ESTIMATION OF COMETARY ROTATION PARAMETERS BASED ON CAMERA IMAGES*

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1. Introduction

The purpose of the Rosetta mission is the *in situ* analysis of a cometary nucleus using both remote sensing equipment and scientific instruments delivered to the comet surface by a lander and transmitting measurement data to the comet-orbiting probe. Following a tour of planets including one Mars swing-by and three Earth swing-bys, the Rosetta probe is scheduled to rendezvous with comet 67P/Churyumov-Gerasimenko in May 2014. The mission poses various flight dynamics challenges, both in terms of parameter estimation and maneuver planning.

Along with spacecraft parameters, the comet's position, velocity, attitude, angular velocity, inertia tensor and gravitational field need to be estimated. The measurements on which the estimation process is based are ground-based measurements (range and Doppler) yielding information on the heliocentric spacecraft state and images taken by an on-board camera yielding information on the comet state relative to the spacecraft. The image-based navigation depends on the identification of cometary landmarks (whose body coordinates also need to be estimated in the process). The paper will describe the estimation process involved, focusing on the phase when, after orbit insertion, the task arises to estimate the cometary rotational motion from camera images on which individual landmarks begin to become identifiable.

2. Scenario Studied

We choose a space-fixed reference system and denote by $\xi(t) = \big(\xi_1(t), \xi_2(t), \xi_3(t)\big)^T \in \mathbb{R}^3$ the coordinate representation of the position vector of the comet's barycenter at time t with respect to this system. Next we choose a coordinate system rigidly attached to the comet, denote by $g_1(t), g_2(t), g_3(t) \in \mathbb{R}^3$ the coordinate representations of these body-fixed directions with respect to the reference system and call the matrix $g(t) = \big(g_1(t) \mid g_2(t) \mid g_3(t)\big) \in \mathrm{SO}(3)$, whose columns are the directions $g_i(t)$, the comet attitude at time t. It is important to realize that this body-fixed system can be arbitrarily chosen. (From a physical point of view it would be most convenient to choose g_1, g_2, g_3 as the directions of the comet's principal axes, but these principal axes are not known initially. From a practical point of view one will rather choose a geometrically defined body-system whose direction vectors are defined in terms of landmarks on the comet surface which can be identified on camera images.) Finally, we denote by $\omega(t) \in \mathbb{R}^3$ the body-referenced angular velocity vector and by $I \in \mathbb{R}^{3\times3}$ the (time-invariant) body-referenced inertia tensor of the spacecraft. To

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summarize, we introduce the following quantities:

$$\begin{aligned} \xi(t) &= \text{position of comet's barycenter at time } t; \\ g(t) &= \text{comet attitude at time } t; \\ \omega(t) &= \text{body-referenced angular velocity vector of comet;} \\ I &= \text{body-referenced inertia tensor of comet.} \end{aligned} \tag{1}$$

Moreover, we associate with each vector $\omega \in \mathbb{R}^3$ the cross-product operator

$$L(\omega) = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \tag{2}$$

which is characterized by the equation $L(\omega)v = \omega \times v$ for all $v \in \mathbb{R}^3$. Under the assumption that there are no external torques acting on the comet (for example due to outgassing), the rotational motion of the comet is governed by the equations

$$\dot{g} = gL(\omega) \quad \text{and} \quad I\dot{\omega} = (I\omega) \times \omega\,. \tag{3}$$

In general, closed-form solutions to Euler's equation $I\dot{\omega} = (I\omega) \times \omega$ are not available in terms of elementary functions. This is different, however, in the special case that ω points in a space-fixed direction which means that $\omega(t) = u(t)a$ where u is a scalar function and where a is a fixed vector. In this case the equation $I\dot{\omega} = (I\omega) \times \omega$ becomes $\dot{u}(Ia) = u^2(Ia) \times a$ which is only possible if $\dot{u} = 0$, i. e., if ω is constant. Then necessarily $I\omega \times \omega = 0$ so that ω is an eigenvector of I, i. e., is aligned with a principal axis direction. For constant ω the equation $\dot{g} = gL(\omega)$ has the explicit solution $g(t) = g_0 \exp\big(tL(\omega)\big)$ where $g_0 = g(0)$ is the attitude at the reference time $t = 0$.

We now proceed to derive the measurement equations. We assume that a landmark located on the comet surface (such as a crater) can be identified on a CCD image taken by the on-board camera. If $b \in \mathbb{R}^3$ is the body-referenced position of the landmark, then the space-referenced position of this landmark at time t is $\ell(t) = \xi(t) + b_1 g_1(t) + b_2 g_2(t) + b_3 g_3(t) = \xi(t) + g(t)b$. We identify the spacecraft position $x(t)$ with the position of the optical center of the on-board camera, denote by f the focal width of this camera and by $c(t) = \big(c_1(t) \mid c_2(t) \mid c_3(t)\big) \in \mathrm{SO}(3)$ the camera attitude (where it is assumed that c_3 points in the direction of the optical axis). To summarize, we introduce the following quantities:

$$\begin{aligned} b &= \text{body-referenced landmark position,} \\ \ell(t) &= \xi(t) + g(t)b = \text{space-referenced landmark position at time } t, \\ x(t) &= \text{spacecraft position (optical center) at time } t, \\ c(t) &= \text{camera attitude at time } t, \\ f &= \text{focal width of camera.} \end{aligned} \tag{4}$$

The image point is given as the point of intersection between the ray from the landmark through the optical center and the image plane, which means that there are real numbers $\lambda > 0$ and $u, v \in \mathbb{R}$ (where u and v denote the horizontal and vertical image coordinates) such that $\ell + \lambda(x - \ell) = x + uc_1 + vc_2 - fc_3$, as is shown in Figure 1.

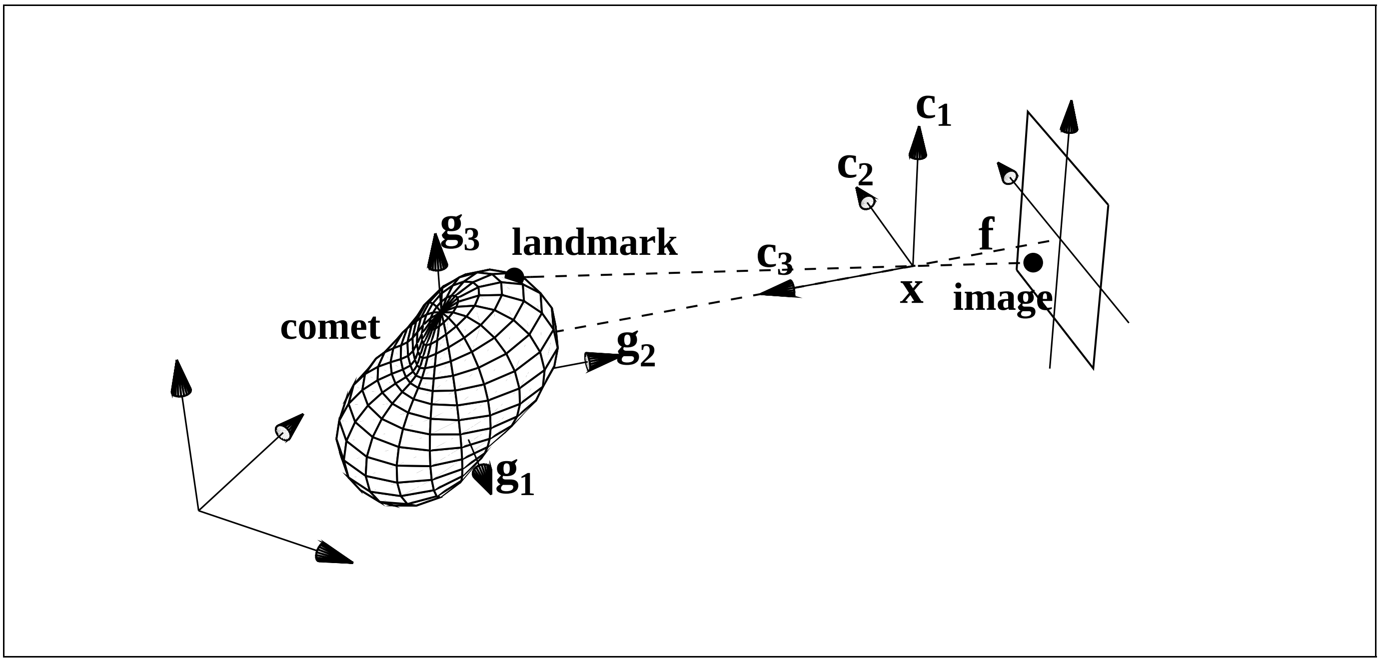

Fig. 1: Derivation of the measurement equations.

