

OPTIMAL LOW-THRUST LIMITED-POWER TRANSFERS BETWEEN ARBITRARY ELLIPTIC COPLANAR ORBITS

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Abstract

In this work, a complete first order analytical solution, which includes the short periodic terms, for the problem of optimal low-thrust limited-power transfers between arbitrary elliptic coplanar orbits in a Newtonian central gravity field is obtained through Hamilton-Jacobi theory and a perturbation method based on Lie series.

1. INTRODUCTION

In the general analysis of optimal space trajectories, two idealized propulsion models are of most frequent use [1]: LP and CEV systems. In the power-limited variable ejection velocity systems – LP systems – the only constraint concerns the power, that is, there exists an upper constant limit for the power. In the constant ejection velocity limited thrust systems – CEV systems – the magnitude of the thrust acceleration is bounded. In both cases, it is usually assumed that the thrust direction is unconstrained. The utility of these idealized models is that the results obtained from them provide good insight about more realistic problems.

The purpose of this work is to present a complete first order analytical solution, which includes the short periodic terms, for the problem of optimal low-thrust limited-power simple transfers (no rendez-vous) between arbitrary elliptic coplanar orbits in a Newtonian central gravity field. This analysis has been motivated by the renewed interest in the use of low-thrust propulsion systems in space missions verified in the last two decades.

The optimization problem is formulated as a Mayer problem of optimal control theory with Cartesian elements – position and velocity vectors – as state variables [1]. After applying the Pontryagin Maximum Principle and determining the maximum Hamiltonian, classical orbital elements are introduced through a canonical transformation – Mathieu transformation. The short periodic terms are then eliminated from the maximum Hamiltonian through an infinitesimal canonical transformation built through Hori method – a perturbation canonical method based on Lie series [2]. The new maximum Hamiltonian, resulting from the infinitesimal canonical transformation, describes the extremal trajectories associated with the long duration maneuvers. Using Hamilton-Jacobi theory [3], closed-form analytical solutions are obtained for the canonical system governed by this new Hamiltonian. For long duration maneuvers, the existence of conjugate points is investigated through the Jacobi condition and the envelope of extremals is obtained considering different configurations of initial and final orbits. Curves representing maneuvers with same fuel consumption for a specified duration, called iso-cost curves, are presented for several configurations of initial and final orbits. The results are very similar to the ones obtained by Edelbaum [4] and Marec and Vinh [5]; but, the theory presented in the paper includes directly the periodic terms such that it can be applied to any transfers independently of its duration. On the other hand, since the theory is based on canonical transformations, a first order solution for the Cartesian elements can also be easily obtained from Lie theorem [2].

2. OPTIMAL SPACE TRAJECTORIES

Let us to consider the motion of a space vehicle M , powered by a limited-power engine in a Newtonian central gravity field. At time t , the state of the vehicle is defined by the position vector $\mathbf{r}(t)$, the velocity vector $\mathbf{v}(t)$ and the consumption variable J [1],

$$J = \frac{1}{2} \int_{t_0}^t \gamma^2 dt, \quad (1)$$

where γ denotes the magnitude of the thrust acceleration $\boldsymbol{\gamma}$, used as control variable. It is assumed that the thrust direction and magnitude are unconstrained.

The optimization problem will be formulated as a Mayer problem of optimal control as follows. It is proposed to transfer the space vehicle M from the initial state $(\mathbf{r}_0, \mathbf{v}_0, 0)$ at the initial time $t_0 = 0$ to the final state $(\mathbf{r}_f, \mathbf{v}_f, J_f)$ at the specified final time t_f , such that the final consumption variable J_f is a minimum. The state equations are

$$\frac{d\mathbf{r}}{dt} = \mathbf{v} \quad \frac{d\mathbf{v}}{dt} = -\frac{\mu}{r^3}\mathbf{r} + \boldsymbol{\gamma} \quad \frac{dJ}{dt} = \frac{1}{2}\gamma^2, \quad (2)$$

where μ is the Gaussian constant.

According to the Pontryagin Maximum Principle [6], the optimal thrust acceleration $\boldsymbol{\gamma}^*$ must be selected from the admissible controls such that the Hamiltonian H reaches its maximum. The Hamiltonian is formed using Eqns (2),

$$H = \mathbf{p}_r \cdot \mathbf{v} + \mathbf{p}_v \cdot \left(-\frac{\mu}{r^3}\mathbf{r} + \boldsymbol{\gamma} \right) + \frac{1}{2} p_J \gamma^2, \quad (3)$$

where \mathbf{p}_r , \mathbf{p}_v and p_J are the adjoint variables and dot denotes the scalar vector product. Since the optimization problem is unconstrained, it follows that

$$\boldsymbol{\gamma}^* = -\mathbf{p}_v / p_J. \quad (4)$$

The optimal thrust acceleration $\boldsymbol{\gamma}^*$ is modulated [1]. Then, the optimal trajectories are governed by the maximum Hamiltonian H^* ,

$$H^* = \mathbf{p}_r \cdot \mathbf{v} - \mathbf{p}_v \cdot \frac{\mu}{r^3}\mathbf{r} - \frac{p_v^2}{2p_J}.$$

Since J is an ignorable variable, it is determined by simple integration, and its adjoint p_J is a first integral, whose value is obtained from the transversality conditions: $p_J = -1$. Thus,

$$H^* = \mathbf{p}_r \cdot \mathbf{v} - \mathbf{p}_v \cdot \frac{\mu}{r^3}\mathbf{r} + \frac{p_v^2}{2}. \quad (5)$$

This Hamiltonian can be decomposed into two parts: $H_0 = \mathbf{p}_r \cdot \mathbf{v} - \mathbf{p}_v \cdot \frac{\mu}{r^3}\mathbf{r}$ is the undisturbed Hamiltonian and

$H_{\gamma^*} = \frac{p_v^2}{2}$ is the disturbing function concerning the optimal thrust acceleration.

3. TRANSFORMATION FROM CARTESIAN ELEMENTS TO ORBITAL ELEMENTS

In order to obtain a first order analytical solution for simple transfers between arbitrary elliptic coplanar orbits, the set of classical orbital elements is introduced through a canonical transformation by using the properties of generalized canonical systems [7]. Consider the extended two-body problem defined by the canonical system of differential equations governed by the undisturbed Hamiltonian H_0 , that is, the problem defined by the differential equations for the state variables \mathbf{r} and \mathbf{v} , which describe the classical two-body problem, and the differential equations for the adjoint variables \mathbf{p}_r and \mathbf{p}_v ,

$$\frac{d\mathbf{r}}{dt} = \mathbf{v} \quad \frac{d\mathbf{p}_r}{dt} = \frac{\mu}{r^3}(\mathbf{p}_v - 3(\mathbf{p}_v \cdot \mathbf{e}_r)\mathbf{e}_r)$$

$$\frac{d\mathbf{v}}{dt} = -\frac{\mu}{r^3}\mathbf{r} \quad \frac{d\mathbf{p}_v}{dt} = -\mathbf{p}_r,$$

where \mathbf{e}_r is the unit vector pointing radially outward of the moving frame of reference. In the coplanar case, the general solution of this system is given by [7]:

