Approximation of Failure Probability Using Conditional Sampling

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In analyzing systems which depend on uncertain parameters, one technique is to partition the uncertain parameter domain into a failure set and its complement, and judge the quality of the system by estimating the probability of failure. If this is done by a sampling technique such as Monte Carlo and the probability of failure is small, accurate approximation can require so many sample points that the computational expense is prohibitive. Previous work of the authors has shown how to bound the failure event by sets of such simple geometry that their probabilities can be calculated analytically. In this paper, it is shown how to make use of these failure bounding sets and conditional sampling within them to substantially reduce the computational burden of approximating failure probability. It is also shown how the use of these sampling techniques improves the confidence intervals for the failure probability estimate for a given number of sample points and how they reduce the number of sample point analyses needed to achieve a given level of confidence.
Keywords: uncertainty propagation, failure probability.

I.   Introduction

This paper considers a system that depends on parameters subject to uncertainty that are modeled probabilistically by modeling the vector of uncertain parameters as a vector of random variables. Acceptability of the system is assessed by determining if the system satisfies a set of requirements which are prescribed as a set of constraints.

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The values of the uncertain parameters which result in an unsatisfactory system may be agglomerated into an event which will be called the *failure domain.* A measure of how satisfactory the system is can be found in the probability of failure. Computation of the exact probability of failure by, for example, computing the integral of the probability density function over the failure domain, is usually a computationally intractable problem. This is true even if the probability distribution of the uncertain parameter vector is known precisely in terms of effectively computable probability density functions or cumulative distribution functions, and even if a given realization of the uncertain parameters may be tested for membership in the failure domain by an effectively computable procedure (which, in a realistic engineering example, often takes a substantial amount of computation).

A typical technique used to approximate the failure probability is based on sampling the uncertain parameter at a large number of points and analyzing the system for each of these sample values to determine which sample values fall in the failure domain. The higher the precision needed from such an approximation, or the closer the failure probability is to zero, the more sample points need to be used; and this technique can also become computationally intractable.

In prior work,[1] the authors have shown how, in this context, to calculate sets whose complements bound the failure domain and which have such simple geometry (hyper-rectangles or hyper-spheres) that their probabilities may be determined analytically for a very useful family of possible distributions of the uncertain parameters. These sets have been used to find upper bounds for the failure probability.

In this paper, two techniques are presented which reduce the number of sample points which must be subjected to the full computational burden of system analysis while using a sampling based method to estimate the failure probability. These are called hybrid methods since they both make use of the failure domain bounding sets and also make use of sampling techniques. In one, the rejection method,[1] an adequate number of sample points is generated in the uncertain parameter space, but a simple test is applied to each to see if it falls outside the failure bounding set. If it does, it is unnecessary to go to the computational expense of subjecting that point to a full system analysis; i.e., that point is rejected for analysis. In the other, the conditional sampling method, the only sample points generated fall within the failure bounding set. These points are distributed according to the portion of the distribution of the uncertain parameters which is supported by the failure bounding set. This reduces the number of points needed (substantially, if the event bounding the failure domain itself has small probability) to achieve a given level of confidence in the failure probability estimate because, in contrast to the rejection based method, no sample points are ever generated that need to be rejected. The conditional distribution technique can be considered to be an *importance sampling* technique[2–4] where the distribution used for sampling is determined using information obtained from a failure bounding event (set) which has been found by solving an optimization problem, and from which samples can be drawn analytically.

It is then shown how using these sampling techniques can reduce the number of samples needed (vs. full space sampling) to achieve a given level of confidence or, conversely, how more confidence can be achieved for an estimate based on a given number of sample points.

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II.   Basic Notions

The objective of this paper is the analysis of systems having a parametric mathematical model. These parameters, denoted by the vector $\boldsymbol{p}$, are considered to be uncertain. This uncertainty is modeled by considering the vector of uncertain parameters to be a vector of random variables whose distribution is specified by a joint probability density function (PDF) $f_{\boldsymbol{p}}(\boldsymbol{p})$ whose support set, also called the *uncertainty set*, is denoted by $\Delta_{\boldsymbol{p}}$. Hereafter, the terms "uncertainty set" and "support set" will be used interchangeably. By specifying the uncertainty model, it is implied that $\Delta_{\boldsymbol{p}}$ contains the true value(s) of $\boldsymbol{p}$. In principle, the uncertainty set can be unbounded and can even be the entire parameter space. In practical applications, the choice of this model is usually made by a discipline expert. However, the theory presented herein acknowledges that such a choice may be fairly arbitrary, and addresses the need for quantifying the level of tolerance of the system to uncertainty in $\boldsymbol{p}$. Any member of the uncertainty set is called a *realization*. The *nominal parameter* value, denoted by $\bar{\boldsymbol{p}}$, is a parameter realization regarded as a good deterministic representation of $\boldsymbol{p}$. Requirements on the system are prescribed by means of the collection of constraint functions, $\boldsymbol{g}(\boldsymbol{p}) \leq \boldsymbol{0}$ (this and all vector inequalities are considered satisfied if they are satisfied in every component).

A system is considered *robust* if the constraints are satisfied for all $\boldsymbol{p} \in \Delta_{\boldsymbol{p}}$. For systems which are non-robust, the failure probability to be approximated subsequently is a metric of the degree of non-robustness.

The *Failure Domain* $\mathcal{F}$ is given by[a]

$$\mathcal{F} \triangleq \bigcup_{j=1}^{q} \mathcal{F}^j,$$

where the scalar components of the vector $\boldsymbol{g}(\boldsymbol{p})$ are $\langle \boldsymbol{g}_1(\boldsymbol{p}), \boldsymbol{g}_2(\boldsymbol{p}), \ldots, \boldsymbol{g}_q(\boldsymbol{p}) \rangle$, and

$$\mathcal{F}^j \triangleq \{ \boldsymbol{p} : \boldsymbol{g}_j(\boldsymbol{p}) > 0 \}.$$

Here, $\mathcal{F}^j$ is the failure domain corresponding to the jth requirement, and $\mathcal{F}$ is the failure domain for all requirements. Usually, the equation $\boldsymbol{g}_j = 0$ partitions the parameter space into two parts, the failure domain $\mathcal{F}^j$ corresponding to the jth requirement, and its complement set, the *Non-failure Domain* corresponding to the jth requirement, $(\mathcal{F}^j)^c$ (the superscript c denotes the set complement operator).

III.   Set Deformations

Failure set bounding is accomplished by expanding or contracting a specially chosen *Reference Set*. The Reference Set will be denoted by Ω and will be chosen as a hyper-sphere or hyper-rectangle centered at the nominal parameter $\bar{\boldsymbol{p}}$. The hyper-sphere of radius R centered at $\bar{\boldsymbol{p}}$, denoted by $\mathcal{S}(\bar{\boldsymbol{p}}, R)$, is defined by

$$S(\bar{\boldsymbol{p}}, R) = \{ \boldsymbol{p} : \|\boldsymbol{p} - \bar{\boldsymbol{p}}\| \leq R \}.$$

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[a]Throughout this paper, super-indices are used to denote a particular vector or set while sub-indices refer to vector components; e.g., $\boldsymbol{p}_i^j$ is the ith component of the vector $\boldsymbol{p}^j$.

