepsilon > 2\varepsilon_c$ .

Let us discuss particle scattering in a range of relativistic energies. Because  $A/m \ll 1$ , we write

$$\tan \chi = v_z/v_t \approx p_z/p_t \approx \varepsilon_z/\varepsilon_t, \quad \chi \in (-\pi/2, \pi/2). \quad (118)$$

at energies above 1 MeV.

Taking account of the invariant of motion (35) we derive the following dependence

$$\tan \chi = (\varepsilon_z/\alpha)^{1/2}. \quad (119)$$

Using (81) along with the measure-preserving transformation  $f(\chi) d\chi = f(\varepsilon) d\varepsilon$  yields the time-independent pitch angle distribution  $f(\chi)$ , namely,

$$f(\chi) d\chi = 2 \tan^{-2} \chi_m \cdot \cos^{-2} \chi \cdot |\tan \chi| d\chi, \\ \chi \in (-\chi_m, \chi_m), \tan \chi_m = (\varepsilon_b/\alpha)^{1/2}, \quad (120)$$

and its dependence on the particle energy

$$f(\chi(\varepsilon)) = 2\varepsilon_b^{-1}(\alpha\varepsilon)^{1/2}(1 + \varepsilon\alpha^{-1}). \quad (121)$$

The function given by (120) is the concave symmetric function of  $\chi$ , which has a maximum at  $\chi = \pi/2$  ( $\chi_p = 0$ ), and its derivatives tend to infinity as  $\chi \rightarrow \pm\pi/2$ . This function describes the so-called U-like distribution.

Since the magnetic field is constant, the gyroradius is a direct measure of the perpendicular electron velocity. Thus the stochastic heating will be accompanied by a radial drift of particles in space. Indeed, in view of the relation  $r = v_t/\omega_B$  and (107), we get the following expression

$$\frac{d \langle r^2 \rangle}{dt} = \omega_B^{-2} \frac{d \langle v_t^2 \rangle}{dt} = D_t, \quad (122)$$

$$D_t = \alpha^2 b^2 N^2 / 16 T \omega_B^2, \quad \varepsilon > \varepsilon_c, \quad (123)$$

where  $D_t$  is the coefficient of collisionless diffusion across the ambient magnetic field. Then we evaluate the radial drift in a range of relativistic particle energies.

First via (9) and (35) we get the relation

$$r = \sqrt{\alpha\varepsilon} \omega_B^{-1}. \quad (124)$$

Now we write down the diffusion coefficient as

$$D_t(\varepsilon) = \frac{\alpha^4 N^2 b^2}{48 T \omega_B^2} \cdot \varepsilon^{-2}, \quad \varepsilon \leq \varepsilon_b, \quad (125)$$

which immediately follows from (84) and above result.

Also relation (124) allows us to represent  $D_t$  as a function on  $r$ ,

$$D_t(r) = \frac{\alpha^4 N^2 b^2}{48 T \omega_B^2} \cdot \left( \frac{r_0}{r} \right)^4, \quad r_0 = \sqrt{\alpha / \omega_B}, \quad r_b = \sqrt{\alpha \varepsilon_b} / \omega_B, \quad r_0 < r < r_b. \quad (126)$$

Equations (123, 125) and (126) are correct if the inequality  $r/L \ll 1$  holds. In view of equation (124), this inequality can be written in the form  $r/L = \sqrt{\alpha \varepsilon_b} \omega / N \omega_B v_{ph} \ll 1$ . For typical parameters this condition is satisfied trivially.

## 6. Application

Enhanced convection electric fields associated with solar wind streams provide the principle mechanism for the intensification of ring current (10 – 100) keV flux, and also leads to the excitation of whistler mode waves *Hastings et al.*, [1996]. The zone of most intense wave activity is spatially localized because of the decrease in resonant energy and wave guiding by strong density gradients associated with the plasmopause *Wolf* [1995]. Typical whistler wave amplitudes are in the range (10 – 100) pT, but occasionally wave amplitude approaches 1 nT. The wave magnitudes of outer zone chorus emissions are usually small enough, resulting in weak diffusion scattering. Consequently, the electron heating should occur gradually over many drift orbits.

To check the performance of the method, at first the parameters of the problem are chosen to reflect a typical whistler wave interacting with electrons at locations outside the plasmopause, where the electron cyclotron frequency  $\omega_B = 6 \cdot 10^4 \text{ s}^{-1}$ , the electron plasma frequency  $\omega_p = 12 \cdot 10^4 \text{ s}^{-1}$ , and the wave frequency  $\omega = 3 \cdot 10^4 \text{ s}^{-1}$ . Thus the dispersion relation (8) corresponds, undoubtedly to the problem. By (8) we find the phase velocity,  $v_{ph} = 0.25$ , and the group velocity,  $v_{gr} = 2 \cdot v_{ph}(\omega_B - \omega / \omega_B)$ ,  $v_{gr} \simeq$