$$\mathbf{r} = \frac{a(1-e^2)}{1+e\cos f}\mathbf{e}_r, \quad (6)$$

$$\mathbf{v} = \sqrt{\frac{\mu}{a(1-e^2)}}[e\sin f\mathbf{e}_r + (1+e\cos f)\mathbf{e}_s], \quad (7)$$

$$\begin{aligned} \mathbf{p}_r = \frac{a}{r^2} & \left\{ 2ap_a + ((1-e^2)\cos E)p_e + \left(\frac{r}{a}\right)\frac{\sin f}{e} \left(p_\omega - \frac{(1-e^3\cos E)}{\sqrt{1-e^2}}p_M \right) \right\} \mathbf{e}_r \\ & + \left\{ \frac{\sin f}{a}p_e - \frac{(e+\cos f)}{ae(1-e^2)}p_\omega + \frac{\sqrt{1-e^2}\cos f}{ae}p_M \right\} \mathbf{e}_s, \end{aligned} \quad (8)$$

$$\begin{aligned} \mathbf{p}_v = \frac{1}{na\sqrt{1-e^2}} & \left\{ \left[2ae\sin f p_a + ((1-e^2)\sin f)p_e - \frac{(1-e^2)\cos f}{e}p_\omega \right. \right. \\ & \left. \left. + \frac{(1-e^2)^{3/2}}{e} \left(\cos f - \frac{2e}{1+e\cos f} \right) p_M \right] \mathbf{e}_r + \left[2a(1-e^2) \left(\frac{a}{r} \right) p_a + (1-e^2)(\cos f + \cos E)p_e \right. \right. \\ & \left. \left. + \frac{(1-e^2)\sin f}{e} \left(1 + \frac{1}{1+e\cos f} \right) (p_\omega - \sqrt{1-e^2}p_M) \right] \mathbf{e}_s \right\}. \end{aligned} \quad (9)$$

where \mathbf{e}_s is the unit vector along circumferential direction of the moving frame of reference; a is the semi-major axis, e is the eccentricity, ω is the pericenter argument, f is the true anomaly, E is the eccentric anomaly, M is the mean anomaly, $n = \sqrt{\mu/a^3}$ is the mean motion, and (r/a) , $(r/a)\sin f$, ... etc are known functions of the elliptic motion, which can be written explicitly in terms of eccentricity and mean anomaly through Lagrange series [8]. The anomalies are related through the equations:

$$\tan \frac{f}{2} = \sqrt{\frac{1+e}{1-e}} \tan \frac{E}{2}, \quad \text{and, } M = E - e\sin E \quad (\text{Kepler's equation}).$$

The unit vectors \mathbf{e}_r and \mathbf{e}_s are written as function of the orbital elements as follows

$$\begin{aligned} \mathbf{e}_r &= [\cos\Omega\cos(\omega+f) - \sin\Omega\sin(\omega+f)\cos I]\mathbf{i} + [\sin\Omega\cos(\omega+f) + \cos\Omega\sin(\omega+f)\cos I]\mathbf{j} \\ \mathbf{e}_s &= -[\cos\Omega\sin(\omega+f) + \sin\Omega\cos(\omega+f)\cos I]\mathbf{i} + [-\sin\Omega\sin(\omega+f) + \cos\Omega\cos(\omega+f)\cos I]\mathbf{j} \end{aligned}$$

where \mathbf{i} and \mathbf{j} are the unit vectors of a fixed frame of reference.

Equations (6) – (9) define a canonical transformation – Mathieu transformation – between the Cartesian elements $(\mathbf{r}, \mathbf{v}, \mathbf{p}_r, \mathbf{p}_v)$ and the orbital ones $(a, e, \omega, M, p_a, p_e, p_\omega, p_M)$. The Hamiltonian H^* is invariant with respect to this canonical transformation. Thus,

$$H_0 = np_M, \quad (10)$$

$$\begin{aligned}
H_{\gamma^*} = & \frac{1}{2n^2 a^2 (1-e^2)} \left\{ \frac{1}{2} (1 - \cos 2f) [2ae p_a + (1-e^2) p_e]^2 \right. \\
& + 2(1-e^2) \sin 2f \left[-ap_a p_\omega - \frac{(1-e^2)}{2e} p_e p_\omega \right] \\
& + 4(1-e^2)^{3/2} \sin f \left(\frac{-2e}{1+e \cos f} + \cos f \right) \left[ap_a p_M + \frac{(1-e^2)}{2e} p_e p_M \right] \\
& + \frac{(1-e^2)^2}{2e^2} (1 + \cos 2f) p_\omega^2 - \frac{2(1-e^2)^{5/2}}{e^2} \left(\frac{-2e}{1+e \cos f} + \cos f \right) \cos f p_\omega p_M \\
& + \frac{(1-e^2)^3}{e^2} \left(\frac{-2e}{1+e \cos f} + \cos f \right)^2 p_M^2 + 4a^2 (1-e^2)^2 \left(\frac{a}{r} \right)^2 p_a^2 \\
& + 4a(1-e^2)^2 \left(\frac{a}{r} \right) (\cos E + \cos f) p_a p_e + (1-e^2)^2 (\cos E + \cos f)^2 p_e^2 \\
& + \frac{4a(1-e^2)^2}{e} \left(\frac{a}{r} \right) \sin f \left(1 + \frac{1}{1+e \cos f} \right) \left[p_a p_\omega - \sqrt{1-e^2} p_a p_M \right] \\
& + \frac{2(1-e^2)^2}{e} (\cos E + \cos f) \left(1 + \frac{1}{1+e \cos f} \right) \sin f \left[p_e p_\omega - \sqrt{1-e^2} p_e p_M \right] \\
& \left. + \left[\frac{(1-e^2)}{e} \left(1 + \frac{1}{1+e \cos f} \right) \sin f \left[p_\omega - \sqrt{1-e^2} p_M \right] \right]^2 \right\}, \tag{11}
\end{aligned}$$

The new Hamiltonian function H^* , defined by Eqns (10) – (11), describes the optimal low-thrust limited-power trajectories in a Newtonian central gravity field for transfers between arbitrary elliptic coplanar orbits.

4. A FIRST ORDER ANALYTICAL SOLUTION

In order to get an approximate formal solution for the problem of transfers between arbitrary elliptic coplanar orbits, a perturbation technique based on Lie series – Hori method – will be applied. For completeness, a brief description of Hori method [2] is presented in the Appendix.

