Anisotropic developments for homogeneous shear flows

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Abstract

The general decomposition of the spectral correlation tensor $R_{ij}(\boldsymbol{k})$ by Cambon *et al.* (*J. Fluid Mech.*, **202**, 295; *J. Fluid Mech.*, **337**, 303) into *directional* and *polarization* components is applied to the representation of $R_{ij}(\boldsymbol{k})$ by spherically averaged quantities. The decomposition splits the deviatoric part $H_{ij}(k)$ of the spherical average of $R_{ij}(\boldsymbol{k})$ into directional and polarization components $H^{(e)}_{ij}(k)$ and $H^{(z)}_{ij}(k)$. A self-consistent representation of the spectral tensor in the limit of weak anisotropy is constructed in terms of these spherically averaged quantities. The directional and polarization components must be treated independently: models that attempt the same representation of the spectral tensor using the spherical average $H_{ij}(k)$ alone prove to be inconsistent with Navier-Stokes dynamics. In particular, a spectral tensor consistent with a prescribed Reynolds stress is not unique.

The degree of anisotropy permitted by this theory is restricted by realizability requirements. Since these requirements will be less severe in a more accurate theory, a preliminary account is given of how to generalize the formalism of spherical averages to higher expansion of the spectral tensor. Directionality is described by a conventional expansion in spherical harmonics, but polarization requires an expansion in tensorial spherical harmonics generated by irreducible representations of the spatial rotation group SO^3. These expansions are considered in more detail in the special case of axial symmetry.

1 Introduction

The most basic statistical property of the fluctuating velocity field in a turbulent flow is its single-time second-order correlation tensor

$$\tilde{R}_{ij}(\boldsymbol{x}, \boldsymbol{x}'; t) = \langle u'_i(\boldsymbol{x}, t) u'_j(\boldsymbol{x}', t) \rangle.$$

In homogeneous turbulence, a simpler description is possible by the second-order spectral tensor $R_{ij}(\boldsymbol{k}, t)$, which is a function of the wavevector argument $\boldsymbol{k}$. Details can be found in Batchelor [1] and Craya [2]. The dependence of R [1] on the entire wavevector $\boldsymbol{k}$ and the consequent angle-dependence is involved in various important dynamical properties like redistribution of energy by the 'rapid' pressure-strain process.

Many simplified models of the wavevector dependence of the correlation tensor R have been proposed; examples include Cambon *et al.* [3], Shih *et al.* [4], Ishihara *et al.* [5], Yoshida *et al.* [6], Thacker *et al.* [7]. These models share the generic form

$$\begin{aligned} R_{ij}(\boldsymbol{k}, t) &= U(k,t) P_{ij}(\hat{\boldsymbol{k}}) + BU(k,t) H_{pq}(k,t) \hat{k}_p \hat{k}_q P_{ij}(\hat{\boldsymbol{k}}) \\ &+ CU(k,t) P_{in}(\hat{\boldsymbol{k}}) P_{jm}(\hat{\boldsymbol{k}}) H_{nm}(k,t) \end{aligned} \qquad (1)$$

Complete explanation of the notation will be given later; for now, we stress the essential point that the anisotropic properties of the correlation are described by a single tensor function $\mathsf{H}(k, t)$ that depends only on the wavenumber $k = |\boldsymbol{k}|$.

These models are revisited here by comparison with an exact decomposition of the spectral tensor (Cambon and Jacquin [8], Cambon *et al.* [9, 10]) into terms that represent distinct properties: *directional* and *polarization* anisotropy. This decomposition has both a physical and a geometrical basis which we review. The replacement of fully anisotropic properties by spherical

[1] Here and throughout, index and index-free notation will both be used as convenient: R and R_{ij} denote the same tensor, and $\boldsymbol{k}$ and k_i the same vector.

averages necessarily restricts the description to weak anisotropy, yet even in this domain, Eq. (1) is inadequate because it implicitly constrains directional and polarization anisotropy in ways that may be inconsistent with Navier-Stokes dynamics. Examples are given of weakly anisotropic flows that cannot be described by a model of this form. Instead, we propose a new model in which directionality and polarization are unconstrained. The consequent description of weak anisotropy by *two* tensor functions of wavenumber k is shown to be consistent with the dynamics.

The paper is organized as follows. General anisotropic correlation functions are discussed in Section 2 without approximations. The fundamental ideas of directional and polarization anisotropy are introduced. The first simplification, the description of anisotropy by spherically averaged tensors, is introduced in Section 3. The undesirable effects of not distinguishing directionality and polarization are described. Two examples of weak shear effects: sheared isotropic turbulence at short times, and the small scales in sheared turbulence at arbitrary times, are analyzed in Section 4, and shown to be consistent with a description in terms of directional and polarization anisotropy. Section 5 discusses the parametrization of spectral anisotropy by single-point moments. A refinement of the anisotropy tensor of single-point turbulence modeling is described, and shown to be equivalent to the 'structure tensor' formalism of Kassinos and Reynolds [11]. Realizability constraints on anisotropic models are discussed in Section 6. They make precise the limitation of these models to weak anisotropy. Section 7 considers how the restriction to weak anisotropy can be mitigated by more accurate, higher order expansions based on representation theory of the rotation group SO^3. Section 8 summarizes the main results.

2 Exact relationship for arbitrary anisotropic second-order statistics

The decomposition of a second-rank tensor into trace and deviator will be generalized to the correlation tensor, taking into account two special features: the solenoidal property

$$k_i R_{ij}(\boldsymbol{k},t) = R_{ij}(\boldsymbol{k},t) k_j = 0 \tag{2}$$

that follows from the incompressibility of the velocity field, and the dependence on the vector argument $\boldsymbol{k}$. In what follows, the helicity of the velocity field will be assumed to vanish; accordingly, the correlation tensor is symmetric: $R_{ij}(\boldsymbol{k},t) = R_{ji}(\boldsymbol{k},t)$. We recall that if the (nonhelical) correlation tensor is isotropic, then elementary arguments (Batchelor [1]) show that it is proportional to the special tensor

$$P_{ij}(\hat{\boldsymbol{k}}) = \delta_{ij} - \hat{k}_i\hat{k}_j \tag{3}$$

where $k = |\boldsymbol{k}|$ and $\hat{k}_i = k_i/k$ is the unit vector along $\boldsymbol{k}$. We also recall the elementary property

$$P_{ij}(\boldsymbol{k})k_i = P_{ij}(\boldsymbol{k})k_j = 0 \tag{4}$$

which states that P is solenoidal, and

$$P_{im}(\boldsymbol{k})P_{mj}(\boldsymbol{k}) = P_{ij}(\boldsymbol{k}) \tag{5}$$

which states that P is a projection. The geometric meaning of Eqs. (4) and (5) is that at any vector $\boldsymbol{k}$, $\mathsf{P}(\boldsymbol{k})$ is the projection onto the plane perpendicular to $\boldsymbol{k}$. We will also use the obvious results

$$P_{mm}(\boldsymbol{k}) = P_{mn}(\boldsymbol{k})P_{mn}(\boldsymbol{k}) = 2 \tag{6}$$

