

3.3 Aberrations for Grazing Incidence Optics

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Grazing incidence telescopes

Large number of grazing incidence telescope configurations have been designed and studied. Wolter¹ telescopes are commonly used in astronomical applications. Wolter telescopes consist of a paraboloidal primary mirror and a hyperboloidal or an ellipsoidal secondary mirror. There are 8 possible combinations of Wolter telescopes². Out of these possible designs only type 1 and type 2 telescopes are widely used. Type 1 telescope is typically used for x-ray applications and type 2 telescopes are used for EUV applications.

Wolter-Schwarzschild (WS) telescopes³ offer improved image quality over a small field of view. The WS designs are stigmatic and free of third order coma and, therefore, the PSF is significantly better over a small field of view. Typically the image is more symmetric about its centroid. As for the Wolter telescopes there are 8 possible combinations of WS telescopes². These designs have not been widely used because the surface equations are complex parametric equations complicating the analysis and typically the resolution requirements are too low to take full advantage of the WS designs.

There are several other design options. Most notable are wide field x-ray telescope designs. Polynomial designs were originally suggested by Burrows⁴ and hyperboloid-hyperboloid designs for solar physics applications were designed by Harvey⁵.

No general aberration theory exists for grazing incidence telescopes that would cover all the design options. Several authors have studied the aberrations of grazing incidence telescopes⁶⁻⁹. A comprehensive theory of Wolter type 1 and 2 telescopes has been developed^{10,11}. Later this theory was expanded to include all possible combinations of grazing incidence and also normal incidence paraboloid-hyperboloid and paraboloid-ellipsoid telescopes¹². In this article the aberration theory of Wolter type telescopes is briefly reviewed.

Surface equations

The surface equations of the grazing incidence telescopes can be combined to a general form covering large number of design options. General surface equations have been developed for grazing incidence telescopes². Unfortunately these equations are too complex to base the aberration theory on. The equations of the Wolter telescopes presented in the cylindrical coordinate system are relatively simple. Assuming an incoming ray hits the telescope entrance aperture at radial and azimuthal location (h, β) , it strikes the primary mirror at a location A, the secondary mirror at a location B, and the focal plane at a location F, as shown in Figures 1 and 2. The extension of the ray which

hits the secondary at the location B would intersect the optical axis at F1, at the common focus of the paraboloid and hyperboloid.

Assuming the primary and the secondary mirror are surfaces of revolution and the primary mirror focus coincides with the secondary mirror focus, then the surface equation of the primary mirror can be written as:

$$1/\rho = (\cos(\theta) - 1)/R_1, \quad (1)$$

where ρ is the distance from the paraboloid-hyperboloid focus F1 to a point on the primary mirror, θ is an angle this ray makes with the optical axis, and R_1 is the vertex radius of curvature of the primary mirror. The secondary hyperboloid can be expressed either as a function of angle α or θ

$$1/q = (\varepsilon \cos(\alpha) - 1)/R_2 \quad (2)$$

$$1/r = (1 - \varepsilon \cos(\theta))/R_2, \quad (3)$$

where q is the distance from the focus F to the point B on the secondary mirror and α is an angle q makes with the optical axis, R_2 is the vertex radius of curvature of the secondary mirror, ε is the eccentricity of the secondary, r is the distance from the point B on the secondary to the focus F1 of the primary.

The principal surface^{13, 14} of the Wolter telescopes is defined by the intersection points of the extensions of the incoming rays and the extensions of the rays reflected on the secondary (q). The principal surface goes through the intersection points of the primary mirror and secondary mirror. The principal surface of the Wolter telescopes is always a paraboloid¹³. It is useful to define the focal length (f) of the telescope to be the distance from the vertex of the principal surface to the telescope focus (F). Quite often the focal length is defined as the axial distance from telescope focus to the primary-secondary surface intersection plane.