In this study, it is assumed that the functions H_0 and H_{γ^*} have different order of magnitude for low-thrust propulsion systems. So, from Eqns (10) – (11), using the algorithm described in the Appendix, one finds the generating function S and the new Hamiltonian F' , given up to the first order by

$$F' = \frac{a'}{2\mu} \left\{ 4a'^2 p_a'^2 + \frac{5}{2} (1-e'^2) p_e'^2 + \frac{(5-4e'^2)}{2e'^2} p_\omega'^2 \right\}, \tag{12}$$

$$\begin{aligned}
S = & \frac{1}{2} \sqrt{\frac{a'^5}{\mu^3}} \left\{ 8e' \sin E' a'^2 p_a'^2 + 8(1-e'^2) \sin E' a' p_a' p_e' - 8 \frac{\sqrt{1-e'^2}}{e'} \cos E' a' p_a' p_\omega' \right. \\
& + (1-e'^2) \left[-\frac{5}{4} e' \sin E' + \frac{3}{4} \sin 2E' - \frac{1}{12} e' \sin 3E' \right] p_e'^2 \\
& + \frac{2\sqrt{1-e'^2}}{e'} \left[\frac{5}{4} e' \cos E' + \frac{1}{4} (e'^2 - 3) \cos 2E' + \frac{1}{12} e' \cos 3E' \right] p_e' p_\omega' \\
& \left. + \frac{1}{e'^2} \left[\left(\frac{5}{4} - e'^2 \right) e' \sin E' - \frac{1}{2} \left(\frac{3}{2} - e'^2 \right) \sin 2E' + \frac{1}{12} e' \sin 3E' \right] p_\omega'^2 \right\} \tag{13}
\end{aligned}$$

Prime denotes the new canonical variables resulting from the infinitesimal canonical transformation generated by S . Terms factored by p_M' do not appear in equations above, since only simple transfers are considered [4]. It should be noted that mixed secular terms can arise in the new Hamiltonian in the case of rendez-vous maneuvers.

The new Hamiltonian F' governs the optimal long duration transfers between arbitrary elliptic coplanar orbits. The general solution of the canonical system described by this Hamiltonian will be obtained through two canonical transformations. First, consider the Mathieu transformation defined by the following equations:

$$a' = a'', \quad p'_a = p''_a, \quad e' = \sin \phi, \quad p'_e = \frac{p_\phi}{\cos \phi}, \quad \omega' = \omega'', \quad p'_\omega = p''_\omega. \quad (14)$$

The Hamiltonian function F' is invariant with respect to this transformation. Thus,

$$F'' = \frac{a''}{2\mu} \left\{ 4a''^2 p''_a{}^2 + \frac{5}{2} p''_\phi{}^2 + \left(\frac{5}{2} \csc^2 \phi - 2 \right) p''_\omega{}^2 \right\}. \quad (15)$$

The canonical system governed by the new Hamiltonian function F'' has three first integrals (t and ω are ignorable variables),

$$p''_\omega = C_1, \quad (16)$$

$$p''_\phi{}^2 + p''_\omega{}^2 \csc^2 \phi = C_2^2, \quad (17)$$

$$\frac{a''}{2\mu} \left\{ 4a''^2 p''_a{}^2 + \frac{5}{2} p''_\phi{}^2 + \left(\frac{5}{2} \csc^2 \phi - 2 \right) p''_\omega{}^2 \right\} = E. \quad (18)$$

These first integrals play important role in determining the general solution of the canonical system governed by the new Hamiltonian function F'' through Hamilton-Jacobi theory.

Let us to consider the problem of determining a canonical transformation defined by a generating function W such that C_1 , C_2 and E become the new generalized coordinates,

$$(a'', \phi, \omega'', p''_a, p''_\phi, p''_\omega) \xrightarrow{W} (C_1, C_2, E; p_{C_1}, p_{C_2}, p_E).$$

Since the new Hamiltonian function F'' is a quadratic form in the conjugate momenta, the separation of variables technique will be applied for solving the Hamilton-Jacobi equation [3].

Consider the transformation equations

$$\begin{aligned} p''_a &= \frac{\partial W}{\partial a''}, & p''_\phi &= \frac{\partial W}{\partial \phi}, & p''_\omega &= \frac{\partial W}{\partial \omega''}, \\ p_{C_1} &= -\frac{\partial W}{\partial C_1}, & p_{C_2} &= -\frac{\partial W}{\partial C_2}, & p_E &= -\frac{\partial W}{\partial E}, \end{aligned} \quad (19)$$

where $W = W(a'', \phi, \omega'', C_1, C_2, E)$ is the generating function. The corresponding Hamilton-Jacobi equation is then given by

$$\frac{a''}{2\mu} \left\{ 4a''^2 \left(\frac{\partial W}{\partial a''} \right)^2 + \frac{5}{2} \left(\frac{\partial W}{\partial \phi} \right)^2 + \left(\frac{5}{2} \csc^2 \phi - 2 \right) \left(\frac{\partial W}{\partial \omega''} \right)^2 \right\} = E. \quad (20)$$

Following the separation of variables method, it is assumed that the generating function W is equal to a sum of functions, each of which depends on a single variable, that is:

$$W(a'', \phi, \omega'', C_1, C_2, E) = W_1(a'', C_1, C_2, E) + W_2(\phi, C_1, C_2, E) + W_3(\omega'', C_1, C_2, E). \quad (21)$$

Therefore, from Eqns (16) through (21), it follows that

$$\frac{\partial W_3}{\partial \omega''} = C_1,$$

$$\left(\frac{\partial W_2}{\partial \phi}\right)^2 + \left(\frac{\partial W_3}{\partial \omega''}\right)^2 \csc^2 \phi = C_2^2,$$

$$\frac{a''}{2\mu} \left\{ 4a''^2 \left(\frac{\partial W_1}{\partial a''}\right)^2 + \frac{5}{2} \left(\frac{\partial W_2}{\partial \phi}\right)^2 + \left(\frac{5}{2} \csc^2 \phi - 2\right) \left(\frac{\partial W_3}{\partial \omega''}\right)^2 \right\} = E.$$

A complete solution of these equations is given by

$$W_1 = -\sqrt{\frac{5C^2}{2}} \left\{ \left[\frac{4\mu E}{5C^2 a''} - 1 \right]^{1/2} - \tan^{-1} \left[\frac{4\mu E}{5C^2 a''} - 1 \right]^{1/2} \right\}, \quad (22)$$

$$W_2 = \int \sqrt{C_2^2 - C_1^2 \csc^2 \phi} \, d\phi, \quad (23)$$

$$W_3 = C_1 \omega'', \quad (24)$$

with $5C_2^2 - 4C_1^2 = 5C^2$. We note that W_2 is given as indefinite integral, since only its partial derivatives are needed (Eqns (19)), as shown in the next paragraphs.

Now, consider the differential equations for the conjugate momenta of the canonical system governed by the new Hamiltonian $F''' = E$, that is:

$$\frac{dp_{C_1}}{dt} = 0 \quad \frac{dp_{C_2}}{dt} = 0 \quad \frac{dp_E}{dt} = -1,$$

whose solution is very simple:

$$p_{C_1} = \alpha_1 \quad p_{C_2} = \alpha_2 \quad p_E = \alpha_3 - t, \quad (25)$$

where $\alpha_i, i=1,2,3$, are arbitrary constants of integration.