The equation $\ell + \lambda(x - \ell) = x + uc_1 + vc_2 - fc_3$ can be rewritten in the form

$$(\lambda - 1)(x - \ell) \;=\; uc_1 + vc_2 - fc_3\,. \tag{5}$$

Taking the inner product with c_3 we find that $(\lambda - 1)\langle x - \ell, c_3\rangle = -f$ and hence that

$$\lambda - 1 \;=\; \frac{-f}{\langle x - \ell, c_3\rangle}\,. \tag{6}$$

Plugging (6) back into (5) and taking the inner product with c_1 and c_2 we find that

$$\begin{aligned} u \;&=\; -f \cdot \frac{\langle x - \ell, c_1\rangle}{\langle x - \ell, c_3\rangle} \;=\; -f \cdot \frac{\langle x - \xi - gb, c_1\rangle}{\langle x - \xi - gb, c_3\rangle}, \\ v \;&=\; -f \cdot \frac{\langle x - \ell, c_2\rangle}{\langle x - \ell, c_3\rangle} \;=\; -f \cdot \frac{\langle x - \xi - gb, c_2\rangle}{\langle x - \xi - gb, c_3\rangle}. \end{aligned} \tag{7}$$

For the purposes of our present analysis we may assume that the spacecraft position x is known from orbit determination based on ground-based measurements (range and Doppler) and that the camera attitude $c(t)$ is known from spacecraft attitude determination. The task is then to estimate from the available measurements (7) (and the underlying dynamical equations) the comet's position, velocity, attitude and angular velocity at some reference time and the body-coordinates of the landmarks involved. We first study the phase in which the comet can be identified as a point on camera images; in this case the comet itself is treated as a single landmark (with $b = 0$), and the task is reduced to estimating the comet position and velocity.

3. Estimation of Comet Position and Velocity

We assume that images are taken during a time interval which is short enough to replace the spacecraft and the comet orbits by their tangent lines. Assuming thus a linear regime, a moment's thought shows that it is not possible to determine the comet state from a sequence of pictures alone, as the spacecraft-comet vector and the speed of the comet are uniquely determined only up to a common scaling factor.

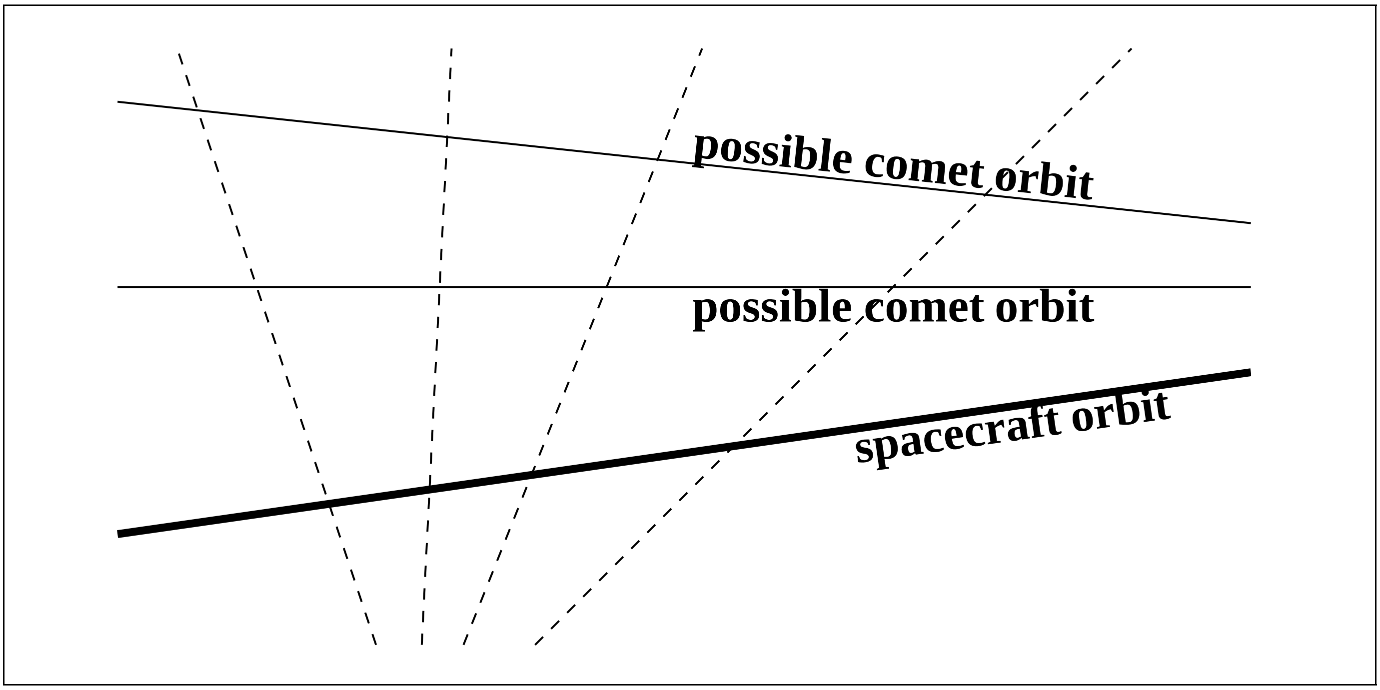

Fig. 2: Impossibility of determining the comet state simply from camera images.

What can be done is performing a maneuver (more or less in the blind) which changes the direction of the spacecraft motion. In this case the comet state can be determined from four images of which at least one each must be taken before and after the maneuver. To derive the necessary formulas, let us fix the reference time $t = 0$ at the time of the maneuver and let us introduce the following data:

$$\begin{aligned} x_0 &:= \text{spacecraft position at maneuver time;} \\ v &:= \text{spacecraft velocity (assumed constant during observation period);} \\ \xi_0 &:= \text{comet position at maneuver time;} \\ u &:= \text{comet velocity (assumed constant during observation period);} \\ d &:= \xi_0 - x_0 \;=\; \text{vector from spacecraft to comet at maneuver time;} \\ w &:= u - v \;=\; \text{velocity of comet relative to spacecraft before maneuver);} \\ \Delta v &:= \text{velocity change caused by maneuver.} \end{aligned} \tag{8}$$

Then at any time $s < 0$ the comet and spacecraft positions are $\xi_0 + su$ and $x_0 + sv$, respectively, whereas at any time $t > 0$ these positions are $\xi_0 + tu$ and $x_0 + t(v + \Delta v)$, respectively. Thus the information to be gleaned from camera images are, for each $s < 0$, the unit vector e_s in the direction of $(\xi_0 + su) - (x_0 + sv) = d + sw$, and, for each $t > 0$, the unit vector e_t in the direction of $(\xi_0 + tu) - (x_0 + tv + t\Delta v) = d + t(w - \Delta v)$, i.e., the unit vectors

$$e_s \;:=\; \frac{d + sw}{\|d + sw\|} \quad \text{and} \quad e_t \;:=\; \frac{d + t(w - \Delta v)}{\|d + t(w - \Delta v)\|}\,. \tag{9}$$