To begin, note from Eq. (2) that 0 is always an eigenvalue of $\mathsf{R}(\boldsymbol{k},t)$, and that $\boldsymbol{k}$ itself is the corresponding eigenvector. It follows that in any frame centered at $\boldsymbol{k}$ in which $\hat{\boldsymbol{k}}$ is one of the basis vectors, $\mathsf{R}(\boldsymbol{k},t)$ can be represented as the matrix

$$\mathsf{R} = \begin{bmatrix} a & b & 0 \\ b & c & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{7}$$

characterized by exactly three real scalars, where we again remark that the absence of helicity implies that R is symmetric in any basis. A trace-deviator decomposition in the plane normal to $\boldsymbol{k}$ yields

$$\mathsf{R} = \begin{bmatrix} e & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} d & b & 0 \\ b & -d & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{8}$$

where

$$e = \tfrac{1}{2}(a+c) \quad d = \tfrac{1}{2}(a-c) \tag{9}$$

In this frame, P, as the projection onto the plane perpendicular to $\hat{\boldsymbol{k}}$ is represented by the matrix

$$\mathsf{P} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{10}$$

Introducing the two independent symmetric matrices

$$\mathsf{M}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \mathsf{M}_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{11}$$

we therefore have

$$\mathsf{R} = e\mathsf{P} + d\mathsf{M}_1 + b\mathsf{M}_2 \tag{12}$$

In terms of the complex quantities

$$Z = d + ib \quad \mathsf{M} = \mathsf{M}_1 - i\mathsf{M}_2 \tag{13}$$

Eq. (12) can be written alternatively as

$$\mathsf{R} = e\mathsf{P} + \Re\left(Z\mathsf{M}\right) \tag{14}$$

An obvious coordinate system in which $\hat{\boldsymbol{k}}$ is a basis vector at every $\boldsymbol{k}$ is the spherical coordinate system in $\boldsymbol{k}$-space, in this context called the Craya-Herring frame (Craya [2], Herring [12]). For the application of this coordinate system to explicit expressions for M_1 and M_2 in terms of the helical mode decomposition using the (complex) eigenvectors of rotations about $\hat{\boldsymbol{k}}$, see Cambon and Jacquin [8] and Waleffe [13]. Eqs. (12) and (14) express R in terms of the minimal number of scalars: the three real quantities $b(\boldsymbol{k},t), d(\boldsymbol{k},t), e(\boldsymbol{k},t)$ or equivalently, $e(\boldsymbol{k},t)$ and the complex scalar $Z(\boldsymbol{k},t)$.

Defining

$$\mathsf{R}^{pol} = d\mathsf{M}_1 + b\mathsf{M}_2 = \begin{bmatrix} d & b & 0 \\ b & -d & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{15}$$

the decomposition in Eq. (12),

$$\mathsf{R} = e\mathsf{P} + \mathsf{R}^{pol} \tag{16}$$

is characterized by the properties

$$e = \tfrac{1}{2}\mathsf{R} : \mathsf{P} = \tfrac{1}{2}\mathrm{tr}\ \mathsf{R} \quad \mathsf{R}^{pol} : \mathsf{P} = 0 \tag{17}$$

Because these properties are independent of the coordinate system, we can also arrive at Eq. (16) by coordinate-free arguments. Thus, define the projection of R along P by

$$\mathsf{R}^{\mathsf{P}} = \tfrac{1}{2}(\mathsf{R} : \mathsf{P})\mathsf{P} \tag{18}$$

where the factor of $1/2$ is due to Eq. (6). The operation so defined is a projection because $[\mathsf{R}^{\mathsf{P}}]^{\mathsf{P}} = \mathsf{R}^{\mathsf{P}}$. Accordingly, the decomposition

$$\mathsf{R} = \tfrac{1}{2}(\mathsf{R} : \mathsf{P})\mathsf{P} + [\mathsf{R} - \tfrac{1}{2}(\mathsf{R} : \mathsf{P})\mathsf{P}] \tag{19}$$

coincides with Eq. (16) after introducing the definition Eq. (17) of e and replacing Eq. (15) by the coordinate-free definition

$$\mathsf{R}^{pol} = \mathsf{R} - \tfrac{1}{2}(\mathsf{R} : \mathsf{P})\mathsf{P} \tag{20}$$

The polarization tensor is geometrically very simple: R^{pol} has one zero eigenvalue because it is solenoidal, and since R and R^{pol} are both solenoidal,

$$\mathrm{tr}\ \mathsf{R}^{pol} = \mathsf{R}^{pol} : \mathsf{P} = 0 \tag{21}$$

Accordingly, its characteristic polynomial is simply

$$p(\lambda) = \lambda^3 - (\tfrac{1}{2}\mathsf{R}^{pol} : \mathsf{R}^{pol})\lambda$$

which also follows directly from the explicit expression Eq. (15). It follows that the eigenvalues of R^{pol} are $0, \pm\sqrt{\tfrac{1}{2}\mathsf{R}^{pol} : \mathsf{R}^{pol}}$. Since the eigenvalues of $e\mathsf{P}$ are obviously just $e, e, 0$, the eigenvalues of R are $e \pm \sqrt{\tfrac{1}{2}\mathsf{R}^{pol} : \mathsf{R}^{pol}}, 0$. The realizability of R is therefore simply the condition

$$e \geq \sqrt{\tfrac{1}{2}\mathsf{R}^{pol} : \mathsf{R}^{pol}} \tag{22}$$

Note from Eqs. (13) and (15) that

$$ZZ^* = \tfrac{1}{2}\mathsf{R}^{pol} : \mathsf{R}^{pol} \tag{23}$$

Thus, although the scalars $d = \Re Z$ and $b = \Im Z$ are coordinate-dependent, the magnitude $|Z|$ is a geometric invariant.

Eq. (19) is a straightforward generalization of the trace-deviator decomposition in which the trace, the projection along δ_{ij}, is replaced by the operation of projection along P.

The general decomposition Eq. (16) can be rewritten in a form that isolates the purely isotropic part by projecting e onto its spherical average,

$$U(k,t) = \frac{1}{4\pi k^2} \oint_{S_k} e(\boldsymbol{k},t)\ d^2\boldsymbol{k} \tag{24}$$

where $\oint_{S_k} (...) d^2\boldsymbol{k}$ denotes integration over a spherical shell of radius k. Note that since trivially $U(k,t) = \frac{1}{4\pi k^2} \oint_{S_k} U(k,t)\ d^2\boldsymbol{k}$, Eq. (24) does define a projection. Defining

$$\mathcal{E}(\boldsymbol{k},t) = e(\boldsymbol{k},t) - U(k,t) \tag{25}$$

and

$$\mathsf{R}^{dir}(\boldsymbol{k},t) = \mathcal{E}(\boldsymbol{k},t)\mathsf{P}(\hat{\boldsymbol{k}}) \tag{26}$$

we have

$$\mathsf{R}(\boldsymbol{k},t) = \underbrace{U(k,t)\mathsf{P}(\hat{\boldsymbol{k}})}_{\text{Isotropic part}} + \underbrace{\mathsf{R}^{dir}(\boldsymbol{k},t)}_{\text{Directional anisotropy}} + \underbrace{\mathsf{R}^{pol}(\boldsymbol{k},t)}_{\text{Polarization anisotropy}} \tag{27}$$

The decomposition of the correlation tensor in Eq. (27) has a simple but important geometrical significance: just as in the trace-deviator decomposition, each component transforms into another term of the like form under arbitrary rotation of axes.

3 Description by spherical averages alone

Even in single-point turbulence modeling, it is often useful to introduce an *ansatz* for the energy spectrum. For example, such assumed spectra have proven useful in closing the 'rapid' pressure-strain correlation [3, 4]. But the obvious practical difficulty of a complete description of spectral anisotropy has motivated a search for simpler descriptions of anisotropy. The approach adopted here is to reconstruct the angular dependence of the spectrum through appropriate tensorial expansions, using either functions of the wavenumber alone, as in the expression $\mathsf{R}(\boldsymbol{k},t) = U(k,t)\mathcal{F}(\hat{\boldsymbol{k}}, \mathsf{H}(k,t))$, or even more simply, functions of single-point moments as in the expression $\mathsf{R}(\boldsymbol{k},t) = U(k,t)\mathcal{F}(\hat{\boldsymbol{k}},\mathsf{b}(t))$: in each case, $\mathcal{F}$ denotes a dimensionless isotropic tensorial function of its arguments. The presence of $\hat{\boldsymbol{k}}$ in the list of arguments distinguishes these expansions from the tensorial expansions familiar in single-point modeling. The Introduction briefly described some models of this type. This type of modeling will be reconsidered in this section in the light of the distinction between directional and polarization anisotropy.

