A Fast Method of Deriving the Kirchhoff Formula for Moving Surfaces

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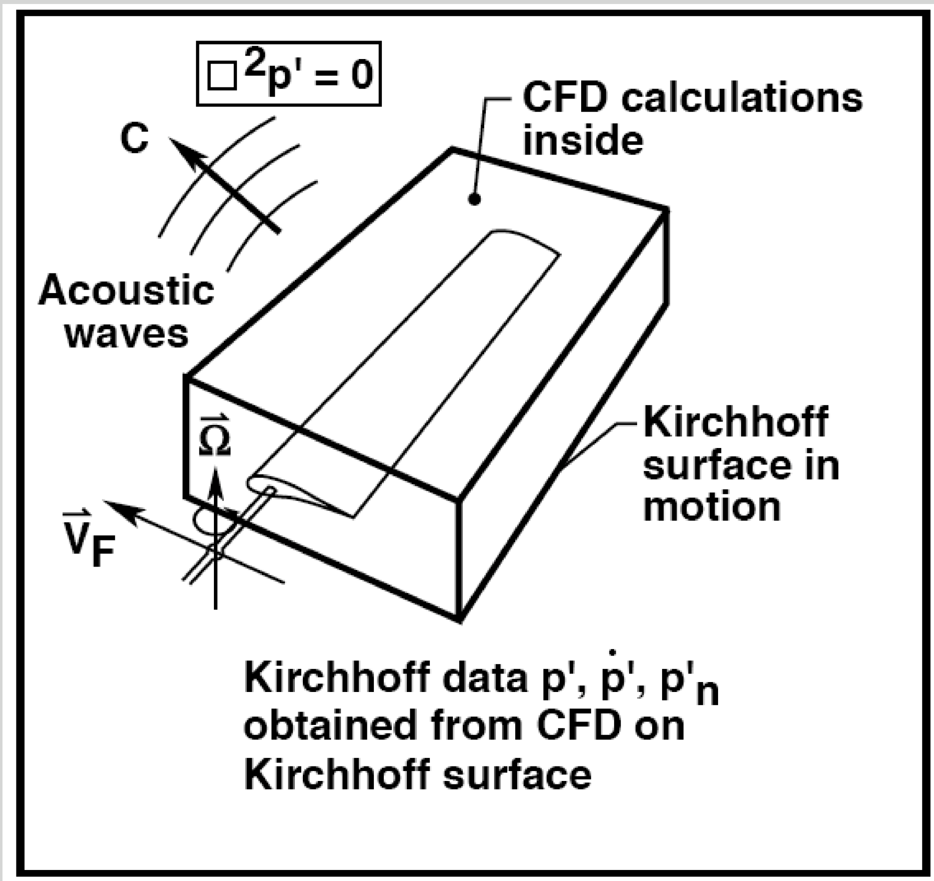
Outline

- Motivation and the Problem
- The Imbedding Method
- Generalized Differentiation Rules
- Classical Kirchhoff Formula
- Kirchhoff Formula for a Moving Surface
- Concluding Remarks

Motivation and the Problem

Helicopter rotor noise prediction



Kirchhoff formula for a moving surface gives p' in the exterior region from surface data.

Derivation of Kirchhoff Formula
for a Moving Surface

Classical Analysis

- 4D Green's Theorem required
- Some ambiguities working with 4D integrals
- Extremely involved algebraic manipulations
- Difficulties with physical meaning of some terms

References:

Classical analysis: W. R. Morgans: The Kirchhoff formula extended to a moving surface, Philosophical Magazine, 9, 1930, 141-161

Modern analysis using generalized functions: F. Farassat and M. K. Myers: Extension of Kirchhoff's formula to radiation from moving sources, Journal of Sound and Vibration, 123 (3), 1988, 451-460

The Imbedding Method- The Idea

- When the Green's function *G (x, y)* of a differential equation $\mathcal{L}$ *u* = *f* is known in the a domain Σ, then that Green's function can be utilized in solving the *same* differential equation in a *smaller* domain $\Sigma' \subset \Sigma$.

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- The problem in the domain Σ' is **imbedded** in the larger domain Σ by giving a ***known*** value to the unknown function ***in the region*** $\Sigma \setminus \Sigma'$ which satisfies the boundary conditions of the problem in Σ.

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- The problem in the domain Σ' is imbedded in the larger domain Σ by giving a *known* value to the unknown function *in the region Σ \ Σ'* which satisfies the boundary conditions of the problem in Σ.

