Time-Varying Expression of the Formation Flying along Circular Trajectories

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Usually, the formation flying associated with circular orbits is discussed through the well-known Hill's or C-W equations of motion. This paper dares to present and discuss the coordinates that may contain time-varying coefficients. The discussion presents how the controller's performance is affected by the selection of coordinates, and also looks at the special coordinate suitable for designating a target bin to which each spacecraft in the formation has only to be guided. It is revealed that the latter strategy may incorporate the J2 disturbance automatically.

1. INTRODUCTION

The formation flying along circular trajectories has been well developed and also well known, and a lot of investifations have been carried out. The most familiar approach to the dynamics is to use so called the Hill's equation or, in other words, the C-W euqation, in which the linearized motion is described as a set of ordinary differential equations with constant coefficients. Its in-plane motion is expressed as :

$$\ddot{x} - 2n\dot{y} - 3n^2 x = u, \quad \ddot{y} + 2n\dot{x} = v \tag{1}$$

As long as the initial-value problem is concerned, the transient behavior is explicitly written and the asymptotic stability is easily verified in this approach, when a conventional feedback scheme with steady constant gains is introduced in the Hill's coordinate domain. However, in the actual formation control aplications, the target to be traced and maintained is persistently moving along with time. The equations of motion associated with circular trajectories naturally contain the *0*-th order derivative term, and maitaining the formation inevitably, in most cases, requires a certain feedforward acceleration control to be input. It is true that the residual error vanishes once the feed-forward control strategy is incorporated. However, even though the stability is assured with no redidual error, there are still some aspects to be examined for practical use. The paper here discusses three coordinates, 1) C-W coordinate, 2) Inertial coordinate and 3) *p*-coordinate that is developed by the author.

The C-W euqation is converted to an altrenative expression corresponding to the inertial motion, and the following equation:

$$\ddot{q} + F(\theta) q = U \tag{2}$$

is obtained. It is a time-varying system with a periodic coefficient matrix F in it. It is not straightforward to verify if the feedback control is stable, and the closed system may result in steady state bias unless an appropriate feed-forward control is introduced, since it contains the *0*-th order derivative term. But the use of this inertial coordinate is objectively suitable for some applications such as astronomy missions, which require the control performance property defined in inertial space.

There is still another aspect in the formation flying. What the formation target should be depends on the control objective, and it might be worthy referring to the special case, in which the formation motion is expressed by the following time-varying equation, in which no *0*-th order derivative term appears. This is what the paper presents:

$$\ddot{p} + A(\theta)\dot{p} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} u \\ \frac{1}{2} v \end{pmatrix} = \hat{U} \tag{3}$$

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As seen in this formation flight, here, no feed forward acceleration input is required apparently to maintain the formation. The paper presents the stability criterion and other properties for the feedback control scheme to this case. The discussion, at the same time, presents the convergence property to the target, either a bin or a slot to which each spacecraft shall be controlled. Note regardless of selection of coordinates, the dynamics itself never changes and what is seen and observed is just apparent behavior with artificial control input.

The paper, first, simply points out that the use of the C-W equation may degrade the convergence performance in the applications that intend the relative geometry to be pointed to the inertial target, simply because the C-W equation is not necessarily suitable for that purpose. This is what the paper discusses first. This degradation is accounted for either from the performance index point of view that directionally weights the formation accuracy, or from the altered dynamics structural point of view. They are stated in the paper. The figure below shows schematically how the system is structurally altered, for instance. Needless to say, the stability needs to be looked at from the time-varying motion's point of view.