By spherical integration, we can construct two obvious tensor measures of anisotropy that depend only on k: they are defined by

$$2E(k,t)\mathsf{H}^{(e)}(k,t) = \oint_{S_k} \mathsf{R}^{dir}(\boldsymbol{k},t)d^2\boldsymbol{k} \tag{28}$$

$$2E(k,t)\mathsf{H}^{(z)}(k,t) = \oint_{S_k} \mathsf{R}^{pol}(\boldsymbol{k},t)d^2\boldsymbol{k} \tag{29}$$

where

$$E(k,t) = \oint_{S_k} e(\boldsymbol{k},t)d^2\boldsymbol{k} = 4\pi k^2 U(k) \tag{30}$$

is the energy spectrum.[2] The notation follows Cambon and Jacquin [8], and is motivated by the characterization in Section 2 of directional anisotropy by the scalar $e(\boldsymbol{k},t)$ and of polarization anisotropy by the complex scalar $Z(\boldsymbol{k},t)$. Obviously,

$$2E(k,t)\mathrm{tr}\ \mathsf{H}^{(e)}(k,t) = \oint_{S_k} \mathrm{tr}\ \mathsf{R}^{dir}(\boldsymbol{k},t)d^2\boldsymbol{k} = \oint_{S_k} (e-U)\mathrm{tr}\ \mathsf{P}(\boldsymbol{k})d^2\boldsymbol{k} = 0 \tag{31}$$

and, in view of Eq. (21),

$$2E(k,t)\mathrm{tr}\ \mathsf{H}^{(z)}(k,t) = \oint_{S_k} \mathrm{tr}\ \mathsf{R}^{pol}(\boldsymbol{k},t)d^2\boldsymbol{k} = 0 \tag{32}$$

so that both of $\mathsf{H}^{(e,z)}$ are trace-free.

We wish to construct a modeled correlation tensor that depends only on the spherical averages $\mathsf{H}^{(e,z)}$. The discussion in Section 2 motivates constructing R^{dir} and R^{pol} separately. To begin, note that R^{dir} depends linearly on $\mathsf{H}^{(e)}$ and is proportional to P. The simplest assumption consistent with with these properties is

$$\mathsf{R}^{dir}(\boldsymbol{k},t) = AU(k,t)\left[\mathsf{H}^{(e)}(k,t) : \mathsf{P}(\hat{\boldsymbol{k}})\right]\mathsf{P}(\hat{\boldsymbol{k}}) \tag{33}$$

[2] Our definition of U is somewhat nonstandard; it is customary to define it so that $E = 2\pi k^2 U$.

with an undetermined constant A. Equivalently, in terms of the solenoidal tensor $\mathsf{PH}^{(e)}\mathsf{P}$, Eq. (33) sets $\mathsf{R}^{dir} = A[\mathsf{PH}^{(e)}\mathsf{P}]^{dir}$. The constant A should be chosen to be consistent with the definition Eq. (28); the spherical average [3] of each side of Eq. (33) gives

$$2E(k,t)\mathsf{H}^{(e)}(k,t) = -\tfrac{2}{15}AE(k,t)\mathsf{H}^{(e)}(k,t) \tag{34}$$

so that $A = -15$.

The treatment of the polarization tensor is somewhat less straightforward. R^{pol} must be solenoidal and linear in $\mathsf{H}^{(z)}$. These requirements suggest the form $\mathsf{R}^{pol} = \mathsf{PH}^{(z)}\mathsf{P}$. But in addition, we must take into account that $\mathsf{R}^{pol} : \mathsf{P} = 0$. A general form consistent with all constraints is therefore

$$R^{pol}_{ij}(\boldsymbol{k},t) = BU(k,t)\left[P_{im}(\hat{\boldsymbol{k}})H^{(z)}_{mn}(k,t)P_{nj}(\hat{\boldsymbol{k}}) - \tfrac{1}{2}\mathsf{H}^{(z)}_{pq}(k,t)P_{pq}(\boldsymbol{k})P_{ij}(\boldsymbol{k})\right] \tag{35}$$

with an undetermined constant B. Again, in terms of the solenoidal tensor $\mathsf{PH}^{(z)}\mathsf{P}$, Eq. (35) sets $\mathsf{R}^{pol} = B[\mathsf{PH}^{(z)}\mathsf{P}]^{pol}$. Spherical averaging as in Eq. (34) gives

$$2E(k,t)\mathsf{H}^{(z)}(k,t) = \tfrac{1}{5}BE(k,t)\mathsf{H}^{(z)}(k,t) \tag{36}$$

so that $B = 5$.

Combining the results of Eqs. (33) and (35), we obtain the required representation

$$\begin{aligned} R_{ij}(\boldsymbol{k},t) &= U(k,t)P_{ij}(\hat{\boldsymbol{k}}) + 15U(k,t)P_{ij}(\hat{\boldsymbol{k}})H^{(e)}_{pq}(k,t)P_{pq}(\hat{k}) \\ &+5U(k,t)\left[P_{in}(\hat{\boldsymbol{k}})P_{jm}(\hat{\boldsymbol{k}})H^{(z)}_{nm}(k,t) - \tfrac{1}{2}P_{mn}(\hat{\boldsymbol{k}})H^{(z)}_{nm}(k,t)P_{ij}(\hat{\boldsymbol{k}})\right] \end{aligned} \tag{37}$$

or equivalently,

$$\begin{aligned} R_{ij}(\boldsymbol{k},t) &= U(k,t)P_{ij}(\hat{\boldsymbol{k}}) - 15U(k,t)P_{ij}(\hat{\boldsymbol{k}})H^{(e)}_{pq}(k,t)\hat{k}_p\hat{k}_q \\ &+5U(k,t)\left[P_{in}(\hat{\boldsymbol{k}})P_{jm}(\hat{\boldsymbol{k}})H^{(z)}_{nm}(k,t) + \tfrac{1}{2}P_{ij}(\hat{\boldsymbol{k}})H^{(z)}_{nm}(k,t)\hat{k}_p\hat{k}_q\right] \end{aligned} \tag{38}$$

This equation is the main result of this paper. It shows that a completely consistent description of weak anisotropy without arbitrary constants is possible using independent descriptors of directionality and polarization by tensor functions of the wavenumber k alone.

[3] The derivation requires the formulas (Cambon *et al.* [3]) $\dfrac{1}{4\pi k^2}\oint \hat{k}_i\hat{k}_j d^2\boldsymbol{k} = \tfrac{1}{3}\delta_{ij}$ and $\dfrac{1}{4\pi k^2}\oint \hat{k}_i\hat{k}_j\hat{k}_m\hat{k}_n d^2\boldsymbol{k} = \tfrac{1}{15}[\delta_{ij}\delta_{mn} + \delta_{im}\delta_{nj} + \delta_{in}\delta_{mj}]$.

Note that since tr $\mathsf{H}^{(e)} = 0$, the quantity $H^{(e)}_{pq}(k,t)\hat{k}_p\hat{k}_q$ that appears in Eq. (38) is a second order spherical harmonic: after choosing a polar axis and introducing spherical coordinates, it would be expressed in terms of Legendre functions in the standard way. The term in brackets containing $\mathsf{H}^{(z)}$ has an analogous interpretation as a *tensor* spherical harmonic (Zemach, [14]). We will follow Zemach [14] in referring to *scalar spherical harmonics* (SSH) and *tensor spherical harmonics* (TSH) henceforth. From this viewpoint, Eq. (38) states the lowest order terms in expansions of R^{dir} and R^{pol} respectively in scalar and tensor harmonics, or in irreducible representations of the rotation group SO^3. This connection will be developed further in Section 7.