A useful relation for the paraboloid-hyperboloid or paraboloid-ellipsoid telescopes is the following relation¹⁵:

$$h = 2f \tan(\alpha/2). \quad (4)$$

This equation ties the telescope object side to the image side. The equation (4) shows that the Wolter telescopes do not satisfy the Abbe's sine condition

$$h = f \sin(\alpha). \quad (5)$$

If the angle α is small, as is the case in x-ray optical systems, then the trigonometric terms could be expanded in α . The difference in Eqs.(4) and (5) would be in the third order term

indicating that the third order coma should be small in the Wolter telescopes and the telescopes nearly satisfy the Abbe's sine condition.

Transverse ray aberration expansions

Transverse ray aberration (TRA) expansions are expressed as functions of entrance aperture coordinates (h, β) shown in Figure 2. The expansions presented here are based on the format introduced by Cox¹⁶. The derivation is simple but rather lengthy. In the derivation an off-axis ray making an angle δ is mathematically traced from the primary mirror to the secondary mirror and to the focal plane. The surface intersection points on the primary and on the secondary are solved with respect to an on-axis ray and expanded in δ and h . The image plane intersection point (H_x, H_y) of this ray are the final image coordinates. They are then expressed as functions of entrance aperture coordinates (h, β) and image height $H_0 (=f \tan(\delta))$. The TRA expansions of Wolter telescopes are:

$$H_x = H_0 A_2 h^2 \sin(2\beta) + H_0^2 \sin(\beta) [A_3 h + B_3 h^3 + C_1 h^5 + \cos^2(\beta) B_4 h^3] + H_0^3 \sin(2\beta) (B_5 h^2 + C_2 h^4) / 2 \quad (6)$$

$$H_y = H_0 [1 + A_2 h^2 (2 + \cos(2\beta))] + H_0^2 \cos(\beta) \{A_4 h + h^3 [B_3 + \lambda_2 + B_4 (1 + \cos^2(\beta))]\} + H_0^3 \cos^2(\beta) [B_5 h^2 + C_2 h^4] \quad (7)$$

In the derivation and resulting expansions only significant terms for grazing incidence telescopes are kept. The image height terms (H_0) higher than third order are dropped. The radial height (h) terms higher than fourth order are also dropped. The parameters of the TRA expansions are listed below in equations (8) through (19) as functions of the telescope basic parameters f, R_1, R_2, ε , and L . The parameter L is the length of the telescope from the entrance aperture to the focal plane.

$$A_2 = 1/(4f^2) \quad (8)$$

$$A_3 = (K/(2f) - 1/R_1 - 1/R_2)/f \quad (9)$$

$$A_4 = (3K/(2f) - 1/R_1 - 1/R_2)/f \quad (10)$$

$$B_2 = 1/(8f^4) \quad (11)$$

$$B_3 = (K/f - 1/R_1 - 1/R_2 - 2f/(R_1 R_2))/(4f^3) \quad (12)$$

$$B_4 = K/(2f^4) \quad (13)$$

$$B_5 = (-K + 1 + 2(R_1/R_2) + (R_1/(2f)))/(f^2 R_1 R_2) \quad (14)$$

$$\lambda_1 = 1/(16f^4) \quad (15)$$

$$\lambda_2 = (-K/(4f) + 1/R_1 + 1/R_2)/f^3 \quad (16)$$

$$C_1 = -1/(4f^4 R_1 R_2) \quad (17)$$

$$C_2 = (-1/R_1 + 1/R_2)/(2f^2 R_1 R_2) \quad (18)$$

$$K = (L - 2\varepsilon R_2 / (\varepsilon^2 - 1)) / f + R_1 / R_2 \quad (19)$$

The Wolter telescopes are stigmatic on-axis and, therefore, the designs are free on all orders of spherical aberration terms. The first term in the expansion is the third order

coma. The coma coefficient is proportional to the inverse square of the focal length and the coma term is inversely proportional to the square of the telescope f-number. It does not depend on the location of the entrance aperture or the other parameters.

The third-order aberration terms A_3 and A_4 are proportional to second order of the image height (H_0) and first order aperture height h . Both terms depend on the location of the entrance aperture. Astigmatism and field curvature can be derived from these terms¹⁰.

