The Aeroacoustics of Turbulent Flows

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Abstract:-Aerodynamic noise prediction has been an important and challenging research area since James Lighthill first introduced his Acoustic Analogy Approach over fifty years ago. This talk attempts to provide a unified framework for the subsequent theoretical developments in this field. It assumes that there is no single approach that is optimal in all situations and uses the framework as a basis for discussing the strengths weaknesses of the various approaches to this topic. But the emphasis here will be on the important problem of predicting the noise from high speed air jets. Specific results will presented for round jets in the 0.5 to 1.4 Mach number range and compared with experimental data taken on the Glenn SHAR rig. It is demonstrated that non-parallel mean flow effects play an important role in predicting the noise at the supersonic Mach numbers. The results explain the failure of previous attempts based on the parallel flow Lilley model (which has served as the foundation for most jet noise analyses during past two decades).

Key-Words: -Aeroacoustics, Jet Noise, Turbulence, Aerodynamic Sound, Turbulent Flow

1 Introduction

Aeroacoustics is concerned with sound generated by forces and stresses resulting from the motion of a fluid rather than the externally applied forces and motions of classical acoustics. Music (i.e., harmonious sound) is often generated in this fashion. But this paper is concerned with aerodynamic noise, (i.e., cacophonous sound), which is of engineering interest because of the demand for quieter transportation vehicles-such as aircraft, busses and automobiles. In particular the noise from jet engine exhausts has been a major concern since the type of engine was introduced over fifty years ago and wind noise from automobiles is now receiving considerable attention since other previously dominant noise sources have been reduced to relatively low levels.

The focus of this paper is on the fundamental theory. But since aerodynamic sound is just a byproduct of fluid motion and since the governing equations of that motion (i.e., the Navier-Stokes equations) are well established and have been for some time now (over 150 years) you might think that the only thing that we need to do here is write down these equations and be done with it. But the solution to these equations—especially for the turbulent flows that are the usual source of aerodynamic noise-- has, to put it mildly, proved to be very elusive. In fact it is probably no exaggeration to say that fluid turbulence is still ranked among the great unsolved problems of classical physics.

So there is a need to obtain useful and interesting information about the sound field without actually solving the full Navier-Stokes equations—which typically involves introducing some sort of reduced order model for these equations —which, in the present context, is usually referred to as an acoustic analogy (following James Lighthill

who was the first to do such analysis)[1]. This is, of course, analogous to the situation in turbulence modeling, which is almost always based on some form of the filtered (usually Favre [2] filtered) Navier-Stokes equations such as the Reynolds averaged Navier-Stokes (RANS) equations[3]. While there is some disagreement about the proper choice of the turbulence modeling equations, the situation in Aeroacoustics is considerably more contentious with significant disagreement about the appropriate starting equations (perhaps because there is no single set of equations that is optimal in all situations).There does, however, seem to be a consensus about some of the requirements for such equations. First of all, they should be derivable from the Navier-Stokes equations and secondly; they should, as argued in the well known text by Ann Dowling & Ffowcs Williams, be formally linear [4].

2 Fundamental Equations

It, therefore, makes sense to begin by dividing the flow variables in the Navier-Stokes equations, the density ρ, the pressure p, the enthalpy h, the velocity v_i, etc, into, say, their base flow components ,
$\bar{\rho}, \bar{p}, \tilde{h},$ and $\tilde{v}_i$ and 'residual' components ρ', p', h' and , v_i', which are essentially defined by

$$\rho = \bar{\rho} + \rho', \quad p = \bar{p} + p',$$

$$(1)$$

$$h = \tilde{h} + h', \quad v_i = \tilde{v}_i + v_i',$$

with the idea being that base flow is to be found from some relatively inexpensive

computational method, which means that it can, at best, be expected to provide an adequate representation of the most energetic part of the flow and perhaps (with enough computational effort) even the low frequency components of the sound field-but certainly not all of the spectral components that are of interest the high Reynolds number flows of practical importance. For example the noise from full jet engine exhausts has a peak frequency in the 200-500 Hz range, but the ear is most sensitive to sound in the 2,000 to 5,000 Hz frequency range. So jet noise prediction methods need to maintain their accuracy over an enormously broad range of frequencies.

So as a practical matter the base flow will have to be determined from some reduced order model of the Navier-Stokes equation. To my knowledge, these models are almost always based on the usual hyperbolic conservation laws that can be written (fairly compactly) as [5,6,7,8]

$$D_o \bar{\rho} = 0, \tag{2}$$

$$D_o \bar{\rho} \tilde{v}_i + \frac{\partial \bar{p}_e}{\partial x_i} = \frac{\partial}{\partial x_j} \tilde{e}_{ij} \tag{3}$$

$$D_o \left(\frac{\gamma \bar{p}_e}{\gamma - 1} + \frac{1}{2} \bar{\rho} \tilde{v}^2 \right) - \frac{\partial \bar{p}_e}{\partial t}$$
$$= \frac{\partial}{\partial x_i} \left[\frac{1}{(\gamma - 1)} \tilde{e}_{4i} + \tilde{e}_{ij} \tilde{v}_j \right] \tag{4}$$

where the Latin indices range from 1 to 3, the summation convention is being used,

$$D_o f \equiv \frac{\partial f}{\partial t} + \frac{\partial}{\partial x_j}\left(\tilde{v}_j f\right) \qquad (5)$$