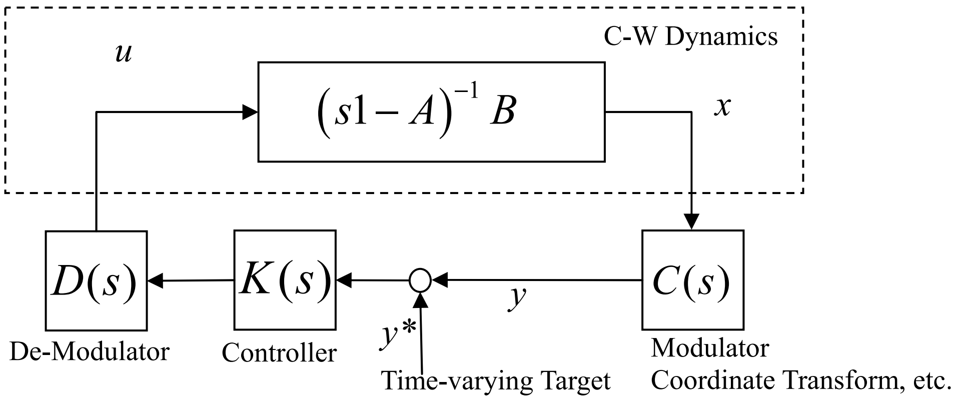

Fig. 1 Control System Transfer Function Characteristics

This diagram shows the relation between the C-W dynamics domain and the time-varying formation domain. What is usually discussed is inside the dotted block, the C-W dynamics, in which the stability is examined easily. However, the formation of concern is associated with the time-varying property y that is modulated not only periodically but sometimes in terms of size and scale. This thing corresponds to the formation maintenance allowance that is directionally different. For instance, the focal length / longitudinal tolerance is much more vigorous than lateral control tolerance. How the controller performance is affected is examined in the paper. Designing the controller has to take this characteristic into account, and might result in the time-varying feed-back strategy for better performance, such as based on the periodic Riccati matrix solution in time-varying Linear Quadratic problem.

The paper, at the same time, presents the in-plane formation maintenance along circular trajectories and shows an example of the strategy through the reference to the time-varying, non-autonomous property mentioned here. The paper tries to reduce them into the alternative equation that is written in a non-autonomous form as below.

$$\ddot{p} + A(\theta)\dot{p} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} u \\ \frac{1}{2}v \end{pmatrix} = \hat{U} \tag{4}$$

Here p-vector denotes the transformed state vectors along a circular trajectory. An importance of the form above is in the direct / trivial provision of the steady state solution that satisfies them automatically. At the cost of non-autonomy, the modified equations here show some advantages especially for the formation maintenance guidance and control, without relying on the feed-forward efforts. The formation maintenance has only to result in controlling the spacecraft position into a virtual bin or slot defined in this p-coordinate. The relative approach trajectory is handled much more easily with the equations, whose transition matrix is also depicted in the paper. The paper presents how the p-coordinate here can incorporate the inclusion of J2 disturbance effect. It expands the usefulness of the coordinate.

The author already reported the first results in regard to this *p*-coordinate in Sedona this February, when the analytical solutions and three kinds of controller designs assuring the stability were presented. This paper does not refer to those results.

Here is shown schematic drawing of three coordinates to be discussed. The control schemes for the formation maintenance rely on the observation measurements with the other spacecraft among the formation. Sometimes, Hill's equations are utilized to describe the relative motion. But it is not directly related to the formation geometry. In most cases, the spacecraft attitude regardless of whether it is a chaser or evader, what is readily available does not appear intimately as long as C-W description is used. The relative motion seen in the inertial coordinate is not simple but complicated as in Fig. 3. On the Hill's coordinate, such closed loci relative motions are along virtual ellipses, and the motion is simple enough. However, what is readily available aboard the spacecraft shows almost frozen geometry quickly illustrated in Fig. 3, while the motion description needs expressed in a not-straightforward manner as on the Hill's coordinate. While the formation is completely frozen, the Hill's state vectors are not frozen. As this implies, the use of inertial coordinate has some practical advantage. On the p-coordinate introduced here, the measurements are nearly the direct observation information obtained on the spacecraft to the adjacent spacecraft. On a complete frozen formation, the *p*-coordinate presents a frozen constant vector. And it apparently looks similar to the relative motion in the inertial motion.

