NASA/TP-2008-215543

Differential Cross Section Kinematics for 3-dimensional Transport Codes

John W. Norbury
Langley Research Center, Hampton, Virginia

Frank Dick
Worcester Polytechnic Institute, Worcester, Massachusetts

November 2008

The NASA STI Program Office . . . in Profile

Since its founding, NASA has been dedicated to the advancement of aeronautics and space science. The NASA Scientific and Technical Information (STI) Program Office plays a key part in helping NASA maintain this important role.

The NASA STI Program Office is operated by Langley Research Center, the lead center for NASA's scientific and technical information. The NASA STI Program Office provides access to the NASA STI Database, the largest collection of aeronautical and space science STI in the world. The Program Office is also NASA's institutional mechanism for disseminating the results of its research and development activities. These results are published by NASA in the NASA STI Report Series, which includes the following report types:

- TECHNICAL PUBLICATION. Reports of completed research or a major significant phase of research that present the results of NASA programs and include extensive data or theoretical analysis. Includes compilations of significant scientific and technical data and information deemed to be of continuing reference value. NASA counterpart of peer-reviewed formal professional papers, but having less stringent limitations on manuscript length and extent of graphic presentations.

- TECHNICAL MEMORANDUM. Scientific and technical findings that are preliminary or of specialized interest, e.g., quick release reports, working papers, and bibliographies that contain minimal annotation. Does not contain extensive analysis.

- CONTRACTOR REPORT. Scientific and technical findings by NASA-sponsored contractors and grantees.

- CONFERENCE PUBLICATION. Collected papers from scientific and technical conferences, symposia, seminars, or other meetings sponsored or co-sponsored by NASA.

- SPECIAL PUBLICATION. Scientific, technical, or historical information from NASA programs, projects, and missions, often concerned with subjects having substantial public interest.

- TECHNICAL TRANSLATION. English-language translations of foreign scientific and technical material pertinent to NASA's mission.

Specialized services that complement the STI Program Office's diverse offerings include creating custom thesauri, building customized databases, organizing and publishing research results ... even providing videos.

For more information about the NASA STI Program Office, see the following:

- Access the NASA STI Program Home Page at ***http://www.sti.nasa.gov***

- E-mail your question via the Internet to help@sti.nasa.gov

- Fax your question to the NASA STI Help Desk at (301) 621-0134

- Phone the NASA STI Help Desk at (301) 621-0390

- Write to:
  NASA STI Help Desk
  NASA Center for AeroSpace Information
  7115 Standard Drive
  Hanover, MD 21076-1320

NASA/TP-2008-215543

Differential Cross Section Kinematics for 3-dimensional Transport Codes

John W. Norbury
Langley Research Center, Hampton, Virginia

Frank Dick
Worcester Polytechnic Institute, Worcester, Massachusetts

National Aeronautics and
Space Administration

Langley Research Center
Hampton, Virginia 23681-2199

November 2008

Available from:

NASA Center for AeroSpace Information (CASI)
7115 Standard Drive
Hanover, MD 21076-1320
(301) 621-0390

National Technical Information Service (NTIS)
5285 Port Royal Road
Springfield, VA 22161-2171
(703) 605-6000

Nomenclature

3-vectors are denoted by bold face quantities such as $\mathbf{p}$.

4-vectors are denoted by non-bold face quantities such as p.

cm is the abbreviation for the center of momentum frame.

The lab frame is always *identical* to the target frame, which is typically the spacecraft frame..

$*$ refers to *either* the cm or projectile frames.

Subscript c refers to the cm frame.

Subscript p refers to the projectile frame.

Subscript l or lab refers to the lab frame.

If equations do not have a $*$, c, p, or l subscript, then it means the equation is true in any frame.

E_* or $\mathbf{p}_*$ or θ_* refer to the energy or momentum or angle *of an arbitrary particle* in the $*$ frame.

E_c or $\mathbf{p}_c$ or θ_c refer to the energy or momentum or angle *of an arbitrary particle* in the cm frame.

E_p or $\mathbf{p}_p$ or θ_p refer to the energy or momentum or angle *of an arbitrary particle* in the projectile frame.

E_l or $\mathbf{p}_l$ or θ_l refer to the energy or momentum or angle *of an arbitrary particle* in the lab frame.

In general, x_* refers to the value of the x quantity *of an arbitrary particle* in the * frame, and similarly for the cm and projectile frames.

E_{j*} or $\mathbf{p}_{j*}$ refers to the energy or momentum of particle j in the $*$ frame, and similarly for the cm and projectile frames.

In general, x_{j*} refers to the value of the x quantity *of particle* j in the $*$ frame, and similarly for the cm and projectile frames.

$\theta jk*$ refers to the angle of the momentum of particle j with respect to the momentum of particle k. The angle is measured in the $*$ frame, and similarly for the cm and projectile frames.

Often, just θ_j or θ_{jl} or θ_{jc} will be written. This is used when we are referring to the angle of particle j but have not yet needed to decide what the angle of particle j is with respect to.

Consider $1+2 \to 3+4$. In the cm frame, $\mathbf{p}_{1c}+\mathbf{p}_{2c} = 0 = \mathbf{p}_{3c}+\mathbf{p}_{4c}$ implying $|\mathbf{p}_{1c}| = |\mathbf{p}_{2c}| \equiv |\mathbf{p}_{ic}|$ and $|\mathbf{p}_{3c}| = |\mathbf{p}_{4c}| \equiv |\mathbf{p}_{fc}|$. E_{ic}, E_{fc} or $\mathbf{p}_{ic}, \mathbf{p}_{fc}$ refer to the energies or momenta of the initial (i.e. particles 1 or 2) or final (i.e. particles 3 or 4) particles in the cm frame.

β_* or γ_* refer to the speed (in units of c) or relativistic γ factor *of an arbitrary particle in the $*$ frame*. They are related via $\gamma^2 = \frac{1}{1-\beta^2}$, and similarly for the cm and projectile frames.

β_{j*} or γ_{j*} refer to the speed or relativistic γ factor *of particle j in the $*$ frame*, and similarly for the cm and projectile frames.

β_{*l} or γ_{*l} refer to the speed or relativistic γ factor *of the $*$ frame with respect to the lab frame*, and similarly for the cm and projectile frames. (Notice that no particular particle is involved here. It's just one frame with respect to another.)

The quantity $\alpha_* \equiv \frac{\beta_{*l}}{\beta_*}$ is defined as the speed of the $*$ frame with respect to the lab divided by the speed of a particular particle in the $*$ frame, and similarly for the cm and projectile frames.

The quantity $\alpha_{j*} \equiv \frac{\beta_{*l}}{\beta_{j*}}$ is defined as the speed of the $*$ frame with respect to the lab divided by the speed of particle j in the $*$ frame, and similarly for the cm and projectile frames.

In general, x_0 refers to zeroth component of the 4-vector x where the components are written $x^\mu = (x^0, \mathbf{x}) = (x_0, \mathbf{x})$. (The 0 is not a particle label. Particle labels are subscripts 1, 2, 3, 4, etc.)

Mandelstam variable $s \equiv (p_1 + p_2)^2$.

Contents

List of Figures

Abstract

In support of the development of 3-dimensional transport codes, this paper derives the relevant relativistic particle kinematic theory. Formulas are given for invariant, spectral and angular distributions in both the lab (spacecraft) and center of momentum frames, for collisions involving 2, 3 and n - body final states.