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The hyper-rectangle centered at $\bar{\boldsymbol{p}}$ whose shape is determined by the vector $\boldsymbol{m} > \boldsymbol{0}$ of the half-lengths of the sides is denoted $R(\bar{\boldsymbol{p}}, \boldsymbol{m})$ and is defined by

$$R(\bar{\boldsymbol{p}}, \boldsymbol{m}) = \{\boldsymbol{p} : \boldsymbol{p}_i \in [\bar{\boldsymbol{p}}_i - \boldsymbol{m}_i, \bar{\boldsymbol{p}}_i + \boldsymbol{m}_i], \quad 1 \leq i \leq s\}$$

where $s \triangleq \dim(\boldsymbol{p})$. Choice of these reference sets is motivated by the analyst's perception of relative uncertainty of the parameters. Hyper-spherical sets consider parameters with similar levels of uncertainty. They could also be used for parameters with dissimilar levels of uncertainty if some scaling is used. However, whether scaling is used or not, a degree of dependence is introduced. For instance, the range of variability of one parameter depends on the values taken on by the others. Rectangular sets permit the consideration of dissimilar levels of uncertainty without the need for scaling and introduce no such dependence. Such levels are attained by making $\boldsymbol{m}_i$ proportional to the level of uncertainty in $\boldsymbol{p}_i$.

In previous work,[1] a measure of robustness has been assigned to the system based on measuring how much the reference set can be deformed before intersecting the failure domain. In that study, deformations of Ω are limited to *Homothetic Deformations*, by which is meant that Ω is expanded or contracted preserving proportionality and orientation, while $\bar{\boldsymbol{p}}$ remains fixed. Thus, if the hyper-sphere $S(\bar{\boldsymbol{p}}, R)$ is deformed by a factor of α, the resulting homothet is $S(\bar{\boldsymbol{p}}, \alpha R)$, while if the hyper-rectangle $R(\bar{\boldsymbol{p}}, \boldsymbol{m})$ is deformed by a factor of α, the resulting homothet is $R(\bar{\boldsymbol{p}}, \alpha\boldsymbol{m})$. Expansions are accomplished when $\alpha > 1$, while contractions result when $\alpha < 1$.

Throughout this paper, two uncertainty sets will be called *Proportional* if one of the two sets can be formed from the other by an homothetic deformation. Call the factor of this deformation the *Similitude Ratio* and denote it by α.

In what follows, it will be assumed that the nominal parameter value is feasible; i.e., $\boldsymbol{g}(\bar{\boldsymbol{p}}) \leq \boldsymbol{0}$. Intuitively, one imagines that a set proportional to the reference set is being deformed with respect to the nominal parameter point until its boundary touches the boundary of the failure domain; i.e., until at least one member of the deformed set is on the verge of violating one or more of the requirements in $\boldsymbol{g}$. Any point where the deforming set touches the failure domain is a *Critical Parameter Value* (CPV). A CPV, which will be denoted hereafter by $\tilde{\boldsymbol{p}}$, might not be unique. The expanded set is called the *Maximal Feasible Set* (MFS) and it is denoted by $\mathcal{M}_{\boldsymbol{p}}$. The *Critical Similitude Ratio*, denoted by $\tilde{\alpha}$, is the Similitude Ratio of that deformation, and the parametric safety margin, to be defined later, is a dimensional metric that quantifies the size of the MFS.

Formulations that enable the deformation of hyper-spherical and hyper-rectangular reference sets are introduced next.

III.A.   Deformation of hyper-spheres

The CPV for the jth requirement is given by

$$\tilde{\boldsymbol{p}}^j = \underset{\boldsymbol{p}}{\operatorname{argmin}} \left\{ \|\boldsymbol{p} - \bar{\boldsymbol{p}}\| : \boldsymbol{g}_j(\boldsymbol{p}) \geq 0 \right\}, \tag{1}$$

while the overall CPV is given by

$$\tilde{\boldsymbol{p}} = \tilde{\boldsymbol{p}}^i, \tag{2}$$

where

$$i = \underset{1 \leq j \leq q}{\operatorname{argmin}} \left\{ \|\tilde{\boldsymbol{p}}^j - \bar{\boldsymbol{p}}\| \right\}. \tag{3}$$

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Hence, the CPV problem is solved for each individual requirement, and, from these, the overall CPV is selected which is closest to the nominal parameter point. Additional constraints can be imposed on Equation (1) to prevent the CPV from assuming values leading to infeasible systems; e.g., if $\boldsymbol{p}_1$ is a mass, the constraint $\boldsymbol{p}_1 > 0$ must be added.

The CPV is the $\boldsymbol{p}$ value on the boundary of the failure domain closest to the nominal parameter point. The Spherical *Parametric Safety Margin* (PSM), denoted by ρ_S, is $\|\tilde{\boldsymbol{p}} - \bar{\boldsymbol{p}}\|$, and the Spherical *Maximal Feasible Set* (MFS) is $\mathcal{M} \triangleq S(\bar{\boldsymbol{p}}, \rho_S)$. $\mathcal{M}$ is the largest sphere centered at $\bar{\boldsymbol{p}}$ which does not intersect the failure domain.

III.B. Deformation of hyper-rectangles

Recall that the infinity norm in a finite dimensional space is defined as $\|\boldsymbol{x}\|^{\infty} = \sup_i\{|\boldsymbol{x}_i|\}$. Let us define the $\boldsymbol{m}$-scaled infinity norm as $\|\boldsymbol{x}\|_{\boldsymbol{m}}^{\infty} = \sup_i\{|\boldsymbol{x}_i|/\boldsymbol{m}_i\}$. A distance between the vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ can be defined as $\|\boldsymbol{x} - \boldsymbol{y}\|_{\boldsymbol{m}}^{\infty}$. Using this distance, the "unit ball" centered at $\bar{\boldsymbol{p}}$ is just $R(\bar{\boldsymbol{p}}, \boldsymbol{m})$.

The CPV corresponding to the deformation of $R(\bar{\boldsymbol{p}}, \boldsymbol{m})$ results from using Equations (1-3) after replacing the Euclidean norm with the $\boldsymbol{m}$-scaled infinity norm defined above. Hence, the CPV is solved for each individual constraint function, and the overall CPV is selected to be the individual CPV which is closest to the nominal parameter point according to the $\boldsymbol{m}$-scaled infinity norm. For instance, using the definition of the $\boldsymbol{m}$-scaled infinity norm in Equation (1) leads to

$$\tilde{\boldsymbol{p}}^j = \operatorname*{argmin}_{\boldsymbol{p}} \left\{ \max_{1 \leq k \leq s} \frac{|\boldsymbol{p}_k - \bar{\boldsymbol{p}}_k|}{\boldsymbol{m}_k} : \boldsymbol{g}_j(\boldsymbol{p}) \geq 0 \right\}.$$

The "max" can be eliminated from the objective function, thereby eliminating potential derivative discontinuities, by introducing the Similitude Ratio α defined earlier:

$$\langle \tilde{\boldsymbol{p}}^j, \tilde{\alpha}^j \rangle = \operatorname*{argmin}_{\boldsymbol{p}, \alpha} \{ \alpha : \boldsymbol{g}_j(\boldsymbol{p}) \geq 0, |\boldsymbol{p}_i - \bar{\boldsymbol{p}}_i| \leq \alpha \boldsymbol{m}_i, \quad 1 \leq i \leq s \} \tag{4}$$

As before, additional constraints can be imposed on Equation (4) to prevent the CPV from assuming values leading to infeasible systems. The interpretation of $\tilde{\boldsymbol{p}}$ given in the previous section can be naturally extended to the rectangular case. Figure 1 shows a sketch with relevant metrics.