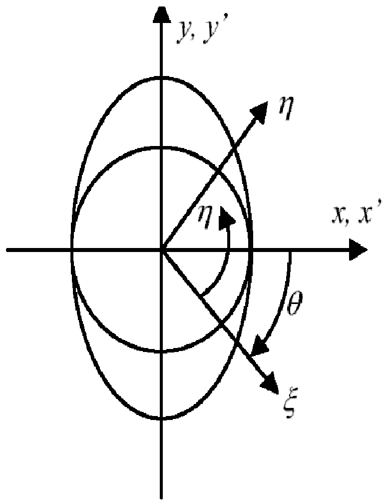

Fig. 2 *p*-Coordinate

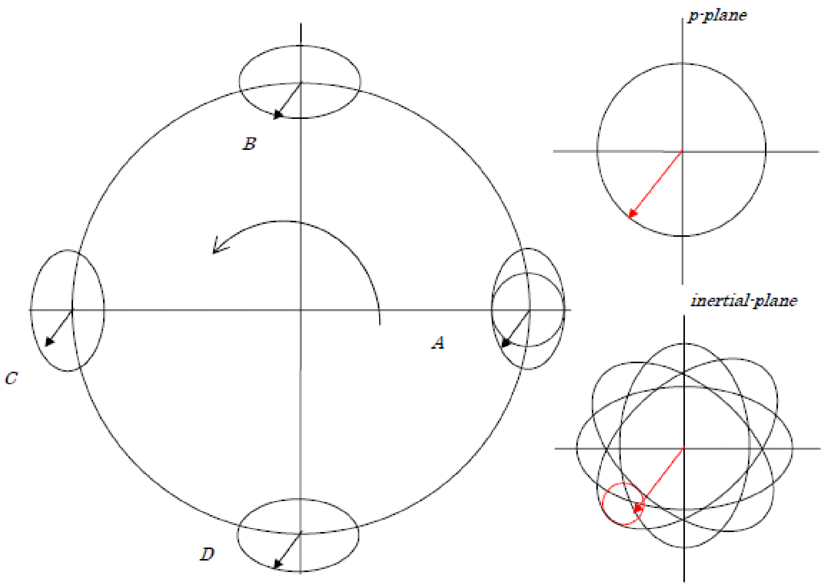

Fig. 3 Measurements Properties and *p*-Coordinate and Inertial Coordinate

2. AUTONOMOUS TO NON-AUTONOMOUS SYSTEMS

2.1 Coordinate Conversion with Projection

Provided an Autonomous system describing the formation motion is given as

$$\dot{x} = Ax + Bu \tag{5}$$

When Time-Varying and Projection Mapping transformation of

$$\hat{x} = \Theta(t)Px, \qquad x = P^{-1}\Theta(t)^T \hat{x} \tag{6}$$

is introduced, an altered system is described by the following Non-Autonomous system.

$$\dot{\hat{x}} = \left\{ \Theta PAP^{-1}\Theta^T - \left[\omega\right] \right\} \hat{x} + \Theta PBu = \hat{A}(t)\hat{x} + \hat{B}(t)u$$

2.2 Feed-Forward and Feed-Back - Stability and Steady State Residual

With the control input combining both Feed-Forward and Feedback Laws of

$$u = u^* - K(t)\left(\hat{x} - \hat{x}(t)^*\right) \tag{7}$$

where u*satisfies

$$\dot{\hat{x}}(t)^* = \hat{A}(t)\hat{x}(t)^* + \hat{B}(t)u^* \tag{8}$$

The state error $\Delta\hat{x} = \hat{x} - \hat{x}(t)^*$ behaves according to

$$\Delta\dot{\hat{x}} = \left\{ \hat{A}(t) - \hat{B}(t)K(t) \right\} \Delta\hat{x} \tag{9}$$

If the stability above converging to the origin is verified in some manner, there is no steady state residual. Note: Not straight-forward in Non-Autonomous systems.

2.3 Stability and Controller Stability

Here are summarized typical characteristics in using three coordinates:

(1) Hill's Coordinate:
  Stability is verified through conventional eigen values.
  Solutions are available analytically.

(2) Inertia Coordinate:
  Stability is verified via examining Lyapnov property.
  Solutions are available analytically.

$$\ddot{q} + F(\theta)q = U \qquad U = -c\dot{q} - (K - F(\theta))q \tag{10}$$

(3) p-Coordinate:
  Stability is verified via examining Lyapnov property.
  Solutions are available analytically.

$$\ddot{p} + A(\theta)\dot{p} = \hat{U} \qquad U = -c\dot{p} - Kp \tag{11}$$

The control laws shown above are some examples.