1 Introduction

One of the main new challenges in the development of radiation transport codes, such as HZETRN, is to provide for fully 3 - dimensional transport. This requires 3 - dimensional cross sections as input. Typically, these differential cross sections are calculated most easily in the center of momentum frame for nucleon - nucleon collisions, or the projectile frame for nucleus - nucleus reactions. However, transport codes require these cross sections in the lab frame. Even though there is extensive literature on transforming these cross sections from one frame to another, there is no reference that provides a complete discussion of this problem or includes special problems that need to be considered for radiation transport. One very common problem encountered in the literature is that lab differential cross sections are often written in terms of center of momentum or projectile variables. This is not useful for radiation transport because everything must be written in terms of lab variables. A major portion of the present paper is devoted to this problem. Significant complications arise because double - valued functions arise and these must be handled with extreme care. For example, two different angles in the center of momentum frame can correspond to a single angle in the lab frame. Another significant problem is that infinities can arise in the transformation of angular distributions to the lab frame. These infinities are rarely discussed in the literature. They are given significant attention in the present work. The aim of this paper is to provide a thorough analysis of relativistic transformations of differential cross sections, so that HZETRN can be upgraded for 3 - dimensional transport.

This section discusses the formulation of differential cross sections in terms of scattering amplitudes, along with the various kinematic factors that are present. The types of differential cross sections that may be formed for 2 and 3 - body final states are also discussed.

1.1 S-matrix and invariant amplitude

The invariant amplitude is determined from the scattering matrix (often referred to as the S-matrix), and along with appropriate phase space factors, is used to determine the decay rate $d\Gamma$ and cross section $d\sigma$ differentials. Cross sections and decay rates are fundamentally calculated from a quantity called the scattering matrix or S-matrix, which is simply the time evolution operator for future and past times going to positive and negative infinity, $S \equiv \mathcal{U}(+\infty, -\infty)$. The Dyson expansion of the S-matrix is

$$\begin{aligned} S &= 1 + \sum_{n=1}^{\infty} \frac{1}{n!}(-i)^n \int d^4x_1 d^4x_2 \cdots d^4x_n \; T[\mathcal{H}_I(x_1)\mathcal{H}_I(x_2)\cdots\mathcal{H}_I(x_n)] \\ &\equiv Te^{-i\int d^4x\, \mathcal{H}} = Te^{-i\int_{-\infty}^{\infty} dt\, H} \;, \end{aligned} \tag{1}$$

where T is the time ordering operator, H is the Hamiltonian and $\mathcal{H}$ is the Hamiltonian density. Cross sections and decay rates are usually expressed in terms of an intermediate quantity called the invariant amplitude $\mathcal{M}$, which is related to the S-matrix. For the reaction $1+2 \to 3+4$, the relation is given by the Particle Data Group [1] and Griffiths [2] as

$$\langle f|S|i\rangle \equiv \langle p_3 p_4|S|p_1 p_2\rangle = 1 - i\mathcal{M}(2\pi)^4\delta^4(p_1+p_2-p_3-p_4)\frac{1}{\sqrt{2E_1 2E_2 2E_3 2E_4}} \;, \tag{2}$$

where p is the 4-momentum, E is the total energy, and the initial and final states are $|i\rangle \equiv (p_1 p_2\rangle$ and $|f\rangle \equiv (p_3 p_4\rangle$. *With the above definition, the Feynman rules give* $-i\mathcal{M}$. In the above equation, the states are normalized according to

$$\langle p'|p\rangle = (2\pi)^3\delta^3(\mathbf{p}-\mathbf{p}') \;. \tag{3}$$

Peskin defines the Transition matrix T (not to be confused with time ordering operator in the Dyson expansion) according to

$$S \equiv 1 + iT \;, \tag{4}$$

and uses a different definition [3] (p. 104),

$$\langle p_1 \ldots p_n|iT|p_A p_B\rangle = i\mathcal{M}(2\pi)^4\delta^4(p_A+p_B-\Sigma p_f) \;, \tag{5}$$

giving

$$\langle f|S|i\rangle \equiv \langle p_1 \ldots p_n|S|p_A p_B\rangle = 1 + i\mathcal{M}(2\pi)^4\delta^4(p_A+p_B-\Sigma p_f) \;. \tag{6}$$

With the above Peskin definition, the Feynman rules give $+i\mathcal{M}$. In the above equation, the states are normalized according to Peskin [3] (p. 23)

$$\langle \mathbf{p}|\mathbf{q}\rangle = 2E_{\mathbf{p}}(2\pi)^3\delta^3(\mathbf{p}-\mathbf{q}) \;. \tag{7}$$

For Dirac spinor states, Peskin [3] (p. 59) uses a similar normalization namely,

$$\langle \mathbf{p}, r|\mathbf{q}, s\rangle = 2E_{\mathbf{p}}(2\pi)^3\delta^3(\mathbf{p}-\mathbf{p}')\delta^{rs} \;. \tag{8}$$

Finally, note that Greiner uses a definition with different normalization for scalar and spinor states. See references [4] (p. 267), [5] (p. 221);

$$\langle f|S|i\rangle = i\mathcal{M}(2\pi)^4\delta^4(p_1+p_2-\sum_{i=1}^{n}p_i')\prod_{i=1}^{2}\sqrt{\frac{N_i}{2E_i(2\pi)^3}}\prod_{i=1}^{n}\sqrt{\frac{N_i'}{2E_i'(2\pi)^3}} \;, \tag{9}$$

where the normalization factors are $N_i = 1$ for scalar bosons and $N_i = 2m$ for fermions. See reference [4] (p. 267).

1.2 Phase space

The phase space of a reaction contains all allowed momentum states of the final state particles. The phase space factor $d\Phi_n$ is used in expressions for $d\Gamma$ and $d\sigma$. The decay rate Γ and total cross section σ are obtained by integrating over phase space.

1.2.1 Units and the invariant amplitude

In this paper, natural units are used, with $\hbar = c = 1$, which means that both length and time have units of GeV^{-1} and mass has units of GeV. Thus, cross sections σ have units of GeV^{-2}. The decay width Γ is related to the lifetime τ of a particle by

$$\tau = \frac{1}{\Gamma} \ . \tag{10}$$

In natural units, the decay width has units of GeV giving the correct units of time, GeV^{-1}, for the lifetime. The invariant amplitude $\mathcal{M}$ used below has units [2] (p. 200) of $\mathrm{GeV}^{4-\mathrm{n}}$ where n is the number of external particles involved in a reaction. The number of external particles is equal to the number of external lines in the corresponding Feynman diagram. For example, when one particle decays into two particles, the number of external lines in the Feynman diagram is 3, and the units of $\mathcal{M}$ are GeV. For a 2 - body reaction producing two bodies in the final state, there are 4 external lines and so $\mathcal{M}$ is dimensionless.