for any function f, we assume that the base flow variables, as well as the original Navier-Stokes variables satisfy an ideal gas law equation of state, with γ being the specific heat ratio, $\overline{p}_e$ denotes a pressure-like variable that can differ from the thermodynamic pressure, and $\tilde{e}_{\lambda j}$ with $\lambda = 1, 2, 3, 4$ denotes an, as yet, arbitrary 4x3 dimensional stress tensor. (Greek indices will always range from 1 to 4.) These equations include, among other things, the Euler equations, the Navier Stokes equations themselves (which correspond to setting the first three components of $\tilde{e}_{\lambda j}$ equal to the Newtonian stress and the fourth component to the heat flux vector). And more generally, the 4x3 dimensional stress tensor can be inputted through a turbulence model, which also determines the difference between the effective pressure $\overline{p}_e$ and the thermodynamic pressure $\overline{p}$ --but in a purely passive fashion. However, this difference is largely irrelevant since these equations will form a closed system of 5 equations in 5 unknowns which can be solved for $\overline{p}_e$ independently of its relation to the thermodynamic pressure $\overline{p}$ once a particular turbulence model has been introduced.

It can now be shown that the remaining residual variables are determined by the five formally linear equations [5,6,7,8]

$$D_o \rho' + \frac{\partial}{\partial x_j} u_j = 0, \qquad (6)$$

$$D_o u_i + u_j \frac{\partial \tilde{v}_i}{\partial x_j} + \frac{\partial}{\partial x_i} p_e'$$

$$-\frac{\rho'}{\overline{\rho}} \frac{\partial}{\partial x_j} \tilde{\tau}_{ij} = \frac{\partial}{\partial x_j} e_{ij}'' \qquad (7)$$

$$D_o p_e' + \frac{\partial}{\partial x_j} \widetilde{c^2} u_j +$$

$$(\gamma - 1)\left(p_e' \frac{\partial \tilde{v}_j}{\partial x_j} - \frac{u_i}{\overline{\rho}} \frac{\partial \tilde{\tau}_{ij}}{\partial x_j} \right)$$

$$= \frac{\partial}{\partial x_j} e_{4j}'' + (\gamma - 1) e_{ij}'' \frac{\partial \tilde{v}_i}{\partial x_j}, \qquad (8)$$

where

$$\widetilde{c^2} = \gamma R \tilde{T} = \gamma \frac{\overline{p}}{\overline{\rho}} \qquad (9)$$

is the base flow sound speed squared and

$$\tilde{\tau}_{ij} \equiv \delta_{ij} \overline{p}_e - \tilde{e}_{ij} \qquad (10)$$

denotes the total base flow stress tensor.
These equations have been put into the linearized conservation law form by introducing the new dependent variables

$$p_e' \equiv p - \overline{p}_e + \frac{\gamma - 1}{2} \rho v'^2 \qquad (11)$$

$$u_i \equiv \rho v_i', \qquad (12)$$

as well as the new source strengths e_{vj}'', $v = 1, 2, 3, 4$ given by the difference

$$e_{j\mu}'' \equiv e_{j\mu}' - \tilde{e}_{j\mu} \qquad (13)$$

between the generalized Reynolds stress

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$$e'_{vj} \equiv -\rho v'_v v'_j +$$
$$\frac{\gamma - 1}{2} \delta_{vj} \rho v'^2 + \sigma_{vj} \qquad (14)$$

(based on the residual velocities and enthalpy) and the base flow stress $\tilde{e}_{vj}$, where

$$v'_4 \equiv (\gamma - 1) h'_o \equiv (\gamma - 1)\left(h' + \frac{1}{2} v'^2 \right) \qquad (15)$$

$$\sigma_{4i} \equiv -(\gamma - 1)\left(q_i - \sigma_{ij} v'_j \right) \qquad (16)$$

with σ_{ij} being the viscous stress, q_i the heat flux vector and δ_{vj} denoting the 4x3 dimensional Kronecker delta in the usual notation.

The true non-linearity of these equations, which can be written more compactly in operator form, as

$$L_{\mu v} u_v = D^{\mu}_{vj} e''_{vj} \quad \text{for} \\ \mu, v = 1, 2, 3, 4, 5 \qquad (17)$$

where $L_{\mu v}$ and D^{μ}_{vj} are first order linear operators and u_v denotes the five dimensional solution vector

$$\{u_v\} \equiv \{u_i, p'_e, \rho'\}, \qquad (18)$$

is hidden in the non-linear dependent variables (11)and (12) as well as in the non-linear source strength (14). The former non-linearity is again largely irrelevant here because the variable of principle interest (the pressure-like variable, p'_e) reduces to the ordinary pressure fluctuation in the far field where the sound field is to be calculated. And in the acoustic analogy approach the latter non-linearity is dealt with by intruding a specific model for the generalized stress tensor (14). This is analogous to the situation in conventional turbulence modeling. But the modeling is now considerably more complex and being able to simplify the requirements by lumping all the modeling into a single tensor is an important consideration. So we might think of requiring that the fundamental acoustic equations satisfy the following two conditions in addition to the two mentioned in the introduction:

(1) The base flow about which the implied linrearization is carried out satisfies the usual hyperbolic conservation laws. And

(2) All of the modeling requirements can be inputted through a single stress tensor

These four requirements taken together appear to be just sufficient to restrict the overall form of the fundamental acoustic equations (i.e., the acoustic analogy equations) to trivial variations of the linearized hyperbolic conservation laws (6) to(8).