3. HOW NON-AUTONOMOUS DESCRIPTION WORKS?

There is no direct distinction among three coordinates, since the dynamics never changes, and the stability and avoiding steady state residual are feasibly assured for each description. However, the paper dares to present two aspects on how such Non-Autonomous description works. What follows lists the subsequent major topics elaborated.

1) Distinction is depicted between C-W and Inertial Coordinates in terms of Performance Index associated with the control laws.
   Discussion also looks at the attitude control aspect, in which control thrust had better be applied along inertial axes.

2) Distinction is shown for *p*-Coordinate, when the Formation Maintenance is intended with no Feed-Forward acceleration.
   Discussion also looks at the inclusion of J2-perturbation.
   (Extension of the previous results this February in Arizona is given here.)

3.1 Evaluating Controller Performance

Assuming an appropriate Feed-Forward Compensation is properly applied, and provided an Autonomous system is given as

$$\dot{x} = Ax + Bu \tag{12}$$

With the rotation transformation of $\hat{x} = \Theta(t)x, \quad x = \Theta(t)^T \hat{x}$, the following Non-Autonomous system is obtained.

$$\dot{\hat{x}} = \left\{ \Theta A \Theta^T - [\omega] \right\} \hat{x} + \Theta B u = \hat{A}(t)\hat{x} + \hat{B}(t)u \tag{13}$$

Here is assumed the Performance Index (criterion) written as a function of linear quadratic products as

$$\hat{J} = \int_0^\infty \left(\frac{1}{2}\hat{x}^T \hat{Q}\hat{x} + \frac{1}{2}u^T R u \right) \tag{14}$$

to have a feed-back law taking the form of $u = -\hat{K}\hat{x}$. This is the most straightforward controller design contemporarily taken.

3.2 Two alternative interpretations as for the Performance Evaluation Problem

Below are summarized how the performance is evaluated under the feedback law adopted.

Case-1: Use of Non-Autonomous Dynamics with Steady LQ Criterion

Under the Non-Autonomous system of

$$\dot{\hat{x}} = \left\{ \Theta A \Theta^T - [\omega] \right\} \hat{x} + \Theta B u = \hat{A}(t)\hat{x} + \hat{B}(t)u \tag{15}$$

with the Performance Index (criterion)

$$\hat{J} = \int_0^\infty \left(\frac{1}{2}\hat{x}^T \hat{Q}\hat{x} + \frac{1}{2}u^T R u \right) \tag{16}$$

to have a Feed-Back law of $u = -\hat{K}\hat{x}$.

Case-2: Use of Autonomous Dynamics with Time-Varying LQ Weighting

Under the Autonomous system:

$$\dot{x} = Ax + Bu \tag{17}$$

with the Performance Index (criterion):

$$\hat{J} = \int_0^\infty \left(\frac{1}{2}x^T \left(\Theta(t)^T \hat{Q} \Theta(t) \right) x + \frac{1}{2}u^T R u \right) \tag{18}$$

to have a Feed-Back law of $u = -\hat{K}\Theta(t)x$.

It should be noted that two alternative interpretations above are basically identical, but that these will provide useful properties that can identify the difference in controllers performance between C-W and Inertial coordinates.

3.3 Case-1: Transfer Function Distinction

Property: [Eigen Values Shift]
Suppose λ_A is an eigen value of matrix *A*, and also suppose λ_C is an eigen value of matrix *C* that is assumed diagonalized to Λ_C by orthogonal transform Q as $C = Q\Lambda_C Q^T$.
When a matrix *(A+C)* has an eigen value λ as $(A+C)x = \lambda x$,
since $\left(Q^T A Q\right)x = (\lambda - \lambda_C)x$, $A(Qx) = (\lambda - \lambda_C)(Qx)$ results.
And this implies $\lambda = \lambda_A + \lambda_C$.