Introducing the generating function W , defined by Eqns (21) through (24), into the transformation equations, Eqns (19), and taking into account the general solution defined by Eqns (25) for the conjugate momenta, one gets

$$p_a'' = \frac{1}{2\sqrt{2}a^2} [4\mu E a - 5C^2 a^2]^{1/2}, \quad (26)$$

$$p_\phi = \sqrt{C_2^2 - C_1^2 \csc^2 \phi}, \quad (27)$$

$$p_\omega'' = C_1, \quad (28)$$

$$\alpha_1 = -\omega - 2\sqrt{\frac{2}{5}} \frac{C_1}{C} \tan^{-1} \left(\left(\frac{4\mu E}{5C^2 a} - 1 \right)^{1/2} \right) + C_1 \int \frac{\csc^2 \phi}{\sqrt{C_2^2 - C_1^2 \csc^2 \phi}} \, d\phi, \quad (29)$$

$$\alpha_2 = -\sqrt{\frac{5}{2}} \frac{C_2}{C} \tan^{-1} \left(\left(\frac{4\mu E}{5C^2 a} - 1 \right)^{1/2} \right) + \sin^{-1} \left(\frac{C_2}{\sqrt{C_2^2 - C_1^2}} \cos \phi \right), \quad (30)$$

$$t - \alpha_3 = -\frac{1}{2\sqrt{2}Ea} [4\mu E a - 5C^2 a^2]^{1/2}. \quad (31)$$

From equations above, one finds the solution of the canonical system governed by the Hamiltonian F'' for a given set of initial conditions:

$$a''(t) = \frac{a_0''}{1 + \frac{4a_0''}{\mu} \left(\frac{1}{2} Et^2 - a_0'' p_{a_0}'' t \right)}, \quad (32)$$

$$a'' \sin^2 k_0 = a_0'' \sin^2(\sqrt{2}\psi + k_0), \quad (33)$$

$$\psi = \frac{1}{5}(\tau - \tau_0) \sqrt{1 + 4 \cos^2 k_1}, \quad (34)$$

$$\cos \phi = \cos k_1 \cos \tau, \quad (35)$$

$$\omega = k_2 + \tan^{-1}(\tan \tau \csc k_1) - \frac{4}{5} \tau \sin k_1, \quad (36)$$

$$p_a'' = \left(\frac{a_0''}{a''} \right)^3 p_{a_0}'' + \frac{1}{8} p_{a_0}''^2 (5 \csc^2 k_1 - 4) \left(\frac{a_0''}{a''^3} - \frac{1}{a''^2} \right), \quad (37)$$

$$p_\phi^2 = p_{a_0}''^2 (\csc^2 k_1 - \csc^2 \phi), \quad (38)$$

$$p_\omega'' = p_{a_0}''^2, \quad (39)$$

with the auxiliary constants k_0 , k_1 and k_2 defined as functions of the initial value of the adjoint variables (conjugate momenta) by

$$\csc^2 k_0 = \frac{8(a_0'' p_{a_0}''^2)^2 + p_{a_0}''^2 (5 \csc^2 k_1 - 4)}{p_{a_0}''^2 (5 \csc^2 k_1 - 4)},$$

$$\csc^2 k_1 = \frac{p_{\phi_0}^2 + p_{a_0}''^2 \csc^2 \phi_0}{p_{a_0}''^2},$$

$$k_2 = \omega_0'' + \frac{4}{5} \tau_0 \sin k_1 - \tan^{-1}(\tan \tau_0 \csc k_1).$$

The constants C , C_1 , C_2 and E in Eqns (26) through (31) can also be written as functions of the initial value of the adjoint variables as follows:

$$C^2 = \frac{1}{5} p_{a_0}''^2 (5 \csc^2 k_1 - 4), \quad C_1 = p_{a_0}''^2,$$

$$C_2^2 = p_{\phi_0}^2 + p_{a_0}''^2 \csc^2 \phi_0, \quad 4\mu E = a_0'' \left(8(a_0'' p_{a_0}''^2)^2 + p_{a_0}''^2 (5 \csc^2 k_1 - 4) \right).$$

The initial conditions for the state variables (generalized coordinates) are given by $a''(0) = a_0''$, $e(0) = \sin \phi_0$ and $\omega''(0) = \omega_0''$, and, τ_0 is obtained from $\cos \phi_0 = \cos k_1 \cos \tau_0$.

Equations (32) through (39) represent the solution of the canonical system concerning the problem of optimal long duration low-thrust limited-power transfers between arbitrary elliptic coplanar orbits in a Newtonian central field.

For maneuvers with arbitrary duration, the periodic terms must be included [9]. Following Hori method, a first order analytical solution is then given by:

$$a(t) = a'(t) + \sqrt{\frac{a'^5}{\mu^3}} \left[8e' \sin E' a'^2 p'_a + 4(1 - e'^2) \sin E' a' p'_e - 4 \frac{\sqrt{1 - e'^2}}{e'} \cos E' a' p'_\omega \right]_{E'_0}^{E'} \quad (40)$$

$$e(t) = e'(t) + \sqrt{\frac{a'^5}{\mu^3}} \left[4(1 - e'^2) \sin E' a' p'_a + (1 - e'^2) \left[-\frac{5}{4} e' \sin E' + \frac{3}{4} \sin 2E' - \frac{1}{12} e' \sin 3E' \right] p'_e \right. \\ \left. + \frac{\sqrt{1 - e'^2}}{e'} \left[\frac{5}{4} e' \cos E' + \frac{1}{4} (e'^2 - 3) \cos 2E' + \frac{1}{12} e' \cos 3E' \right] p'_\omega \right]_{E'_0}^{E'} \quad (41)$$

$$\omega(t) = \omega'(t) + \sqrt{\frac{a'^5}{\mu^3}} \left[-4 \frac{\sqrt{1 - e'^2}}{e'} \cos E' a' p'_a + \frac{\sqrt{1 - e'^2}}{e'} \left[\frac{5}{4} e' \cos E' + \frac{1}{4} (e'^2 - 3) \cos 2E' \right. \right. \\ \left. \left. + \frac{1}{12} e' \cos 3E' \right] p'_e + \frac{1}{e'^2} \left[\left(\frac{5}{4} - e'^2 \right) e' \sin E' - \frac{1}{2} \left(\frac{3}{2} - e'^2 \right) \sin 2E' + \frac{1}{12} e' \sin 3E' \right] p'_\omega \right]_{E'_0}^{E'} \quad (42)$$

with a' , e' , ..., p'_ω given through Eqns (14). Equations (40) – (42) are in agreement to the ones obtained in [9] through Bogoliubov-Mitropolsky method (a non-canonical perturbation technique). The eccentric anomaly E' is computed from Kepler's equation with the mean anomaly M' given by

$$M'(t) = M'_0 + \int_{t_0}^t \left[\sqrt{\frac{\mu}{a'^3}} - \left(\frac{5 + 2e'^2}{2} \right) \frac{a' \sqrt{1 - e'^2}}{\mu e'^2} p'_\omega \right] dt,$$

where $M'_0 = M'(t_0)$. The differential equation for the mean anomaly M' is derived from the undisturbed Hamiltonian F_0 and an additional term $\Delta F'$ factored by p'_M ,

$$F' = \left(n' - \left(\frac{5 + 2e'^2}{2} \right) \frac{a' \sqrt{1 - e'^2}}{\mu e'^2} p'_\omega \right) p'_M + \dots,$$

where dots denote the terms defined by Eqn (12).