0.25. The theory is based on three parameters, namely,  $\alpha, N, b$ . First we evaluate  $\alpha = 2v_{ph}\omega_B/\omega$ ,  $\alpha = 1.0$ . Typical magnitudes of whistler mode waves are in a range of  $10^{-6} - 10^{-7}$  G, consequently the parameter  $b$  is of order  $10^{-4} - 10^{-3}$ . We set  $b = 1.3 \cdot 10^{-4}$  below. To evaluate the parameter  $N = \omega T$  for the time-like wave packet, we need to estimate the transit time of the wave packet through the domain of resonant interaction (the zone of intense wave activity). Thus the characteristic size of this region is of the same order of magnitude as Earth's radius, we have accepted  $L = 5 \cdot 10^8$  cm. By that the value of  $T$ ,  $T = L/v_{gr}$ , is about  $7 \cdot 10^{-2}$  s, and  $N = 2 \cdot 10^3$ . In the following we will use these values to obtain an estimate of some quantities. First we calculate the threshold value of electron energy,  $E_c$ , at a given value of  $b$ . Applying (71) to (100) yields an estimate of  $E_c$ ,  $E_c = mc^2\varepsilon_c$ ,  $E_c \simeq 16$  keV. Then we calculate by (52) the upper value of the energetic spectrum,  $E_b \simeq 3.5$  MeV. Considering (123, 109), we evaluate the rate of stochastic diffusion across the ambient magnetic field,  $D_t \simeq 3.5 \cdot 10^{10}$  cm<sup>2</sup>s<sup>-1</sup>, and the heating rate,  $D_E = D_\varepsilon mc^2$ ,  $D_E \simeq 70$  keVs<sup>-1</sup>. In the relativistic region of particle energies, the expressions (125, 84) yield  $D_t \simeq 1.1 \cdot 10^{10}(m/E)^2$  cm<sup>2</sup>s<sup>-1</sup>,  $D_E \simeq 25(m/E)^2$  keVs<sup>-1</sup>. Note that  $D_E$  as well as  $D_t$  are a decreasing function of  $E$  in a relativistic range of the energy spectrum, while  $D_E$  and  $D_t$  in a non-relativistic domain of particle energy are independent on  $E$ . The following dependencies,  $\omega_p \propto \sqrt{n_e}$ ,  $v_{ph} \propto B/\sqrt{n_e}$  and  $\alpha N \propto B$ , allow us to express  $D_E$  as a function of wave amplitude  $B^w$ , strength of an ambient magnetic field  $B$ , and plasma particle density  $n_e$  and arrange the resulting expression as the scaling law  $D_E = const.(B/\sqrt{n_e})(B^w)^2$ . Using above result in (123), we reveal that  $D_t$  hinges on  $B$  as  $B^{-1}$ .

Now the effect of pitch angle scattering of relativistic electrons should be evaluated. From (120) at  $\varepsilon_b = 7$  and  $\alpha = 1$  follows that in this case whistler waves can diffuse electrons in a cone with the vertex angle,  $\chi_v = 2\chi_b$ , which is about  $140^\circ$ . The equation (121) indicates a degree of scattering anisotropy,  $\deg A = f(\chi)(\varepsilon = \varepsilon_b)/f(\chi)(\varepsilon = 1)$ , increases with  $\varepsilon$  up to  $\deg A \simeq 20$ . Now we respect to the pitch angle scattering of non-relativistic electrons. The results (116) and (117) show that the wave packet to effectively scatter electrons in pitch angle leading to establishing the distribution peaked at  $\chi_p \simeq 90^\circ$  with the degree of anisotropy  $\deg A = 2$ .

Now to see that our approach correctly interprets the facts, it is necessary to find the range of validity of our results. Thus the basic equations are applicable provided  $kL \gg 1$ , and  $\omega_B T \gg 1$ . Then it is necessary to establish that the characteristic time for the energy the distribution over the spectrum is small as compared with the period of an intense wave activity. According to (90) and (85) the energization time for non-relativistic electrons is about 7 s, while the time for establishing the energy distribution in a range of relativistic energies is approximately 0.5 h. Note that the efficiency of this process is high enough, because observations show that the power of whistler-mode waves is enhanced at substorm injection and decays over a period of hours or over a period of a few days during the storm recovery (*Baker et al.*, [1986], *Horne et al.*, [2003]).

Now the threshold value of the wave field given by (65) may be evaluated as  $b_c \simeq 10^{-7}$ . The equations of motion (38) in the region of relativistic energies are obtained in the faithful representation, and these include the ultra-relativistic approximation,  $\varepsilon^2 \gg 1$ , that has been employed in deriving (41). According to (35), this inequality may be

written in the form

$$2(\omega_B/\omega)v_{ph}\varepsilon^{-1} < 1. \quad (127)$$

Owing to (52) the latter can be transformed into the requiring

$$b = B^w/B > 4N^{-2} \ (\simeq 10^{-6}). \quad (128)$$

These conditions are fairly easy to check. Recalling direct amplitude measurements (*Nagano et al.*, [1996]) and the extremes of spectral density measurements (*Parrot et al.*, [1994]) yield  $(B^w/B) = (10^{-4} - 10^{-3})$ , we conclude that (128) trivially satisfied, therefore, the lower- and the upper limits for the energetic spectrum are correct.