Note: in the system $\dot{\hat{x}} = \left\{\Theta A \Theta^T - [\omega]\right\}\hat{x} + \Theta B u = \hat{A}(t)\hat{x} + \hat{B}(t)u$,
$[\omega]$ has the eigen values of pure imaginary and is diagonalized by orthogonal transformation.

Thus the transfer function $\hat{G}(s) = \left(s1 - \hat{A}\right)^{-1} B$ shifts its poles of the transfer function $G(s) = (s1 - A)^{-1} B$ in C-W coordinate to higher frequency direction.
This causes the performance index distinction between C-W and Inertial coordinates.

3.4 Case-2 : Performance Index Evaluation

Here is evaluated the performance under the following index, for the controller designed in autonomous (C-W) coordinate.

$$\hat{J} = \int_0^\infty \left(\frac{1}{2}x^T Q(t) x + \frac{1}{2}u^T R u\right) = \frac{1}{2}x(t_0)^T S(t_0) x(t_0) \qquad Q(t) = \Theta(t)^T \hat{Q} \Theta(t) \tag{19}$$

The performance index is obtained by solving the following equation with a pre-determined gain *K(t)*

$$-\dot{S} = \left(A - BK(t)\right)^T S + S\left(A - BK(t)\right) + K(t)^T R K(t) + Q(t) \qquad S(\infty) = 0 \tag{20}$$

While the gain K_0 is designed in C-W coordinate based on

$$-\dot{S}_0 = 0 = \left(A - BK_0\right)^T S_0 + S_0\left(A - BK_0\right) + K_0{}^T R K_0 + Q_0 \qquad K_0 = R^{-1} B^T S_0 \tag{21}$$

The evaluation results in solving

$$-\dot{S} = \left(A - BK_0\right)^T S + S\left(A - BK_0\right) + K_0{}^T R K_0 + Q(t) \qquad S(\infty) = 0 \tag{22}$$

And the performance difference is obtained by

$$-\left(\dot{S} - \dot{S}_0\right) = \left(A - BK_0\right)^T \left(S - S_0\right) + \left(S - S_0\right)\left(A - BK_0\right) + \left(Q(t) - Q_0\right) \tag{23}$$

In case the last term in the equation above is positive definite, the performance difference is positive and this means the performance is degrade. The astronomy pointing performance is defined in Inertial Coordinate, and it is obviously directional, and is not described as a constant weighting in C-W coordinate. Thus designing in Inertial Coordinate must be better for the astronomy purpose.
Note, in case a Time-Varying weighting is introduced in C-W coordinate, the design results in the same performance and the same gain. But such design process is identical to design in Inertial Coordinate.

4. CONVERSION INTO *p*-COORDINAE

When a contraction of $x' = x,\ \ y' = y/2$ is used, and the rotation coordinate transformation of

$$p = \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} \tag{24}$$

is introduced, the equations of motion is altered to the following time-varying expression.

$$\ddot{p} + A(\theta)\dot{p} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} u \\ \frac{1}{2}v \end{pmatrix} = \hat{U} \tag{25}$$

where

$$A(\theta) = \begin{pmatrix} 3\sin\theta\cos\theta & 1-3\cos^2\theta \\ -1+3\sin^2\theta & -3\sin\theta\cos\theta \end{pmatrix} \tag{26}$$

It is clear that a trivial solution, a constant vector always satisfies the equation above. This corresponds to the state in which the formation is frozen along similar ellipse trajectories, i.e. like cart wheel orbits. A question was raised in Sedona this February on whether the *p*-coordinate can represent the inclusion of *J2* disturbance. Here is given an answer to it.

4.1 J_2 effect appears in C-W equation.

Previous researches found the J_2 effect can be incorporated in the C-W equation. (Ref. 3, 4, 5)

$$\begin{aligned} &\ddot{x} - 2\omega\dot{y} - \omega^2 x = 2n^2 x + 4n^2 sx - \kappa\left(1 - 3\sin^2 i \sin nt\right) \\ &\ddot{y} + 2\omega\dot{x} - \omega^2 y = -n^2 y - n^2 sy - \kappa\left(\frac{1}{2}\sin^2 i \sin 2nt\right) \\ &\ddot{z} = -n^2 z - 3n^2 sz - \kappa\left(\frac{1}{2}\sin 2i \sin nt\right) \end{aligned} \tag{27}$$

Here, $\kappa = \dfrac{3J_2 R_e^{\,2} n^2}{r^*}$, $s = \dfrac{3J_2 R_e^{\,2}}{8r^{*2}}\left(1 + 3\cos 2i\right)$ and $c^2 = 1 + s$, $\omega = nc$.