Eqs. (28) and (29) give expressions for the spherically averaged correlation function

$$\varphi_{ij}(k,t) = \oint_{S_k} R_{ij}(\boldsymbol{k},t)d^2\boldsymbol{k} = 2E(k,t)\left(\tfrac{1}{3}\delta_{ij} + H^{(e)}_{ij}(k,t) + H^{(z)}_{ij}(k,t)\right). \quad (39)$$

Eq. (38) states the important conclusion that a solenoidal tensor R can indeed be constructed from the spherical averages $\mathsf{H}^{(e,z)}$; this conclusion is not obvious because the solenoidal property is lost on spherical averaging. In modeling, it has been the general practice to characterize anisotropy by one tensor instead of two. Thus, many models are based on

$$\varphi_{ij}(k,t) = \oint_{S_k} R_{ij}(\boldsymbol{k},t)d^2\boldsymbol{k} = 2E(k,t)\left(\tfrac{1}{3}\delta_{ij} + H_{ij}(k,t)\right) \quad (40)$$

where $\mathsf{H} = \mathsf{H}^{(e)} + \mathsf{H}^{(z)}$. Such modeling mixes directional and polarization anisotropy. We next ask whether a solenoidal correlation function can be constructed consistent with H alone.

Proceding as before, we set

$$\begin{aligned} \mathsf{R}^{dir} &= AU(k,t)(\mathsf{H}:\mathsf{P})\mathsf{P} \\ \mathsf{R}^{pol} &= BU(k,t)\left[\mathsf{PHP} - \tfrac{1}{2}(\mathsf{H}:\mathsf{P})\mathsf{P}\right] \end{aligned} \quad (41)$$

We already note that these equations implicitly impose some relation between R^{dir} and R^{pol} and therefore cannot be entirely satisfactory. On spherical averaging, we find

$$\mathsf{H} = (\tfrac{1}{15}A + \tfrac{1}{5}B)\mathsf{H} \quad (42)$$

The solution is not unique: it is

$$A = 15 + 3a \quad B = -a \quad (43)$$

where a can be a function $a = a(k,t)$. Thus, the spectral tensor is

$$\mathsf{R}(\boldsymbol{k},t) = U(k,t)\mathsf{P}(\hat{\boldsymbol{k}}) + (1+\tfrac{1}{5}a)U(k,t)(\mathsf{H}(\boldsymbol{k},t) : \mathsf{P}(\hat{\boldsymbol{k}})\mathsf{P}(\hat{\boldsymbol{k}}) \\ -\tfrac{1}{5}aU(k,t)\left[\mathsf{P}(\hat{\boldsymbol{k}})\mathsf{H}(\boldsymbol{k},t)\mathsf{P}(\hat{\boldsymbol{k}}) - \tfrac{1}{2}(\mathsf{H}(\boldsymbol{k},t) : \mathsf{P}(\hat{\boldsymbol{k}}))\mathsf{P}(\hat{\boldsymbol{k}})\right] \tag{44}$$

and

$$\mathsf{H}^{(e)} = \left(1+\tfrac{2}{5}a\right)\mathsf{H}, \quad \mathsf{H}^{(z)} = -\tfrac{2}{5}a\mathsf{H} \tag{45}$$

which implies the relation

$$\tfrac{2}{5}a\mathsf{H}^{(e)} = -(1+\tfrac{2}{5}a)\mathsf{H}^{(z)}. \tag{46}$$

Eq. (44) is exactly the proposal of Cambon *et al.* [3] for the first general model linking R with $\boldsymbol{\varphi}$. The aim was to derive an equation for the spherical average $\boldsymbol{\varphi}(k,t)$ from the evolution of R in the presence of mean flows with constant mean velocity gradients. The basic problem is that spherical averaging introduces a closure problem for the pressure-strain and transfer terms, so that a model for $\mathsf{R}(\hat{\boldsymbol{k}})$ was needed, parametrized by functions of k only, and so related in a straightforward way to $\boldsymbol{\varphi}(k,t)$. Starting from an equation for $\mathsf{R}(\boldsymbol{k},t)$ that includes the exact linear terms (recalled at the beginning of the next section) and contributions from triple correlations closed by an anisotropic EDQNM theory, Eq. (44) allowed Cambon *et al.* [3] to systematically derive a closed equation for $\boldsymbol{\varphi}(k,t)$, but involving $a(k,t)$ from (44) as an ajustable parameter. Other more empirical models in which anisotropy is parametrized entirely by a tensor function of wavenumber k have been proposed [15, 16]; they explicitly use some approaches of single-point turbulence models to close the pressure-strain correlation.

4 Application: short and long time behavior of weakly sheared turbulence

We have noted that Eq. (38) can be understood as a lowest order expansion in anisotropy valid only for small departures from isotropy; the restrictions will be clarified shortly by analysis of the realizability of the correlation proposed in Eq. (38). This section will give examples of weakly anisotropic flows that permit direct analysis, and will show that the correlation tensor is indeed described by Eq. (38). The first example is homogeneous shear flow with isotropic initial conditions treated by rapid distortion theory (RDT) in the short-time limit in which anisotropic effects are necessarily small.

Using the notation of Cambon and Scott [17], we begin with the general RDT equation

$$\frac{\partial R_{ij}}{\partial t}(\boldsymbol{k},t) = -\frac{dk_n}{dt}\frac{\partial}{\partial k_n}R_{ij}(\boldsymbol{k},t) - M_{in}(\boldsymbol{k})R_{nj}(\boldsymbol{k},t) - M_{jn}(\boldsymbol{k})R_{in}(\boldsymbol{k},t) \quad (47)$$

where $M_{ij} = (\delta_{im} - 2\hat{k}_i\hat{k}_m)A_{mj}$, $A_{im} = \partial\bar{U}_i/\partial x_m$ is the mean velocity gradient, and the wavevector k_n satisfies $dk_i/\,dt = -A_{ji}k_j$. We will consider evolution away from an isotropic initial condition $\mathsf{R}(\boldsymbol{k},0) = U(k)\mathsf{P}(\hat{\boldsymbol{k}})$; we analyze the evolution at very short times, when the effects of shear remain weak.

Consider the first order Taylor series expansion $\mathsf{R}(\boldsymbol{k},t) = U(k)\mathsf{P} + t\dot{\mathsf{R}}(\boldsymbol{k},0)$. Evaluating Eq. (47) at $t = 0$ using the isotropic initial condition $\mathsf{R}(\boldsymbol{k},0) = U(k)\mathsf{P}(\boldsymbol{k})$ leads easily to

$$\begin{aligned}
&\frac{\partial R_{ij}}{\partial t}(\boldsymbol{k},0) = A_{pn}\hat{k}_p\hat{k}_n kU'(k)P_{ij}(\hat{\boldsymbol{k}}) + U(k)\hat{k}_i\hat{k}_p A_{pn}P_{nj}(\hat{\boldsymbol{k}}) \\
&+U(k)\hat{k}_j\hat{k}_p A_{pn}P_{ni}(\hat{\boldsymbol{k}}) - U(k)A_{in}P_{nj}(\hat{\boldsymbol{k}}) - U(k)A_{jn}P_{ni}(\hat{\boldsymbol{k}}) \\
&= S_{pn}\hat{k}_p\hat{k}_n kU'(k)P_{ij}(\hat{\boldsymbol{k}}) - U(k)P_{ip}(\boldsymbol{k})P_{nj}(\boldsymbol{k})S_{pn} \qquad (48)
\end{aligned}$$

where $S_{ij} = \frac{1}{2}(A_{ij} + A_{ji})$ is the strain rate. It follows that

$$R_{ij}^{dir}(\boldsymbol{k},t) = \tfrac{1}{2}tS_{pq}\hat{k}_p\hat{k}_q\left[kU'(k) + U(k)\right]P_{ij}(\boldsymbol{k}) \quad (49)$$

$$R_{ij}^{pol}(\boldsymbol{k},t) = -tU(k)\left[P_{im}(\boldsymbol{k})P_{jn}(\boldsymbol{k})S_{mn} + \tfrac{1}{2}S_{pq}\hat{k}_p\hat{k}_q P_{ij}(\boldsymbol{k})\right] \quad (50)$$

We remark that the U' term comes from the $\partial/\partial\boldsymbol{k}$ effect in the RDT equations: it is a conservative linear energy transfer mechanism in $\boldsymbol{k}$-space, which therefore appears as a directionality effect; polarization effects instead arise when energy is transferred between different tensor components of the correlation. We see from Eqs. (49)–(50) that both effects are relevant.