The Rectangular *Parametric Safety Margin* (PSM), denoted by ρ_R, is $\rho_R \triangleq \tilde{\alpha}\|\boldsymbol{m}\|$, and the Rectangular *Maximal Feasible Set* (MFS) is $\mathcal{M} = R(\bar{\boldsymbol{p}}, \tilde{\alpha}\boldsymbol{m})$. $\mathcal{M}$ is the largest homothet of $R(\bar{\boldsymbol{p}}, \boldsymbol{m})$ which does not intersect the failure domain.

IV. Failure Probability: Upper Bounds and Hybrid Methods for Estimation

In general, the estimation of failure probabilities is difficult since it requires evaluating a multi-dimensional integral over a complex integration domain. Since the direct computation of $P[\mathcal{F}]$ is usually impractical, one needs to use approximations such as First Order Reliability Method (FORM)[2] or Monte Carlo sampling.[5] Sampling techniques tend to be computationally intensive.
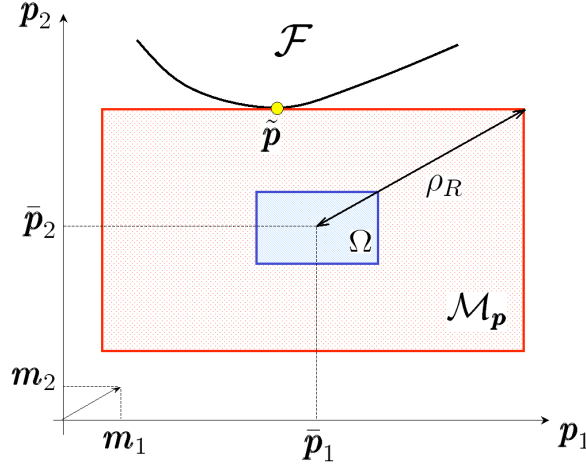
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Figure 1. Relevant metrics for a rectangular reference set.

Recall that a probabilistic uncertainty model is prescribed by the joint PDF $f_{\boldsymbol{p}}(\boldsymbol{p})$, whose support set is $\Delta_{\boldsymbol{p}}$; or, equivalently, by the joint cumulative distribution function (CDF) $F_{\boldsymbol{p}}(\boldsymbol{p})$. Strategies for the calculation of upper bounds to the probability of failure based on the deformations of sets in $\boldsymbol{p}$-space and $\boldsymbol{u}$-space (to be defined) are presented next.

IV.A. Upper bounds to Failure Probability

Let
$$\psi \overset{\Delta}{=} P[\mathcal{M}^c] = 1 - P[\mathcal{M}], \tag{5}$$
where $\mathcal{M}$ is the maximal feasible set resulting from a homothetic deformation of a reference set Ω. This number ψ is an upper bound to $P[\mathcal{F}]$.

When the components of $\boldsymbol{p}$ are independent random variables and Ω is hyper-rectangular, ψ can be calculated by a simple formula[1] which requires using evaluations at only two points of each of the CDFs of the individual components of $\boldsymbol{p}$.

One important special case of parametric uncertainty arises when the parameter vector has the standard normal distribution. This is not only a useful model in itself, but is also used in analyzing other parameter uncertainty models when a probability preserving transformation U can be found so that, when $\boldsymbol{u} = U(\boldsymbol{p})$, then $f_{\boldsymbol{u}}$ is the standard normal distribution. Under such a transformation, if $\mathcal{F}$ is the failure set in $\boldsymbol{p}$-space, $U(\mathcal{F})$ is the failure set in the corresponding $\boldsymbol{u}$-space, and the constraint function $\boldsymbol{g}$ in $\boldsymbol{p}$-space transforms into the constraint function $\boldsymbol{G}(\boldsymbol{u}) \overset{\Delta}{=} \boldsymbol{g}(U^{-1}(\boldsymbol{u}))$. Such a transformation is used in many standard techniques such as FORM. In this paper, the term $\boldsymbol{u}$-*space* refers to an uncertain parameter space in which the uncertainty is the standard normal distribution. Note that if a system has both $\boldsymbol{p}$-space and $\boldsymbol{u}$-space representations connected by a transformation U, reference set choices are made independently in the two spaces and are not connected through the transformation U, since U will usually fail to preserve the geometric form needed for the failure probability bounding calculations made in this paper. This also means different bounds may be achieved in the two representations, and no general rule is known to determine in advance which is better.

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IV.B. Hybrid Methods for Estimating Failure Probability

The developments that follow are applicable to both $\boldsymbol{p}$- and $\boldsymbol{u}$-spaces.

In Monte Carlo sampling, a representative sample $\{\boldsymbol{p}^1, \boldsymbol{p}^2, \ldots, \boldsymbol{p}^N\}$ of the random vector $\boldsymbol{p}$ is generated, the constraint function values $\boldsymbol{g}(\boldsymbol{p}^i)$, $i = 1, \ldots N$, are calculated, and $P[\mathcal{F}]$ is estimated as the fraction of these constraint vector values which place $\boldsymbol{p}^i$ in the failure domain. The value of N is dictated by the confidence desired in the resulting approximation (the higher the confidence, the bigger N must be) and $P[\mathcal{F}]$ (the closer $P[\mathcal{F}]$ is to 0, the bigger N must be). Since $P[\mathcal{F}]$ is presumably unknown in advance, it might be necessary to pick a trial value for N, generate N samples, and estimate $P[\mathcal{F}]$ and calculate the confidence interval at some level of confidence (frequently, 95%) for this estimate, and decide if more samples must be analyzed to gain an acceptable level of confidence. (As an aside, note that the same problem arises if $P[\mathcal{F}]$ is close to 1. However, as a practical matter, it is not anticipated that failure probability estimates will be made in cases where failure is almost certain.)

Very small probabilities arise in practical engineering examples. For example, the FAA Federal Aviation Regulations (FARS, 14 CFR) Part 25 – Airworthiness Standards: Transport category airplanes, Section 1309 – Equipment, systems, and installations includes the specification that

> . . .
>
> (b) The airplane systems and associated components, considered separately and in relation to other systems, must be designed so that –
>
> (1) The occurrence of any failure condition which would prevent the continued safe flight and landing of the airplane is extremely improbable . . .

In Advisory Circular No. 25.1309-1A, the FAA quantifies their phrase "Extremely Improbable" as "having a probability on the order of 10^{-9} or less."