Interpretation of the right side of these equations as acoustic sources implies that that the left sides account for all of the interaction of this sound with the base flow-- including the propagation of the sound through that flow. The apparent linearity of the equations can be exploited to separate out these interaction/propagation effects from the unsteady source fluctuations that produce the sound by using the 4th component of the adjoint vector Green's function $g^a_{v\mu}(y, \tau | x, t)$, which is related to the 4th component of the ordinary vector Greens' function $g_{4v}(x, t | y, \tau)$ by the reciprocity relation $g^a_{v4}(y, \tau | x, t) = g_{4v}(x, t | y, \tau)$ and satisfies the inhomogeneous linear equation [9]

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$$\left(L_{\mu\nu}^{a}\right)_{\mathbf{y},\tau} g_{\nu4}^{a}\left(\mathbf{y},\tau|\mathbf{x},t\right) =$$

$$\delta_{\mu4}\delta\left(\mathbf{x}-\mathbf{y}\right)\delta\left(t-\tau\right)$$

(19)

with Delta-function source term where $L_{\mu\nu}^{a}$ is the adjoint of the operator $L_{\mu\nu}$ that appears in equation **Error! Reference source not found.**. And setting aside solid surface interaction effects, for simplicity sake, we can use Green's formula to show that the pressure like variable p_{e}' is given by the tensor product

$$p_{e}'\left(\mathbf{x},t\right) = -\int_{V}\int_{-\infty}^{\infty}\gamma_{\mu j}\left(\mathbf{y},\tau|\mathbf{x},t\right)e_{\mu j}''\left(\mathbf{y},\tau\right)d\mathbf{y}d\tau$$

(20)

of the source strength (13) with the propagator $\gamma_{\mu j}\left(\mathbf{y},\tau|\mathbf{x},t\right)$ which is related to the adjoint Green's function $g^{a}{}_{4\mu}\left(\mathbf{y},\tau|\mathbf{x},t\right)$ by

$$\gamma_{\mu j}\left(\mathbf{y},\tau|\mathbf{x},t\right) \equiv \frac{\partial g^{a}{}_{4\mu}\left(\mathbf{y},\tau|\mathbf{x},t\right)}{\partial y_{j}} -$$
$$\left(\gamma-1\right)\frac{\partial\tilde{v}_{\mu}}{\partial y_{j}}g^{a}{}_{44}\left(\mathbf{y},\tau|\mathbf{x},t\right),$$

(21)

with $\tilde{v}_{4}\equiv 0$,and, therefore, accounts for all of the base flow interaction effects.
Notice that $p_{e}'\left(\mathbf{x},t\right)$ reduces to the ordinary pressure fluctuation $p'\left(\mathbf{x},t\right)$ when the observation point $\mathbf{x}$ is in the far field.

3 Specific Analogies

The present results are, of course, general enough to allow for many possible analogies--each of which corresponds to a different choice of base flow--with the most

rudimentary being obtained by setting the base flow stresses $\tilde{e}_{\nu j}$ and the velocity $\tilde{v}_{i}$ to zero. The hyperbolic conservation laws then imply that the remaining base flow dependent variables $\overline{\rho},\overline{p}_{e}$ will be equal to constants, say to their values at infinity. This is essentially the original Lighthill [1] approach as modified by Lilley [10,11] to separate out the isentropic and non-isentropic components of the pressure/density term that appears in Lighthill's source function. This turns out to be important because it is only the latter component that can be associated with viscous and heat conduction effects that are almost universally neglected at the high Reynolds numbers of interest in most Aerodynamic noise problems. The former can be an important noise source that has to be modeled in this type of acoustic analogy.

But the main criticism of the Lighthill approach is that the residual velocities v_{i}' that appear in the residual equation source term have steady flow components that introduce steady and linear terms into the residual stress tensor. But the former cannot generate any sound and therefore do not belong in the source function, while the latter are at least in part associated with propagation effects and might, therefore, be best book kept on the left hand side of the equations.

This criticism is at least partially overcome by the next most complicated form of the analogy which amounts to again setting the stress tensor $\tilde{e}_{\nu j}$ to zero but now requiring that the base flow dependent variables $\tilde{v}_{i}$, $\overline{\rho},\overline{p}_{e}$ satisfy the resulting Euler's equations. However, the interest is usually in the steady flow solutions to these equations because for many turbulent flows the turbulent stresses act only slowly over long distances and the Euler equation solutions are, therefore, expected to provide a good local approximation to the actual mean flow. This is especially true for the important class of nearly parallel flows, such

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as jets wakes and shear layers. In which case the relevant Euler equation solution is the uni-directional transversely sheared flow

$$\tilde{v}_i = \delta_{ij} U\left(\boldsymbol{x_T}\right), \bar{\rho} = \bar{\rho}\left(\boldsymbol{x_T}\right)$$
$$, \bar{p}_e = \text{constant} \tag{22}$$

where subscript T denotes transverse components, the velocity $\tilde{v}_i = \delta_{ij} U\left(\boldsymbol{x_T}\right)$ is in a single direction but, along with the density $\bar{\rho} = \bar{\rho}\left(\boldsymbol{x_T}\right)$, can vary in planes perpendicular to that direction. The hyperbolic conservation laws (or in this case the Euler equations) then imply that the pressure like variable $\bar{p}_e$ will be constant.