All fifth-order terms are represented in the expansions relative to Cox's work¹⁶. As the third-order spherical aberration, the fifth-order spherical aberration term is zero since the Wolter telescopes are stigmatic. Coefficients B_2 and λ_1 represent fifth order circular coma. The term including the B_3 , B_4 , and λ_2 is the so-called astralate aberration¹⁶. If coefficient $B_4 = 0$, the term is called fifth-order oblique spherical aberration. The terms represented by B_5 coefficient is the fifth-order elliptical coma aberration. Approximations suitable for grazing incidence telescopes were made in the derivation of B_5 coefficient.

Two seventh-order terms C_1 and C_2 proportional to $H_0^2 h^5$ and $H_0^3 h^4$ are approximations. Exact solutions are very complex formulas of the basic parameters.

Typically the seventh-order terms and the fifth order terms become more important when the grazing angles of the surfaces decrease. Expanding the TRA equations as a function of radial height h is not the best choice in case of grazing incidence systems. For example, expanding the TRA equations as a function of Δh ($=h - h_{\text{int}}$), where h_{int} is the radial height at the primary secondary intersection plane, could lead to fewer terms and aberration coefficients more meaningful for the grazing incidence telescopes.

The RMS image size can be represented as a function of the aberrations coefficients¹⁷. The resulting equation is rather complex function of the aberration coefficients and the field angle.

Curvature of the best focal surface

All combinations of Wolter telescopes suffer from large curvature of the best focal surface. This limits the field of view of the telescopes. The shape of the best focal surface is parabolic. The radius of curvature R_d of this surface can be estimated from the TRA equations¹¹

$$1/R_d = f[A_3 + A_4 + (R_{\text{max}}^2 + R_{\text{min}}^2)(B_3 + B_4 + \lambda_2) + \frac{C_1}{3} \frac{R_{\text{max}}^6 - R_{\text{min}}^6}{R_{\text{max}}^2 - R_{\text{min}}^2}], \quad (20)$$

where R_{max} and R_{min} are the maximum and minimum radial heights of the entrance aperture, respectively. If only the third-order terms A_3 and A_4 are included, the equation represents third-order field curvature¹⁰. In case of grazing incidence telescopes the fifth-order term ($B_3+B_4+\lambda_2$) is comparable to the third-order term. If the grazing angles are small, even the seventh-order term C_1 cannot be omitted.

Alternative equation for the shape of the best focal surface is given by Shealy¹⁷. In this paper the RMS image radius is formally calculated from the TRA equations (6) and (7).

Aberration balancing

In case of Wolter type 1 telescopes the aberration equations suggest that for the optimum design the primary and the secondary should be as close to each other as possible. Separating the primary and the secondary, increase the radial heights on the primary and, therefore the image size.

The best focal surface of the Wolter type 1 telescopes always curves towards the telescope. The largest term in the aberration coefficients is the sum of the inverse of the radii of curvatures ($1/R_1+1/R_2$). For Wolter type 1 telescopes, the radii of curvatures are both negative and these quantities in the terms A_3 , A_4 , and B_3 cannot cancel each other.

In case of Wolter type 2 telescopes, R_1 is negative and R_2 is positive. The radius of curvature can be optimized. It turns out that for all the practical designs the R_2 is always smaller than R_1 and the radius of curvature of the best focal surface is negative and curving towards the telescope. The aberrations are optimized when the primary and secondary are as close to the primary-secondary surface intersection point as possible.

The field curvature could be improved by moving the entrance aperture away from the telescope. The K parameter would get bigger since the length L of the telescope would increase. Having the entrance aperture far in front of the telescope may not be practical design. For example, the vignetting would increase rapidly as a function of the off-axis angle.