1.2.2 $d\Gamma$ and $d\sigma$ in terms of the phase space factor

The partial decay width (rate) of a particle of rest mass m decaying into n other particles is given by [1]

$$d\Gamma = \frac{\mathcal{S}}{2m} |\mathcal{M}|^2 (2\pi)^4 d\Phi_n(P; p_1, \cdots, p_n) \ , \tag{11}$$

or, in compact form as

$$d\Gamma = \frac{\mathcal{S}}{2m} |\mathcal{M}|^2 (2\pi)^4 d\Phi_n \ , \tag{12}$$

where $\mathcal{M}$ is the invariant amplitude and $\mathcal{S}$ is a product [2] (p. 376) of statistical factors, $1/j!$ for each group of j identical particles in the final state, i.e. $\mathcal{S} = \Pi_a \frac{1}{j_a!}$. These statistical factors are discussed in references [3] (pp. 108, 151), [6] (p. 259), [7] (p. 110), [8] (p. 200), [9] (p. 285) and [10] (p. 99). Gross [6] (p. 259) gives the most extensive discussion concerning the statistical factor versus limiting the regions of integration. The phase space factor is [1]

$$d\Phi_n(P; p_1, \cdots, p_n) = \delta^4(P - \sum_{i=1}^{n} p_i) \prod_{i=1}^{n} \frac{d^3 p_i}{(2\pi)^3 2E_i} \ , \tag{13}$$

where n is the number of final state particles, and

$$P = \sum_{i=1}^{k} p_i \ , \tag{14}$$

where k is the number of initial state particles. Equation (13) can be generated recursively [1],

$$d\Phi_n(P; p_1, \cdots, p_n) = d\Phi_j(q; p_1, \cdots, p_j) d\Phi_{n-j+1}(P; q, p_{j+1}, \cdots, p_n)(2\pi)^3 dq^2 \ , \tag{15}$$

where

$$q^2 = (\sum_{i=1}^{j} E_i)^2 - |\sum_{i=1}^{j} \mathbf{p}_i|^2 \ . \tag{16}$$

Consider the particle reaction,

$$1 + 2 \ \rightarrow \ 3 + 4 + \cdots + (n+2) \ , \tag{17}$$

where the numbers label the particles. The microscopic differential cross section is [1]

$$\begin{aligned} d\sigma &= \frac{\mathcal{S}}{4F} |\mathcal{M}|^2 (2\pi)^4 d\Phi_n(p_1 + p_2; p_3, \cdots, p_{n+2}) \\ &= \frac{\mathcal{S}}{4F} |\mathcal{M}|^2 (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4 \ldots - p_{n+2}) \\ &\quad \times \frac{d^3p_3}{(2\pi)^3 2E_3} \frac{d^3p_4}{(2\pi)^3 2E_4} \cdots \frac{d^3p_{n+2}}{(2\pi)^3 2E_{n+2}} \ , \end{aligned} \tag{18}$$

or, in compact form [10] (p. 99),

$$d\sigma = \frac{\mathcal{S}}{4F} |\mathcal{M}|^2 (2\pi)^4 d\Phi_n \ , \tag{19}$$

where, again, $\mathcal{S}$ is a product [2] (p. 376) [10] (p. 99) of statistical factors, $1/j!$ for each group of j identical particles in the final state. The denominator flux factor F is defined as [10] (p. 98), [11]

$$F \ \equiv \ \sqrt{(p_1.p_2)^2 - m_1^2 m_2^2} \ , \tag{20}$$

where $p_1.p_2$ denotes a 4-vector product. F can be written,

$$F_{\text{any frame}} = \frac{1}{2} \sqrt{\lambda_{12}} \ , \tag{21}$$

where

$$\begin{aligned} \lambda_{jk} &\equiv \lambda(s, m_j^2, m_k^2) \\ &\equiv (s - m_j^2 - m_k^2)^2 - 4 m_j^2 m_k^2 \ , \end{aligned} \tag{22}$$

with the Mandelstam variable,

$$s \equiv (p_1 + p_2)^2 \ . \tag{23}$$

Note that

$$\lambda_{jk} = \lambda_{kj} \ . \tag{24}$$

Also, note that [10] (p. 98),

$$F_l = m_2 \, |\mathbf{p}_{1l}| \ , \tag{25}$$

and

$$F_c = \sqrt{s} \, |\mathbf{p}_{1c}| = (E_1 + E_2)|\mathbf{p}_{1c}| \ . \tag{26}$$

These equations agree with equation (5.113) in reference [11] and equations (4.52) and (4.53) in reference [10]. Note that F can be written in terms of the relative velocity $v_{rel} = |v_1 - v_2|$. This method is used in references [3] (pp. 105, 106) and [12] (pp. 17, 18). In the above equation, the 4-dimensional delta function is

$$\delta^4(p - p_0) \quad \equiv \quad \delta(E - E_0)\delta^3(\mathbf{p} - \mathbf{p}_0) \ , \tag{27}$$

or, for example,

$$\delta^4(p_1 + p_2 - p_3 - p_4) \quad = \quad \delta(E_1 + E_2 - E_3 - E_4)\delta^3(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_3 - \mathbf{p}_4) \ , \tag{28}$$

where $\delta^3(\mathbf{p} - \mathbf{p}_0)$ cancels a 3 - dimensional integral, with $f(\mathbf{p})$ being an arbitrary function, as

$$\int d^3p f(\mathbf{p}) \, \delta^3(\mathbf{p} - \mathbf{p}_0) \equiv f(\mathbf{p}_0) \ , \tag{29}$$

or

$$\int d^3p \, \delta^3(\mathbf{p} - \mathbf{p}_0) \equiv 1 \ . \tag{30}$$

1.3 Differential cross sections

From equation (18), it can be seen that, in general, one will be able to form Lorentz invariant differential cross sections, such as $d^3\sigma/(d^3p/E)$, by bringing the relevant phase space factor to the left hand side. From such Lorentz invariant cross sections, one can form doubly differential cross sections such as,

$$\frac{d^2\sigma}{dEd\Omega} = |\mathbf{p}|\frac{d^3\sigma}{d^3p/E} \ , \tag{31}$$

which follows from

$$\frac{d^3p}{dE} = \frac{\mathbf{p}^2}{E} d|\mathbf{p}| d\Omega \; , \tag{32}$$

and

$$\mathbf{p}^2 = E^2 - m^2 \; , \tag{33}$$

and

$$2|\mathbf{p}| \; d|\mathbf{p}| = 2E \; dE \; . \tag{34}$$

1.3.1 2 - body final state

For a 2 - body final state, such doubly differential cross sections are meaningless, since E and θ are functions of each other. Therefore, either $d\sigma/dE$ or $d\sigma/d\Omega$ may be formed, but not $d^2\sigma/dE \; d\Omega$ and therefore, not $Ed^3\sigma/d^3p$. Consider the reaction,

$$1 + 2 \rightarrow 3 + 4 \; , \tag{35}$$

where the numbers represent particles. From equation (18), the cross section is of the form,

$$d\sigma \propto |\mathcal{M}|^2 \delta(E_1 + E_2 - E_3 - E_4)\delta^3(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_3 - \mathbf{p}_4)\frac{d^3p_3}{(2\pi)^3 2E_3}\frac{d^3p_4}{(2\pi)^3 2E_4} \; . \tag{36}$$

The delta function, $\delta^3(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_3 - \mathbf{p}_4)$, cancels the integral over d^3p_3, resulting in,

$$d\sigma \propto |\mathcal{M}|^2 \delta(E_1 + E_2 - E_3 - E_4)\frac{1}{(2\pi)^3 2E_3}\frac{d^3p_4}{(2\pi)^3 2E_4} \; . \tag{37}$$

However, since $d^3p_4 = p_4^2 dp_4 d\Omega_4 \propto dE_4 d\Omega_4$, and the energy delta function cancels the integral over dE_4, the result is

$$d\sigma \propto |\mathcal{M}|^2 d\Omega_4 \; . \tag{38}$$

Thus, for a 2 - body final state, one can form the single differential cross section $d\sigma/d\Omega_4$, or *alternatively* $d\sigma/dE_4$, because energy is a function of angle.

Finally, note that for a 2 - body final state, the angular distribution $d\sigma/d\Omega$ can be formed in both the lab and cm frames. However, the spectral distributions $d\sigma/dE$ and $d\sigma/dT$, where T is kinetic energy, can *only* be formed in the lab frame and *not* in the cm frame. The reason is that both T and $E = m + T$ are fixed in the cm frame, but vary with angle in the lab frame. An exception to the above statement is if the produced particle has a variable mass. An example is the Δ particle. (See reference [13].) However, a variable mass implies that the particle decays, which means that ultimately the final state is not a 2 - body state.