Hybrid Methods, to be introduced subsequently, approximate the failure probability by combining sampling methods with the failure set bounds. Applications of the rejection based method and a conditional sampling method to problems in $\boldsymbol{u}$-space have been made previously.[1] This paper extends these ideas to problems in $\boldsymbol{p}$-space, and presents improved and extended versions of the conditional sampling techniques. The present $\boldsymbol{p}$-space conditional sampling techniques can be applied in $\boldsymbol{u}$-space. Using the present techniques in $\boldsymbol{u}$-space constitutes an improvement over the methods given in the Ref. 1.

One advantage of doing the analysis in $\boldsymbol{p}$-space is that, as long as the constraint function $\boldsymbol{g}$ and the reference set Ω have not changed, the $\boldsymbol{p}$-space MFS, denoted here by $\mathcal{M}_{\boldsymbol{p}}$, remains the same even if the uncertainty distribution $f_{\boldsymbol{p}}$ assigned to $\boldsymbol{p}$ changes. However, if the $\boldsymbol{u}$-space has been defined as the image of $\boldsymbol{p}$-space under a probability preserving transformation U and a $\boldsymbol{u}$-space MFS, denoted here by $\mathcal{M}_{\boldsymbol{u}}$, has been calculated using the derived constraint function $\boldsymbol{G}(\boldsymbol{u}) = \boldsymbol{g}(U^{-1}(\boldsymbol{u}))$, changing $f_{\boldsymbol{p}}$ changes U and consequently $\boldsymbol{G}$, so a new MFS $\mathcal{M}_{\boldsymbol{u}}$ must be calculated.

Sampling based methods depend on the generation of sample sets from specified distributions. One general technique is reviewed here. Suppose that $X = (X_1, X_2, \ldots, X_s)$ is a vector of s mutually independent random variables whose CDF is $F(x_1, x_2, \ldots, x_s) = F_1(x_1)F_2(x_2) \cdots F_s(x_s)$. It is desired to generate a sample set of N vectors from the distribution F. First, generate a sample $\{v^1, v^2, \ldots, v^N\}$ distributed uniformly over an s-dimensional

hypercube which is the Cartesian product of s copies of the interval $(0, 1)$. There is an abundance of software to find such samples either using pseudo-random number generators (as would be done in Monte Carlo analysis) or using deterministic schemes such as Hammersley Sequence Sampling or Halton Sequence Leaped[6] for generating so-called *low discrepancy* sequences. Then, if $x_i^j = F_i^{-1}(v_i^j)$, the set $\{x^1, x^2, \ldots, x^N\}$ is such a sample set. Standard libraries for statistical analysis, such as the Matlab® Statistics Toolbox, contain software to evaluate the PDF, CDF, and inverse CDF of many commonly used distributions.

IV.B.1. Rejection Method

The simplest of the hybrid methods is the rejection method. In the rejection method, a $\boldsymbol{p}$-space sample $\{\boldsymbol{p}^1, \boldsymbol{p}^2, \ldots, \boldsymbol{p}^N\}$ is first generated according to the distribution $f_{\boldsymbol{p}}(\boldsymbol{p})$.

Once the MFS $\mathcal{M}$ has been determined, an arbitrary sample $\boldsymbol{p}^i$ can be easily tested for membership in this MFS. If $\boldsymbol{p}^i \in \mathcal{M}$, it is unnecessary to go to the computational expense of evaluating $\boldsymbol{g}$ at $\boldsymbol{p}^i$, since the outcome is now known. For hyper-rectangular reference sets, where $\mathcal{M} = R(\bar{\boldsymbol{p}}, \tilde{\alpha}\boldsymbol{m})$, the membership test for $\boldsymbol{p}^i \in \mathcal{M}$, is that every component of the vector $|\boldsymbol{p}^i - \bar{\boldsymbol{p}}|$ be bounded by the corresponding component of $\tilde{\alpha}\boldsymbol{m}$; i.e., $\|\boldsymbol{p}^i - \bar{\boldsymbol{p}}\|_{\boldsymbol{m}}^{\infty} \leq \tilde{\alpha}$. The only situation in which we use hyper-spherical reference sets is in $\boldsymbol{u}$-space with its standard normal distribution. Consideration is limited to hyper-spheres centered at the origin. In this case, $\mathcal{M} = S_{\boldsymbol{u}}(\boldsymbol{0}, \tilde{\alpha}R)$, and the membership test for $\boldsymbol{u}^i \in \mathcal{M}$, is $\|\boldsymbol{u}^i\| \leq \tilde{\alpha}R$.

For ease of discussion, it is assumed that the sample points in $\{\boldsymbol{p}^1, \boldsymbol{p}^2, \ldots, \boldsymbol{p}^N\}$ have been renumbered so that $\{\boldsymbol{p}^1, \ldots, \boldsymbol{p}^n\} \subset \mathcal{M}^c$ and $\{\boldsymbol{p}^{n+1}, \ldots, \boldsymbol{p}^N\} \subset \mathcal{M}$. Since $\mathcal{F} \subset \mathcal{M}^c$, and recalling the definition of ψ in Equation (5), the conditional probability equation $P[A \mid B] = P[A \cap B]/P[B]$ implies (with $A = \mathcal{F}$ and $B = \mathcal{M}^c$) that

$$P[\mathcal{F}] = \psi P[\mathcal{F} \mid \mathcal{M}^c].$$

$P[\mathcal{F} \mid \mathcal{M}^c]$ is estimated by $\sum_{i=1}^{n} \mathcal{I}(\boldsymbol{p}^i)/n$ where $\mathcal{I}$ is the indicator function of $\mathcal{F}$ which is given by:

$$\mathcal{I}(\boldsymbol{p}) = \left\{ \begin{array}{ll} 1 & \text{if } \boldsymbol{p} \in \mathcal{F} \\ 0 & \text{otherwise} \end{array} \right.$$

Notice that determining which value $\mathcal{I}(\boldsymbol{p})$ takes on requires evaluating $\boldsymbol{g}(\boldsymbol{p})$. This provides the estimate:

$$P[\mathcal{F}] \approx \frac{\psi}{n} \sum_{i=1}^{n} \mathcal{I}(\boldsymbol{p}^i) \tag{6}$$

This sum involves only the members of the sample set which are in $\mathcal{M}^c$. Thus, it can be considered that the points of the original sample which fall in $\mathcal{M}$ have been rejected. This motivates the use of the name *Rejection Method*. Notice also that, since ψ can be calculated directly from knowledge of $\mathcal{M}$ and $f_{\boldsymbol{p}}$, and since n/N is an estimate of ψ, the expected value of n, the number of sample points falling inside $\mathcal{M}^c$, can be controlled by the choice of N.

Equation (6) provides an approximation to $P[\mathcal{F}]$ when ψ is calculated using the overall CPV $\tilde{\boldsymbol{p}}$, and to $P[\mathcal{F}^j]$ when ψ is calculated using the individual CPV $\tilde{\boldsymbol{p}}^j$. Note that $P[\mathcal{F}] \geq \max\{P[\mathcal{F}^j]\}$. The membership test above allows identifying the n samples out of N to be evaluated by the indicator function. Since the MFS is used in this fashion, this variant of the hybrid method is called rejection-based. Other variants[7] of the hybrid method are available.