The aberration equations presented in this paper are derived in terms of conventional parameters. The equations are shown as functions of the entrance aperture coordinates using the formulation introduced by Cox¹⁶. The TRA polynomials and the OPD-polynomial have also been derived as functions of the exit pupil coordinates¹² using the traditional formulation shown for example in Handbook of Optics¹⁸. The derivation includes all the terms shown in this paper (as a function of entrance aperture coordinates). The expansions are valid for all the combinations of Wolter telescopes and also for all the combinations of normal incidence paraboloid-hyperboloid or paraboloid-ellipsoid telescopes.

On-axis aberrations

Rigid-body motions

Rigid-body motions of the primary and the secondary and the low spatial frequency errors of the primary and secondary are the most important on-axis image aberrations. These errors typically degrade the on-axis resolution of the grazing incidence telescopes and limit the encircled energy performance of the telescopes. Glenn¹⁹ introduced an orthonormal set of Legendre-Fourier (L-F) polynomials for cylindrical mirrors which are used to describe the low order errors of the primary and secondary mirrors. The L-F

polynomials have been implemented in the Optical Surface Analysis Code (OSAC) ray trace code²⁰.

The TRA aberration expansions have been derived for the rigid body motions and low order L-F polynomials²¹. The rigid-body motions of the primary and secondary mirror are despace, decenter, tilt, and defocus errors. The TRA equations can be derived following the similar principle used in the derivation of TRA expansions of off-axis aberrations shown in the previous section.

The defocus term (Δz) can be expressed as^{11, 21}:

$$H_x = -\frac{\Delta z}{f} h \sin(\beta) \quad (21)$$

$$H_y = -\frac{\Delta z}{f} h \cos(\beta). \quad (22)$$

In the equations only the first order term in radial height h is kept and the higher order terms are omitted. The defocus terms are proportional to the radial height h . Therefore, this term can be used to optimize the off-axis aberrations terms that are also proportional to $h \sin(\beta)$ - $h \cos(\beta)$ pair. This principle was used to find the best focal surface and the radius of curvature of the best focal surface.

Despace surface errors of the primary ($n=1$) and secondary ($n=2$) can be expressed as surface radial height errors Δh_n and surface axial errors Δz_n as:

$$\Delta h_n = 0 \quad (23)$$

$$\Delta z_n = \text{constant}, \quad (24)$$

The despace image terms of the primary mirror can be approximated by:

$$H_x = -\Delta z_1 \frac{4fR_1^2}{h^3} \sin(\beta) \quad (25)$$

$$H_y = -\Delta z_1 \frac{4fR_1^2}{h^3} \cos(\beta), \quad (26)$$

The despace image terms of the secondary mirror are:

$$H_x = \Delta z_2 \left(\frac{4fR_1^2}{h^3} + \frac{h}{f} \right) \sin(\beta) \quad (27)$$

$$H_y = \Delta z_2 \left(\frac{4fR_1^2}{h^3} + \frac{h}{f} \right) \cos(\beta). \quad (28)$$

The leading term in the expansions is now proportional to inverse of h^3 . The off-axis aberrations shown above do not have terms proportional to inverse of h . This is because the primary mirror and secondary mirror are stigmatic and confocal.

Decentering the mirror leads to radial height error of the primary (=1) and secondary (n=2) that can be written in terms of radial error as:

$$\Delta h_n = e_{01n} \cos(\beta) + f_{01n} \sin(\beta), \quad (29)$$

where e_{01n} and f_{01n} are the amount of decenter error. The approximate TRA aberrations of the decentered primary mirror for the cosine component are:

$$H_x = e_{011} \frac{2fR_1}{h^2} \sin(2\beta) \quad (30)$$

$$H_y = e_{011} \frac{2fR_1}{h^2} \cos(2\beta) \quad (31)$$

The equations of decentered secondary are similar:

$$H_x = -e_{012} \frac{2fR_1}{h^2} \sin(2\beta) \quad (32)$$

$$H_y = e_{012} \left[1 - \frac{2fR_1}{h^2} \cos(2\beta)\right] \quad (33)$$

The tilt error of the mirrors can also be expressed in terms of radial height error and axial translation of the primary (n=1) and secondary (n=2). The radial and axial errors are:

$$\Delta h_n = z_n (E_{11n} \cos(\beta) + F_{11n} \sin(\beta)) \quad (34)$$

$$\Delta z_n = -h_n (E_{11n} \cos(\beta) + F_{11n} \sin(\beta)), \quad (35)$$

where E_{11n} and F_{11n} are the amount of tilt error in radians when the mirror is rotated about x-axis and y-axis, respectively. The approximate expansions of the primary mirror are:

$$H_x = -E_{111} f \left(\frac{h_{10}}{h}\right)^2 \sin(2\beta) \quad (36)$$

$$H_y = -E_{111} f \left[1 + \left(\frac{h_{10}}{h}\right)^2 \cos(2\beta)\right], \quad (37)$$

where h_{10} is the radial height of the primary mirror at the center of the mirror. The equations of the secondary mirror are:

$$H_x = E_{112} \frac{fh_{10}h_{20}}{h^2} \sin(2\beta) \quad (38)$$

$$H_y = E_{112} f \frac{h_{10}}{h_{20}} \left[1 + \left(\frac{h_{10}}{h}\right)^2 \cos(2\beta)\right], \quad (39)$$

where h_{20} is the radial height of the secondary mirror at the center of the mirror.

The $\sin(2\beta)$ - $\cos(2\beta)$ relationship of the primary and secondary components shown in the equations for the decenter and tilt errors is typical of coma. Note that the components are now inversely proportional to h^2 . The off-axis coma term is directly proportional to h^2 .

Axial and circumferential slope errors and TRA equations

Assuming the primary mirror figure error Δh_1 and the secondary mirror figure error Δh_2 are known as functions of the surface axial coordinate and circumferential coordinate. Then, if the grazing angles of the mirrors are small, the on-axis aberrations can be evaluated easily from the surface slope errors. The TRA equations of the primary mirror of the axial surface errors are:

$$H_x = -2f \frac{\partial \Delta h_1}{\partial z_1} \sin(\beta) \quad (40)$$

$$H_y = -2f \frac{\partial \Delta h_1}{\partial z_1} \cos(\beta), \quad (41)$$

where $\partial \Delta h_1 / \partial z_1$ is its axial slope error. The secondary mirror TRA equations for the axial slope errors are:

$$H_x = 2q \frac{\partial \Delta h_2}{\partial z_2} \sin(\beta) \quad (42)$$

$$H_y = 2q \frac{\partial \Delta h_2}{\partial z_2} \cos(\beta), \quad (43)$$

where q is the distance from the secondary to the telescope focus and $\partial \Delta h_2 / \partial z_2$ is the axial slope error of the secondary. The q variable can be approximated by q_0 that is the distance from the center point of the surface to the telescope focus.

The approximated TRA equations for the primary mirror circumferential slope errors are:

$$H_x = f \sin(2I_1) \frac{1}{h} \frac{\partial \Delta h_1}{\partial \beta} \cos(\beta) \quad (44)$$

$$H_y = -f \sin(2I_1) \frac{1}{h} \frac{\partial \Delta h_1}{\partial \beta} \sin(\beta), \quad (45)$$

where I_1 is the grazing angle on the primary and $\partial \Delta h_1 / (h \partial \beta)$ is the circumferential slope error of the primary mirror. The approximate TRA equations of the secondary for the circumferential slope error are:

$$H_x = -2f \sin(I_2) \frac{h_2}{h} \frac{\partial \Delta h_2}{\partial \beta} \cos(\beta) \quad (46)$$

$$H_y = 2f \sin(I_2) \frac{h_2}{h} \frac{\partial \Delta h_2}{\partial \beta} \sin(\beta), \quad (47)$$

Where I_2 is the grazing angle of the surface, h_2 is the radial height of the surface, and $\partial \Delta h_2 / (h_2 \partial \beta)$ is the circumferential slope of the surface. The variables I_2 and h_2 can be approximated by the grazing angle and radial height at the midpoint on the surface.