The relative Formation Flying motion is now replaced expressed by

$$\begin{aligned} &\ddot{x} - 2nc\dot{y} - \left(5c^2 - 2\right)x = 0 \\ &\ddot{y} + 2nc\dot{x} = 0 \end{aligned} \tag{28}$$

4.2 Elimination of Restoring Force Terms

In order that this p-coordinate is useful even when the J2 disturbance is applied, the restoring force terms shall be eliminated. Starting from the modifies C-W equation of

$$\ddot{x} + C\dot{x} + Dx = u \tag{29}$$

Where *C* is a skew symmetric and *D* is a diagonal matrix respectively. Here is introduced Coordinate Rotation and Projection Transforms :

$$\hat{x} = \Theta(t)Px, \qquad x = P^{-1}\Theta(t)^T \hat{x} \tag{30}$$

The system is now converted to

$$\ddot{\hat{x}} + \left\{2[\omega] + \Theta PAP^{-1}\Theta^T\right\}\dot{\hat{x}} + \Theta\left\{PCP^{-1}[\omega] + PDP^{-1} + [\omega]^2\right\}\Theta^T\hat{x} = \hat{u} \tag{31}$$

In order that $PCP^{-1}[\omega] + PDP^{-1} + [\omega]^2 = 0$,

$$[\omega] = \begin{pmatrix} 0 & \chi nc \\ -\chi nc & 0 \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 0 \\ 0 & 1/k \end{pmatrix} \tag{32}$$

are tested and here is concluded that

$$\chi = \frac{\sqrt{2-c^2}}{c}, \quad k = \frac{2c}{\sqrt{2-c^2}} \tag{33}$$

eliminates the restoring force terms and trivial constant vectors express the formation slots / bins where each spacecraft shall be controlled.

5. HILL'S (C-W), INERTIAL & *p*-COORDINATES

So far, the paper has discussed about how non-autonomous systems work and what kind of distinction appears. What follows summarizes the characteristics of it.

- Formation measurement does not relate to the Hill's coordinate, since in most cases the spacecraft is stabilized in three-axis manner.
- Inertial frame is useful as for astronomy missions in this regard. But the relative motion is always varying even though the formation gets settled to natural trajectories.
- p-coordinate retains measurement characteristics and shows the frozen property for settled formation.

Table-1 Summary of Comparison among C-W, Inertial and *p*-Coordinates

	C-W (Hill's)	Inertial	*p*-coordinate
Eq. of Motion	Available	Available	Available
Time-Invariance	Invariant	Variant	Variant
Analytic Solutions	Available	Available	Available
Observability*	Not Appropriate	Appropriate	Suitable
Formation Target	Time-Varying	Time-Varying	Frozen
Steady State Control Input	Persistently Varying	Persistently Varying	Zero
Appropriate for	NADIA Tracking	Astronomy	Formation Maintenance
Advantage	Comprehensive Stability Augmentation	Minimizes suitable Performance Index	Defines a specific bin/slot easily that ea s/c to be controlled

6. CONCLUDING REMARKS

- The paper dares to introduce Non-Autonomous Coordinates to make difference more clearly from the C-W coordinate.
- Three Coordinates were examined: 1) C-W, 2) Inertial, and 3) p-Coordinate.
- Here is described first : fundamental Stability, avoidance of Steady State Residual error are assured for all coordinates, as long as appropriate Feed-Forward acceleration is applied.
- Two illustrations were given as to how Non-Autonomous systems work: 1) Control Performance Index Degradation once designed in C-W coordinate, 2) Formation Maintenance property is still easily handled by p-coordinate even in case J2 perturbation is incorporated.

The paper wants to encourage engineers to visit Non-Autonomous coordinates in some practical applications rather than resorting to the C-W coordinates.

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