Since this method avoids evaluating $\boldsymbol{g}$ at all samples within the MFS, its efficiency increases with the magnitude of the parametric safety margin. The accuracy of the hybrid

method using n samples is comparable to the accuracy of the Monte Carlo method using $N = [n/\psi]$ samples, where $[x]$ is the integer for which $[x] \le x < [x] + 1$. Hence, the larger the MFS, the larger the numerical gain.

IV.B.2. Conditional Sampling

The rejection method can save a lot of computational effort on system analysis (evaluation of the constraint functions). However, if the failure probability is very small, it is still necessary to generate a very large number of samples (the N of the previous section) in order to have a sufficiently large number (the n of the previous section) of samples falling in $\mathcal{F}$ to provide a good estimate of $P[\mathcal{F}]$. The same considerations for determining N for Monte Carlo sampling apply to determining N for the rejection method. This would require an unmanageable number of samples if $P[\mathcal{F}]$ were on the order of, say, 10^{-9} as in the FAA specification mentioned earlier.

The next technique presented is *Conditional Sampling*. In this technique, only sample points outside the MFS $\mathcal{M}$ are generated. These are generated according to the portion of the distribution of the uncertain parameters which has $\mathcal{M}$ as its support (see Equation (7)). In this way, the expense of generating $N-n$ points which end up falling within $\mathcal{M}$ and of testing them for membership in $\mathcal{M}$ is avoided. The computational cost of generating each sample point using the proposed conditional sampling techniques for either the hyper-rectangular or hyper-spherical $\mathcal{M}$ can be expected to be larger than the cost of generating a sample point unconditionally from the distribution of the uncertain parameter. So conditional sampling becomes preferable to the rejection method when $P[\mathcal{M}]$ is sufficiently close to 1.

Versions are given for MFSs which are hyper-rectangles and MFSs which are hyper-spheres. The version for hyper-rectangles requires only that the components of the random vector be independent random variables, and is stated for $\boldsymbol{p}$-space. It could be applied to $\boldsymbol{u}$-space since the random variables there also have independent components. The hyper-spherical version applies only to $\boldsymbol{u}$-space hyper-spheres centered at the origin.

If components of a random vector $\boldsymbol{p}$ are *independent* random variables, then samples of the vector can be generated by sampling each component independently. The problem to be overcome with the conditional sampling techniques given here is that the conditional distributions used no longer describe independent random variables.

The two conditional sampling techniques given here are examples of what is called in the literature *importance sampling*. See Refs. [2–4] and their bibliographies. In importance sampling, the uncertain parameters are sampled, not according to the distribution f which actually describes their uncertainty, but according to a different distribution h chosen (typically) to give more samples in areas important to the analysis. In order to make the resulting analysis valid for the distribution f, the contribution of each sample point x generated according to the h distribution is weighted by the correction factor $f(x)/h(x)$. For this to be a valid procedure, care must be exercised in the choice of h; for example, consideration must be given to the support set of f and to the probabilistic estimate being accomplished by the sampling procedure. In the present case, the h distributions are the conditional distributions used for picking the conditional samples, and the ratio $f(x)/h(x)$ is constant over the range of x values of interest, and that constant is the reciprocal of the probability of some bounding

set for the failure domain. Specifically, if $\mathcal{M}^c$ is a failure bounding set and $\psi = P[\mathcal{M}^c]$:

$$h(x) = \begin{cases} 0 & x \in \mathcal{M} \\ \frac{f(x)}{\psi} & x \notin \mathcal{M} \end{cases} \tag{7}$$

Some authors have referred to this special case of importance sampling as "Partition of the Region."

Hyper-rectangles:

Recall that s has been defined to be $\dim(\boldsymbol{p})$. Here it is assumed that the components $\langle \boldsymbol{p}_1, \ldots, \boldsymbol{p}_s \rangle$ of the random vector $\boldsymbol{p}$ of uncertain parameters are independent random variables. This means that the joint PDF of $\boldsymbol{p}$ and the joint CDF of $\boldsymbol{p}$ are products of the respective marginal functions:

$$f_{\boldsymbol{p}}(\boldsymbol{p}) = f_{\boldsymbol{p}_1}(\boldsymbol{p}_1) \cdots f_{\boldsymbol{p}_s}(\boldsymbol{p}_s) \tag{8}$$

$$F_{\boldsymbol{p}}(\boldsymbol{p}) = F_{\boldsymbol{p}_1}(\boldsymbol{p}_1) \cdots F_{\boldsymbol{p}_s}(\boldsymbol{p}_s) \tag{9}$$

The MFS is a hyper-rectangle:

$$\mathcal{M} = R(\bar{\boldsymbol{p}}, \alpha \boldsymbol{m})$$

For purposes of this section, it is more convenient to represent $\mathcal{M}$ as a Cartesian product of intervals:

$$\mathcal{M} = I_1 \times \cdots \times I_s$$

where

$$I_j = [a_j, b_j], \qquad j = 1, \ldots, s$$

with $a_j = \bar{\boldsymbol{p}}_j - \alpha \boldsymbol{m}_j$ and $b_j = \bar{\boldsymbol{p}}_j + \alpha \boldsymbol{m}_j$.

The conditional PDF to be sampled from is

$$h(\boldsymbol{p}) \stackrel{\Delta}{=} f_{\boldsymbol{p}}(\boldsymbol{p} \mid \mathcal{M}^c),$$

(we write h instead of $h_{\boldsymbol{p}}$ to reduce notational clutter subsequently when subscripts are added to represent various marginal distributions) and is represented by the equation:

$$h(\boldsymbol{p}) = \begin{cases} 0 & \boldsymbol{p} \in \mathcal{M} \\ \dfrac{f_{\boldsymbol{p}}(\boldsymbol{p})}{P[\mathcal{M}^c]} & \text{otherwise} \end{cases}$$

Denote the corresponding CDF by $H(\cdot)$.

A sample point from this conditional distribution is generated one coordinate at a time, with coordinate s being generated first, then $s-1$, and so on until a value has been generated for each coordinate. To do this, it is necessary to refer to the "partial hyper-rectangles"

$$\mathcal{M}_k = I_1 \times \cdots \times I_k$$

so that $\mathcal{M}_s = \mathcal{M}$.