The linear operator on the left hand side of the residual equations now reduces to the usual Rayleigh operator of linear stability theory fame and, as is well known, the residual equations can be reduced to a single inhomogeneous equation for the pressure like variable p'_e which, in the Aeroacoustics context, is usually referred to as a Lilley's equation [10]. And while it can be argued that this equation has served as the basis of most of the theoretical jet noise research for the past two decades, it has a number of issues which will be discussed subsequently. But it is first necessary to finish cataloguing the various specific analogies.

The true reduced order modeling comes into play when specific turbulence models are introduced for the base flow stress tensor. As noted in the introduction, these models are usually based on the filtered Navier Stokes equations, which are obtained by applying a linear filter to these equations. From our point of view a linear filter is nothing more than a linear transformation of the continuous functions into a simpler subspace that commutes with differentiation. The resulting filtered Navier-Stokes equations are special cases of the general base flow equations with the stress tensor now given explicitly by

$$\tilde{e}_{ij} \equiv -\bar{\rho}\left(\widetilde{v_i v_j} - \tilde{v}_i \tilde{v}_j\right) +$$
$$\frac{\gamma-1}{2}\delta_{ij}\bar{\rho}\left(\widetilde{v^2} - \tilde{v}^2\right) \tag{23}$$

$$\tilde{e}_{4j} \equiv$$
$$-(\gamma-1)\left[\bar{\rho}\left(\widetilde{h_0 v_j} - \tilde{h}_0 \tilde{v}_j\right) + \frac{1}{2}\tilde{e}_{ii}\tilde{v}_j + \tilde{e}_{ij}\tilde{v}_i\right] \tag{24}$$

and with $\bar{p}_e$ related to $\bar{p}$ by

$$\bar{p}_e = \bar{p} + \frac{\gamma-1}{2}\bar{\rho}\left(\widetilde{v^2} - \tilde{v}^2\right) \tag{25}$$

where the overbars denote filtered variables, the tildes denote the Favre (ref.2) filtered variables

$$\tilde{\bullet} \equiv \left(\overline{\rho \bullet}\right)/\bar{\rho} \tag{26}$$

and the usually unimportant viscous terms have, for simplicity, been omitted.

It is possible to choose a filter that only filters out the components of the motion that actually radiate to the far field [12]. So the base flow will be completely silent and all of the residual flow will radiate away as sound. In which case, it would not be completely unreasonable to interpret the apparent source terms on the right side of the residual equations as the "true sources of sound". But no one has as yet succeeded in developing a base flow closure model for this type of filter. While equations (24) and (24) are very suggestive, they do not actually provide a means for constructing such a model—beyond, perhaps, providing some constraints that they have to satisfy.

4 Actual Mean Flow Analogy

There is much more that can be done with this. But I'm definitely not going to get into the vast subject of turbulence modelling here, except to point out that the most well known filter, namely the time average

$$\bar{\bullet} \equiv \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \bullet\left(\boldsymbol{x}, t\right) dt, \tag{27}$$

also produces a completely silent base flow, but it also filters out much of the non-radiating (i.e., non-acoustic) components of the motion. The effective pressure is now given by

$$p'_e \equiv p' + \frac{\gamma-1}{2}\left(\rho v'^2 - \overline{\rho v'^2}\right) \qquad (28)$$

and more importantly, the base flow stress tensor $\tilde{e}_{\nu j}$ is now equal to the time average $\overline{e'_{\nu j}}$ of the generalized residual velocity/enthalpy Reynolds stress. So the actual stress tensor (acoustic source strength) that appears in the residual equations (6) to (8) is just the difference

$$e''_{\nu j} = e'_{\nu j} - \overline{e'_{\nu j}} \qquad (29)$$

between this Reynolds stress and its time average--which means that it has zero time average as would be expected for a true acoustic source.

The base flow equations now turn out to be the usual Reynolds averaged Navier-Stokes equations for which the turbulence modelling is certainly highly developed [3]. This form of the analogy completely removes the mean flow propagation effects from the source term, which is expected to significantly improve the medium and lower frequency predictions, but a steady base flow analogy cannot provide the requisite structure for inputting the long range turbulent scattering effects, which tend to become more important at the higher frequencies

The Greens' function solution (20) can now be used to show that the far field pressure autocovariance

$$\overline{p^2}\left(x,t_0\right) \equiv \frac{1}{2T}\int_{-T}^{T} p'\left(x,t\right)p'\left(x,t+t_0\right)dt, \qquad (30)$$

which is the quantity that is usually measured in Aeroacoustic experiments, can be expressed as the convolution product

$$\overline{p^2}\left(x,t\right) = \int_{-\infty}^{\infty}\iint_{V}\overline{\gamma}_{\nu j\mu l}\left(x|y;\eta,t+\tau\right)\mathcal{R}_{\nu j\mu l}$$
$$\left(y;\eta,\tau\right)dy\,d\eta\,d\tau \qquad (31)$$

of a "propagator" $\overline{\gamma}_{\nu j\mu l}$, which is related to the original propagator $\gamma_{\mu j}$ by the convolution