Spherical integration gives

$$H_{ij}^{(e)}(k,t) = -\frac{1}{15}\left(-1 + \frac{k}{E}\frac{dE}{dk}\right)S_{ij}t \quad \text{and} \quad H_{ij}^{(z)}(k,t) = -\frac{2}{5}S_{ij}t \quad (51)$$

This leads to $a(k) = 15/(5 + (k/E)dE/dk)$ in Eq. (45), which is constant in any k-space region where E obeys power-law scaling.

It is useful to supplement this short-time analysis by long-time nonlinear analysis, as proposed by [18, 19, 20] and others. In these computations, it is assumed that at least for some range of scales, the shear may be treated as a weak perturbation of isotropic turbulence: the calculation itself will identify an appropriate small parameter. The most recent such analysis, by Yoshida *et al.* [5, 6] overcomes some limitations of previous work, and introduces a Lagrangian viewpoint, with important conceptual and computational advantages. Yoshida *et al.* refer to this calculation as 'linear response theory' for turbulence, since it is accomplished by linearizing about an isotropic nonlinear state. As in the RDT problem above, anisotropy is therefore weak, and we can again expect Eq. (38) to describe the spectral tensor.

General kinematic considerations (the same as in [3, 4]) lead again to

$$R_{ij}(\hat{\boldsymbol{k}}) = U(k)P_{ij}(\boldsymbol{k}) + b(k)P_{ij}(\hat{\boldsymbol{k}})S_{pq}\hat{k}_p\hat{k}_q + 2a'(k)P_{in}(\hat{\boldsymbol{k}})P_{jm}(\hat{\boldsymbol{k}})S_{nm}. \quad (52)$$

from which we immediately deduce

$$\begin{aligned} R_{ij}^{dir} &= (b-a')(S_{pq}\hat{k}_p\hat{k}_q)P_{ij}(\hat{\boldsymbol{k}}) \\ R_{ij}^{pol} &= 2a'\left(P_{im}(\hat{\boldsymbol{k}})P_{jn}(\hat{\boldsymbol{k}})S_{mn} + \tfrac{1}{2}S_{pq}\hat{k}_p\hat{k}_qP_{ij}(\hat{\boldsymbol{k}})\right) \end{aligned} \quad (53)$$

exactly the same structure as in Eqs. (49)–(50) except for the scalar functions of k; consequently,

$$H_{ij}^{(e)} = -\tfrac{1}{15}\theta^{(e)}S_{ij} \quad H_{ij}^{(z)} = -\tfrac{2}{5}\theta^{(z)}S_{ij} \quad (54)$$

where $\theta^{(e)}(k) = [b(k) - a'(k)]/U(k)$ and $\theta^{(z)}(k) = 2a'(k)/U(k)$ are time scales. In a Kolmogorov inertial range, we will have

$$\theta^{(e)} = (A - B)\epsilon^{-1/3}k^{-2/3} \quad \theta^{(z)} = A\epsilon^{-1/3}k^{-2/3} \quad (55)$$

where A and B are universal constants. In this case,

$$\tfrac{2}{5}A\mathsf{H}^{(e)} = \tfrac{1}{15}(B - A)\mathsf{H}^{(z)} \quad (56)$$

and the anisotropic part of the spectrum satisfies

$$E\mathsf{H}^{(e)} \sim E\mathsf{H}^{(z)} \sim k^{-7/3} \quad (57)$$

the scaling suggested by dimensional analysis since $\mathsf{H}^{(e)} \propto \mathsf{S}$ (see Yoshida *et al.* [6] for more details and references to earlier work).

Yoshida *et al.* find theoretical values from spectral closure theory $A \approx -0.16$ and $B \approx -0.40$. Experimental and DNS measurements give values closer to $A \approx -0.12$ and $B \approx -0.009$. It is found that making realistic corrections to the theoretical values to account for the finite inertial range in the measurements, results in much closer agreement. However, for our purposes, the actual values are not so crucial; the important observation is that a weakly anisotropic spectral model based on a single tensor H imposes some fixed proportionality $\mathsf{H}^{(e)} = \lambda \mathsf{H}^{(z)}$ as shown by Eq. (46); such models cannot be consistent with *both* the short- and the long-time results Eqs. (51) and (56). On the other hand, the model Eq. (38) can be consistent with both limits.

In the case of pure rotation, where A is antisymmetric and hence the strain S vanishes, no information is given by the previous 'linear' approaches. Nevertheless, rotating turbulence provides another example of how directionality and polarization must be separated in general. The inviscid RDT equation (e.g. Cambon and Jacquin [8]) for this problem implies

$$\mathsf{R}^{dir}(\boldsymbol{k}, t) = \mathsf{R}^{dir}(\boldsymbol{k}, 0), \quad \mathsf{R}^{pol}(\boldsymbol{k}, t) = \exp(4\imath\Omega\hat{k}_3 t)\mathsf{R}^{pol}(\boldsymbol{k}, 0) \tag{58}$$

which yields

$$H^{e}_{ij}(k, t) = H^{(e)}_{ij}(k, 0), \quad H^{(z)}_{ij}(k, t) \to 0. \tag{59}$$

The damping of polarization anisotropy reflects the angular phase mixing due to the anisotropic dispersivity of inertial waves. We see then that the kinematics of turbulence under rapid rotation is dominated by directional anisotropy alone. Spherically averaged polarization can be neglected even in the presence of nonlinearity, but directional anisotropy can be created by nonlinearity, even if it is initially zero, in an incomplete transition from 3D to 2D structure [8, 10, 21].

5 Parametrization by single-point moments

In Reynolds stress modeling, the basic descriptor is the *anisotropy tensor* b which can be recovered in the present notation by

$$\int_0^\infty dk\ \mathsf{b}(t)E(k, t) = \int_0^\infty dk\ E(k, t)\mathsf{H}(k, t) \tag{60}$$

with the classical definitions for kinetic energy $\mathcal{K}$ and Reynolds stress anisotropy tensor b:

$$\mathcal{K}(t) = \frac{1}{2}\overline{u_n' u_n'} = \int_0^\infty E(k,t)dk, \quad b_{ij}(t) = \frac{\overline{u_i' u_j'}}{\overline{u_n' u_n'}} - \frac{1}{3}\delta_{ij}$$

An example of a parametrization of the correlation tensor by single-point moments is the proposal of Shih *et al.* [4]; written in terms of directional and polarization anisotropy components, it is

$$\begin{aligned} R_{ij}(\boldsymbol{k}) &= U(\boldsymbol{k},t)P_{ij}(\boldsymbol{k}) + \psi_3(\boldsymbol{k},t)P_{ij}(\hat{\boldsymbol{k}})(\gamma - \tfrac{1}{2})b_{pq}\hat{k}_p\hat{k}_q \\ &+\psi_3(\boldsymbol{k},t)P_{in}(\hat{\boldsymbol{k}})P_{jm}(\hat{\boldsymbol{k}})\left(b_{nm} + \tfrac{1}{2}\delta_{nm}b_{pq}\hat{k}_p\hat{k}_q\right). \end{aligned} \tag{61}$$

This expression can be identified with Eq. (38) with the particular choices

$$\mathsf{H}^{(e)} = (\gamma - \tfrac{1}{2})15\frac{\psi_3(k,t)}{E(k,t)}\mathsf{b} \quad \mathsf{H}^{(z)} = 5\frac{\psi_3(k,t)}{E(k,t)}\mathsf{b} \tag{62}$$

so that

$$5\mathsf{H}^{(e)} = 3(\gamma - \tfrac{1}{2})\mathsf{H}^{(z)} \tag{63}$$

Since this result fixes a definite proportionality between $\mathsf{H}^{(e)}$ and $\mathsf{H}^{(z)}$, however, it is really a special case of Eq. (44), so that the constant γ could be related to the parameter a. It is not necessary to give the explicit relation; the important fact is such models cannot be consistent with the equations of motion or the general kinematics. Considerations of realizability which will be discussed later, led to the choice $\gamma = \frac{1}{2}$, which removes directional anisotropy entirely, which is clearly inconsistent with the problems of shear and rotating turbulence just analyzed. The important conclusion is that *the spectral tensor cannot be uniquely reconstructed from the stress anisotropy alone.*

Another model of this type recently proposed by Thacker *et al.* [7] has the form

$$R_{ij}(\boldsymbol{k},t) = U(k,t)P_{ij}(\boldsymbol{k}) + P_{in}(\hat{\boldsymbol{k}})P_{jm}(\hat{\boldsymbol{k}})U^a(k,t)[b_{nm}(t) + \frac{1}{2}\delta_{nm}b_{pq}(t)\hat{k}_p\hat{k}_q]. \tag{64}$$

This model is essentially the same as Eq. (61) with the special choice $\gamma = \frac{1}{2}$, which removes directional anisotropy.