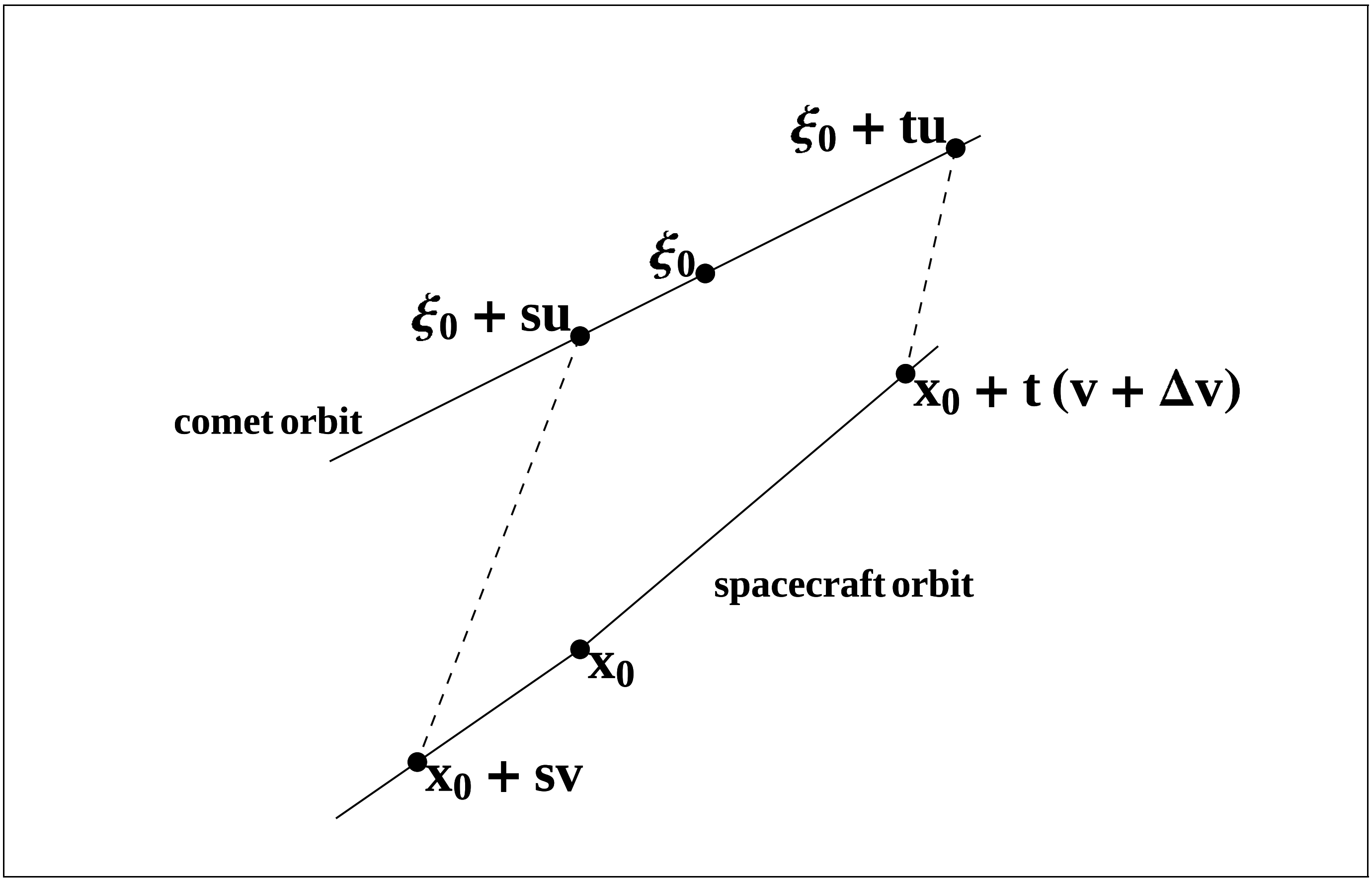

Fig. 3: Effect of orbit maneuver.

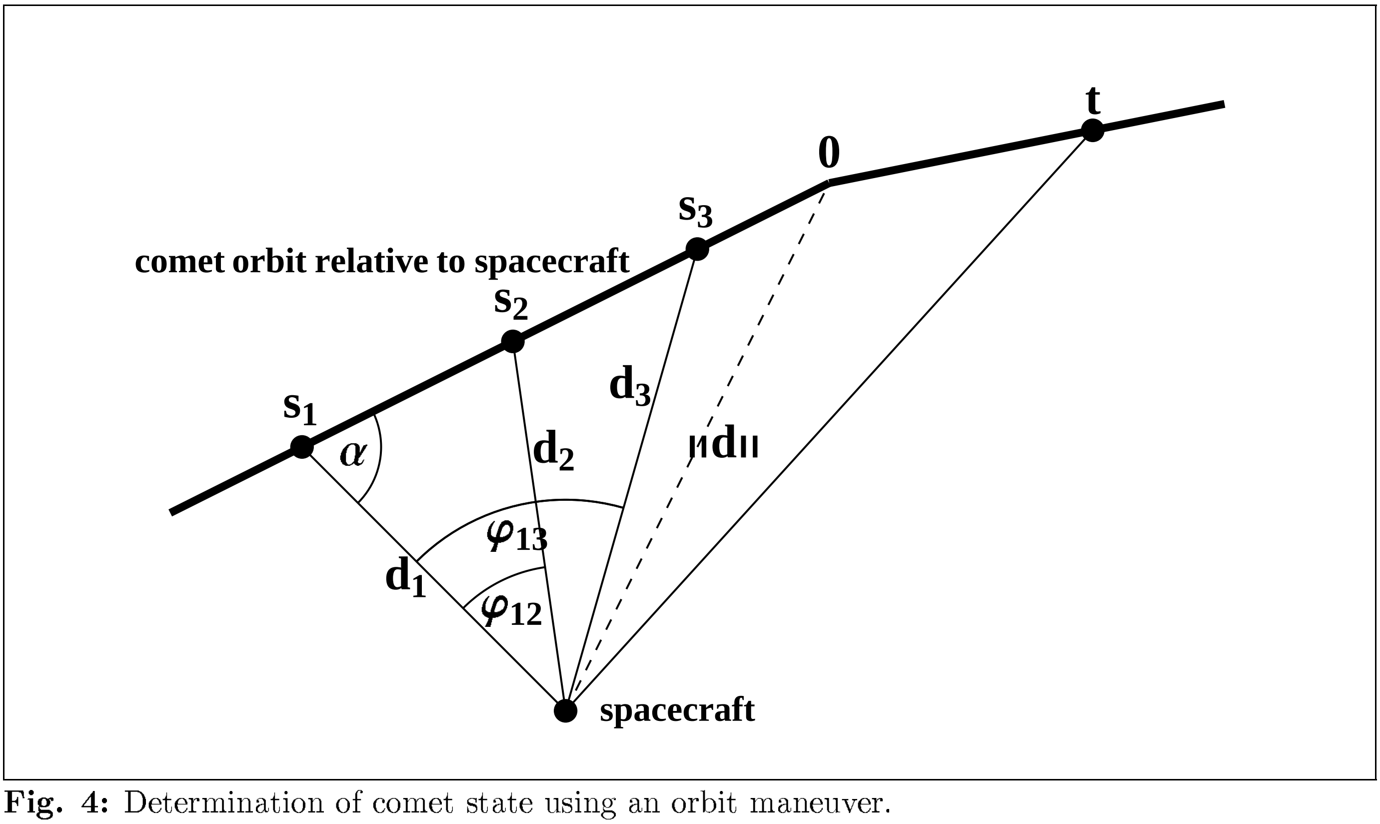

Fig. 4: Determination of comet state using an orbit maneuver.

Let us assume that images of the comet are obtained at times $s_1 < s_2 < s_3 < 0 < t$. With the notation introduced above, let us write $e_i := e_{s_i}$ for the unit vector from the spacecraft to the comet at time s_i (where $i = 1, 2, 3$). Knowledge of e_1 and e_2 fixes the plane in which d and w lie. Let us write $d_i := \|d + s_i w\|$ for the spacecraft-comet distance at time s_i and let us denote by φ_{ij} the angle from e_i to e_j; then the Theorem of Sines yields

$$\begin{aligned} \frac{\sin(\varphi_{12})}{(s_2 - s_1)\|w\|} &= \frac{\sin(\pi - \alpha - \varphi_{12})}{d_1} = \frac{\sin(\alpha + \varphi_{12})}{d_1} \quad \text{and} \\ \frac{\sin(\varphi_{13})}{(s_3 - s_1)\|w\|} &= \frac{\sin(\pi - \alpha - \varphi_{13})}{d_1} = \frac{\sin(\alpha + \varphi_{13})}{d_1}. \end{aligned} \tag{10}$$

Taking quotients, we find that

$$\frac{\sin(\varphi_{12}) \cdot (s_3 - s_1)}{\sin(\varphi_{13}) \cdot (s_2 - s_1)} = \frac{\sin(\alpha + \varphi_{12})}{\sin(\alpha + \varphi_{13})} = \frac{\tan(\alpha)\cos(\varphi_{12}) + \sin(\varphi_{12})}{\tan(\alpha)\cos(\varphi_{13}) + \sin(\varphi_{13})} \tag{11}$$

from which $\tan(\alpha)$ can be determined (and hence also α with a single ambiguity to be resolved). Once α is known, (10) relates the speed $\|w\|$ with the distance d_1, say $\|w\| = \lambda d_1$ with a known factor λ. Then the Theorem of Cosines yields

$$\begin{aligned} \|d\|^2 &= d_1^2 + s_1^2\|w\|^2 - 2d_1|s_1|\,\|w\|\cos(\alpha) \\ &= d_1^2 + s_1^2 d_1^2 \lambda^2 - 2d_1^2|s_1|\lambda\cos(\alpha) \\ &= d_1^2\left(1 + s_1^2\lambda^2 - 2|s_1|\lambda\cos(\alpha)\right) \end{aligned} \tag{12}$$

so that $\|d\| = \Theta d_1$ with a known factor Θ. Moreover, for $k = 2, 3$ the Theorem of Sines yields

$$\begin{aligned} \frac{\sin(\alpha)}{d_k} &= \frac{\sin(\pi - \alpha - \varphi_{1k})}{d_1} = \frac{\sin(\alpha + \varphi_{1k})}{d_1} \quad \text{and hence} \\ d_k &= \theta_k d_1 \quad \text{where} \quad \theta_k := \frac{\sin(\alpha)}{\sin(\alpha + \varphi_{1k})}. \end{aligned} \tag{13}$$