$$\overline{\gamma}_{\nu j\mu l}\left(x|y;\eta,t+\tau\right) \equiv$$
$$\int_{-\infty}^{\infty} \gamma_{\nu j}\left(x|y,t_1+t+\tau\right)\gamma_{\mu l}\left(x|y+\eta,t_1\right)dt_1 \qquad (32)$$

with the two point time-delayed correlation

$$\mathcal{R}_{\nu j\mu l}\left(y;\eta,\tau\right) \equiv \frac{1}{2T}\int_{-T}^{T} e''_{\nu j}\left(y,\tau_0\right)e''_{\mu l}\left(y+\eta,\tau_0+\tau\right)d\tau_0 \qquad (33)$$

of the source strength $e''_{\nu j}\left(y,\tau\right)$, which is directly related to the generalized velocity /enthalpy Reynolds stress autocovariance tensor

$$R_{\nu j\mu l}\left(y;\eta,\tau\right) \equiv \frac{1}{2T}\int_{-T}^{T}\left[\rho v'_\nu v'_j - \overline{\rho v'_\nu v'_j}\right]\left(y,\tau_0\right)$$
$$\left[\rho v'_\mu v'_l - \overline{\rho v'_\mu v'_l}\right]\left(y+\eta,\tau_0+\tau\right)d\tau_0 \qquad (34)$$

by the relatively simple linear transform

$$\mathcal{R}_{\nu j\mu l} = R_{\nu j\mu l} -$$
$$\frac{\gamma-1}{2}\left(\delta_{\nu j}R_{kk\mu l} + \delta_{\mu l}R_{\nu jkk}\right) + \left(\frac{\gamma-1}{2}\right)^2\delta_{\nu j}\delta_{\mu l}R_{iikk}. \qquad (35)$$

Since this autocovariance tensor is about as close as you can get to what is actually measured in turbulent flows, these results provide an essentially exact relation between the quantities that are typically measured in Aeroacoustic experiments—as would be expected from any good theory.

The tensor $R_{\nu j \mu l}\left(\boldsymbol{y}; \boldsymbol{\eta}, \tau\right)$ is also the quantity that ultimately has to be modeled in the acoustic analogy approach. The hope is that most of the non-local "propagation effects" which would be very difficult to distinguish from the turbulent fluctuations-- and, therefore, very difficult to model-- have been removed from these stresses.

But the result (31) to (33) is actually more general than this and applies to any steady base flow, including the uni-directional mean flow. This greatly simplifies the computation of the propagator $\overline{\gamma}_{\nu j \mu l}$, but, of course, only applies to nearly parallel shear flows. And even in that, admittedly important, case has, as mentioned in the previous section, a number of serious issues. First of all the source term contains steady as well mean flow interaction terms because v'_i still has a mean (steady) flow component. A more serious difficulty is that the Rayleigh operator that now appears on the left hand side of the residual equations can support linear instability waves that grow without bound far downstream in the flow. And when the Greens' function is required to satisfy causality, which seems like a reasonable thing to do, the corresponding homogeneous solutions to the Rayleigh equation contribute to the adjoint Greens' function and therefore to the propagator $\gamma_{\mu j}\left(\boldsymbol{y}, \tau \mid \boldsymbol{x}, t\right)$ causing them to become infinite there as well. An even more serious difficulty at supersonic speeds is that the Rayleigh operator has a singularity at the so called critical layer which produces a much stronger non-integrable singularity in the propagator $\gamma_{\mu j}\left(\boldsymbol{y}, \tau \mid \boldsymbol{x}, t\right)$ when the observation point $\boldsymbol{x}$ is in the far field—leading to the ridiculous conclusion that the radiated sound is infinitely loud.

But none of these difficulties would occur it the base flow were taken to be the actual mean flow field—even in the important case of a nearly parallel shear flow. Because, while the linear instabilities can initially grow in such flows, the slow divergence of the flow will eventually cause them to decay—keeping the Green's function and, therefore, the propagator finite. And even more importantly at supersonic speeds, these flows do not posses critical layers and, therefore, have no critical layer singularities. But the propagator is much more expensive to compute in this case---in fact, prohibitively so for many applications.

5 Application to Slowly Diverging Mean Flows

There is, however, a way to obtain the best of both worlds—which amounts to using a perturbation approach that takes advantage of the small spread rate, say ε, of the mean flow. The base flow velocities will then expand like [7,8]

$$\tilde{v}_1 = U\left(Y, \boldsymbol{y}_T\right) + \varepsilon U^{(1)}\left(Y, \boldsymbol{y}_T\right) + \ldots \quad (36)$$

$$\tilde{\boldsymbol{v}}_T = \varepsilon V\left(Y, \boldsymbol{y}_T\right) + \varepsilon^2 V^{(1)}\left(Y, \boldsymbol{y}_T\right) + \ldots (37)$$

where

$$Y \equiv \varepsilon \, y_1 \qquad\qquad (38)$$

denotes a slow streamwise (source) variable that varies on the streamwise length scale of the mean flow , and

$$\boldsymbol{y}_T = \left\{y_{2,} y_3\right\} \qquad\qquad (39)$$

$$\tilde{\boldsymbol{v}}_T = \left\{\tilde{v}_2, \tilde{v}_3\right\} \qquad\qquad (40)$$

denote cross flow variables. Of course, the other base flow dependent variables will

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have similar expansions. And it is assumed that all lengths have been normalized by some characteristic cross flow dimension and that all velocities have been normalized by some appropriate characteristic streamwise velocity with similar obvious normalizations for the density, pressure and temperature. The lowest order terms in this expansion correspond to the unidirectional transversely sheared flow with the slow streamwise variable entering only parametrically.