The fundamental decomposition of anisotropy into directionality and polarization suggests introducing two additional single-point descriptors

$$\begin{aligned}\int_0^\infty dk\ \mathsf{b}^{(e)}(t)E(k,t) &= \int_0^\infty dk\ E(k,t)\mathsf{H}^{(e)}(k,t)\\ \int_0^\infty dk\ \mathsf{b}^{(z)}(t)E(k,t) &= \int_0^\infty dk\ E(k,t)\mathsf{H}^{(z)}(k,t)\end{aligned} \tag{65}$$

so that

$$\mathsf{b} = \mathsf{b}^{(e)} + \mathsf{b}^{(z)} \tag{66}$$

which can be considered a refined decomposition of the Reynolds stress anisotropy.

As an example of Eq. (66), consider the short time RDT results of the previous section. Multiplying each result in Eq. (51) by $2E$ and integrating over k gives

$$b_{ij} = -\tfrac{4}{15}S_{ij}t \quad b_{ij}^{(e)} = \tfrac{2}{15}S_{ij}t \quad b_{ij}^{(z)} = -\tfrac{2}{5}S_{ij}t \tag{67}$$

so that $\mathsf{b}^{(e)} = -(1/3)\mathsf{b}^{(z)}$, or equivalently, $-2\mathsf{b}^{(e)} = \mathsf{b}$. This condition, referred to by Kassinos *et al.* [11] as 'dimensionality and componentality having the same anisotropy' although strictly derived only at short times, is considered to be valid at large time provided that the mean flow is irrotational. Comparison with Eq. (51) shows that Eq. (67) certainly does not hold for spectral quantities, because the ratio of $\mathsf{H}^{(e)}$ to $\mathsf{H}^{(z)}$ at the first order in time depends on the initial spectrum through the 'linear transfer' generated by the time dependence $\boldsymbol{k}(t)$.

We can also compute $\mathsf{b}^{(e,z)}$ for the results of Yoshida *et al.* [6]. To obtain a definite result, we will follow previous calculations of this type ([18, 19, 20] to name just a few) and integrate the inertial range spectrum $E(k) = C_k\epsilon^{2/3}k^{-5/3}$ over a range of scales $k \geq k_0$, where k_0^{-1} is an integral scale of the turbulence. The result is

$$\begin{aligned}b_{ij}^{(e)} &= \tfrac{1}{30}C_k^{-1}(B-A)\epsilon^{-1/3}k_0^{-2/3}S_{ij}\\ b_{ij}^{(z)} &= \tfrac{1}{10}C_k^{-1}A\epsilon^{-1/3}k_0^{-2/3}S_{ij}\end{aligned} \tag{68}$$

In comparing the results Eqs. (67) and (68), we see that the time-scale in short-time RDT is simply elapsed time t, whereas it is a turbulent time-scale $\propto \epsilon^{-1/3}k_0^{-2/3}$ in the linear response theory. Moreover, the ratios of components $b_{ij}^{(e)}/b_{ij}^{(z)}$ are not the same.

We would like to add some brief remarks on the use of the refined single-point anisotropy measures $\mathsf{b}^{(e,z)}$ in turbulence modeling. One attempt to improve the prediction of RANS models, and especially to enforce greater consistency with rapid distortion theory, has been to introduce new tensors in the models. After Cambon *et al.* [9], the most prominent example is the *structure tensor* formalism proposed in Kassinos *et al.* [11]. Two kinds of anisotropy, called 'componentality' and 'dimensionality' were considered. Since 'componentality' is nothing else than the Reynolds stress tensor anisotropy identified by b_{ij}, new information is carried only by the 'dimensionality' tensor

$$D_{ij} = 2\int e(\boldsymbol{k},t)\hat{k}_i\hat{k}_j d^3\boldsymbol{k}.$$

A trace-deviator decomposition for this tensor can be expressed as

$$D_{ij} = 2\mathcal{K}\left(\frac{1}{3}\delta_{ij} - 2b^{(e)}_{ij}\right), \tag{69}$$

so that the 'dimensionality', measured by the deviatoric part of D_{ij}, is induced by directional anisotropy alone, and 'componentality' mixes directionality and polarization, via $\mathsf{b} = \mathsf{b}^{(e)} + \mathsf{b}^{(z)}$. The reader is referred to Salhi and Cambon [22] for more complete relationship with 'circulicity' and 'stropholysis' tensors from Kassinos *et al.* [11].

6 Realizability

The construction of a modeled correlation tensor in terms of spherical averages leaves open the question whether the result Eq. (38) in fact describes a possible correlation. This is the issue of realizability. Obviously, in the context of weak anisotropy, the anisotropic part is a small perturbation of the isotropic part $U(k,t)\mathsf{P}(\hat{\boldsymbol{k}})$, and as this part is realizable, it remains realizable under sufficiently small perturbations.

Realizability is imposed in RANS models through the positivity of the Reynolds stress. But in a model in which directionality and polarization are treated separately, it is natural to impose realizability conditions on each component. Recall in the case of Eq. (38), that $e = U\left(1 - 15H^{(e)}_{pq}\hat{k}_p\hat{k}_q\right)$. Thus the simple necessary realizability condition $e > 0$ is violated if the largest positive eigenvalue of $H^{(e)}_{ij}$ is larger than 1/15: this tensor being tracefree, at least one positive eigenvalue must exist; [4] therefore, this realiz-

[4] In fact, from $\oint_{s_k} e(\boldsymbol{k},t)P_{ij}(\hat{\boldsymbol{k}})d^2\boldsymbol{k} = 2E(k)\left(\frac{1}{3}\delta_{ij} + H^{(e)}_{ij}\right)$, these eigenvalues are

ability constraint quantifies precisely how 'small' the anisotropy must be to admit description by Eq. (38).

These difficulties are not unexpected, because we have formulated an expression applicable only to weak anisotropy. The most direct remedy is to introduce higher order angular harmonics in the expression for R. This problem is considered in the next section. We thus find that strong anisotropy is reflected not only in large values of the lowest order harmonics but also by a large number of angular harmonics expansion needed for a correct description of $e - Z$ angle-distribution.

So far, we have only treated the condition $e > 0$; the stronger condition $e \geq |Z|$ is not discussed here for the sake of brevity. We can just note that the condition $e > 0$ which depends only on the magnitude of $H^{(e)}_{ij}$ in (38) is the most sensitive condition; this could explain why the directional anisotropy is neglected in some *ad hoc* models, for examply by choosing $\gamma = 1/2$ in Eq. (61): even if $\mathsf{H}^{(e)}$ is never zero in 'true' anisotropic homogeneous flows, it is safer to suppress its contribution in some spectrum models to avoid too much sensitivity to realizability issues. Even the value of the constant in the quasi-isotropic model by Launder et al. (1975) [23] seems (fortuitously ?) to reflect this condition, as shown by [3, 4].