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Recursive formulas give the marginal probabilities of these partial hyper-rectangles and their complements. This means that probabilities of $\mathcal{M}_k$ and $\mathcal{M}_k^c$ are given with respect to the marginal distribution in k-dimensional space of the partial random vector $\langle \boldsymbol{p}_1, \ldots, \boldsymbol{p}_k \rangle$. The basis step of these recursive formulas is:

$$P[\mathcal{M}_1] = F_{\boldsymbol{p}_1}(b_1) - F_{\boldsymbol{p}_1}(a_1)$$

$$P[\mathcal{M}_1^c] = F_{\boldsymbol{p}_1}(a_1) + \left(1 - F_{\boldsymbol{p}_1}(b_1)\right)$$

For $k > 1$, the recursive step is:

$$P[\mathcal{M}_k] = P[\mathcal{M}_{k-1}]\left(F_{\boldsymbol{p}_k}(b_k) - F_{\boldsymbol{p}_k}(a_k)\right)$$

$$P[\mathcal{M}_k^c] = P[\mathcal{M}_{k-1}^c] + P[\mathcal{M}_{k-1}]\left(F_{\boldsymbol{p}_k}(a_k) + (1 - F_{\boldsymbol{p}_k}(b_k))\right)$$

This form of the formulas is chosen rather than calculating $P[\mathcal{M}_k^c]$ as $1 - P[\mathcal{M}_k]$ to avoid the numerical pitfalls of subtracting two nearly equal numbers which could arise if the MFS had probability nearly equal to 1. Of course, the same problem can arise in calculations of the form $1 - F(b)$. Any relief from numerical problems of this form must depend on specific properties of the individual CDFs. For example, if F is the standard normal distribution, $1 - F(b)$ can be calculated as $F(-b)$ to avoid the problem. Using software for the standard normal CDF from the Matlab$^{\circledR}$ Statistics Toolbox on a 64 bit computer with ANSI standard floating point arithmetic for F and $b = 8.25$, $F(-b)$ is computed to at least 12 digits accuracy while the computation represented by $1 - F(b)$ is in error by about 40%.

The sample construction process also uses the marginal distributions of h on each of the spaces formed by the first k components of $\boldsymbol{p}$. So, if

$$f_k(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_k) = f_{\boldsymbol{p}_1}(\boldsymbol{p}_1) \cdots f_{\boldsymbol{p}_s}(\boldsymbol{p}_k),$$

this marginal conditional PDF is

$$h_k(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_k) \stackrel{\Delta}{=} f_k(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_k \mid \mathcal{M}_k^c),$$

and is represented by the equations:

$$h_k(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_k) = \begin{cases} 0 & \boldsymbol{p} \in \mathcal{M}_k \\ \dfrac{f_{\boldsymbol{p}}(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_k)}{P[\mathcal{M}_k^c]} & \text{otherwise} \end{cases}$$

Denote the corresponding CDF by $H_k(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_k)$.

The generation of a given coordinate of a sample vector depends on where the previously generated coordinates lie. Suppose that the coordinates $\langle \boldsymbol{p}_{k+1}, \ldots, \boldsymbol{p}_s \rangle$ have been generated. If any one of the coordinates $\boldsymbol{p}_j \notin I_j$, $k+1 \leq j \leq s$, no matter how the remaining coordinates are picked, the sample point will fall in $\mathcal{M}^c$, as desired. This means that sample value $\boldsymbol{p}_k$ may be picked by sampling from the distribution $F_{\boldsymbol{p}_k}$. If, however, all of the coordinates $\boldsymbol{p}_j \in I_j$, $k+1 \leq j \leq s$ (and this case applies vacuously in case $k = s$), the sample must be chosen according to a distribution which apportions the proper weight to whether the sample point $\boldsymbol{p}_k$ falls within or outside I_k. This requires sampling from $H_{k,k}$ which is the marginal

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distribution of H_k for its last coordinate. $h_{k,k}(\boldsymbol{p}_k^*)$ is calculated by integrating $h_k(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_k)$ over the half-space

$$\{\langle \boldsymbol{p}_1, \ldots, \boldsymbol{p}_k \rangle \mid -\infty < \boldsymbol{p}_1 < \infty, \ldots, -\infty < \boldsymbol{p}_{k-1} < \infty, -\infty < \boldsymbol{p}_k \le \boldsymbol{p}_k^* \}.$$

This produces the formula:

$$H_{k,k}(\boldsymbol{p}_k^*) = \begin{cases} \dfrac{F_{\boldsymbol{p}_k}(\boldsymbol{p}_k^*)}{P[\mathcal{M}_k^c]} & \boldsymbol{p}_k^* < a_k \\[2ex] \dfrac{F_{\boldsymbol{p}_k}(a_k)P[\mathcal{M}_{k-1}] + F_{\boldsymbol{p}_k}(\boldsymbol{p}_k^*)P[\mathcal{M}_{k-1}^c]}{P[\mathcal{M}_k^c]} & a_k \le \boldsymbol{p}_k^* < b_k \\[2ex] \dfrac{F_{\boldsymbol{p}_k}(\boldsymbol{p}_k^*) - P[\mathcal{M}_k]}{P[\mathcal{M}_k^c]} & b_k \le \boldsymbol{p}_k^* \end{cases}$$

By using $F_{\boldsymbol{p}_k}^{-1}$, $H_{k,k}^{-1}$ may be evaluated as needed, and the required sample generated as noted previously.

As previously remarked, this technique can be applied in $\boldsymbol{u}$-space for arbitrary hyper-rectangles.

Origin centered hyper-spheres in $\boldsymbol{u}$-space:

The problem considered here is to sample from the s-dimensional standard normal distribution conditional on the sample points falling outside an origin centered hyper-sphere in $\boldsymbol{u}$-space having radius R. One sampling technique for this case has been presented in earlier work.[1] The present work is an improvement over the previous in that sample points are not confined to fall on a limited number of lines through the origin, so are distributed in a more representative fashion. The key to this is making use of the CDF Λ_s of the random variable $\|\boldsymbol{u}\|$. This is known as the χ-distribution with s degrees of freedom. Λ_s is given[1] (see also Ref. [8], p. 168, for the corresponding χ^2-distribution CDF formula) by:

$$\Lambda_s(r) = \begin{cases} \text{erf}\left(\frac{r}{\sqrt{2}}\right) - \sqrt{\frac{2}{\pi}}\left(\frac{r^{s-2}}{(s-2)!!} + \frac{r^{s-4}}{(s-4)!!} + \cdots + \frac{r}{1!!}\right)e^{-r^2/2} & \text{if } r \ge 0, \ s \text{ odd} \\[2ex] 1 - \left(\frac{r^{s-2}}{(s-2)!!} + \frac{r^{s-4}}{(s-4)!!} + \cdots + \frac{r^2}{2!!} + 1\right)e^{-r^2/2} & \text{if } r \ge 0, \ s \text{ even} \\[2ex] 0 & \text{otherwise} \end{cases}$$

Here, $n!!$ is the double factorial, defined as:

$$n!! = \begin{cases} n \cdot (n-2) \cdots 5 \cdot 3 \cdot 1 & n > 0 \text{ and odd} \\ n \cdot (n-2) \cdots 6 \cdot 4 \cdot 2 & n > 0 \text{ and even} \\ 1 & n = -1, 0 \end{cases}$$

The conditional distribution of the magnitude of the s-dimensional standard normal random vector $\boldsymbol{u}$ subject to the condition that $r \overset{\Delta}{=} \|\boldsymbol{u}\| > R$ is given by:

$$\Lambda_{s,R}(r) = \begin{cases} 0 & r \le R \\[2ex] \dfrac{\Lambda_s(r) - \Lambda_s(R)}{1 - \Lambda_s(R)} & r > R \end{cases}$$

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To generate n samples of the standard normal random vector of dimension s, first generate n unit vectors uniformly distributed over the surface of the unit sphere. One way to accomplish that is to first generate n vectors $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ in s-dimensional Euclidean space which are sampled from the standard normal distribution. The rows of the matrix generated by the Matlab® command `v = randn(n,s);` can be used here. Then normalize them by setting $\boldsymbol{w}_i = \boldsymbol{v}_i/\|\boldsymbol{v}_i\|$. Then sample n numbers $\{r_1, \ldots, r_n\}$ from the distribution $\Lambda_{s,R}$. The desired conditional sample is then calculated as $\boldsymbol{u}_i = r_i \boldsymbol{w}_i$, $1 \le i \le n$.