It might be expected that the 4th component of the adjoint Greens' function would posses a corresponding expansion of the form

$$g_{\nu\mu}^a = g_{\nu\sigma}^{a,0} + \varepsilon g_{\nu\sigma}^{a,1} + \dots \qquad (41)$$

But in so far as this expansion is concerned the lowest order base flow solution still acts like a uni-directional transversely sheared mean flow. So nothing much has been accomplished here because the lowest order term satisfies Rayleigh's equation and, therefore, still posses a critical layer singularity. But embedding this solution in this more global context allows us to construct a new "inner solution" in the vicinity of the critical layer that brings in nonparallel mean flow effects to eliminate the singularity there. Standard singular perturbation techniques [13] can then be used to combine these two solutions into a single uniformly valid result that remains finite everywhere in the flow and is not that much more expensive to compute than the original parallel flow result. The resulting "propagator" still becomes large when the source point is at the critical layer but, unlike the parallel flow result, now remains finite there.

This result can now be inserted into the formula (31) for the pressure autocovariance which can then be Fourier transformed to show, upon assuming only that the transverse length scale of the turbulence is short compared to the transverse length scale of the mean flow, that far field acoustic spectrum is given be the following purely algebraic expression[8]

$$I_{\omega}\left(\boldsymbol{x}\big|\boldsymbol{y}\right) \rightarrow$$

$$\left(\frac{2\pi}{x}\right)^2 \frac{2\pi\omega}{c_{\infty}} \sin\theta \, \overline{\Gamma}_{\kappa j}\left(\boldsymbol{x}\big|\boldsymbol{y}_T,Y\right) \overline{\Gamma}_{\lambda l}^{*}\left(\boldsymbol{x}\big|\boldsymbol{y}_T,Y\right)$$

$$\Phi^{*}{}_{\kappa j\lambda l}\left(\boldsymbol{y}; \frac{\omega}{c_{\infty}}\cos\theta, \frac{\omega}{c_{\infty}}\nabla S, \omega\left(1 - M_c\cos\theta\right)\right)$$

$$\text{as } x \rightarrow \infty \qquad (42)$$

where S denotes the Eikonal of the corresponding geometric acoustics solution, the turbulence source correlations enter only through the spectral tensor

$$\Phi_{\nu j\mu l}^{*}\left(\boldsymbol{y}; k_1, \boldsymbol{k}_T, \omega\right)$$

$$\equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega\tau} \int_V e^{i\left(k_1\xi_1 + \boldsymbol{k}_T \cdot \boldsymbol{\xi}_T\right)} \mathcal{R}_{\nu j\mu l}^{M}\left(\boldsymbol{y}, \boldsymbol{\xi}, \tau\right) d\boldsymbol{\xi}\, d\tau$$

$$(43)$$

where the asterisk is being used to denote complex conjugates, $\overline{\Gamma}_{\nu j}\left(\boldsymbol{x}\big|\boldsymbol{y}_T, Y\right)$ is proportional the far field expansion of the Fourier transform of $\gamma_{\nu j}$, $\mathcal{R}_{\nu j\mu l}^{M}\left(\boldsymbol{y}, \boldsymbol{\xi}, \tau\right)$ is the moving frame correlation defined in terms of the fixed frame correlation (33) by [8]

$$\mathcal{R}_{\nu j\mu l}^{M}\left(\boldsymbol{y}, \boldsymbol{\xi}, \tau\right) \equiv$$
$$\mathcal{R}_{\nu j\mu l}\left(\boldsymbol{y}; \boldsymbol{\xi} + \hat{\boldsymbol{i}}U_c\tau, \tau\right) \qquad (44)$$

(where U_c denotes the convection velocity of the turbulence) and I have introduced $I_{\omega}\left(\boldsymbol{x}\,|\,\boldsymbol{y}\right)$, the acoustic spectrum at the observation point $\mathbf{x}$ due to a unit volume of turbulence at the source point $\mathbf{y}$, for transparency sake. Obviously, this quantity

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has to be integrated over the entire noise producing region of the flow to calculate the actual acoustic spectrum

$$I_\omega(\boldsymbol{x}) = \int_V I_\omega(\boldsymbol{x}|\boldsymbol{y})d\boldsymbol{y} \qquad (45)$$

And finally I have neglected the contribution of the linear instability waves primarily because they do not seem to be all that important at the relatively low supersonic Mach numbers that we will be dealing with here.

6 Application to the Jet Noise Problem

The results of the previous section apply to any parallel shear flow but, for definiteness, the reminder of the paper will be restricted to the technologically important jet noise problem. The propagator $\overline{\Gamma}_{\kappa j}(\boldsymbol{x}|\boldsymbol{y}_T, Y)$ ultimately depends only on the mean flow which, in today's environment, would almost certainly be determined from a RANS computation. But the spectral tensor $\Phi^*_{\nu j\mu l}(\boldsymbol{y}; k_1, \boldsymbol{k}_T, \omega)$ depends on the turbulence statistics, which can not be obtained from any steady flow solution. It, therefore, has to be modeled, but the model can be paramatized and the parameters can be determined from some reduced order flow computation such as a RANS solution.