We now make use of the equations $d + s_i w = d_i e_i$ where $i = 1, 2, 3$. We take two of these equations, say with indices i and j; subtracting these equations and also subtracting the s_j-fold of the first equation from the s_i-fold of the second equation, we find that

$$w = \frac{d_i e_i - d_j e_j}{s_i - s_j} = d_1 \frac{\theta_i e_i - \theta_j e_j}{s_i - s_j} \quad \text{and} \quad d = \frac{s_i d_j e_j - s_j d_i e_i}{s_i - s_j} = d_1 \frac{s_i \theta_j e_j - s_j \theta_i e_i}{s_i - s_j}. \tag{14}$$

This shows that the measurements taken before the maneuver enable us to determine w and d up to a common scaling factor d_1. (In particular, the unit vectors $e_d = d/\|d\|$ and $e_w = w/\|w\|$ can be determined from the measurements.) The scaling factor d_1 can now be determined from one single measurement taken after the maneuver; if e_t is the unit vector from the spacecraft to the comet at some time $t > 0$, we have $d + t(w - \Delta v) = \mu e_t$ for some factor μ. Since $d = \|d\|e_d = \Theta d_1 e_d$ and $w = \|w\|e_w = \lambda d_1 e_w$ this means $\Theta d_1 e_d + t(\lambda d_1 e_w - \Delta v) = \mu e_t$, i.e.,

$$d_1(\Theta e_d + t\lambda e_w) - \mu e_t = t\Delta v \tag{15}$$

where the coefficients d_1 and μ are unknown whereas everything else is known. Thus d_1 and μ can be found by decomposing $t\Delta v$ into its components relative to the known vectors $\Theta e_d + t\lambda e_w$ and e_t. (The velocity change Δv can be assumed to be known from spacecraft orbit determination using ground-based measurements after the maneuver.) Hence from now on we can assume that the comet's position and velocity are known and concentrate on estimating the comet's rotational parameters.

4. Estimation of Cometary Rotation Parameters

Here we consider the phase in which the spacecraft is in orbit around the comet and the first individual landmarks can be identified on CCD images; we try to estimate the rotational motion of the comet and the locations of the observed landmarks. We make the simplifying assumption that the comet rotates with a constant spin rate ω about an axis given by a space-fixed direction d; this assumption is legitimate at least for short observation periods. Choosing a body-fixed system whose third direction coincides with the (unknown) spin axis direction and denoting by g_0 the comet attitude at initial time $t_0 = 0$, the attitude at any time t is given by

$$g(t) \; = \; g_0 \begin{bmatrix} \cos(\omega t) & -\sin(\omega t) & 0 \\ \sin(\omega t) & \cos(\omega t) & 0 \\ 0 & 0 & 1 \end{bmatrix} . \tag{16}$$

Hence if a landmark is located at radius r, longitude φ und latitude θ with respect to this body-fixed system, then its space-referenced position is given by

$$\ell(t) \; = \; \xi(t) + r \cdot g(t) \begin{bmatrix} \cos\theta\cos\varphi \\ \cos\theta\sin\varphi \\ \sin\theta \end{bmatrix} \; = \; \xi(t) + r \cdot g_0 \begin{bmatrix} \cos\theta\cos(\varphi+\omega t) \\ \cos\theta\sin(\varphi+\omega t) \\ \sin\theta \end{bmatrix} . \tag{17}$$

Let us denote by n the unit vector from the optical center to the landmark, by e the unit vector from the comet center to the landmark and by $d := \xi - x$ the vector from the spacecraft to the comet center; note that both d and n are known, n being given by

$$n \; = \; \frac{-uc_1 - vc_2 + fc_3}{\sqrt{u^2+v^2+f^2}} . \tag{18}$$

Then there is a number $\lambda > 0$ such that $re = -d + \lambda n$. Taking norms on both sides of this equation, we find $r^2 = \|d\|^2 - 2\lambda\langle n, d\rangle + \lambda^2$ and hence

$$\lambda_{1,2} = \langle n, d\rangle \pm \sqrt{\langle n, d\rangle^2 - \|d\|^2 + r^2} . \tag{19}$$

Since $\langle n, d\rangle > 0$ and $r < \|d\|$, both solutions λ_i are positive; it is geometrically clear that, if the comet shape is not too different from that of a sphere, the smaller of the two values λ_i is the sought solution. Hence $\lambda = \langle n, d\rangle - \sqrt{\langle n, d\rangle^2 - \|d\|^2 + r^2}$ so that the equation $re = -d + \lambda n$ becomes

$$e \; = \; \frac{1}{r}\left(-d + \langle n, d\rangle n - \sqrt{\langle n, d\rangle^2 - \|d\|^2 + r^2} \cdot n \right) . \tag{20}$$

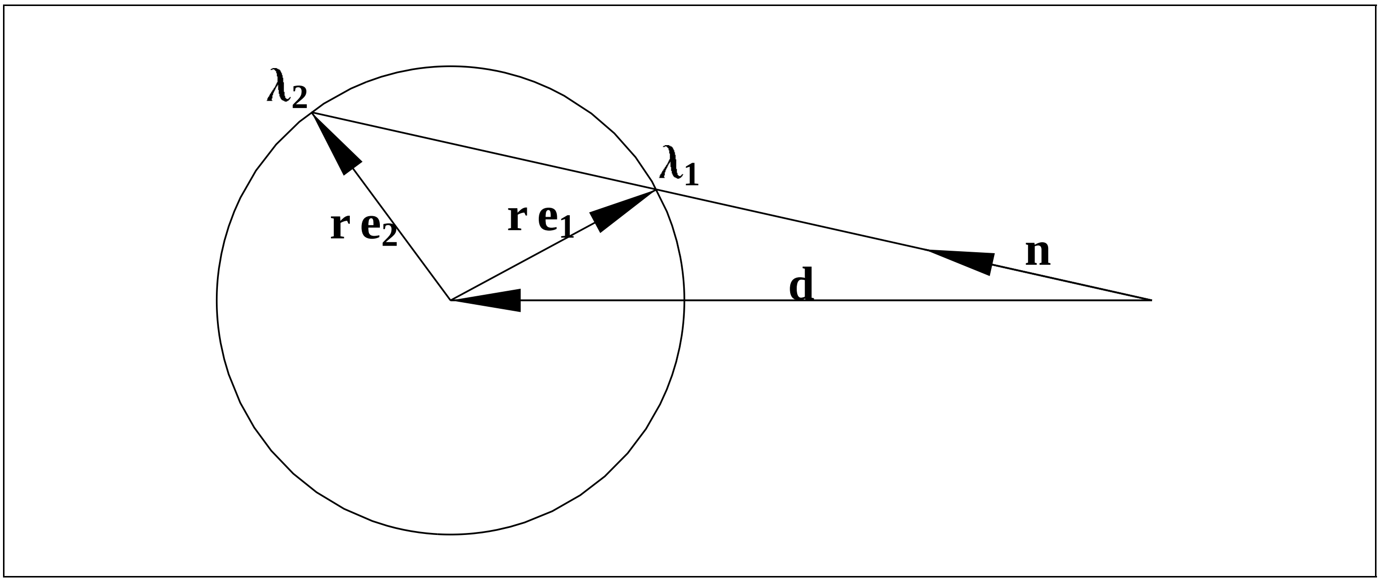

Fig. 5: Determination of the unit vector from the comet center to the landmark

Let us now assume that the comet can be approximated by a sphere of radius R where R is at least approximately known. Then $r = R$, and the upshot of (20) is that the unit vector $e(t)$ can be computed from a measurement taken at time t. This observation will now be exploited to determine the spin rate ω, the comet attitude g_0 at the reference time and the landmark coordinates θ and φ from given landmark vectors $e_i = e(t_i)$ resulting from measurements at times t_i.