For transfers between coaxial orbits with no change in the argument of pericenter, the analytical solution described in the preceding paragraphs simplifies. Since $p'_{\omega_0} = 0$, some equations must be rewritten:

$$a'' \sin^2 k_0 = a_0'' \sin^2 \left(\sqrt{\frac{2}{5}} (\phi - \phi_0) + k_0 \right), \quad (43)$$

$$p_a'' = \left(\frac{a_0''}{a''} \right)^3 p_{a_0}'' + \frac{5}{8} p_\phi'' \left(\frac{a_0''}{a''^3} - \frac{1}{a''^2} \right), \quad (44)$$

$$p_\phi = p_{\phi_0} = C, \quad (45)$$

with $\csc^2 k_0 = \frac{8(a_0'' p_{a_0}'')^2 + 5p_\phi''^2}{5p_\phi''^2}$ and $4\mu E = a_0'' (8(a_0'' p_{a_0}'')^2 + 5p_\phi''^2)$.

5. ANALYSIS OF LONG DURATION TRANSFERS

In this section, a complete analysis of long duration transfers described by the Hamiltonian F' is presented. The investigation of conjugate points and the solution of the two-point boundary value problem are considered.

To perform this analysis, a new consumption variable is introduced [4], $u = \sqrt{2E}t$, and Eqns (32) and (33) can be put in the form:

$$\frac{a''}{a_0''} = \left[1 - 2 \left(\frac{u}{v_0} \right) \cos k_0 + \left(\frac{u}{v_0} \right)^2 \right]^{-1}, \quad (46)$$

$$\left(\frac{u}{v_0} \right) = \frac{\sin(\sqrt{2}\psi)}{\sin(\sqrt{2}\psi + k_0)}, \quad (47)$$

where $v_0 = \sqrt{\mu/a_0''}$. By eliminating k_0 from these equations, one finds:

$$\left(\frac{u}{v_0} \right) = \sqrt{1 - 2 \sqrt{\frac{a_0''}{a''}} \cos(\sqrt{2}\psi) + \left(\frac{a_0''}{a''} \right)}. \quad (48)$$

On the other hand, $J = Et$, thus:

$$J = \frac{v_0^2}{2t} \left[1 - 2 \sqrt{\frac{a_0''}{a''}} \cos(\sqrt{2}\psi) + \left(\frac{a_0''}{a''} \right) \right]. \quad (49)$$

Now, consider Eqns (33) – (36) which define a two-parameter family of extremals in the phase space (a'', ϕ, ω'') for a given initial phase point $(a_0'', \phi_0, \omega_0'')$ corresponding to an initial orbit. By eliminating the auxiliary variables τ and ψ , $\alpha = a''/a_0''$ and ω'' can be written as explicit functions of ϕ , that is, $\alpha = \alpha(\phi, \phi_0, k_0, k_1)$ and $\omega'' = \omega''(\phi, \phi_0, \omega_0, k_1)$. The conjugate points to the phase point $(a_0'', \phi_0, \omega_0'')$ are determined through the roots of the equation [10]:

$$\frac{\partial \alpha}{\partial k_0} \frac{\partial \omega''}{\partial k_1} - \frac{\partial \alpha}{\partial k_1} \frac{\partial \omega''}{\partial k_0} = 0.$$

Since ω'' does not depend on k_0 , $\partial \omega'' / \partial k_0 = 0$. On the other hand, from Eqns (33) and (47), one finds

$\frac{\partial \alpha}{\partial k_0} = -\frac{2\alpha}{\sin k_0} \left(\frac{u}{v_0} \right) \neq 0$, for $u > 0$. Thus, the problem of determining the conjugate points reduces to the analysis of the roots of the following equation:

$$\frac{\partial \omega''}{\partial k_1} = 0. \quad (50)$$

From Eqns (35) and (36), one finds the explicit form of $\omega'' = \omega''(\phi, \phi_0, \omega_0, k_1)$, that is,

$$\omega'' = \omega_0'' - \frac{4}{5} \sin k_1 \left\{ \cos^{-1} \left(\frac{\cos \phi}{\cos k_1} \right) - \cos^{-1} \left(\frac{\cos \phi_0}{\cos k_1} \right) \right\} + \cos^{-1} \left(\frac{\cotg \phi}{\cotg k_1} \right) - \cos^{-1} \left(\frac{\cotg \phi_0}{\cotg k_1} \right).$$

The partial derivative $\partial \omega'' / \partial k_1$ is given by

$$\frac{\partial \omega''}{\partial k_1} = -\frac{4}{5} \cos k_1 (\tau - \tau_0) - \frac{4}{5} \tan^2 k_1 \left[\frac{\cos \phi_0}{\sin \tau_0} - \frac{\cos \phi}{\sin \tau} \right] + \sec^2 k_1 \left[\frac{1}{\sqrt{\tan^2 \phi_0 - \tan^2 k_1}} - \frac{1}{\sqrt{\tan^2 \phi - \tan^2 k_1}} \right]. \quad (51)$$

The auxiliary variable τ is re-introduced to simplify the expression. Therefore, from Eqns (50) and (51), after some simplifications, it follows that the conjugate points are given by the roots of following equation [4],

$$\left(1 - \frac{4}{5} \sin^2 k_1\right) \sin(\tau - \tau_0) - \frac{4}{5} \cos^2 k_1 (\tau - \tau_0) \sin \tau \sin \tau_0 = 0. \quad (52)$$

As discussed in [11,12], the conjugate points determined through Eqn (52) occur for transfers during which the direction of motion in the orbit reverses; that is, for transfers between direct orbits (no direction reversals) the extremals computed for the canonical system governed by the Hamiltonian F' are globally minimizing.

To complete the analysis, a brief description of an iterative algorithm for solving the two-point boundary value problem of going from an initial orbit $O_0 : (a_0'', e_0'', \omega_0'')$ to a final orbit $O_f : (a_f'', e_f'', \omega_f'')$ is presented.

The steps of this algorithm can be described as:

1. Guess a starting value of k_1 ;
2. Determine ϕ_0, ϕ_f, τ_0 and τ_f through Eqns (14) and (35);
3. Compute ω'' through Eqn (36).
4. If $\omega'' \neq \omega_f''$, adjust the value of k_1 and repeat steps 2 and 3 until $|\omega'' - \omega_f''| < \varepsilon$, where ε is a prescribed small quantity;
5. Compute ψ_f and u_f/v_0 through Eqns (34) and (48);
6. Compute k_0 using equation $\tan k_0 = \frac{\sin(\sqrt{2}\psi_f)}{\sqrt{\alpha_f - \cos(\sqrt{2}\psi_f)}}$, where $\alpha_f = a_f''/a_0''$;
7. Compute successively p_{a_0}'' , p_{e_0}'' and p_{ϕ_0}'' through the following equations:

$$p_{a_0}'' = \frac{v_0^2}{2a_0'' e_f} \left(\frac{u_f}{v_0} \right) \cos k_0,$$

$$p_{e_0}'' = \frac{2\sqrt{2}a_0'' p_{a_0}''}{\cot k_0 \sqrt{5 \csc^2 k_1 - 4}},$$

$$p_{\phi_0}'' = p_{e_0}''^2 (\csc^2 k_1 - \csc^2 \phi_0).$$

The adjust of the value of k_1 can be obtained through the well-known Newton-Raphson method, that is,

$$k_{1n+1} = k_{1n} - \frac{\omega''(k_{1n}) - \omega_f}{(\partial \omega'' / \partial k_1)_{k_1=k_{1n}}},$$

with the partial derivative $(\partial \omega'' / \partial k_1)_{k_1=k_{1n}}$ computed from Eqn (51). In the case of coaxial orbits, $k_1 = 0$ and the solution of the two-point boundary value problem involves only the solution of Eqn (43), which is given through step 6 with ψ replaced by $\frac{1}{\sqrt{5}}(\phi - \phi_0)$. The initial value of the adjoint variables is given as follows: p_{a_0}'' is

computed through step 7 and p_{ϕ_0}'' is given by $p_{\phi_0}'' = \frac{8 \tan^2 k_0 (a_0'' p_{a_0}'')^2}{5}$.