Ideally, we would like to model the spectral tensor Φ^*_{ijkl} itself, but the models must, at least at present, be based on experimental data and the experimentalists are unlikely to measure this quantity any time in the near future. A fall back position might be to develop models for the spectra

$$\Psi_{ijkl}(\boldsymbol{y}; k_1, \boldsymbol{k}_T, \omega)$$

$$\equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega\tau} \int_V e^{i(k_1\xi_1 + \boldsymbol{k}_T \cdot \boldsymbol{\xi}_T)} R^M_{ijkl}(\boldsymbol{y}, \boldsymbol{\xi}, \tau)d\boldsymbol{\xi}\,d\tau$$

$$(46)$$

of the generalized Reynolds stress autocovariance tensor, which can be related to Φ^*_{ijkl} by using the linear transform (35). But even this quantity has only been infrequently measured (see, for example, Harper-Bourne)[14] -and then only for very low Mach number flows. The only recourse is to model the Reynolds stress autocovariance tensor itself and to work backward through these formulas to calculate the spectrum Φ^*_{ijkl} that actually appears in the acoustic equation(42).

There are two main requirements for such models. The first is that they must reduce the large number of independent spectral components to a manageable level (there are 45 of these in all even when the enthalpy fluctuations are neglected, there are 78 when they are not) and the second is that the models for the remaining components must be relatively inexpensive to compute.

The first of these is usually met by introducing some sort of symmetry and/or statistical assumptions such as local isotropy and quasi-normality. But these assumptions do not seem to be all that viable at the high Mach number end of the range of interest. We, therefore, assume only that the turbulence anisotropy is Mach number independent and that the turbulence itself is axysymmetric [15]—which is consistent with the experimental observation that the turbulence statistics in the various cross flow directions are much more similar to one another than they are to those in the streamwise direction.

The second requirement usually implies that the integrations that arise in the Φ^*_{ijkl} calculation can be done in closed form, or nearly so, because of the relative expense in doing numerical integrations over and over again. But the models must also be flexible enough to provide an accurate representation of the turbulence structure because the strong streamwise retarded time variations cause the radiated sound field to be very sensitive to that structure.

Appropriate models were introduced in reference [8]. They involve a large number

of adjustable parameters (actually infinitely many) which, as noted above, have to be determined from some reduced order flow computation—which in today's world would again have to be a RANS computation. But this type of calculation can only provide enough information to determine a small number of these parameters. However, newer higher fidelity reduced order flow computations, such as URANS, VLES or even hybrid RANS/LES, are rapidly coming on line and these codes should be able to determine many more of these constants in the near future.

But the results that will be presented in this paper are based on a pure RANS code, namely the NPARK WIND code, which is widely used in the United States [16]. This code was used to calculate the mean flow from cold (i.e., unheated) round jets with acoustic Mach numbers $M_J \equiv U_J / c_\infty$ of 0.50, 0.90, and 1.4, where U_J is the jet exit velocity and c_∞ is the speed of sound at infinity. The upstream conditions were specified in terms of the nozzle temperature and pressure ratios. The results were then used in equation (42) to calculate their far-field acoustic spectra on the arc $x / D_J = 100$ and compared with jet noise measurements taken NASA Glenn SHJAR rig [17] with the same upstream conditions and with the atmospheric absorption removed from the data in order to make the comparisons on a lossless basis. The results are shown in Fig.1. The lower curves are the acoustic spectra at 90° to the downstream axis and the upper curves are the spectra at 30° (which is close to the peak radiation direction).

The overall agreement appears to be quite good but there is a tendency to under predict the higher frequency components of the 90° spectrum in the supersonic case. This is because the flow is not correctly expanded here and the present analysis does not account for the resulting shock associated noise (shown cross hatched in the figure).

Computations were only carried out at the relatively low (<1.5) supersonic (acoustic) Mach numbers where non-linear propagation effects are believed to be unimportant. Fortunately, this also corresponds to the Mach number range of most technological interest. The critical layer only appears when the observation angle (as measured from the downstream axis) is fairly small ($<45^o$) and gradually moves inboard from the nozzle lip line with increasing downstream distance until it reaches a point beyond the end of the potential core where it quickly moves onto the jet axis and suddenly disappears. But the propagator is still very close to being singular, and can consequently be relatively large, for a significant distance downstream of this point. It was, therefore, necessary to construct an additional inner solution for this region as well—especially since much of the small angle sound field will actually be generated in this relatively localized portion of the jet when the acoustic Mach number is close to unity.

Finally, in order to put these results into perspective Fig. 2 shows a comparison with the best previous calculation using the Glenn JeNo code with an ad hoc correction to eliminate the critical layer singularity [18]. The results clearly demonstrate that there is enormous improvement in predictive capability with the present code.