To calculate $\{r_1, \ldots, r_n\}$, one can start with numbers $\{x_1, \ldots, x_n\}$ sampled from the uniform distribution on the real interval $(0, 1)$, and solve each of the (non-linear) equations $\Lambda_{s,R}(r_i) = x_i$ for r_i. This can be accomplished using available software (for example, in the Matlab® Statistics Toolbox) for the inverse of the χ^2 CDF.

IV.B.3. Confidence Intervals

A few remarks on the numerical features of the hybrid method are now in order. If $\hat{P}$ is a sampling-based estimate of $P[\mathcal{F}]$, the corresponding $100(1-\alpha)\%$ confidence interval $[L, U]$, satisfies

$$P[L < \hat{P} < U] = 1 - \alpha.$$

Confidence intervals relate $\hat{P}$ with $P[\mathcal{F}]$ by bounding the true, unknown value of the approximation with defined certainty. This technique is illustrated in the literature.[9] Here, confidence interval studies for traditional Monte Carlo are compared with those for the hybrid method. A model problem is chosen with a two dimensional vector of uncertain parameters having a standard normal distribution; i.e., a two dimensional $\boldsymbol{u}$-space. There is a single constraint function which is affine; i.e., linear plus a constant. With this simple model, the probability of failure can be determined analytically, and can be adjusted to any desired level by adjusting the coefficients in the constraint function. Since $\hat{P}$ is a binomial random variable, we can solve for L and U after applying the binomial test.

For the first study, the constraint function coefficients are chosen so that the MFS $\mathcal{M}$ is an origin centered circle (2-dimensional hypersphere) of radius is 3.25. This dictates that $P[\mathcal{M}] = 0.9949$, $P[\mathcal{M}^c] = 5.086 \times 10^{-3}$, and $P[\mathcal{F}] = 5.72 \times 10^{-4}$. Figure 2 illustrates how the width of the confidence interval varies with the required number of $\boldsymbol{g}$ evaluations, namely n, for this problem, using both Monte Carlo sampling and the hybrid method.

The computational improvements of the hybrid method are apparent. The improvement in efficiency, denoted by δn, is given by the horizontal distance between the two curves for a fixed $U - L$ value. δn at a given $U - L$ value is the additional number of samples a Monte Carlo simulation requires to get the same confidence interval width. The improvement in accuracy, denoted by δa, is given by the vertical distance between the curves for a fixed number of $\boldsymbol{g}$ evaluations. δa at a given n value is how much wider the confidence interval of the Monte Carlo estimate is as compared to the confidence interval of the hybrid method. For the particular case shown, improvements of about two orders of magnitude in δn and δa are attained.

Metrics for evaluating the relative improvements in efficiency and accuracy, denoted by η_n and η_a hereafter, are introduced next. For a fixed $U - L$ value, the number of Monte Carlo samples at which the constraint function must be evaluated is η_n times as large as the corresponding number for the hybrid method. On the other hand, for a given value of n,
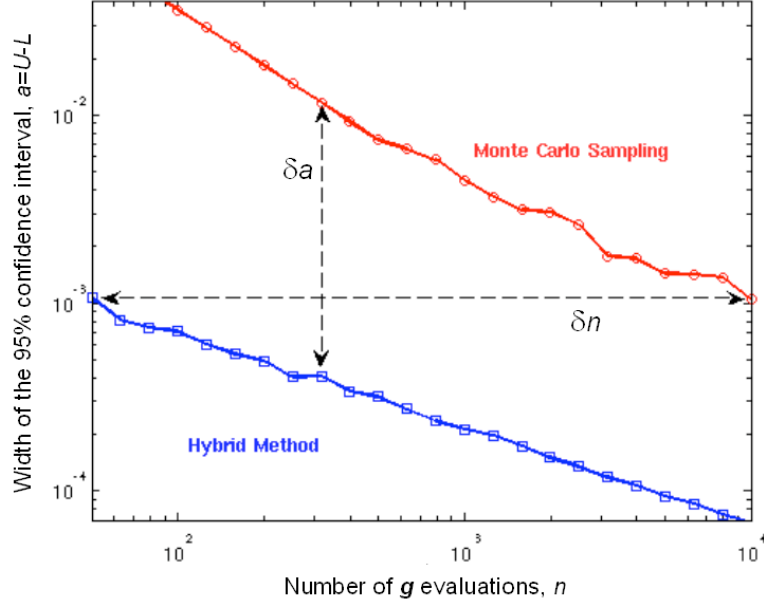
Figure 2. Number of *g* evaluations vs. width of the 95% confidence interval for $P[\mathcal{F}] = 5.72 \times 10^{-4}$.

the width of the confidence interval of the Monte Carlo estimate is η_a times as large as the width of the confidence interval for the Hybrid Method.

Figure 3 illustrates the dependence of η_n and η_a on $P[\mathcal{F}]$. In the calculation of η_n, we use $\alpha = 0.05$ and $U - L = P[\mathcal{F}]/10$ for $P[\mathcal{F}] < 0.5$, and $U - L = (1 - P[\mathcal{F}])/10$ otherwise. Hence, we set the width of 95% confidence interval to be 10% of the failure probability. In the calculation of η_a we assume $n = 1000$. It can be seen that the numerical gain of the hybrid method for both η_n and η_a increases with the distance of $P[\mathcal{F}]$ from 0.5. While η_a reaches 6 when $P[\mathcal{F}] = 0.005$, η_n grows even faster reaching 16.8 at $P[\mathcal{F}] = 0.01$. In general, the numerical improvements are proportional to the size of the MFS. Such improvements are substantial when $P[\mathcal{F}]$ is relatively large or relatively small, the latter being the targeted range in practical, safety-driven applications. When $\mathcal{M} = \emptyset$, which is the worst case, the hybrid method is equivalent to Monte Carlo. In the last two figures we have purposely left out the number of ***g*** evaluations required to find the CPV since this number depends on the initial condition of the numerical search. For this representative problem the number of ***g*** evaluations to find the CPV ranged from 9 to 31 and averaged 15. As compared to the number of function evaluations due to sampling, this contribution to n is practically negligible. Even though the same trends have been observed in large dimensional problems, a formal analysis of the method is beyond the scope of this paper.