The thrust acceleration required to perform the maneuver is given through Eqns (4) and (9), that is,

$$\begin{aligned} \mathbf{p}_v = & \frac{1}{na\sqrt{1-e^2}} \left\{ 2ae \sin f p_a + ((1-e^2)\sin f) p_e - \frac{(1-e^2)\cos f}{e} p_\omega \right\} \mathbf{e}_r \\ & + \left\{ 2a(1-e^2) \left(\frac{a}{r} \right) p_a + (1-e^2)(\cos f + \cos E) p_e + \frac{(1-e^2)\sin f}{e} \left(1 + \frac{1}{1+e\cos f} \right) p_\omega \right\} \mathbf{e}_s. \end{aligned}$$

with the state and adjoint variables computed from Eqns (32) through (39).

6. RESULTS

In this section some numerical results are presented through Figs 1 – 5. Figures 1 and 2 show the field of extremals and iso-cost curves for long duration transfers between coaxial orbits, considering two cases: in the first one, the space vehicle departs from a circular orbit and, in the second one, it departs from an elliptic orbit with eccentricity $e_0 = 0.4$. In both cases, the semi-major axis is $a_0 = 1$, such that the results are presented in canonical units. The extremals are plotted for different values of the constant k_0 and the iso-cost curves for different values of the new consumption variable u/v_0 .

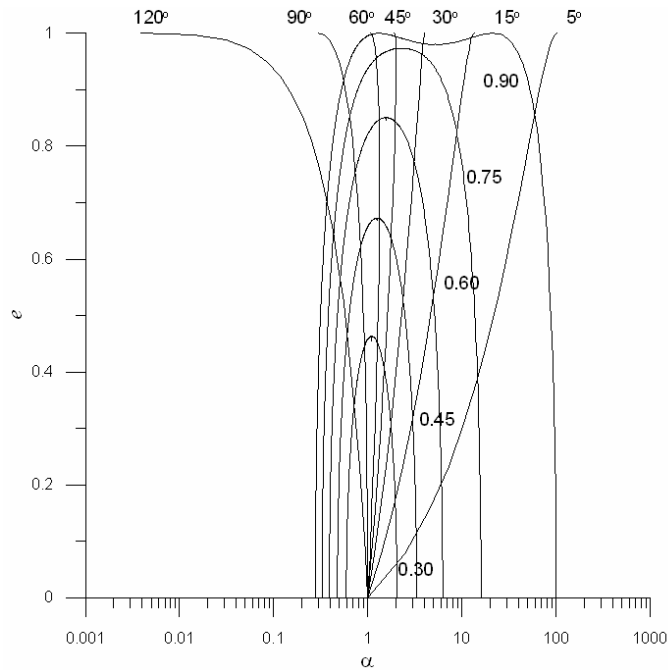


Figure 1 – Field of extremals and iso-cost curves for transfers between coaxial orbits for $e_0 = 0.0$.

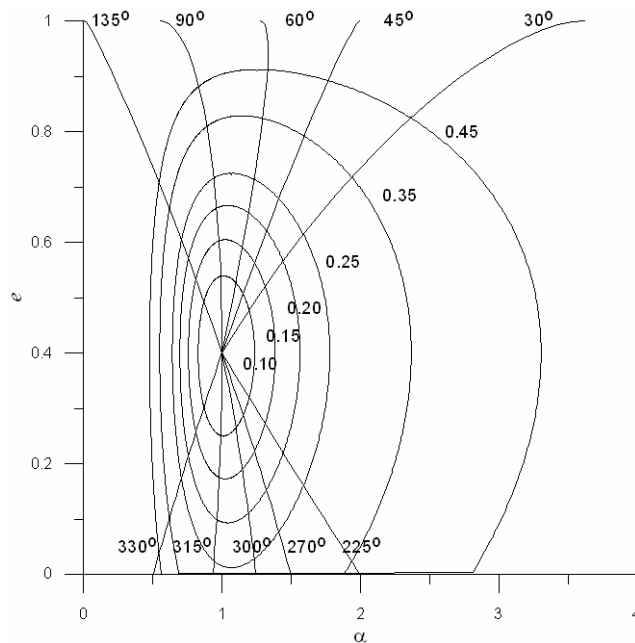


Figure 2 – Field of extremals and iso-cost curves for transfers between coaxial orbits for $e_0 = 0.4$.

Figures 3 and 4 show the field of extremals and iso-cost curves for long duration transfers between non-coaxial orbits. Note that in the general case the family of extremals is described by two constants k_0 and k_1 . In the plots of Figure 3, $k_0 = 18^\circ$ has been chosen in order to illustrate the general aspect of the field of extremals and compare the results with the ones obtained in [4,5,11,12]. The projection of extremals on the (e, ω) -plane shows clearly the non-existence of conjugate points for the extremals computed from Eqns (32) – (36) in the case of transfers between direct orbits (no direction reversals). Figure 4 shows a sample of iso-cost curves for different values of the new consumption variable u/v_0 and a fixed value of $k_1 = 18^\circ$. Note that these curves lie on an iso-cost surface with complex geometry. By simplicity, the symmetric interval $[-180^\circ, 180^\circ]$ for ω is considered.

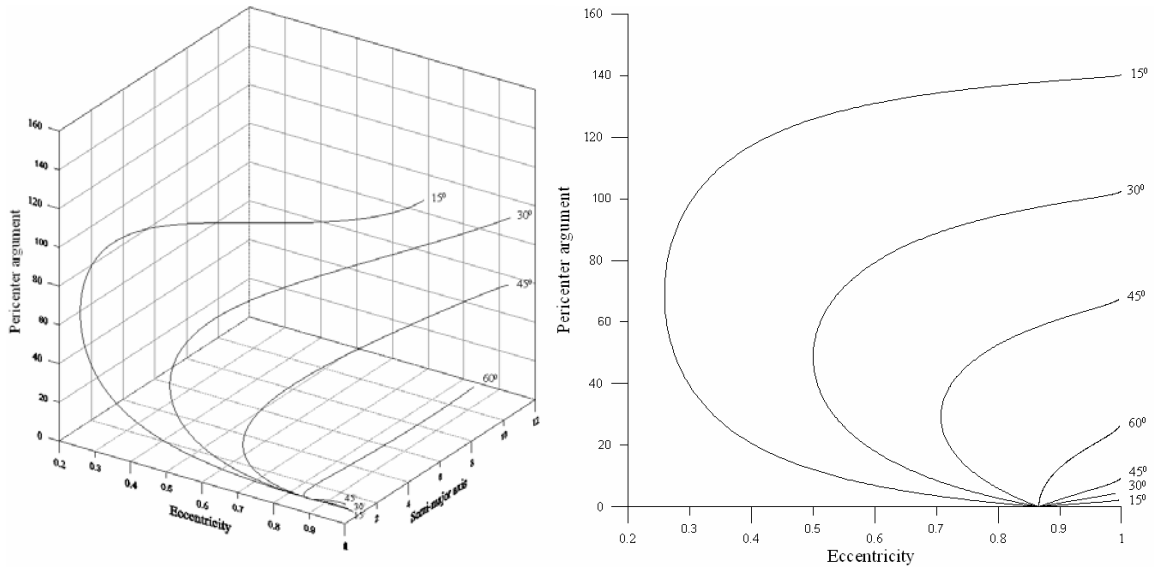


Figure 3 – Field of extremals for transfers between non-coaxial orbits for $e_0 = \sqrt{3}/2$ with $k_0 = 18^\circ$.