1.3.2 3 - body final state

Now, consider the 3 - body final state,

$$1 + 2 \rightarrow 3 + 4 + 5 \ . \tag{39}$$

From equation (18), the cross section is of the form,

$$\begin{aligned} d\sigma \propto |\mathcal{M}|^2 \delta(E_1 + E_2 - E_3 - E_4 - E_5)\delta^3(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_3 - \mathbf{p}_4 - \mathbf{p}_5) \\ \times \frac{d^3p_3}{(2\pi)^3 2E_3} \frac{d^3p_4}{(2\pi)^3 2E_4} \frac{d^3p_5}{(2\pi)^3 2E_5} \ , \end{aligned} \tag{40}$$

which becomes,

$$d\sigma \propto |\mathcal{M}|^2 \delta(E_1 + E_2 - E_3 - E_4 - E_5) \frac{1}{(2\pi)^3 2E_3} \frac{d^3p_4}{(2\pi)^3 2E_4} \frac{d^3p_5}{(2\pi)^3 2E_5} \ , \tag{41}$$

because the momentum delta function cancels an integral. The energy delta function cancels another integral giving,

$$d\sigma \propto |\mathcal{M}|^2 \frac{1}{(2\pi)^3 2E_3} \frac{d\Omega_4}{(2\pi)^3 2E_4} \frac{d^3p_5}{(2\pi)^3 2E_5} \ , \tag{42}$$

or

$$d\sigma \propto |\mathcal{M}|^2 d\Omega_4 \frac{d^3p_5}{E_5} \ , \tag{43}$$

so that the differential cross section *can* now be formed as,

$$\frac{d^3\sigma}{d^3p_5/E_5} \propto \int |\mathcal{M}|^2 d\Omega_4 \ . \tag{44}$$

1.3.3 Unit conversion to mb

As shown above, the units of $d\sigma/dt$ are GeV^{-4}. These units are converted to $\mathrm{mb}/\mathrm{GeV}^2$ by the conversion factor $\mathrm{GeV}^{-2} = 0.389379$ mb.

1.4 Proof that the differential cross section $Ed^3\sigma/d^3p$ is Lorentz invariant

A proof of the Lorentz invariance of $Ed^3\sigma/d^3p$ can also be found in reference [14] (p. 52). Differentiating $p^2 = E^2 - \mathbf{p}^2$ gives,

$$d^4p = dE \ d^3\mathbf{p} \ . \tag{45}$$

Expanding $\delta(p^2 - m^2)d^4p$ gives,

$$\delta(p^2 - m^2)d^4p = \delta(E^2 - \mathbf{p}^2 - m^2)dE \ d^3\mathbf{p} \ . \tag{46}$$

Applying the formula

$$\delta(f(E)) = \sum_i \frac{1}{f'(E)} \delta(E - E_i) \; , \tag{47}$$

with $f(E) = (E + \sqrt{\mathbf{p}^2 + m^2})(E - \sqrt{\mathbf{p}^2 + m^2})$, and keeping only the positive E root, gives

$$\delta(p^2 - m^2) d^4 p = \delta(E - \sqrt{\mathbf{p}^2 + m^2}) dE \frac{d^3\mathbf{p}}{2E} \; . \tag{48}$$

Integrating over dE cancels the delta function, leaving

$$\frac{d^3 p}{2E} = \delta(p^2 - m^2) d^4 p \; . \tag{49}$$

Since p^2 and m^2 are Lorentz invariant, the right hand side is invariant, and therefore $d^3p/2E$ is invariant. The total cross section, σ, is invariant since $d\sigma$ is transverse to the direction of motion z, and transforms like an area. Consequently, $Ed^3\sigma/d^3p$ is invariant.

As an alternate proof, expand the momentum integration measure,

$$d^3 p = dp_z dp_x dp_y \; . \tag{50}$$

Here, dp_x and dp_y are Lorentz invariant, being transverse to the direction of motion z. Since σ is Lorentz invariant, it remains to show that $\frac{E}{dp_z}$ is invariant. Rapidity y, is defined through

$$E = m_T \cosh y \qquad \text{and} \qquad p_z = m_T \sinh y \; , \tag{51}$$

and

$$\begin{aligned} m_T &\equiv m^2 + px^2 + p_y^2 \\ &= E^2 - pz^2 \; , \end{aligned} \tag{52}$$

which gives

$$\frac{dp_z}{dy} = m_T \cosh y = E \; . \tag{53}$$

Thus,

$$\frac{E}{dp_z} = \frac{1}{dy} \; . \tag{54}$$

Under Lorentz transformations, rapidities add, so that $y = y' + y_{RF}$, where y_{RF} is the rapidity of a certain reference frame, y' is the rapidity in that reference frame, and y is the transformed rapidity. Thus, $dy = dy'$, showing that $dy = dp_z/E$ is invariant under Lorentz transformations. Thus, $Ed^3\sigma/d^3p$ is Lorentz invariant.

1.5 Denominator flux factor in any frame

The denominator flux factor F is defined in reference [10] (p. 98) as

$$F \equiv \sqrt{(p_1.p_2)^2 - m_1^2 m_2^2} \, . \tag{55}$$

This can be written

$$F_{\text{any frame}} = \frac{1}{2}\sqrt{\lambda_{12}} \, , \tag{56}$$

where [14] (p. 23), [15]

$$\begin{aligned} \lambda(x,y,z) &\equiv (x-y-z)^2 - 4yz \\ &= x^2 - 2(y+z)x + (y-z)^2 \\ &= x^2 + y^2 + z^2 - 2yx - 2zx - 2yz \, , \end{aligned} \tag{57}$$

and

$$\lambda_{ij} \equiv \lambda(s, m_i^2, m_j^2) = (s - m_i^2 - m_j^2)^2 - 4m_i^2 m_j^2 \, . \tag{58}$$

Now calculate s as,

$$\begin{aligned} s &\equiv (p_1 + p_2)^2 = p_1^2 + p_2^2 + 2p_1 \cdot p_2 \\ &= m_1^2 + m_2^2 + 2(E_1 E_2 - \mathbf{p}_1 \cdot \mathbf{p}_2) \quad \text{in any frame} \, . \end{aligned} \tag{59}$$

Then,

$$\begin{aligned} \lambda_{12} = \lambda(s, m_1^2, m_2^2) &= (s - m_1^2 - m_2^2)^2 - 4m_1^2 m_2^2 \\ &= 4[(E_1 E_2 - \mathbf{p}_1 \cdot \mathbf{p}_2)^2 - m_1^2 m_2^2] \\ &= 4[(p_1 \cdot p_2)^2 - m_1^2 m_2^2] \, . \end{aligned} \tag{60}$$

1.6 Denominator flux factor in the lab frame

For the reaction $1 + 2 \to 3 + 4 + \cdots$, let the rest frame of particle 2 be the lab (target) frame. Then, $\mathbf{p}_2 \equiv 0$, and

$$\begin{aligned} (p_1 \cdot p_2)^2 - m_1^2 m_2^2 &= (E_1 E_2 - \mathbf{p}_1 \cdot \mathbf{p}_2)^2 - m_1^2 m_2^2 \\ &= E_1^2 E_2^2 - m_1^2 m_2^2 \qquad \text{(because} \quad \mathbf{p}_2 \equiv 0) \\ &= (\mathbf{p}_{1lab}^2 + m_1^2) m_2^2 - m_1^2 m_2^2 \\ &= m_2^2 \mathbf{p}_{1lab}^2 \, , \end{aligned} \tag{61}$$

giving

$$F_{lab} = m_2 |\mathbf{p}_{1llab}| \, . \tag{62}$$

2 Decay into two particles

This section is concerned with the decay of one particle into two particles. The fundamental quantity is the decay width. Formulas are given that enable this to be calculated from the invariant amplitude. The kinematics of the final state particles is also discussed. Finally, the decay width is related to the cross section for the reverse process in which two particles react to form a single particle.