Note that in accordance with *Faith et al.*, [1997] the magnitude of wave field required to induce chaos by a single wave is about  $10^{-1}$ , which is quite larger compared with those actually observed.

It should be noted that the geometry of an ambient magnetic field does not play a role in the given problem, thus the lengthscale of an interaction region is typically smaller than the travelling path, i.e. an ambient magnetic field is always locally an uniform field. To this may be added that the analytic model describing the motion of a particle out of the interaction region must include the radial diffusion of electrons conserving the first and second adiabatic invariants.

These results demonstrate the possibility of electron heating over a wide energy range between  $\geq 10$  keV and a few MeV. This is in reasonable agreement with the experimental observations.



## 7. Summary

A canonical Hamiltonian approach has been employed to deal with the whistler wave-electron interaction and the stochastic heating of high-energy electrons in magnetized plasmas. The time-like and space-like wave packet representations have been used in deriving the equations of motion for non-relativistic and respectively relativistic electrons. It is shown that phase dynamics of the system is realized on stochastic attractor, all means on this attractor are stable and irrelevant to any initial conditions.

The results afford a basis for conclusions.

Solutions of these equations have revealed the existence of an energetic threshold below which electron motion is regular. This threshold can be expressed in terms of the normalized magnitude of wave field,  $b$ , as  $\varepsilon_c = \text{const.} \cdot b^{-4}$  as  $b \rightarrow 0$ , and  $(\varepsilon_c - v_{gr}^2/2) = \text{const.} \cdot b^{-2}$  as  $b \rightarrow \infty$ . When the initial energy is greater than the energetic threshold, an electron moves stochastically. Thus, nonlinear electron acceleration by a whistler wave packet is always a stochastic process. Chaotic motion gives rise to diffusion in energy, and leads to establishing an energy spectrum, bounded above by  $\varepsilon_b$ . The upper value of energy spectrum has a weak dependence on  $b$  ( $\propto b^{2/5}$ ). The energy spectrum in a range of non-relativistic electron energies obeys the law  $f(\varepsilon) = \text{const.}/(\varepsilon - \varepsilon_c)$  which describes the order-chaos bifurcation transition at  $\varepsilon = \varepsilon_c$ . While the relativistic electron distribution obeys the uniform distribution.

The obtained energetic spectra have been used for evaluating pitch angle distribution functions over different energies. Whistler mode waves can scatter non-relativistic electrons with pitch angles of up to  $90^\circ$  and are responsible for the formation of the pancake distribution in the  $(v_z, v_t)$  phase space. Relativistic electrons are mainly scattered by these waves.

almost along the direction of an external magnetic field. The high-energy electrons with chaotic motion are equally likely to scatter in the direction with or against the wave, and there exists a certain probability that some particles will be scattered into a loss cone and will precipitate into the polar regions along magnetic field lines. Chaotic motion of an electron in a constant and homogeneous magnetic field is accompanied by radial drift. This effect must have associated with stochastic heating of the particle. High-energy electrons observed in Earth's radiation belts can be regarded as tails of the electron distributions in magnetosphere. According to our analysis, for a spectrum of a whistler modes interacting nonlinearly with Earth's radiation belts electrons, substantial energization can occur in a range of energy from a few keV up to a few MeV. Thus stochastic processes occurring in the plasma can play an important role in the evolution of the energy spectra of radiation belts electrons.

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**Figure 1.** The phase space of the map  $g^n$ . (a) One single trajectory of length  $10^6$  for  $Q = 0.01\pi$ . The trajectory was started from the point  $u_0 = 10^{-3}$ ,  $\psi_0 = 10^{-4}$ . The identification of points is indicated by the arrows. (b) Close to zero, approximation  $g^n$  loses its validity. Therefore in the vicinity of  $u = 0$  we must use the solution of the original equations (52) and (38) expressed in terms of the variables of  $u$ ,  $\psi$  given by (42). The figure shows the boundary of the chaotic region is well approximated by the condition (50). The parameters are the the same as in Figure 1a.

**Figure 2.** Joint distribution (density of an imaging points)  $F(\psi, u)$  obtained numerically from (41) with the parameters as in Figure 1.

**Figure 3.** The phase space of the map  $G^n$ . (a) A single trajectory of length  $10^6$  for  $Q = 10^{-3}$ ,  $s = 5 \cdot 10^3$ ,  $(u_0, \psi_0) = (0.01, 0.0001)$ . (b) A phase curve for the same values of  $Q$  and  $s$  as in (a). The trajectory was started from the point  $(0.11, 0.0001)$ . (c) One trajectory of length  $7 \cdot 10^6$  for  $Q = 0.5 \cdot 10^{-3}$ ,  $s = 510^{-3}$ ,  $(u_0, \psi_0) = (0.21, 0.0001)$