V. Example

In order to be able to evaluate the results of a conditional sampling numerical experiment, a simple model is chosen which allows analytic determination of $P[\mathcal{F}]$. The uncertain parameter vector is a 4-dimensional standard normal random vector. A single constraint
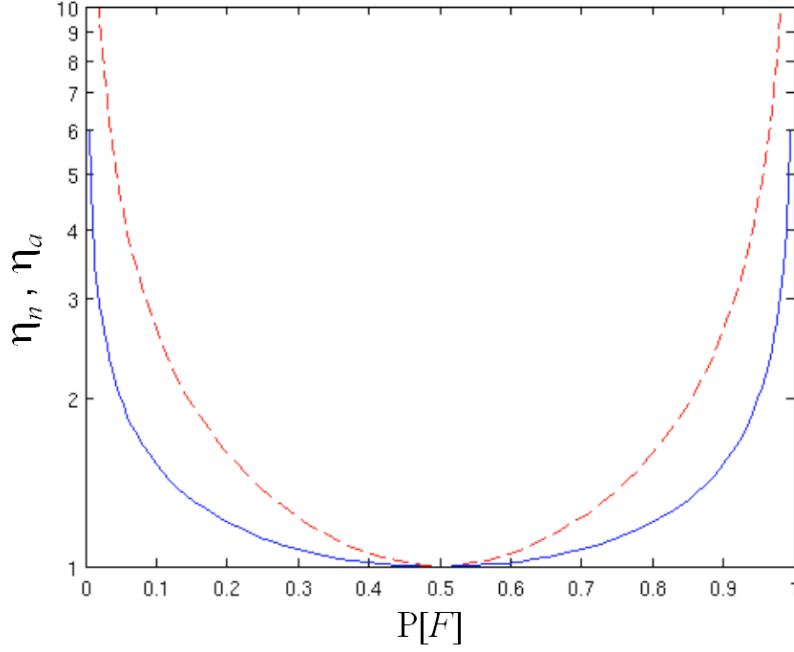
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Figure 3. $P[\mathcal{F}]$ vs. η_n (dashed line) and η_a (solid line).

function is used, $G(\boldsymbol{u}) = \langle \boldsymbol{a}, \boldsymbol{u} \rangle - b$ where $\langle \boldsymbol{a}, \boldsymbol{u} \rangle$ denotes the vector inner product of $\boldsymbol{a}$ and $\boldsymbol{u}$, $\boldsymbol{a}$ is chosen to be $(10, 1, 1, 1)$, and $\boldsymbol{b} \triangleq \beta \|\boldsymbol{a}\|$ where $\beta = 5.9978$. These choices make $P[\mathcal{F}] = 10^{-9}$. This probability is so small that it is not practical to estimate using full space sampling methods such as the hybrid rejection method or the Monte Carlo method. For example, it would take 1,442,695,000 random samples to have a 50% chance of finding even one failure point. To estimate this $P[\mathcal{F}]$ with any significance would take tens or hundreds of billions of random samples. This is not currently practical.

However, $P[\mathcal{F}]$ can be estimated using the spherical MFS version of the conditional sampling technique with reasonable precision with much fewer sample points. In this case, $\mathcal{M} = S_{\boldsymbol{u}}(\boldsymbol{0}, \beta)$, so $P[\mathcal{M}^c] = 293.0 \times 10^{-9}$. Thus, it would be expected that about one in every 293 conditional samples is a failure point. The failure probability can be determined with a manageable number of sample points.

The conditional sampling estimate of $P[\mathcal{F}]$ was made using a conditional sample of 100,000 points. A total of 100 independent repetitions of the sampling estimate were made. A plot, shown in Fig. 4, of the relative errors between the estimates and the true failure probability shows that the Conditional Sampling technique has captured the failure probability to a reasonable level. For example, 92 of the 100 trials show a relative error within the $\pm 10\%$ range and the worst relative error is about 14%.

Confidence interval analyses at the 95% confidence level for each of these 100 trials shows that the true failure probability falls within the confidence interval in 93 of the trials.

On a desktop system running Matlab® 7.4.0.336 (R2007a) on an Intel® Pentium® 4 CPU 3.40GHz computer, it took about 6 minutes to generate and analyze the 10 million sample points. One must note that the analysis in this simple example is computationally negligible. It is also of interest to note that, if all 10 million samples are analyzed as a single sample set,
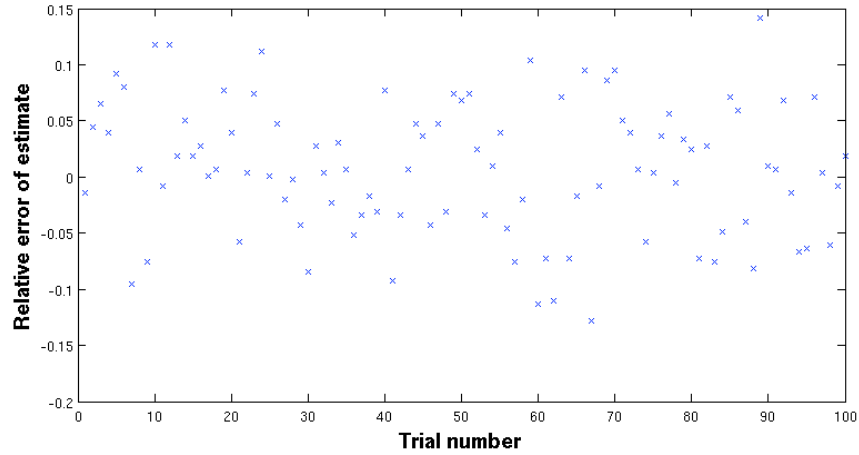
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Figure 4. Relative error in Conditional Sampling estimates of $P[\mathcal{F}]$.

the resulting estimate of failure probability is 0.9926×10^{-9} for a relative error of -0.74%. The 95% confidence interval for the 10 million point sample is $[0.982 \times 10^{-9}, 1.003 \times 10^{-9}]$. Note that, to get equivalent results to this 10 million point conditional sampling analysis using the Monte Carlo method or even the hybrid rejection method would require about 34.13 trillion sample points. Based on generating and analyzing 10 million points every 6 minutes, this would take about 39 years.

VI. Summary

Techniques from the literature have been reviewed for using optimization methods to find upper bounds of failure probabilities by bounding the failure set (constraint violation set) by a geometrically simple set whose probability can be calculated analytically. We have shown how to use that failure bounding set to reduce the computational burden for estimating failure probability by sampling the uncertain parameters. Some of these techniques may be used in p-space while previous methods were limited to u-space. Improvements have been made to previous u-space methods. The rejection method generates a full uncertainty set sample, but rejects any points which fall outside the failure domain bounding set before subjecting the remaining points to the analysis necessary to determine of they are constraint violation points. Conditional sampling techniques are given in which only sample points within the failure domain bounding set are generated, saving the expense of generating points outside the failure domain bounding set. Probability estimates based on these samples constitute an application of importance sampling. It is shown how the use of either technique based on failure domain bounding improves the ratio of confidence in the failure probability approximation per sample point subject to full system analysis. Finally, an example shows that Conditional Sampling can be enabling technology in case of need to estimate extremely small probabilities.

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