2.1 Decay width

Consider the reaction,

$$N_1 + N_2 \to N_1 + \Delta \to N_1 + \pi + N_3 \;, \tag{63}$$

where N_i are nucleons (either neutron or proton), Δ is the Δ particle and π denotes a pion. Pilkuhn [16] (pp. 37, 38) and de Wit [7] (p. 113) show that the cross section for this two step process can be approximately written as,

$$\sigma(N_1 + N_2 \to N_1 + \pi + N_3) \approx \sigma(N_1 + N_2 \to N_1 + \Delta)\frac{\Gamma(\Delta \to \pi + N_3)}{\Gamma} \;. \tag{64}$$

In the above, Γ is the total decay width for the process $\Delta \to$ anything. We seek expressions for 2 - body cross sections, as well as the decay of a single body into 2 bodies. Consider the decay,

$$3 \to 1 + 2 \;, \tag{65}$$

where the numbers denote particles. The 2-particle decay width is given by

$$d\Gamma = \frac{\mathcal{S}}{2m}|\mathcal{M}|^2(2\pi)^4 d\Phi_2(P; p_1, p_2) \;, \tag{66}$$

with $m \equiv m_3$. The phase space factor is

$$\begin{aligned} d\Phi_2(P; p_1, p_2) &= \delta^4(p_3 - p_1 - p_2)\frac{d^3p_1}{(2\pi)^3 2E_1}\frac{d^3p_2}{(2\pi)^3 2E_2} \\ &= \delta(E_3 - E_1 - E_2')\frac{d^3p_1}{e_1 e_2'} \\ &= \delta(E_3 - E_1 - E_2')\frac{|\mathbf{p}_1|^2 d|\mathbf{p}_1| d\Omega_1}{e_1 e_2'} \;, \end{aligned} \tag{67}$$

with

$$e \equiv (2\pi)^3 2E \;, \tag{68}$$

where it is understood that

$$\mathbf{p}_2 \equiv \mathbf{p}_3 - \mathbf{p}_1 \;, \tag{69}$$

so that in the above equation, with $|\mathbf{p}_1|^2 \equiv \mathbf{p}_1^2$,

$$\begin{aligned} E_1 &= \sqrt{\mathbf{p}_1^2 + m_1^2} && (70) \\ E_2' &\equiv \sqrt{(\mathbf{p}_3 - \mathbf{p}_1)^2 + m_2^2} \, . && (71) \end{aligned}$$

Two equivalent methods to eliminate the δ function are now given. These are the Ho-Kim phase space method [10] and the Griffiths phase space method [2].

2.1.1 Ho-Kim phase space method

Ho-Kim [10] eliminates the δ function by making a change of variables as follows. Simplify equation (67) by writing

$$d|\mathbf{p}_1| = \frac{d|\mathbf{p}_1|}{d(E_1 + E_2')} d(E_1 + E_2') \,, \tag{72}$$

so that the integral over $d(E_1 + E_2')$ will cancel the deltal function $\delta(E_3 - E_1 - E_2')$. Ultimately, the quantity $\frac{d(E_1+E_2')}{d|\mathbf{p}_1|}$ will be evaluated. The quantity, $d(E_1 + E_2')$, is used rather than dE_1, because *both* E_1 and E_2' contain the term $|\mathbf{p}_1|$. The phase space factor becomes, (with the energy δ function eliminated),

$$d\Phi_2(P; p_1, p_2) = \frac{|\mathbf{p}_1|^2}{e_1 e_2'} \frac{d|\mathbf{p}_1|}{d(E_1 + E_2')} d\Omega_1 \qquad \text{(in general)}, \tag{73}$$

giving the partial width as,

$$d\Gamma = \frac{\mathcal{S}}{32\pi^2 m} |\mathcal{M}|^2 \frac{|\mathbf{p}_1|^2}{E_1 E_2'} \frac{d|\mathbf{p}_1|}{d(E_1 + E_2')} d\Omega_1 \qquad \text{(in general)}. \tag{74}$$

2.1.2 Griffiths phase space method

Griffiths [2] eliminates the δ function by making use of the formula,

$$\delta[g(x)] = \sum_{j=1}^{n} \frac{\delta(x - x_j)}{|g'(x_j)|} \,, \tag{75}$$

where the sum over j represents the sum over the *zeroes* x_j of the function $g(x)$. The term $g'(x_i)$ is the derivative of the function $g(x)$, evaluated at the zero, x_j. In the delta function, $\delta(E_3 - E_1 - E_2') \equiv \delta[g(x)]$, we set,

$$\begin{aligned} g(x) \equiv g(|\mathbf{p}_1|) &= E_3 - E_1 - E_2' \\ &= E_3 - \sqrt{\mathbf{p}_1^2 + m_1^2} - \sqrt{(\mathbf{p}_3 - \mathbf{p}_1)^2 + m_1^2} \, . && (76) \end{aligned}$$

The zeroes, $\mathbf{p}_{1i}$, are obtained by setting

$$g(|\mathbf{p}_1|) = E_3 - \sqrt{\mathbf{p}_1^2 + m_1^2} - \sqrt{(\mathbf{p}_3 - \mathbf{p}_1)^2 + m_1^2} \quad \equiv \quad 0 \; . \tag{77}$$

The derivative is

$$g'(x_i) \equiv \frac{dg(|\mathbf{p}_1|)}{d|\mathbf{p}_1|} = -\frac{d(E_1 + E_2')}{d|\mathbf{p}_1|} \; , \tag{78}$$

so that

$$\delta(E_3 - E_1 - E_2')d|\mathbf{p}_1| = \frac{\delta(|\mathbf{p}_1| - |\mathbf{p}_{1i}|)d|\mathbf{p}_1|}{\frac{d(E_1+E_2')}{d|\mathbf{p}_1|}} = \frac{d|\mathbf{p}_1|}{d(E_1 + E_2')} \; , \tag{79}$$

which is the same result obtained with the Ho-Kim phase space method. The phase space factor and the decay width are evaluated in a particular frame by evaluating $\frac{d|\mathbf{p}_1|}{d(E_1+E_2')}$ in that frame. The cm frame, in which the decaying particle is at rest, is the most logical choice of frame. Thus,

$$\mathbf{p}_3 = 0 \qquad\qquad \text{and} \qquad\qquad \mathbf{p}_1 = -\mathbf{p}_2 \; , \tag{80}$$

and $E_2' = \sqrt{|\mathbf{p}_1|^2 + m_2^2}$, giving,

$$\left[\frac{d(E_1 + E_2')}{d|\mathbf{p}_1|}\right]_c = |\mathbf{p}_1| \frac{E_1 + E_2'}{E_1 E_2'} \; . \tag{81}$$

Since $E_1 + E_2' = E_3 = m$ and $|\mathbf{p}_1| = |\mathbf{p}_2| \equiv |\mathbf{p}_f|$ in the cm frame,

$$\begin{aligned} d\Phi_2(P; p_1, p_2) &= \frac{|\mathbf{p}_1|^2}{e_1 e_2'} \frac{d|\mathbf{p}_1|}{d(E_1 + E_2')} d\Omega_1 \\ &= \frac{|\mathbf{p}_1|^2}{e_1 e_2'} \frac{E_1 E_2'}{m} d\Omega_1 \\ &= \frac{|\mathbf{p}_1|}{4(2\pi)^6 \, m} d\Omega_1 \; . \end{aligned} \tag{82}$$