This problem does not occur for simpler models based only on H or b, both of which can be computed for any anisotropy whatsoever, although there is no guarantee that the predicted dynamics will be realistic (Rubinstein and Girimaji [24]). Thus, the inclusion of $\mathsf{b}^{(e,z)}$ will improve a model only for weak anisotropy, as the low order suggests. Although, as noted earlier, it is possible to derive from Eq. (38) closed equations without adjustable parameter for $E(k,t)$, $\mathsf{H}^{(e)}(k,t)$, $\mathsf{H}^{(z)}(k,t)$ from any closed equation for R_{ij}, and the same procedure also yields closed equations for single-point moments $\mathcal{K}$, $\mathsf{b}^{(e)}$, $\mathsf{b}^{(z)}$, the disappointing result, found independently by Cambon *et al.* [9] and Kassinos *et al.* [11], is that the resulting model behaves in the RDT limit worse than a conventional RST model, because of possible loss of realizability when the anisotropy is not necessarily small.

bounded by $\pm 1/3$, as they are for any deviatoric tensor derived from a definite-positive matrix.

7 SO^3 decomposition

The realizability constraint restricts the application of a theory based on spherical averages alone; this section will describe how to construct more accurate approximations using higher order expansions based on irreducible representations of the rotation group SO^3. This representation theory has recently been found very useful in clarifying the scaling properties of correlation functions in turbulent flows (Arad *et al.*, [25]).

7.1 Directional anisotropy

The spectrum models that have been considered so far suggest how the anisotropic part of the correlation can be expanded in powers of $\hat{\boldsymbol{k}}$. For example, we can continue the development of the first term on the right side of Eq. (38) beyond the second order by writing

$$\mathcal{E} = U^2_{mn}(k,t)\hat{k}_m\hat{k}_n + U^4_{mnrs}(k,t)\hat{k}_m\hat{k}_n\hat{k}_r\hat{k}_s + ... \tag{70}$$

where $\mathsf{U}^2 = -15\mathsf{H}^{(e)}$. The expansion is restricted to polynomials of even order because of the parity property $R_{ij}(-\hat{\boldsymbol{k}}) = R_{ij}(\hat{\boldsymbol{k}})$.

Although the notation suggests that Eq. (70) proceeds in powers of $\hat{k}_i$, it does not do so without some restrictions on the coefficients. For example, $U^4_{mnrs} = \delta_{mn}A_{rs} + \cdots$ will generate a term $A_{rs}k_rk_s$ that could be included in the U^2 term. Without presenting any details, suffice it to say that if all such redundancies are eliminated, then U^4 will belong to a 9-dimensional representation of SO^3 on homogeneous quartic polynomials satisfying $\nabla^2 U^4_{ijmn}k_ik_jk_mk_n = 0$. This is discussed in detail in standard references such as Weyl [26]. Equivalently, after choosing a polar axis $\boldsymbol{n}$, the expansion could be described in terms of Legendre functions as a spherical harmonics decomposition following Cambon and Teissèdre [27],

$$\mathcal{E}(\boldsymbol{k}) = \sum_{n=1}^{N_0}\sum_{m=-n}^{n} e^m_{2n}(k,t)\,\underbrace{P^m_{2n}(\cos\theta)\exp(\imath m\varphi)}_{Y^m_{2n}(\theta,\varphi)}\,. \tag{71}$$

where $\theta = \arccos(\boldsymbol{n}\cdot\hat{\boldsymbol{k}})$ is the polar angle and φ is the azimuthal angle in a system of polar-spherical coordinates with axis $\boldsymbol{n}$, and P^m_{2n} are the associated Legendre polynomials of degree $2n$ and order m.

7.2 Polarization tensor

The analogous higher order expansion of the second term on the right side of Eq. (38) is

$$R_{ij}^{pol}(\boldsymbol{k},t) = \frac{1}{2}\left[P_{im}(\hat{\boldsymbol{k}})P_{jn}(\hat{\boldsymbol{k}}) + P_{in}(\hat{\boldsymbol{k}})P_{jm}(\hat{\boldsymbol{k}}) - P_{ij}(\hat{\boldsymbol{k}})P_{mn}(\hat{\boldsymbol{k}})\right] \times \left\{T_{mn}^{0}(k,t) + T_{mnrs}^{2}(k,t)\hat{k}_r\hat{k}_s + \cdots\right\} \quad (72)$$

where $\mathsf{T}^0 = 5\mathsf{H}^{(z)}$.

The structure of the higher order terms in the expansion of polarization is not so simple as the higher order terms in the expansion of directional anisotropy, and the steps which make Eq. (72) an orthogonal expansion in irreducible representations are less obvious and standard than the steps leading from Eq. (70) to Eq. (71). Although T^0 ($\propto \mathsf{H}^{(z)}$) is simply a constant trace-free second-rank tensor, T^2 consists of matrices with quadratic polynomial entries; their decomposition into irreducible representations or *tensor spherical harmonics* (TSH) can be summarized as follows (compare also the discussion in Arad *et al.* [25] for more general tensor quantities): invariant tensors of differential operators can be constructed. Their action on homogeneous polynomials belonging to a representation on scalar functions generate the appropriate representations on tensors. In the case just mentioned, we find that T^2 belongs to a 25-dimensional representation that is decomposed into irreductible representations of dimensions 1,3,5,7,9. The representation of dimension 1 cannot contribute to polarization, and the representation of dimension 3 does not survive solenoidal projection. The remaining representations of dimensions 5,7,9 define solenoidal TSH. In the interest of concreteness, we list the TSH that correspond to the irreducible representation of dimension 5. To compute it, we use the infinitesimal generators of SO^3:

$$\begin{aligned} L_x &= k_y\partial/\partial k_z - k_z\partial/\partial k_y \\ L_y &= k_z\partial/\partial k_x - k_x\partial/\partial k_z \\ L_z &= k_x\partial/\partial k_y - k_y\partial/\partial k_x \end{aligned} \quad (73)$$

Let P_2^I denote any five harmonic quadratics, for example, $\{k_x^2 - k_y^2, k_y^2 - k_z^2, k_xk_y, k_yk_z, k_zk_x\}$ The representation of dimension 5 is defined by

$$N_{ij}^{I} = \left(L_iL_j + L_jL_i - \frac{1}{3}L_pL_p\delta_{ij}\right)P_2^I \quad I = 1, ...5 \quad (74)$$

Examples are the matrix so generated from $k_x^2 - k_y^2$:

$$\begin{bmatrix} 2(k_x^2 - k_z^2) & 2k_xk_y & 3k_xk_z \\ 2k_xk_y & 2(k_z^2 - k_y^2) & -3k_yk_z \\ 3k_xk_z & -3k_yk_z & 2(k_y^2 - k_x^2) \end{bmatrix} \tag{75}$$

and two others obtained by simultaneous cyclic permutations of x, y, z and of tensor indices (note that the three tensors so generated sum to zero), and the matrix generated from k_xk_y

$$\begin{bmatrix} 2k_xk_y & k_x^2 + k_y^2 - 2k_z^2 & 3k_yk_z \\ k_x^2 + k_y^2 - 2k_z^2 & 2k_xk_y & 3k_xk_z \\ 3k_yk_z & 3k_xk_z & -4k_xk_y \end{bmatrix} \tag{76}$$

and two others obtained by the same cyclic permutations.

This analysis shows that it is only a coincidence that at the lowest order of anisotropy, directionality and polarization are both described by a constant trace-free second-rank tensor. At even the next order, the descriptions are quite different: as noted above, directionality is described by a homogeneous fourth degree polynomial, but polarization is described by three distinct types of second rank tensors with quadratic polynomial entries.

We believe that these expansions in scalar and tensor spherical harmonics, which clearly have very different mathematical origins, again underscore the difficulties of confounding directional and polarization anisotropy along the lines of the standard anisotropic models.