Method 1: One landmark from three images. Take three measurements on which the same landmark can be identified and let e_i, e_j and e_k be the associated landmark unit vectors. The comet's rotation axis d must be perpendicular to both $e_j - e_i$ and $e_k - e_j$, hence is given by

$$d \;=\; \pm \frac{(e_j - e_i) \times (e_k - e_j)}{\|(e_j - e_i) \times (e_k - e_j)\|}\,. \tag{21}$$

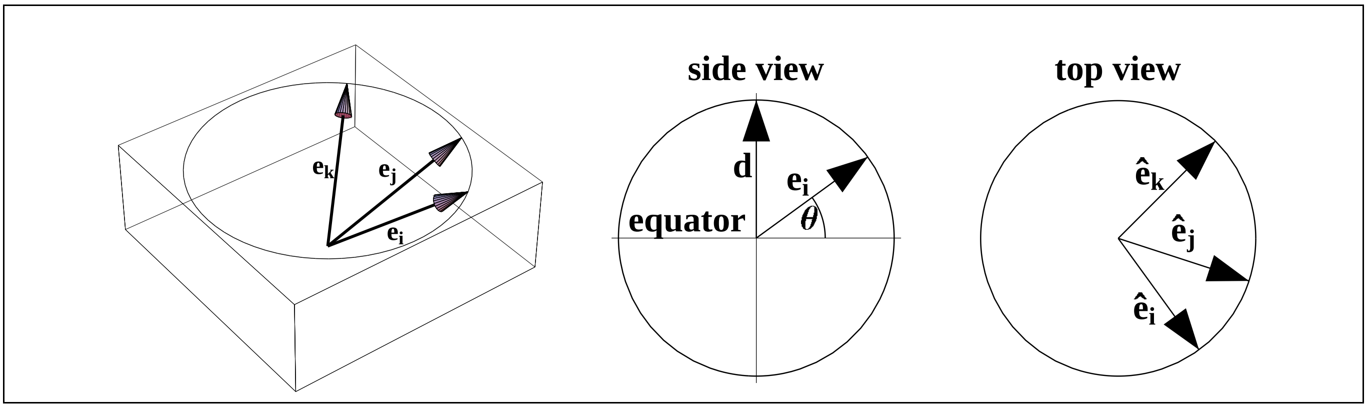

Fig. 6: Determination of spin axis, landmark latitude and spin rate.

The landmark latitude θ must then satisfy $\sin^2\theta = \cos^2(90^0 - \theta) = \langle d, e_i\rangle^2$ so that

$$\sin\theta \;=\; \pm\langle d, e_i\rangle \tag{22}$$

for all i. Moreover, writing

$$\widehat{e_i} \;:=\; \frac{e_i - \langle e_i, d\rangle d}{\|e_i - \langle e_i, d\rangle d\|}, \tag{23}$$

and denoting by φ_{ij} the angle by which the landmark was rotated about the axis of rotation, we have

$$\langle \hat{e}_i, \hat{e}_j \rangle \;=\; \cos\varphi_{ij} \;=\; \cos\bigl(\omega(t_j - t_i)\bigr) \tag{24}$$

which determines the spin rate ω. Consequently, the comet attitude at time t_i is given by the column vectors

$$g_3 = d, \quad g_2 = \frac{d \times e_i}{\|d \times e_i\|} \quad \text{and} \quad g_1 = g_2 \times g_3. \tag{25}$$

It is clear that the landmark longitude cannot be estimated from the available measurements; however, we can assume that the first axis in the body-fixed system is pointing in the direction of the landmark longitude so that $\varphi = 0$ (i.e., we define zero longitude by the landmark location).

Method 2: General equations. Let us assume that a number of landmarks can be identified on a sequence of images. Let $e^{(i,k)}$ be the unit vector from the comet center to the i-th landmark at the time the k-th image is taken. Dividing (17) by $r = R$ we find that

$$e^{(i,k)} \;=\; g_0 \begin{bmatrix} \cos\theta_i \cos(\varphi_i + \omega t_k) \\ \cos\theta_i \sin(\varphi_i + \omega t_k) \\ \sin\theta_i \end{bmatrix} \tag{26}$$

where t_k is the time (counted from the epoch) at which the k-th picture is taken. Since g_0 is a rotation matrix and hence preserves inner products, we find from (26) that

$$\langle e^{(i,k)}, e^{(j,\ell)} \rangle \;=\; \cos\theta_i \cos\theta_j \cos\bigl(\varphi_i - \varphi_j + \omega(t_k - t_\ell)\bigr) \;+\; \sin\theta_i \sin\theta_j \tag{27}$$

so that the comet attitude g_0 is decoupled from the other unknowns. (We note that it is not possible to determine all longitudes φ_i from these equations, since only the differences $\varphi_i - \varphi_j$ occur; this shows again that, within our present considerations, zero longitude can be defined arbitrarily for one of the landmarks, and one can only hope to determine the longitude separations from all other landmarks to the chosen one.) Let us see which conclusions can be drawn from (27) in special cases.

Method 2a: One single landmark from several images. Considering only one single landmark (say the i-th), we find that

$$\begin{aligned} \langle e^{(i,k)}, e^{(i,\ell)} \rangle \;&=\; \cos^2\theta_i \cos(\omega t_k - \omega t_\ell) + \sin^2\theta_i \\ &=\; \cos(\omega t_k - \omega t_\ell) + \bigl(1 - \cos(\omega t_k - \omega t_\ell)\bigr)\sin^2\theta_i \end{aligned} \tag{28}$$

Eliminating θ_i via

$$\sin^2\theta_i \;=\; \frac{\langle e^{(i,k)}, e^{(i,\ell)} \rangle - \cos(\omega t_k - \omega t_\ell)}{1 - \cos(\omega t_k - \omega t_\ell)} \tag{29}$$

we obtain the compatibility conditions

$$\frac{\langle e^{(i,k)}, e^{(i,\ell)} \rangle - \cos(\omega t_k - \omega t_\ell)}{1 - \cos(\omega t_k - \omega t_\ell)} \;=\; \frac{\langle e^{(i,K)}, e^{(i,L)} \rangle - \cos(\omega t_K - \omega t_L)}{1 - \cos(\omega t_K - \omega t_L)} \tag{30}$$

which must hold for all pairs (k,ℓ) and (K,L). Note that these can be rewritten in the form $a_{k\ell}\cos(\omega\Delta_{KL}) - a_{KL}\cos(\omega\Delta_{k\ell}) + a_{KL} - a_{k\ell} = 0$ where $\Delta_{k\ell} := t_k - t_\ell$ and $a_{k\ell} := 1 - \langle e^{(i,k)}, e^{(i,\ell)} \rangle$; hence ω can be found as a root of functions of the form $F(\omega) := a\cos(A\omega) - b\cos(B\omega) + b - a$ with known constants a, b, A, B. Thus the following algorithm can be used to reconstruct the unknown parameters from one single landmark which is seen on at least three different images: We first determine the spin rate ω from (30) (possibly with ambiguities), then the landmark latitude θ_i from (29) (with a single ambiguity) and finally the comet attitude g_0 from (26) (to be solved in the least-squares sense after letting $\varphi_i := 0$).

Method 2b: Several landmarks on one single image. Let us see which information can be inferred from one single image (say the k-th). Letting $\ell = k$ in (27), we find that

$$\langle e^{(i,k)}, e^{(l,k)} \rangle = \cos\theta_i \cos\theta_j \cos(\varphi_i - \varphi_j) + \sin\theta_i \sin\theta_j. \tag{31}$$

If the landmark latitudes θ_i are already known (for example from the method described before), we can simply use (31) to determine the longitude separations via

$$\cos(\varphi_i - \varphi_j) = \frac{\langle e^{(i,k)}, e^{(l,k)} \rangle - \sin\theta_i \sin\theta_j}{\cos\theta_i \cos\theta_j}. \tag{32}$$

If (32) is used without a priori information then, if s landmarks can be identified, this constitutes a system of $s(s-1)/2$ equations for the $2s-1$ unknowns $\theta_1, \ldots, \theta_s$ and $\varphi_2 - \varphi_1, \ldots, \varphi_s - \varphi_1$, which can be solved if $s(s-1)/2 \geq 2s - 1$ (which is the case for $s \geq 5$).