Assuming no angular dependence in $\mathcal{M}$, the integral $\int d\Omega_1 = 4\pi$, gives the width in the cm frame,

$$\Gamma_c = \frac{\mathcal{S}}{2m} |\mathcal{M}|^2 (2\pi)^4 d\Phi_2(P; p_1, p_2) = \frac{\mathcal{S}|\mathbf{p}_f|}{8\pi m^2} |\mathcal{M}|^2 \; . \tag{83}$$

This is only valid if M has no angular dependence. This agrees with references [2] (equations 6.30, 6.32) and [10] (p. 103) (with $d\Omega = 4\pi$). The units can now be verified. . There are 3 external lines in the Feynman diagram, so that $\mathcal{M}$ has units of GeV. The units of $\mathcal{M}^2$ cancel m^2 in the denominator, leaving Γ with the correct units of GeV. Using the result from the following section,

$$|\mathbf{p}_2| = |\mathbf{p}_f| = \frac{1}{2m_3}\sqrt{m_1^4 + m_2^4 + m_3^4 - 2m_1^2m_2^2 - 2m_1^2m_3^2 - 2m_2^2m_3^2}\ , \tag{84}$$

which is equivalent to solving equation (77) for the zeroes. The decay rate is written

$$\Gamma_c = \frac{\mathcal{S}}{16\pi m_3^3}\ |\mathcal{M}|^2\ \sqrt{m_1^4 + m_2^4 + m_3^4 - 2m_1^2m_2^2 - 2m_1^2m_3^2 - 2m_2^2m_3^2}\ . \tag{85}$$

with $m \equiv m_3$. Again, *this is only valid if M has no angular dependence.* This agrees with reference [10] (p. 103).

2.2 Decay energies and momenta

The energy E_2, and 3-momentum $\mathbf{p}_2$, of the decay,

$$3 \rightarrow 1 + 2 \tag{86}$$

may be expressed in terms of the masses of the three particles. Conservation of 4-momentum gives

$$\begin{aligned} p_3^2 &= (p_1 + p_2)^2 \\ = m_3^2 &= p_1^2 + p_2^2 + 2p_1.p_2 = m_1^2 + m_2^2 + 2(E_1E_2 - \mathbf{p}_1.\mathbf{p}_2) \\ &= m_1^2 + m_2^2 + 2(E_1E_2 + \mathbf{p}_1^2)\ \ \text{(because } \mathbf{p}_1 = -\mathbf{p}_2 \text{ in the cm frame)} \\ &= m_1^2 + m_2^2 + 2(E_1E_2 + E_2^2 - m_2^2) \\ &= m_1^2 + m_2^2 + 2E_2(E_1 + E_2) - 2m_2^2 = m_1^2 - m_2^2 + 2E_2(E_1 + E_2)\ . \end{aligned} \tag{87}$$

Also,

$$\begin{aligned} p_3^2 &= (p_1 + p_2)^2 \\ = m_3^2 &= (E_1 + E_2)^2 - (\mathbf{p}_1 + \mathbf{p}_2)^2\ , \end{aligned} \tag{88}$$

giving,

$$m_3 = E_1 + E_2\ , \tag{89}$$

since $\mathbf{p}_1 + \mathbf{p}_2 = 0$ in the cm frame. Substituting equation (89) into equation (87) leads to

$$m_3^2 = m_1^2 - m_2^2 + 2E_2m_3\ , \tag{90}$$

$$\Rightarrow\ E_2 = \frac{m_2^2 + m_3^2 - m_1^2}{2m_3}\ . \tag{91}$$

The magnitude of the 3-momentum is given by

$$|\mathbf{p}_2| = \sqrt{E_2^2 - m_2^2}$$

$$= \sqrt{\left(\frac{m_2^2 + m_3^2 - m_1^2}{2m_3}\right)^2 - m_2^2} \,. \tag{92}$$

Expanding gives

$$|\mathbf{p}_2| = \frac{1}{2m_3}\sqrt{m_1^4 + m_2^4 + m_3^4 - 2m_1^2m_2^2 - 2m_1^2m_3^2 - 2m_2^2m_3^2} \,. \tag{93}$$

2.3 Two - body reaction producing one body

We can relate the decay width of the process,

$$3 \rightarrow 1 + 2 \,,$$

to the cross section for

$$1 + 2 \rightarrow 3 \,.$$

The cross section is given by equation (19),

$$\begin{aligned}
d\sigma &= \frac{\mathcal{S}}{4F}|\mathcal{M}|^2(2\pi)^4 d\Phi_n(p_1 + p_2; p_3) \\
&= \frac{\mathcal{S}}{4F}|\mathcal{M}|^2(2\pi)^4\delta^4(p_1 + p_2 - p_3)\frac{d^3p_3}{(2\pi)^3 2E_3} \qquad \text{(in any frame)}, \\
&= \frac{\mathcal{S}\pi}{4|\mathbf{p}_1|\sqrt{s}}|\mathcal{M}|^2\delta(E_1 + E_2 - E_3)\delta^3(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_3)\frac{d^3p_3}{E_3} \qquad \text{(cm frame)},
\end{aligned} \tag{94}$$

using equation (26). In the cm frame, we have $E_3 = m_3 \equiv m = \sqrt{s}$ and $\mathbf{p}_1 = -\mathbf{p}_2$, finally giving the cross section, (with $\mathcal{S} = 1$ for the above reaction $1 + 2 \rightarrow 3$),

$$d\sigma = \frac{\pi}{4m^2|\mathbf{p}_1|}|\mathcal{M}|^2\delta(E_1 + E_2 - E_3) \,, \tag{95}$$

with $E_1 = \sqrt{\mathbf{p}_1^2 + m_1^2}$ and $E_2 = \sqrt{\mathbf{p}_1^2 + m_2^2}$. We now relate this cross section to the decay width. Substituting $|\mathcal{M}|^2$ from equation (85), into equation (95), gives the relation between decay width and cross section as,

$$d\sigma = \frac{2\pi^2}{\mathcal{S}|\mathbf{p}_1|^2}\,\Gamma\,\delta(E_1 + E_2 - E_3) \,. \tag{96}$$

The units of $\delta(E_1 + E_2 - E_3)$ are GeV^{-1}, cancelling the GeV units of Γ, so that the units of σ come out correctly as GeV^{-2}.

Sometimes the delta function is written a little differently, in terms of $\delta(s - m^2)$. In the cm frame, $\sqrt{s} = E_1 + E_2 \equiv W^2$. Thus $\delta(s - m^2) = \delta(W^2 - m^2)$. Using the result [2],

$$\delta(x^2 - a^2) = \frac{1}{2|a|}[\delta(x-a) + \delta(x+a)] \ , \tag{97}$$

gives

$$\delta(s - m^2) = \delta[(E_1+E_2)^2 - m^2] = \frac{1}{2m}[\delta(E_1+E_2-m) + \delta(E_1+E_2+m)] \ . \tag{98}$$

However, we cannot have $E_1 + E_2 = -m$, so that the second term does not contribute, [16] (p. 15), [3] (p. 151), giving only

$$\delta(s - m^2) = \frac{1}{2m}\delta(E_1+E_2-m) \ . \tag{99}$$

Thus, the relation between the decay width and the cross section can also be written

$$d\sigma \quad = \quad \frac{4\pi^2 m}{\mathcal{S}|\mathbf{p}_1|^2}\,\Gamma\,\delta(s-m^2) \ . \tag{100}$$