7.3 Axisymmetric turbulence

The problems raised by extending the description from weak anisotropy to arbitrary anisotropy can be better understood in the comparatively simple case of axial symmetry. Let the unit vector $\boldsymbol{n}$ be the axis of symmetry. In this case, any tracefree tensor function of k alone obtained by spherical averaging, or any single-point moment obtained by integration over all Fourier modes can be expressed as $H_{ij} = \frac{1}{2}H_n(3n_in_j - \delta_{ij})$ in terms of the single axial component $H_n = H_{ij}n_in_j$. As for the $\boldsymbol{k}$-dependent spectra, a polar-spherical system of coordinates can be introduced, so that Eq. (71) reduces to

$$\mathcal{E}(\boldsymbol{k}, t) = \sum_{n=1}^{N_0} e_{2n}(k, t) P_{2n}^0(\cos\theta) \tag{77}$$

in which only terms with $m = 0$ appear. In addition, the polarization anisotropy also can be expanded as

$$Z(\boldsymbol{k}, t) = \sin^2\theta \sum_{n=0}^{N_1} z_n(k,t) P_n^0(\cos\theta) \tag{78}$$

in which the complex-valued Z is defined as in Cambon and Jacquin [8], using the symmetry axis $\boldsymbol{n}$ to define the helical modes. In Eq. (78), z_n is real for even n and purely imaginary for odd n due to the hermitian symmetry property $Z(-\boldsymbol{k}) = Z^*(\boldsymbol{k})$; imaginary terms can represent the breaking of mirror symetry present, for example, in rotating turbulence and also yield some 'stropholysis' terms (Kassinos *et al.* [11]). If axial symmetry is understood to include invariance under reflections through planes containing the polar vector $\boldsymbol{n}$, then Z is real, and only terms of even order appear in the expansion of Z. Restricting to even orders with $N_1 = 2N_0$, Eqs. (77) and (78) can be recovered from Sreenivasan and Narasimha [28] and from Cambon & Teissèdre [27].

Note that the factor $\sin^2\theta$ is essential in the expansion of Z in Eq. (78) because polarization anisotropy must vanish when the wave-vector is parallel to the axial vector $\boldsymbol{n}$: reference to Eq. (15) shows that R^{pol} can only be axisymmetric if $b = d = 0$. The expansion of Z in Eq. (78) therefore is somewhat special. The general spherical harmonics expansion for Z in [12] is not consistent with this property. At the lowest order ($N_0 = 1$, $N_1 = 0$), $H_{ij}^{(e)}\hat{k}_i\hat{k}_j = H_n^{(e)} P_2^0$ with $P_2^0 = \frac{1}{2}(3\cos^2\theta - 1)$, and $Z = \frac{3}{4}H_n^{(z)}\sin^2\theta$, defining Z as in Cambon and Jacquin [8].

Since the spherical harmonic decomposition required for the expansion of directionality to higher order is entirely standard, we consider the expansion of polarization to higher order. We will construct the axisymmetric tensor spherical harmonics of the first order beyond $\mathsf{H}^{(z)}$, without however presenting derivations. We noted at the end of Section 7.2 the existence of TSH belonging to the irreducible representations of dimensions 5,7, and 9. We find exactly one axisymmetric TSH belonging to each of these representations. That corresponding to the irreducible representation of dimension 5 is generated by the (unique) axisymmetric quadratic polynomial $2k_z^2 - k_x^2 - k_y^2$ by the differentiation process described in Section 7.2. The corresponding TSH is

$$\mathsf{A}_5 = \begin{bmatrix} -2k_x^2 + 4k_y^2 - 2k_z^2 & -6k_xk_y & 3k_xk_z \\ -6k_xk_y & 4k_x^2 - 2k_y^2 - 2k_z^2 & 3k_yk_z \\ 3k_xk_z & 3k_yk_z & -2k_x^2 - 2k_y^2 + 4k_z^2 \end{bmatrix} \tag{79}$$

so that

$$\mathsf{A}_5 : \mathsf{P} = 2k^{-2}((k_x^2 + k_y^2) - 2k_z^2) \tag{80}$$

Recalling the relation Eq. (23),

$$Z = k^{-4}\left(6(k_x^2 + k_y^2)^2 + 6(k_x^2 + k_y^2)k_z^2\right) = 6\sin^2\theta \tag{81}$$

in agreement with (78) for $n = 0$.

The TSH belonging to the irreducible representation of dimension 7 prove to change sign under inversion through planes containing the polar axis. We simply note the (again unique) axisymmetric TSH

$$\mathsf{A}_7 = \begin{bmatrix} 2k_xk_y & -k_x^2 + k_y^2 & -4k_yk_z \\ -k_x^2 + k_y^2 & -2k_xk_y & 4k_xk_z \\ -4k_yk_z & 4k_xk_z & 0 \end{bmatrix} \tag{82}$$

for which $e = 0$ and $Z = 10\imath k^{-3}k_z(k_x^2 + k_y^2) = 10\imath \sin^2\theta\cos\theta$, in agreement with (78) for $n = 1$. The inversion antisymmetry implies that Z is purely imaginary.

The axisymmetric TSH belonging to the irreducible representation of dimension 9 is

$$\mathsf{A}_9 = \begin{bmatrix} -4k_z^2 + 3k_x^2 + k_y^2 & 2k_xk_y & -8k_xk_z \\ 2k_xk_y & -4k_z^2 + 3k_y^2 + k_x^2 & -8k_yk_z \\ -8k_xk_z & -8k_yk_z & 8k_z^2 - 4(k_x^2 + k_y^2) \end{bmatrix} \tag{83}$$

so that

$$\mathsf{A}_9 : \mathsf{P} = k^{-4}\left(24k_z^2(k_x^2 + k_y^2) - 3(k_x^2 + k_y^2)^2 - 8k_z^4\right) \tag{84}$$

and

$$Z = k^{-4}\left(30k_z^2(k_x^2 + k_y^2) - 5(k_x^2 + k_y^2)^2\right) = 5\sin^2\theta(7\cos^2\theta - 1) \tag{85}$$

in agreement with (78) provided terms with both $n = 0$ and $n = 2$ are included.

8 Conclusions and perspectives

We summarize the main points of this paper as follows:

1. The description of weak anisotropy by tensor functions of wavenumber k alone requires *two* tensors as in Eq. (38). These tensors represent the *distinct* effects of directional anisotropy and polarization anisotropy as defined by Cambon *et al.* [8, 9, 10]. The reduction to a single tensor function of k compromises the kinematics by introducing implicit assumptions about anisotropy, usually that directional anisotropy vanishes. We emphasize in particular that the correlation tensor cannot be uniquely and self-consistently reconstructed in terms of the Reynolds stresses alone.

2. Some special cases of anisotropy were considered: short time response of turbulence to arbitrary strain and the long time nonlinear response of turbulence to small strain. Polarization and directional anisotropy are related differently in each limit; consequently, a model based on a single spherical average cannot be consistent with both limits. Turbulence under rapid rotation leads to a conclusion opposite to that usually adopted in models, namely dominant directional anisotropy and vanishing polarization.

3. The description of anisotropy by two tensor functions $\mathsf{H}^{(e,z)}(k,t)$ was exhibited as the lowest order in an infinite expansion in scalar and tensor spherical harmonics generated by the SO^3 decomposition. In this case, polarization and directional anisotropy are both described by a traceless second rank tensor, but the description becomes more complex at higher order.

To conclude, we note some important open issues for anisotropic turbulence. The first is the question of connecting the higher order coefficients in SSH and TSH expansions to spherically averaged higher order moments of the spectral tensor. A related question is whether it is possible to model the higher order terms by tensor products of $H^{(e)}$, $H^{(z)}$, and the mean velocity gradient $\partial\overline{U}_i/\partial x_j$ if a mean flow is present. Recent studies in rotating (Bellet *et al.* [21]) and/or in stably stratified turbulence have confirmed that the anisotropy identified by the angle distribution of $\mathcal{E}(\boldsymbol{k})$ can be very large, and is only reflected by small or moderate values of H^e_{ij} or $b^{(e)}_{ij}$. In such cases, the SSH expansion of R would need to be carried out to extremely high order. This expansion is unlikely to be practical; this raises the question whether a different characterization of large anisotropy than spherical harmonics expansions may be required.

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