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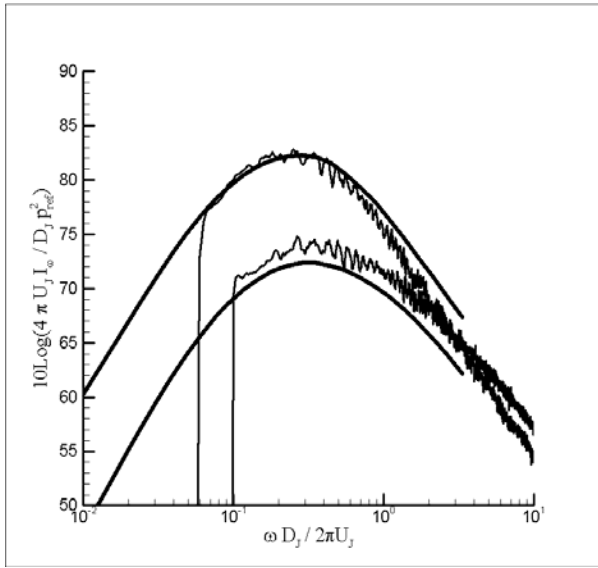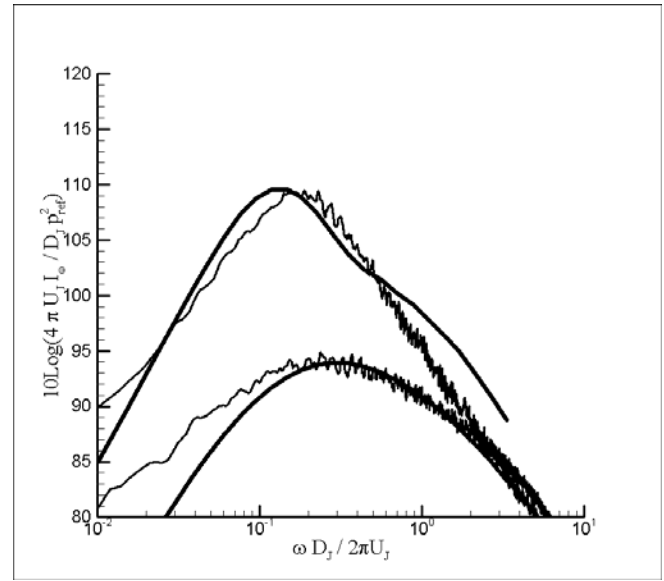
Part a) $M_J = 0.5$ Part b) $M_J = 0.9$
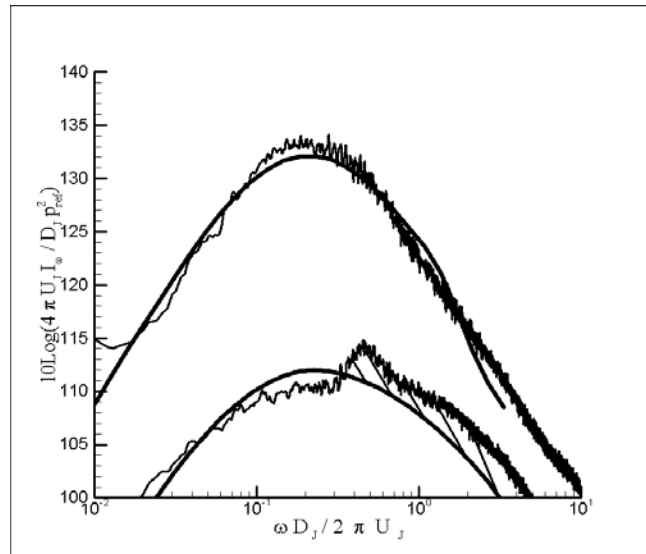
Part c) $M_J = 1.4$

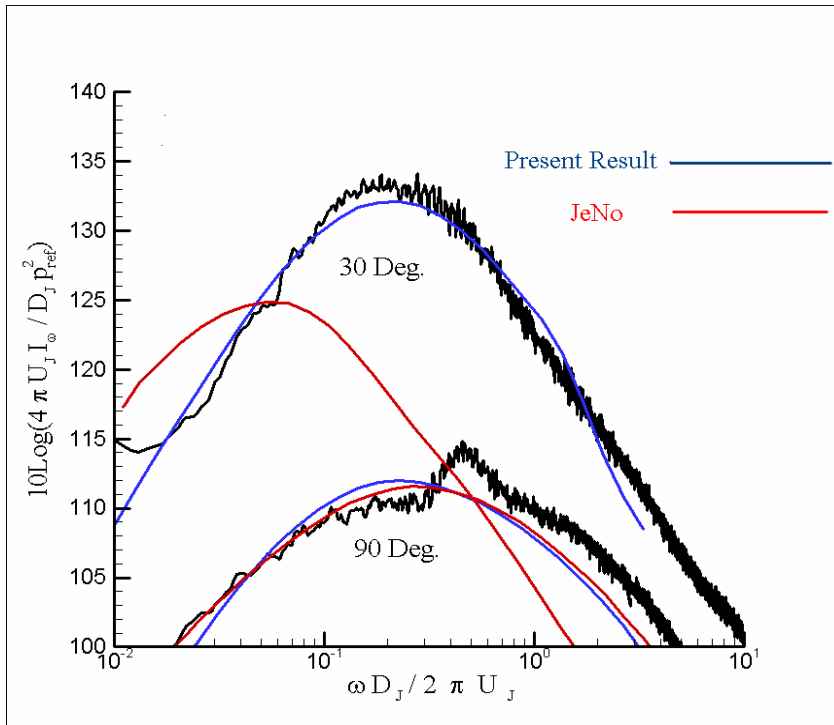Fig. 1) Comparison of Jet Noise Predictions with Measurements (Taken from [8])

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Fig. 2) Comparison with JeNo Results. (Taken From [8])