Suppose the decay products are both massless (e.g. $H \to \gamma\gamma$). Recall that in the cm frame, the momentum of the decay products is $|\mathbf{p}_1| = |\mathbf{p}_2|$. Also, the mass of the decaying particle is $E_3 = m_3 \equiv m$. By energy conservation (with $E = |\mathbf{p}|$ for photons), we have

$$m = 2|\mathbf{p}| \ . \tag{101}$$

Substituting $|\mathbf{p}| = m/2$ gives,

$$d\sigma \quad = \quad \frac{16\pi^2}{\mathcal{S}m}\,\Gamma\,\delta(s-m^2) \ . \tag{102}$$

Now, if the particles have spin, then it must be averaged over the initial spin states S_1 and S_2 and summed over the final spin state S_3 [7] (p. 114). This brings in an extra factor

$$d\sigma \quad = \quad \frac{2S_3+1}{(2S_1+1)(2S_2+1)}\frac{16\pi^2}{\mathcal{S}m}\,\Gamma\,\delta(s-m^2) \ , \tag{103}$$

which is our final result. This gives the results of several other authors. For example, Norbury [17] considered the decay of a Higgs boson into two photons,

$$H \to \gamma + \gamma \ ,$$

where the statistical factor is $\mathcal{S} = 1/2!$, because there are 2 particles in the final state. The spin of the Higgs is $S_H \equiv S_3 = 0$, and the spin of the photons are $S_1 = S_2 = S_\gamma = 1$. However, the factor, $\frac{1}{2S_\gamma+1} = 1/3$, is only correct for *massive* spin 1 states. Photons loose one degree of freedom, meaning that the factor should actually be 1/2. Substituting gives

$$d\sigma = \frac{8\pi^2}{m} \Gamma \, \delta(s - m^2) \; , \tag{104}$$

in agreement with [17]. Another example is the decay of a vector boson B into an electron-positron pair [3] (p. 151),

$$B \to e^+ + e^- \; .$$

Here, $\mathcal{S} = 1$ because there are no identical particles in the final state. Also, the spins are $S_A \equiv S_B = 1$, and $S_1 \equiv S_{e^+} = 1/2$, and $S_2 \equiv S_{e^-} = 1/2$. Substituting, gives

$$d\sigma = \frac{12\pi^2}{m} \Gamma \, \delta(s - m^2) \; , \tag{105}$$

in agreement with [3] (p. 151). Finally, note that Vidovic et al [18] have an extra factor of 2 in their equation (46). However, this results from the definition of their equation (21).

2.3.1 Calculation using de Wit and Smith formula

De Wit and Smith [7] (p. 114) have also derived a relation between the cross section and decay width that is more general than our result. In this section, we show that the de Wit and Smith formula, reduces exactly to our result. They consider the reaction

$$a + b \to A \to X \; , \tag{106}$$

where A has decayed into X, i.e. $A \to X$. Their result is

$$\sigma(a + b \to X) = \frac{16\pi m_A^4}{\mathcal{S}\lambda(m_A^2, m_a^2, m_b^2)} \frac{\Gamma(A \to X)\Gamma(A \to a + b)}{(s_A^2 - m_A^2)^2 + m_A^2\Gamma^2} \; , \tag{107}$$

where $\mathcal{S}$ is the same statistical factor as the present work and $\lambda(m_A^2, m_a^2, m_b^2)$ is also the same as used herein, namely

$$\lambda(m_A^2, m_a^2, m_b^2) = 4|\mathbf{p}_{1c}|^2 s = m_A^4 \; , \tag{108}$$

where we have used equation (101). The above result is valid for arbitrary width. Now, examine the narrow width approximation. A *narrow width corresponds to a large lifetime.* In other words, the narrow width approximation has as its limit the decaying particle *not* decaying. That is, the decay $A \to X$ does not proceed. Thus, the symbol X in the above equations should be replaced with A. In the narrow width approximation, one has [7] (p. 112)

$$\frac{1}{\pi} \frac{m_A \Gamma(A \to X)}{(s_A^2 - m_A^2)^2 + m_A^2\Gamma^2} \approx \delta(s_A - m_A^2) \; . \tag{109}$$

Substituting equations (108) and (109) into equation (107) gives

$$\sigma(a + b \to A) = \frac{16\pi^2}{\mathcal{S} m_A} \Gamma(A \to a + b)\, \delta(s_A - m_A^2) \;, \tag{110}$$

which is the same as our result in equation (103).

3 Non - relativistic two - body kinematics

Some key results in this paper will be in finding the relation between the lab angle and the lab momenta for 2 - body final state particles. An important outcome is that for a given lab angle, there will be *two* corresponding momentum values, which result from two angles in the center of momentum or projectile frames. See equations (146) and (147). This is a crucial point for 3 - dimensional transport codes, because for a given angle in the lab (spacecraft) frame, there will be *two* particles arriving from the center of momentum or projectile frames with different energies. The differential cross section for the production of each particle must be added incoherently to form the differential cross section in the lab frame. This section examines these results from a non - relativistic point of view. This will enable easier understanding of the corresponding relativistic results, which will be derived later.

3.1 Reactions between elementary particles

First, examine collisions from a more general point of view. Consider the collision of elementary particles given in the reaction

$$1 + 2 \to 3 + 4 \;, \tag{111}$$

where the numbers represent the particles involved in the collision. At energies comparable to the masses of the particles, relativistic kinematics is needed to accurately analyze the collisions. This section considers low energy collisions, and will use non-relativistic kinematics. By conservation of mass

$$m_1 + m_2 = m_3 + m_4 \;. \tag{112}$$

Momentum is always conserved,

$$\mathbf{p}_1 + \mathbf{p}_2 = \mathbf{p}_3 + \mathbf{p}_4 \tag{113}$$

We don't need initial i and final f subscripts now because particles 1 and 2 are automatically in the initial state and particles 3 and 4 are automatically in the final state. For *elastic* collisions, kinetic energy will be conserved,

$$T_1 + T_2 = T_3 + T_4 \;. \tag{114}$$

Remember that the kinetic energy and momentum can be expressed in terms of each other via

$$T = \frac{\mathbf{p}^2}{2m} \ , \tag{115}$$

which is the non-relativistic version of kinetic energy. Also, note that we will often use the identity

$$|\mathbf{p}|^2 \equiv \mathbf{p}^2 \ , \tag{116}$$

where $|\mathbf{p}|$ is the magnitude of the momentum vector $\mathbf{p}$. In particle reactions like (111), particles 1 and 2 are usually produced in a particle accelerator and are made to collide with each other. *The mass and energy or momentum of particles 1 and 2 is always known.* Thus, $m_1, m_2, T_1, T_2, \mathbf{p}_1, \mathbf{p}_2$ are always known.

In a *fixed target accelerator*, the target particle 2 is at rest and the projectile particle 1 is fired into it. The collection of projectile particles is often called the beam. The collection of target particles is often just called the target. Thus, the moving accelerator beam is fired into a target at rest. The whole accelerator complex is called a laboratory or simply a lab. Thus, the target particle is not moving in the lab. We often call this reference frame in which particle 2 is at rest simply the lab frame or the target frame. So, in a fixed target accelerator we have

$$\mathbf{p}_2 = 0 \qquad \text{and} \qquad T_2 = 0 \ . \tag{117}$$

In a *colliding beam accelerator*, often called a *collider*, both particles 1 and 2 are moving in opposite directions and they are fired into each other. However, again we will always know the masses and energies of both of these particles.

In either case, particles 3 and 4 are the reaction products and their properties are studied in great detail. These properties include their mass, charge, spin, and other quantum numbers, and also their kinematic properties such as their energy or momentum and also the angle from which they emerge from the collision.