- The imbedded problem, in general, has a *discontinuous solution* at the boundary surface ∂Σ' of the domain Σ'. This means that the best tool to solve the imbedded problem is the generalized function (GF) theory.

The Imbedding Method (cont'd)

- The Green's function method is applicable in finding the discontinuous solution of differential equation as long as all the derivatives are viewed as generalized derivatives.

References:

F. Farassat, Introduction to Generalized Functions With Applications in Aerodynamics and Acoustics, NASA Technical Paper 3428, 1994

F. Farassat: The Kirchhoff Formulas for Moving Surfaces in Aeroacoustics - The Subsonic and Supersonic Cases, NASA Technical Memorandum 110285, September 1996

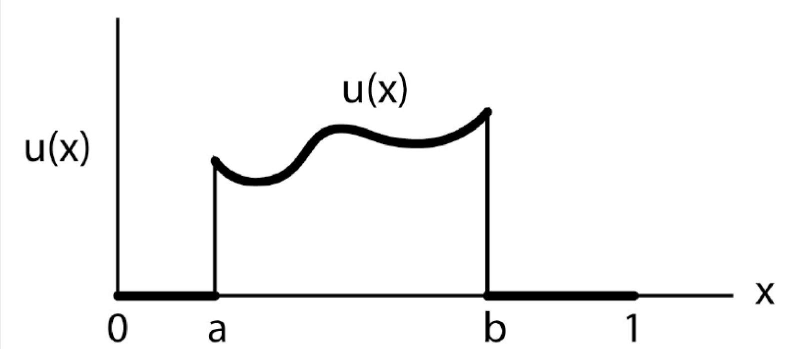
F. Farassat, et. al: Working With the Wave Equation in Aeroacoustics- The Pleasures of Generalized Functions, AIAA-2007-3562, 2007

An Example of the Imbedding Method

Consider $l\,u(x) = u''$ on [0, 1], BCs: $u(0) - 2\,u'(0) = 0$ & $u(1) + u'(1) = 0$. The Green's function is

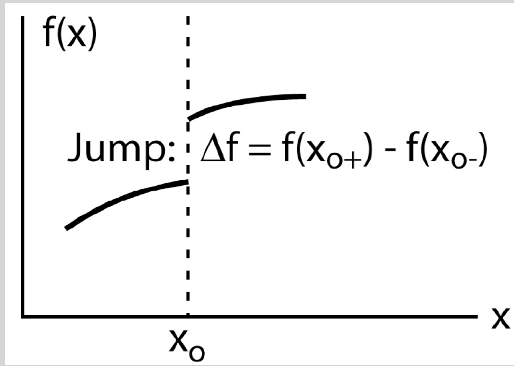
$$g(x,\,y) = \begin{cases} \dfrac{(y-2)\,(x+2)}{4} & x < y \\ \dfrac{(y+2)\,(x-2)}{4} & x > y \end{cases}$$

Use above Green's function to solve the same ODE $l\,u(x) = u''(x) = k(x)$ on [a, b] ⊂ [0, 1] by the imbedding method. Let unknown function take value 0 on [0, 1] \ [a, b].



Then use generalized function (GF) theory to find new ODE.

Generalized Differentiation Rules



Generalized derivative (GD) **of a differentiable function with a single jump discontinuity at x_0 is**

$$\overline{f}'(x) = f'(x) + \Delta f\, \delta(x - x_0)$$

GD of a function k (x) in multidimensions with a discontinuity Δk across a surface f = 0, ∇f = n, is

$$\overline{\nabla} k(\boldsymbol{x}) = \nabla k(x) + \Delta k\, \boldsymbol{n}\, \delta(f)$$

An Example of the Imbedding Method (cont'd)

New ODE of example imbedded problem is:

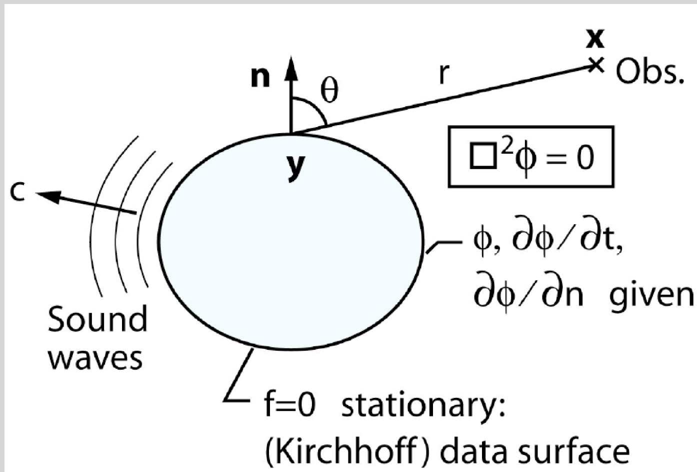
$$\bar{\tilde{u}}''(x) = \tilde{k}(x) + u'(a)\,\delta(x-a) - u'(b)\,\delta(x-b) + u(a)\,\delta'(x-a) - u(b)\,\delta'(x-b)$$

with the following definitions:

$$\tilde{u}(x) = \begin{cases} u(x) & x \in [a, b] \\ 0 & x\,\varepsilon\,[0, 1] \setminus [a, b] \end{cases}$$

$$\tilde{k}(x) = \begin{cases} k(x) & x \in [a, b] \\ 0 & x\,\varepsilon\,[0, 1] \setminus [a, b] \end{cases}$$

Classical Kirchhoff Formula



Imbed this problem in unbounded 3D space by assuming that the unknown function is 0 inside the surface f = 0 (∇ f = n). Let

$$\tilde{\varphi}(\boldsymbol{x}, t) = \begin{cases} \varphi(\boldsymbol{x}, t) & f > 0 \\ 0 & f < 0 \end{cases}$$

Classical Kirchhoff Formula (cont'd)

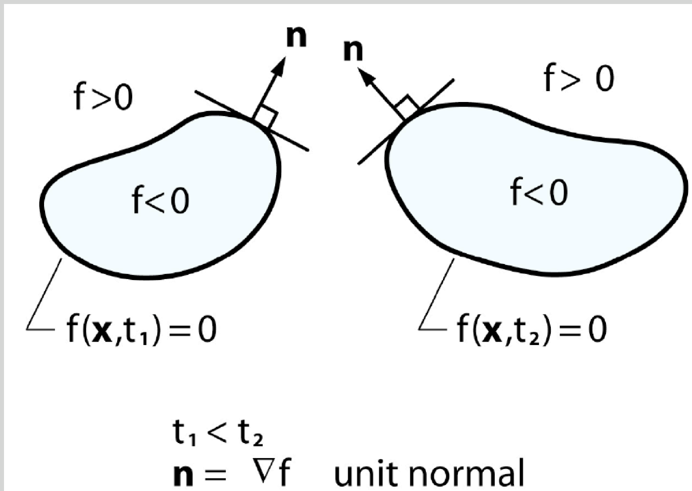
New PDE to derive the classical Kirchhoff formula is:

$$\bar{\Box}^2 \tilde{\varphi} = -\varphi_n \delta(f) - \nabla \cdot [\varphi \boldsymbol{n} \delta(f)]$$

Next use the Green's function of the wave equation in the unbounded space to get the result.

The rest is simple algebraic manipulations!

Kirchhoff Formula for a Moving Surface



To solve $\square^2 \varphi = Q(x, t)$ **in the exterior of the moving surface f = 0, imbed this problem in unbounded 3D space taking the value 0 inside the moving surface. Let:**

$$\tilde{\varphi}(x, t) = \begin{cases} \varphi(x, t) & f > 0 \\ 0 & f < 0 \end{cases}$$

$$\tilde{Q}(x, t) = \begin{cases} Q(x, t) & f > 0 \\ 0 & f < 0 \end{cases}$$

Kirchhoff Formula for a Moving Surface (cont'd)

New PDE to derive Kirchhoff formula for a moving surface is:

$$\bar{\Box}^2 \tilde{\varphi} = \tilde{Q}(\boldsymbol{x}, t) - \frac{1}{c} \varphi_t M_n \delta(f) - \frac{1}{c} \frac{\partial}{\partial t} [\varphi M_n \delta(f)] - \varphi_n \delta(f) - \nabla \cdot [\varphi \boldsymbol{n} \delta(f)]$$

Use the Green's function of the wave equation in the unbounded space to get the result.

The rest is again simple algebraic manipulations!

Concluding Remarks

The imbedding method increases the utility of a known Green's function in solving new BV and IV problems of acoustics and other applied fields.

Generalized function theory provides a fast method of finding both the classical Kirchhoff formula and the Kirchhoff formula for a moving surface.