The TRA equations of the circumferential errors include the grazing angle I_1 or I_2 . Typically, the grazing angles are small (0.5 to few degrees). Therefore, the circumferential errors have miniscule effect on the image at the focal plane compared to the axial slope errors and, in many cases, can be ignored.

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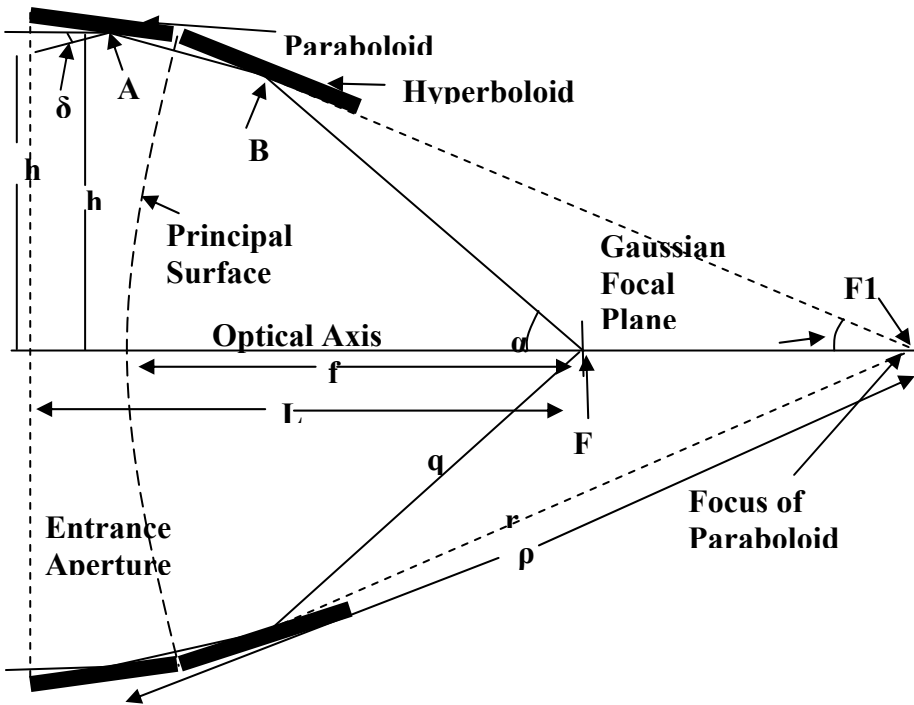


Figure 1. Cross-section of Wolter type 1 telescope showing the optical components, ray paths and the parameters.

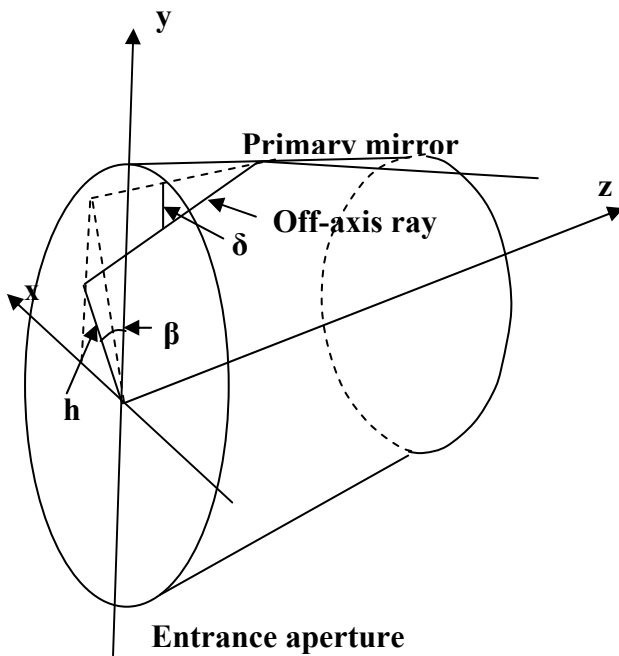


Figure 2. Primary mirror of the grazing incidence telescope showing entrance aperture and incoming off-axis ray.