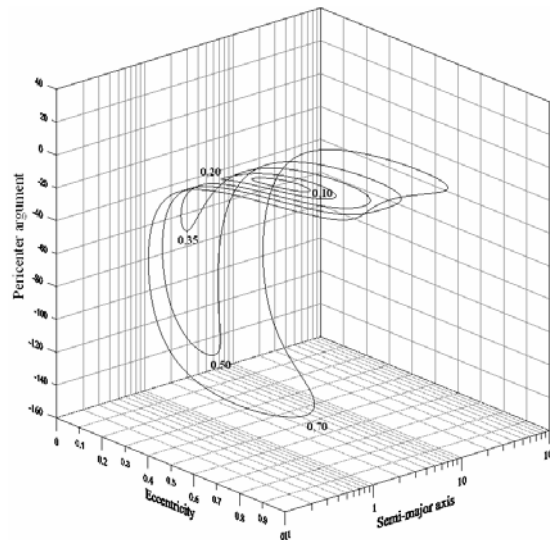


Figure 4 – Iso-cost curves for $a_0 = 1.0$, $e_0 = 0.5$ and $\omega_0 = 0^\circ$, with $k_1 = 18^\circ$.

Figure 5 shows a comparison between the complete analytical solution, the average solution and the one obtained through numerical integration of the set of canonical equations which describes the optimal trajectories (system of differential equations governed by new Hamiltonian function H^* , defined by Eqns (10) – (11)) for a transfer between coaxial orbits with the departure orbit O_0 defined by the set of initial conditions $a_0 = 1.0$, $e_0 = 0.20$, $\omega_0 = 0^\circ$ and the arrival orbit defined by the set of terminal conditions $a_0 = 2.0$, $e_0 = 0.25$, $\omega_0 = 0^\circ$,

and a transfer duration $t_f - t_0 = 500$ canonical units. The two-point boundary value problem is solved through a Newton-Raphson algorithm based on the complete analytical solution presented in Section 4 which uses the algorithm described in the preceding section for long duration transfers to generate a starting approximation of the adjoint variables. Note that there exists a good agreement between the complete analytical solution and the numerical one; but, the average solution loses accuracy for large values of the time when compared to the others solutions.

Finally, we note that a first order solution for the Cartesian elements can also be easily obtained from Lie theorem [2], since the theory presented in the paper is based on canonical transformations.

6. CONCLUSION

A first order analytical solution for the problem of optimal low-thrust limited-power transfers between arbitrary elliptic coplanar orbits in a Newtonian central gravity field has been obtained through Hamilton-Jacobi theory and a perturbation method based on Lie series. A complete analysis of long duration transfers, which includes the investigation of conjugate points and the solution of the two-point boundary value problem, is presented. Results show the good agreement between the complete analytical solution and the one obtained through numerical integration of the set of canonical equations which describes the optimal trajectories for a transfer between coaxial orbits. On the other hand, the average solution loses accuracy for large values of the time when compared to the complete analytical solution.

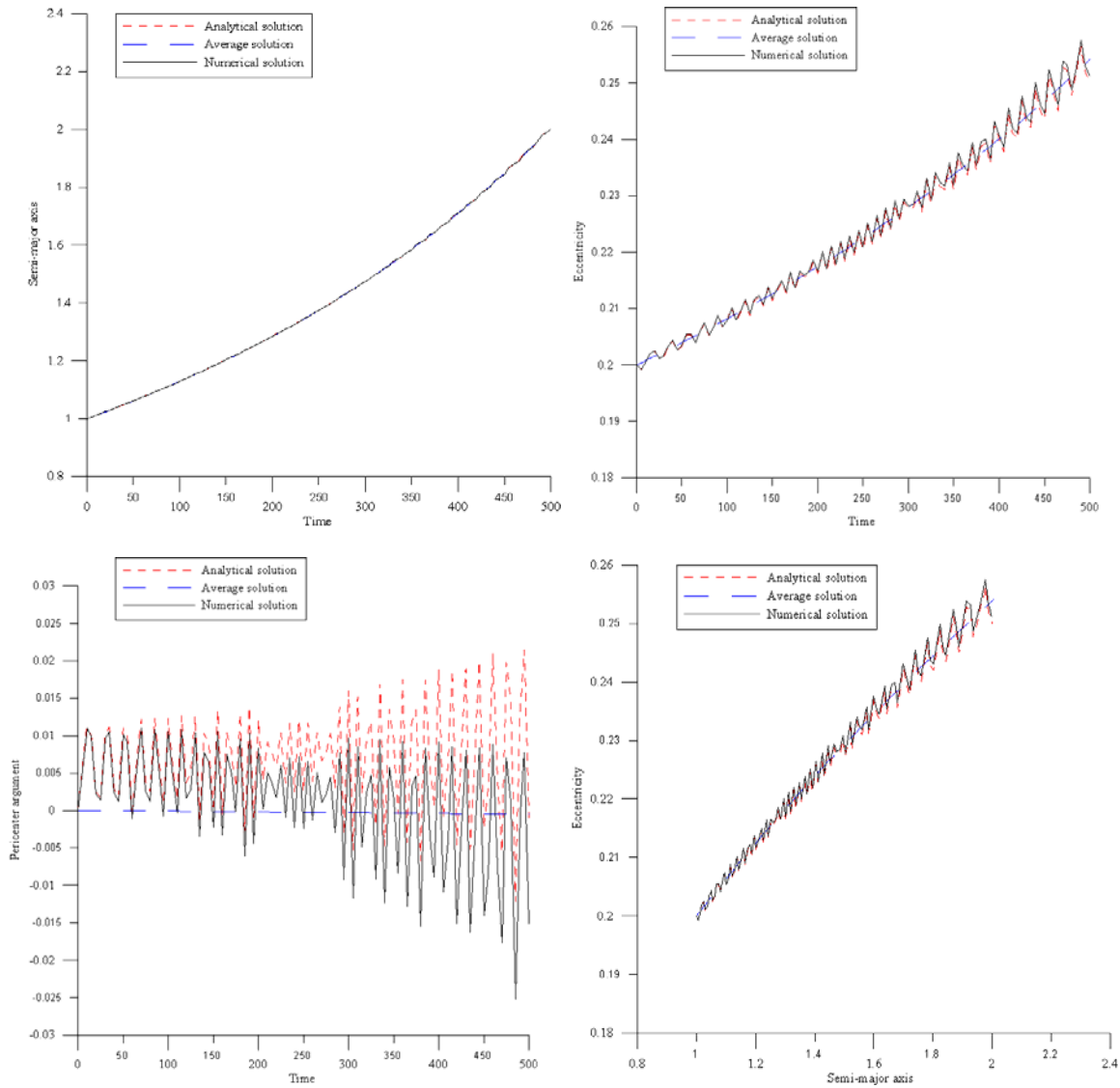


Figure 5 – Comparison between numerical, analytical and average solutions for a transfer between coaxial orbits.