Method 2c: Two landmarks from two images. Let us consider the case that two different landmarks can be identified on two different images. Writing $\Delta t := t_2 - t_1$ and $\Delta\varphi := \varphi_2 - \varphi_1$, the nontrivial equations of the form (27) $\big($i.e., the ones for which $(i,k) \neq (j,\ell)\big)$ take the following forms:

$$\begin{aligned}
\langle e^{(1,1)}, e^{(1,2)} \rangle &= \cos^2\theta_1 \cos(\omega\Delta t) + \sin^2\theta_1; \\
\langle e^{(1,1)}, e^{(2,1)} \rangle &= \cos\theta_1 \cos\theta_2 \cos(\Delta\varphi) + \sin\theta_1 \sin\theta_2; \\
\langle e^{(1,1)}, e^{(2,2)} \rangle &= \cos\theta_1 \cos\theta_2 \cos(\Delta\varphi + \omega\Delta t) + \sin\theta_1 \sin\theta_2; \\
\langle e^{(1,2)}, e^{(2,1)} \rangle &= \cos\theta_1 \cos\theta_2 \cos(\Delta\varphi - \omega\Delta t) + \sin\theta_1 \sin\theta_2; \\
\langle e^{(1,2)}, e^{(2,2)} \rangle &= \cos\theta_1 \cos\theta_2 \cos(\Delta\varphi) + \sin\theta_1 \sin\theta_2; \\
\langle e^{(2,1)}, e^{(2,2)} \rangle &= \cos^2\theta_2 \cos(\omega\Delta t) + \sin^2\theta_2.
\end{aligned} \tag{33}$$

A comparison of the second and the fifth equation yields the compatibility condition

$$\langle e^{(1,1)}, e^{(2,1)} \rangle = \langle e^{(1,2)}, e^{(2,2)} \rangle; \tag{34}$$

since (33) consists of six equations for four unknowns there must be (at least) one other compatibility condition. From the first and the last equation we find that

$$\cos^2\theta_1 = \frac{1 - \langle e^{(1,1)}, e^{(1,2)} \rangle}{1 - \cos(\omega\Delta t)} \quad \text{and} \quad \cos^2\theta_2 = \frac{1 - \langle e^{(2,1)}, e^{(2,2)} \rangle}{1 - \cos(\omega\Delta t)}; \tag{35}$$

since the landmark latitudes must satisfy $-\pi/2 < \theta_i < \pi/2$ and hence have a positive cosine, roots can be extracted unambiguously in (35) so that

$$\cos\theta_1 = \sqrt{\frac{1 - \langle e^{(1,1)}, e^{(1,2)} \rangle}{1 - \cos(\omega\Delta t)}} \quad \text{and} \quad \cos\theta_2 = \sqrt{\frac{1 - \langle e^{(2,1)}, e^{(2,2)} \rangle}{1 - \cos(\omega\Delta t)}}. \tag{36}$$

Plugging in (33) the second into the third and the fifth into the fourth equation, we find that

$$\begin{aligned} \langle e^{(1,1)}, e^{(2,2)} \rangle &= \cos\theta_1 \cos\theta_2 \cos(\Delta\varphi + \omega\Delta t) + \langle e^{(1,1)}, e^{(2,1)} \rangle - \cos\theta_1 \cos\theta_2 \cos(\Delta\varphi), \\ \langle e^{(1,2)}, e^{(2,1)} \rangle &= \cos\theta_1 \cos\theta_2 \cos(\Delta\varphi - \omega\Delta t) + \langle e^{(1,2)}, e^{(2,2)} \rangle - \cos\theta_1 \cos\theta_2 \cos(\Delta\varphi) \end{aligned} \tag{37}$$

and hence that

$$\begin{aligned} \langle e^{(1,1)}, e^{(2,2)} - e^{(2,1)} \rangle &= \cos\theta_1 \cos\theta_2 \cdot \big(\cos(\Delta\varphi + \omega\Delta t) - \cos(\Delta\varphi)\big), \\ \langle e^{(1,2)}, e^{(2,1)} - e^{(2,2)} \rangle &= \cos\theta_1 \cos\theta_2 \cdot \big(\cos(\Delta\varphi - \omega\Delta t) - \cos(\Delta\varphi)\big). \end{aligned} \tag{38}$$

Plugging (36) into (38), we obtain

$$\begin{aligned} U &:= \frac{\langle e^{(1,1)}, e^{(2,2)} - e^{(2,1)} \rangle}{\sqrt{1 - \langle e^{(1,1)}, e^{(1,2)} \rangle}\sqrt{1 - \langle e^{(2,1)}, e^{(2,2)} \rangle}} = \frac{\cos(\Delta\varphi + \omega\Delta t) - \cos(\Delta\varphi)}{1 - \cos(\omega\Delta t)}, \\ V &:= \frac{\langle e^{(1,2)}, e^{(2,1)} - e^{(2,2)} \rangle}{\sqrt{1 - \langle e^{(1,1)}, e^{(1,2)} \rangle}\sqrt{1 - \langle e^{(2,1)}, e^{(2,2)} \rangle}} = \frac{\cos(\Delta\varphi - \omega\Delta t) - \cos(\Delta\varphi)}{1 - \cos(\omega\Delta t)}. \end{aligned} \tag{39}$$

Addition of the last two equations yields $U + V = -2\cos(\Delta\varphi)$, i.e.,

$$\cos(\Delta\varphi) = \frac{1}{2} \cdot \frac{\langle e^{(1,2)} - e^{(1,1)}, e^{(2,2)} - e^{(2,1)} \rangle}{\sqrt{1 - \langle e^{(1,1)}, e^{(1,2)} \rangle}\sqrt{1 - \langle e^{(2,1)}, e^{(2,2)} \rangle}}. \tag{40}$$

Once $\Delta\varphi$ is known, we can determine $\omega\Delta t$ (and hence ω) from any of the two equations in (39), which can be rewritten in the form

$$\begin{aligned} \cos(\Delta\varphi)\cos(\omega\Delta t) - \sin(\Delta\varphi)\sin(\omega\Delta t) - \cos(\Delta\varphi) &= U - U\cos(\omega\Delta t), \\ \cos(\Delta\varphi)\cos(\omega\Delta t) + \sin(\Delta\varphi)\sin(\omega\Delta t) - \cos(\Delta\varphi) &= V - V\cos(\omega\Delta t). \end{aligned} \tag{41}$$

A particularly elegant solution (which yields $\omega\Delta t$ directly in terms of the measurement data, without a need to determine $\Delta\varphi$ beforehand) is obtained by rewriting (41) in the form

$$\begin{bmatrix} \cos(\omega\Delta t) - 1 & -\sin(\omega\Delta t) \\ \cos(\omega\Delta t) - 1 & \sin(\omega\Delta t) \end{bmatrix} \begin{bmatrix} \cos(\Delta\varphi) \\ \sin(\Delta\varphi) \end{bmatrix} = \big(1 - \cos(\omega\Delta t)\big) \begin{bmatrix} U \\ V \end{bmatrix} \tag{42}$$

which, upon inversion, results in

$$\begin{bmatrix} \cos(\Delta\varphi) \\ \sin(\Delta\varphi) \end{bmatrix} = \frac{-1}{2\sin(\omega\Delta t)} \begin{bmatrix} \sin(\omega\Delta t) & \sin(\omega\Delta t) \\ 1 - \cos(\omega\Delta t) & \cos(\omega\Delta t) - 1 \end{bmatrix} \begin{bmatrix} U \\ V \end{bmatrix}, \tag{43}$$

i.e.,

$$-2\sin(\omega\Delta t) \begin{bmatrix} \cos(\Delta\varphi) \\ \sin(\Delta\varphi) \end{bmatrix} = \begin{bmatrix} (U + V)\sin(\omega\Delta t) \\ (U - V)\big(1 - \cos(\omega\Delta t)\big) \end{bmatrix}. \tag{44}$$

Taking norms on both sides yields $4\sin^2(\omega\Delta t) = (U + V)^2 \sin^2(\omega\Delta t) + (U - V)^2\big(1 - \cos(\omega\Delta t)\big)^2$ which can be rewritten as $\big(4 - (U + V)^2\big)\big(1 - \cos^2(\omega\Delta t)\big) = (U - V)^2\big(1 - \cos(\omega\Delta t)\big)^2$. Dividing this equation by $1 - \cos(\omega\Delta t)$ yields $\big(4 - (U + V)^2\big)\big(1 + \cos(\omega\Delta t)\big) = (U - V)^2\big(1 - \cos(\omega\Delta t)\big)$, which results in $\big(4 + (U - V)^2 - (U + V)^2\big) \cos(\omega \Delta t) = (U - V)^2 + (U + V)^2 - 4$. This can be rewritten in the form

$$\cos(\omega \Delta t) \;=\; \frac{2(U^2 + V^2) - 4}{4 - 4UV} \;=\; \frac{\frac{1}{2}(U^2 + V^2) - 1}{1 - UV}\,. \tag{45}$$

Subsequently, the latitudes θ_1 and θ_2 can be obtained from (36).