REFERENCES

- [1] Marec, J. P. – *Optimal Space Trajectories*, Elsevier, New York, 1979.
- [2] Hori, G. – *Theory of General Perturbations with Unspecified Canonical Variables*, Publications of the Astronomical Society of Japan, Vol. 18, No. 4, pp. 287 – 295, 1966.
- [3] Lanczos, C. – *The Variational Principles of Mechanics*, Fourth Edition, Dover, New York, 1970.
- [4] Edelbaum, T. N. – *Optimum Power-Limited Orbit Transfer in Strong Gravity Fields*, AIAA Journal, Vol 3, No. 5, pp. 921 – 925, 1965.
- [5] Marec, J. P. & Vinh, N. X. – *Optimal Low-Thrust, Limited Power Transfers between Arbitrary Elliptical Orbits*, Acta Astronautica, Vol. 4, pp. 511 – 540, 1977.
- [6] Pontryagin, L. S., Boltyanskii, V. G., Gamkrelidze, R. V. & Mishchenko, E. F. – *The Mathematical Theory of Optimal Processes*, John Wiley & Sons, New York, 1962.
- [7] Da Silva Fernandes, S. – *Generalized Canonical Systems – III. Space Dynamics Applications: Solution of the Coast-Arc Problem*, Acta Astronautica, Vol. 32, No. 5, pp. 347 – 354, 1994.
- [8] Battin, R. H. – *An Introduction to the Mathematics and Methods of Astrodynamics*, American Institute of Aeronautics and Astronautics, New York, 1987.
- [9] Edelbaum, T. N. – *An Asymptotic Solution for Optimum Power Limited Orbit Transfer*, AIAA Journal, Vol. 4, No. 8, pp. 1491 – 1494, 1966.
- [10] Bliss, G. A. – *Lectures on the Calculus of Variations*, University of Chicago Press, Chicago, 1946.
- [11] Marec, J.P. & Vinh, N.X. – *Etude Generale des Transferts Optimaux a Poussee Faible et Puissance Limitee entre Orbites Elliptiques Quelconques*, Publication ONERA, No. 1980 – 2, 1980.
- [12] Haissig, C.M., Mease, K.D. & Vinh, N.X. – *Minimum-Fuel Power-Limited Transfers Between Coplanar Elliptical Orbits*, Acta Astronautica, Vol 29, No. 1, pp. 1 – 15, 1993.
- [13] Da Silva Fernandes, S. – *Notes on Hori Method for Canonical Systems*, Celestial Mechanics & Dynamical Astronomy, Vol. 85, No. 1, pp. 67 – 77, 2003.

APPENDIX

Hori method is based on a Lie theorem [2].

Let $H(x, y, \varepsilon)$ be a Hamiltonian written as a power series of a small parameter ε .

$$H(x, y, \varepsilon) = H_0(x, y) + \sum_{k=1}^{\infty} \varepsilon^k H_k(x, y),$$

where the undisturbed Hamiltonian $H_0(x, y)$ describes an integrable system. Here, x and y are $n \times 1$ -vectors of generalized coordinates and conjugate momenta, respectively.

Consider an infinitesimal canonical transformation $(x, y) \rightarrow (\xi, \eta)$ defined by a generating function $S(\xi, \eta, \varepsilon)$, also developed as a power series of the small parameter ε ,

$$\varepsilon S(\xi, \eta) = \sum_{k=1}^{\infty} \varepsilon^k S_k(\xi, \eta).$$

This transformation is such that the new dynamical system has some advantages for the solution. ξ and η are also $n \times 1$ -vectors. The new Hamiltonian $F(\xi, \eta, \varepsilon)$, resulting from this canonical transformation, is also developed as a power series of the small parameter ε ,

$$F(\xi, \eta, \varepsilon) = F_0(\xi, \eta) + \sum_{k=1}^{\infty} \varepsilon^k F_k(\xi, \eta).$$

The transformation equations are given explicitly by

$$x_i = \xi_i + \sum_{n=1}^{\infty} \frac{\varepsilon^n}{n!} D_s^{n-1} \frac{\partial S}{\partial \eta_i} \quad y_i = \eta_i + \sum_{n=1}^{\infty} \frac{\varepsilon^n}{n!} D_s^{n-1} \frac{\partial S}{\partial \xi_i}, \quad i = 1, \dots, n,$$

where $D_s^0 f = f$, $D_s^1 f = \{f, S\}$, $D_s^n f = D_s^{n-1}(D_s^1 f)$, $n \geq 2$, $f(\xi, \eta)$ denotes an arbitrary function and braces stand for Poisson brackets.

According to the algorithm of Hori method, the new Hamiltonian $F(\xi, \eta, \varepsilon)$ and the generating function $S(\xi, \eta, \varepsilon)$ are obtained, at each order of the small parameter ε , from the equations:

- order 0: $H_0(\xi, \eta) = F_0(\xi, \eta),$ (A.1)

- order 1: $\{H_0, S_1\} + H_1 = F_1,$ (A.2)

- order 2: $\{H_0, S_2\} + \frac{1}{2}\{H_1 + F_1, S_1\} + H_2 = F_2, \dots$ (A.3)

Functions in above equations are written in terms of the new set of canonical variables (ξ, η) .

The m -th order equation of the algorithm can be put in the form

$$\{H_0, S_m\} + \Theta_m = F_m, \quad (\text{A.4})$$

where Θ_m is a function obtained from the preceding orders, involving H_0, H_m, S_k, F_k and $H_k, k = 1, \dots, m-1$.

The determination of the functions F_m and S_m is based on the general solution of the undisturbed system

$$\frac{d\xi_i}{dt} = \frac{\partial F_0}{\partial \eta_i}, \quad \frac{d\eta_i}{dt} = -\frac{\partial F_0}{\partial \xi_i}, \quad i = 1, \dots, n, \quad (\text{A.5})$$

and it is performed through an averaging principle. Following the integration theory proposed in [13], Eqn (A.4) can be put in the form

$$\frac{\partial S_m}{\partial t} = \Theta_m - F_m,$$

with Θ_m written in terms of the arbitrary constants of integration, c_1, \dots, c_{2n} , of the general solution of the canonical system (A.5). S_m and F_m are unknown functions.

Following Hori [2], suppose that the canonical transformation generated by the function S is such that the time t is eliminated from the new Hamiltonian F . This is accomplished taking F_m as the mean value of the function Θ_m . Therefore, functions S_m and F_m are given through the equations

$$F_m = \langle \Theta_m \rangle, \quad (\text{A.6})$$

$$S_m = \int [\Theta_m - \langle \Theta_m \rangle] dt, \quad (\text{A.7})$$

where $\langle \rangle$ stands for the mean value of the function. It should be noted that the averaging process and the integration in Eqns (A.6) and (A.7) are performed considering the explicit time dependence of the functions.