Method 2d: Determination of longitude separations. If the landmark latitudes θ_i and θ_j and the spin rate ω are already known, we can rewrite (27) in the form

$$\cos\big(\varphi_i - \varphi_j + \omega(t_k - t_\ell)\big) \;=\; \frac{\langle \xi^{(i,k)}, \xi^{(j,\ell)} \rangle - \sin\theta_i \sin\theta_j}{\cos\theta_i \cos\theta_j} \tag{46}$$

to determine the longitude separation $\varphi_i - \varphi_j$ if landmark i can be seen on the k-th image and landmark j on the ℓ-th image.

5. Averaging Initial Estimates

In the previous paragraph it was described how initial estimates for the estimation parameters can be obtained under simplifying assumptions. However, different methods and different images used may lead to different raw estimates, and to obtain an initial estimate which is good enough to ensure convergence of the subsequent estimation process, it is necessary to form an "average" of the raw estimates obtained. This leads to a general problem: Given elements $p_1, \ldots, p_n$ on a manifold M, what is the "average value" of these elements? (In our case the manifold M is the three-dimensional rotation group, and the elements to be averaged are rotation matrices representing different raw estimates of the comet attitude. Similarly, in spin axis attitude determination, the manifold M is the unit sphere in three-dimensional space, and the elements to be averaged are unit vectors representing different raw estimates of the spin axis direction of a spacecraft.) Average values of elements $p_1, \ldots, p_n$ on a Riemannian manifold were first discussed by Maurice Fréchet⋆ who defined them as those elements ξ which minimize $\sum_{i=1}^{n} d(\xi, p_i)^2$ where $d(p, q)$ is the Riemannian distance between points p and q, i.e., the length of the shortest curve joining p and q. In most practical cases such a mean can be found by an iterative scheme whereby an initial guess ξ for the average of $p_1, \ldots, p_n$ leads to an improved guess $\widehat{\xi}$ via the general formula

$$\widehat{\xi} \;:=\; \exp_\xi \left(\frac{1}{n} \sum_{i=1}^{n} \log_\xi(p_i) \right) \tag{47}$$

where $\exp_\xi$ denotes the exponential function at a point ξ (defined by $\exp_\xi(v) = \alpha(1)$ where α is the unique geodesic with $\alpha(0) = \xi$ and $\dot\alpha(0) = v$) and where $\log_\xi$ is its inverse.

⋆ L'intégrale abstraite d'une fonction abstraite d'une variable abstraite et son application à la moyenne d'un élément aléatoire de nature quelconque, *Revue Scientifique* 1944, pp. 483-512. – Les éléments aléatoires de nature quelconques dans un espace distancié, *Annales de l'Institut Henri Poincaré* **10** (1948), pp. 215-310.

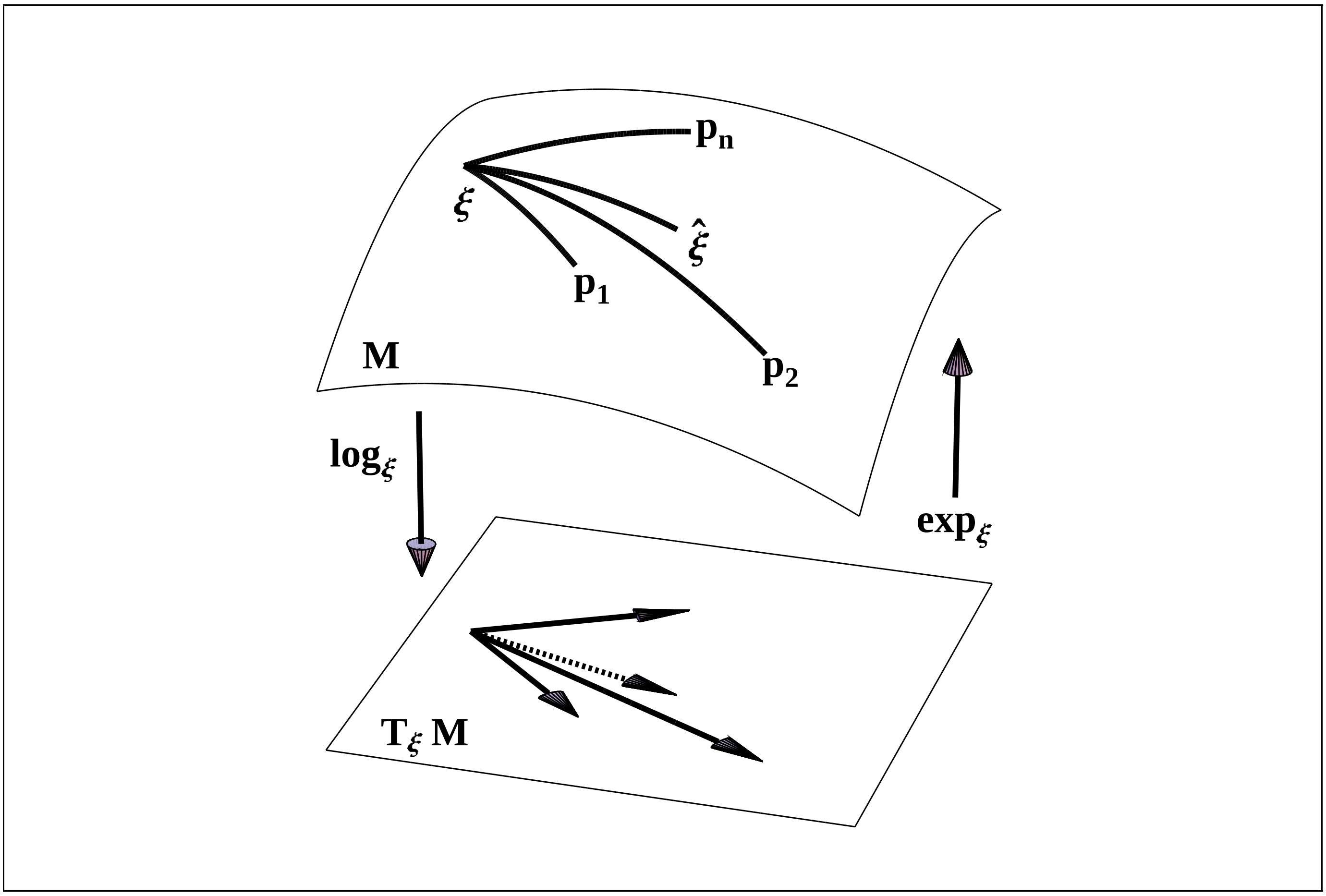

Fig. 7: Improving an approximate mean ξ of points $p_1, \ldots, p_n$ on a Riemannian manifold M.

Examples. (a) If $M = \mathbb{S}^{n-1} := \{x \in \mathbb{R}^n \mid \|x\| = 1\}$ is the unit sphere in n-dimensional space, then the exponential function and the logarithm are given by

$$\begin{aligned} \exp_\xi(v) &= \cos(\|v\|)\xi + \sin(\|v\|)\frac{v}{\|v\|}, \\ \log_\xi(p) &= \frac{\arccos(\langle p, \xi\rangle)}{\sqrt{1 - \langle p, \xi\rangle^2}} \Big(p - \langle p, \xi\rangle\xi\Big). \end{aligned} \tag{48}$$

(b) If $M = G$ is a Lie group (such as the rotation group), then everything can be reduced to a neighborhood of the identity element e; i.e., the iteration scheme takes the form

$$\widehat{\xi} = \xi \exp_e\left(\frac{1}{n}\sum_{i=1}^{n} \log_e(\xi^{-1}p_i)\right). \tag{49}$$

(c) If $M = \mathrm{SO}(3) := \{g \in \mathbb{R}^{3\times3} \mid g^T g = \mathbf{1}\}$ is the rotation group in three-dimensional space (with the identity matrix $\mathbf{1}$ playing the role of the neutral element e), then the Rodrigues formula yields

$$\begin{aligned} \exp_e\big(L(\omega)\big) &= \cos(\|\omega\|)\mathbf{1} + \frac{1 - \cos(\|\omega\|)}{\|\omega\|^2}\,\omega \otimes \omega + \frac{\sin(\|\omega\|)}{\|\omega\|}L(\omega), \\ \log_e g &= \frac{\arccos\big((\mathrm{tr}(g) - 1)/2\big)}{\sqrt{3 - \mathrm{tr}(g)^2 + 2\,\mathrm{tr}(g)}}(g - g^T). \end{aligned} \tag{50}$$

6. Iterative Improvement of Estimates

In the previous paragraphs we showed how rough estimates of the comet attitude and angular velocity can be obtained. These estimates need to be improved in an iterative scheme whereby in each step the measurement equations are linearized about the current estimates and then a least-squares minimization of the resulting residuals is performed to update these estimates. For the purpose of this paper we assume that both the spacecraft position and the camera attitude are known. Then the only uncertain term in the measurement equations

$$u \;=\; -f\cdot\frac{\langle L,c_1\rangle}{\langle L,c_3\rangle}, \quad v \;=\; -f\cdot\frac{\langle L,c_1\rangle}{\langle L,c_3\rangle} \quad (\text{where } L:=\ell-x) \tag{51}$$

is the landmark location ℓ. A change $\delta\ell=\delta L$ in the nominal value ℓ results in changes δu and δv in the expected measurements which are given by

$$\begin{aligned} \delta u &= -f\,\frac{\langle\delta L,c_1\rangle\langle L,c_3\rangle-\langle L,c_1\rangle\langle\delta L,c_3\rangle}{\langle L,c_3\rangle^2} = -f\frac{\langle c_1\times c_3,(\delta L)\times L\rangle}{\langle L,c_3\rangle^2} = f\,\frac{\langle L\times c_2,\delta L\rangle}{\langle L,c_3\rangle^2}, \\ \delta v &= -f\,\frac{\langle\delta L,c_2\rangle\langle L,c_3\rangle-\langle L,c_2\rangle\langle\delta L,c_3\rangle}{\langle L,c_3\rangle^2} = -f\,\frac{\langle c_2\times c_3,(\delta L)\times L\rangle}{\langle L,c_3\rangle^2} = -f\,\frac{\langle L\times c_1,\delta L\rangle}{\langle L,c_3\rangle^2}. \end{aligned} \tag{52}$$

Now a change in $L=\ell-x=\xi+gb-x$ results from changes in ξ, g and b. Let ξ_0, u_0, g_0 and ω_0 be the position, velocity, attitude and angular velocity at a reference time $t_0=0$. While in operational software we have to derive ξ and g at the measurement time from these initial data by numerically integrating the dynamical equations, we assume here for simplicity's sake that the comet velocity and angular velocity can be treated as constants (which is not too unrealistic for a sufficiently short observation interval). Then $\xi(t)=\xi_0+tu$ and $g(t)=g_0\exp\big(tL(\omega)\big)$ and hence

$$\begin{aligned} \delta\xi &= \delta\xi_0+t\cdot\delta u, \\ \delta g &= (\delta g_0)\exp\big(tL(\omega)\big)+t\cdot g_0\exp\big(tL(\omega)\big)L(\delta\omega). \end{aligned} \tag{53}$$

Now due to the structure of the rotation group SO(3) as a nonlinear manifold, a change (i. e., infinitesimal increment) δg_0 in the estimate g_0 takes the form $\delta g_0=g_0L(\Delta)$ with a vector $\Delta=(\Delta_1,\Delta_2,\Delta_3)^T\in\mathbb{R}^3$. Then, writing $g=g_0\exp\big(tL(\omega)\big)$, the second equation in (53) becomes

$$\begin{aligned} \delta g &= g_0L(\Delta)\exp\big(tL(\omega)\big)\;+\;t\cdot g_0\exp\big(tL(\omega)\big)L(\delta\omega) \\ &= g_0L(\Delta)g_0^{-1}g\;+\;t\cdot gL(\delta\omega)\;=\;L(g_0\Delta)g\;+\;t\cdot gL(\delta\omega)\,. \end{aligned} \tag{54}$$

Thus changes in ξ_0, u, g_0, ω and the body-referenced landmark position b result in a change in $L=\xi+gb-x$ which is given by

$$\delta L \;=\; (\delta\xi)+(\delta g)b+g(\delta b) \;=\; (\delta\xi_0)+t\cdot(\delta u)+L(g_0\Delta)gb+t\cdot gL(\delta\omega)b+g(\delta b)\,. \tag{55}$$

Now if a sequence of images is taken on which a number of s landmarks (out of which r are different) can be identified, then plugging (55) into (52) results in an equation of the form

$$\begin{bmatrix} \delta u_1 \\ \delta v_1 \\ \vdots \\ \delta u_s \\ \delta v_s \end{bmatrix} = A \begin{bmatrix} \delta\xi_0 \\ \delta u \\ \Delta \\ \delta\omega \\ \delta b^{(1)} \\ \vdots \\ \delta b^{(r)} \end{bmatrix} \tag{56}$$

where A is a matrix of size $(2s) \times (12 + 3r)$ involving the comet states at the measurement times. As it is desired to match, as well as possible, the theoretically expected with the actually obtained measurements, this leads to solving the overdetermined linear system

$$A \begin{bmatrix} \delta\xi_0 \\ \delta u \\ \Delta \\ \delta\omega \\ \delta b^{(1)} \\ \vdots \\ \delta b^{(r)} \end{bmatrix} = \begin{bmatrix} u_1^{\text{obtained}} - u_1^{\text{expected}} \\ v_1^{\text{obtained}} - v_1^{\text{expected}} \\ \vdots \\ u_s^{\text{obtained}} - u_s^{\text{expected}} \\ v_s^{\text{obtained}} - v_s^{\text{expected}} \end{bmatrix} \tag{57}$$

in the least-squares sense. (Here the expected measurement values are obtained by evaluating (7) using the current estimates for the estimation parameters.) Once this is done, the computed parameter increments $\delta\xi_0$, δu, Δ, $\delta\omega$ and $\delta b^{(i)}$ are used to update the current estimates. In the case of the parameter g_0, this update state is nonlinear, being given by the formula

$$\widehat{g}_0 = g_0 \exp\big(L(\Delta)\big) \,. \tag{58}$$

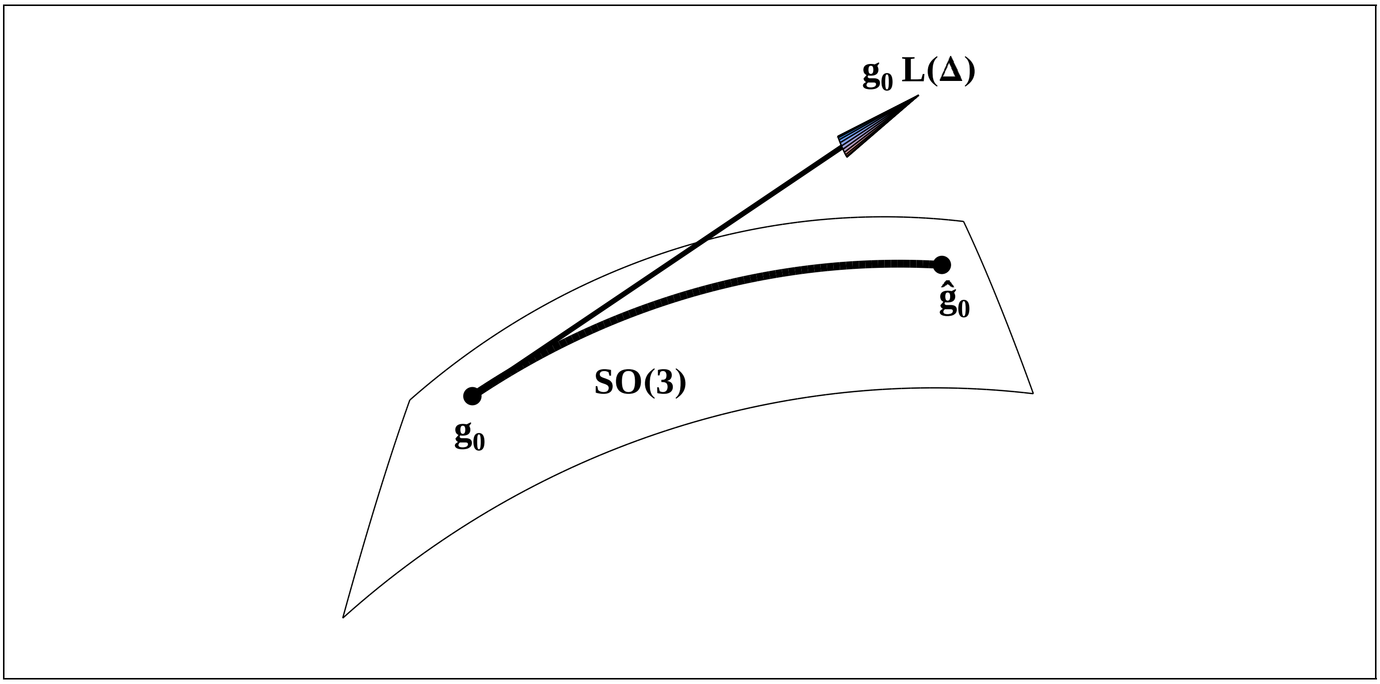

Fig. 8: Nonlinear update step for the estimate of the initial comet attitude.