3.1.1 Mandelstam variables

If we write down the x and y components of the momentum conservation equation (113), we can then deduce the angles with which particles 3 and 4 emerge. However, this can involve quite a lot of algebra. An easier method is to use the so-called *Mandelstam variables*, which are defined in their non - relativistic form as

$$s \equiv (\mathbf{p}_1 + \mathbf{p}_2)^2 = (\mathbf{p}_3 + \mathbf{p}_4)^2 \ , \tag{118}$$
$$t \equiv (\mathbf{p}_1 - \mathbf{p}_3)^2 = (\mathbf{p}_4 - \mathbf{p}_2)^2 \ , \tag{119}$$
$$u \equiv (\mathbf{p}_1 - \mathbf{p}_4)^2 = (\mathbf{p}_3 - \mathbf{p}_2)^2 \ . \tag{120}$$

These are just the usual relativistic Mandelstam variables with 4-vectors replaced by 3-vectors. Note that s is related to the total energy of the system and t and u are momentum transfers.

The square of the sum of any two 3-vectors is

$$(\mathbf{A}+\mathbf{B})^2 = \mathbf{A}^2+\mathbf{B}^2+2\mathbf{A}\cdot\mathbf{B}\,,$$
$$= \mathbf{A}^2+\mathbf{B}^2+2|\mathbf{A}||\mathbf{B}|\cos\theta \quad (121)$$

where θ is the angle between the vectors, as seen in figure 1. The Mandelstam variables can then can be written

$$s = \mathbf{p}_1^2+\mathbf{p}_2^2+2|\mathbf{p}_1||\mathbf{p}_2|\cos\theta_{12} = \mathbf{p}_3^2+\mathbf{p}_4^2+2|\mathbf{p}_3||\mathbf{p}_4|\cos\theta_{34}\,, \quad (122)$$
$$t = \mathbf{p}_1^2+\mathbf{p}_3^2-2|\mathbf{p}_1||\mathbf{p}_3|\cos\theta_{13} = \mathbf{p}_2^2+\mathbf{p}_4^2-2|\mathbf{p}_2||\mathbf{p}_4|\cos\theta_{24}\,, \quad (123)$$
$$u = \mathbf{p}_1^2+\mathbf{p}_4^2-2|\mathbf{p}_1||\mathbf{p}_4|\cos\theta_{14} = \mathbf{p}_2^2+\mathbf{p}_3^2-2|\mathbf{p}_2||\mathbf{p}_3|\cos\theta_{23}\,, \quad (124)$$

where θ_{jk} is the angle between particle j and k, as shown in figure 2.

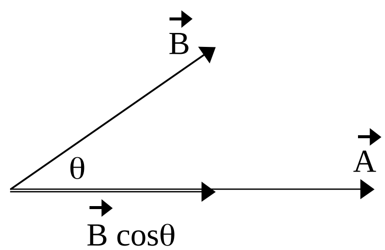

Figure 1: Scalar product between two vectors $\mathbf{A}$ and $\mathbf{B}$ is given by $\mathbf{A}.\mathbf{B} = AB\cos\theta$, where A and B are the vector magnitudes. Note that the angle θ is defined as the angle between the vectors arranged tail to tail.

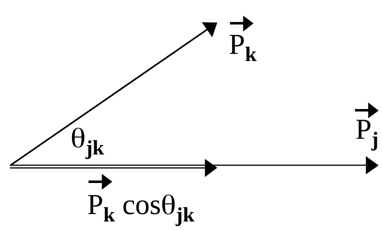

Figure 2: The angle θ_{jk} is the angle between the vectors $\mathbf{p}_j$ and $\mathbf{p}_k$ connected tail to tail as in figure 1. The scalar product is given by $\mathbf{p}_j.\mathbf{p}_k = |\mathbf{p}_j||\mathbf{p}_k|\cos\theta_{jk}$.

3.1.2 Lab frame

As discussed above, the lab frame is the frame in which the target particle 2 is at rest as in equation (117). With $\mathbf{p}_{2l} = T_{2l} = 0$, our conservation equations become

$$\mathbf{p}_{1l} = \mathbf{p}_3+\mathbf{p}_4 \quad (125)$$

and

$$m_1 + m_2 = m_3 + m_4 \ , \tag{126}$$

and if the collision is elastic,

$$T_{1l} = T_3 + T_4 \ . \tag{127}$$

The momentum conservation equation can be written in terms of its components as

$$\begin{aligned} \text{x component :} \qquad m_1 v_{1l} &= m_3 v_3 \cos\theta_{13} + m_4 v_4 \cos\theta_{14} \ , & (128) \\ \text{y component :} \qquad 0 &= m_3 v_3 \sin\theta_{13} - m_4 v_4 \sin\theta_{14} \ , & (129) \end{aligned}$$

or as

$$\begin{aligned} \text{x component :} \qquad |\mathbf{p}_{1l}| &= |\mathbf{p}_3| \cos\theta_{13} + |\mathbf{p}_4| \cos\theta_{14} \ , & (130) \\ \text{y component :} \qquad 0 &= |\mathbf{p}_3| \sin\theta_{13} - |\mathbf{p}_4| \sin\theta_{14} \ . & (131) \end{aligned}$$

Given that the target particle 2 is at rest, then it is *meaningless* to define an angle relative to particle 2, since angles are defined between *lines* and not a line and a point. Thus, in the frame in which particle 2 is at rest, there will be no angle appearing that contains particle 2. That is, there will be no angle θ_{j2}. Setting $\mathbf{p}_{2l} = 0$ gives the lab frame Mandelstam variables

$$\begin{aligned} s &= \mathbf{p}_{1l}^2 = \mathbf{p}_3^2 + \mathbf{p}_4^2 + 2|\mathbf{p}_3||\mathbf{p}_4| \cos\theta_{34} \ , & (132) \\ t &= \mathbf{p}_1^2 + \mathbf{p}_3^2 - 2|\mathbf{p}_1||\mathbf{p}_3| \cos\theta_{13} = \mathbf{p}_{4l}^2 \ , & (133) \\ u &= \mathbf{p}_1^2 + \mathbf{p}_4^2 - 2|\mathbf{p}_1||\mathbf{p}_4| \cos\theta_{14} = \mathbf{p}_{3l}^2 \ . & (134) \end{aligned}$$

The portions of these equations that do not have a l subscript are true in any frame. The angles are shown in figures 1 - 4. Note that

$$\theta_{34} = \theta_{13} + \theta_{14} \ . \tag{135}$$

3.1.3 Proof of the relation $t = (\mathbf{p}_1 - \mathbf{p}_3)^2 = |\mathbf{p}_{4l}|^2$

Using the x and y components of the momentum conservation equation, one can prove the Mandelstam variable relation

$$t \equiv (\mathbf{p}_1 - \mathbf{p}_3)^2 = |\mathbf{p}_{4l}|^2 \ , \qquad \text{in the lab frame where } \mathbf{p}_{2l} = 0. \tag{136}$$

This statement will now be proved. Conservation of momentum is

$$\mathbf{p}_1 + \mathbf{p}_2 = \mathbf{p}_3 + \mathbf{p}_4 \ . \tag{137}$$

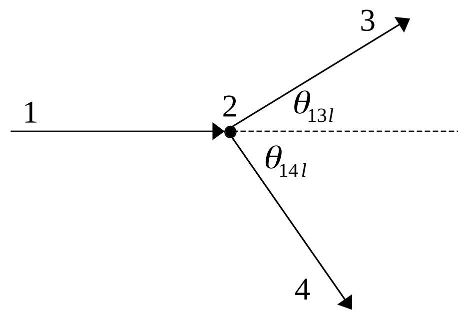

Figure 3: Definition of angles in the *lab* frame. θ_{14l} is the recoil angle of the target. The angle between the final state particles is $\theta_{34} = \theta_{13} + \theta_{14}$. Note that the angle θ is defined as the angle between the vectors arranged tail to tail.

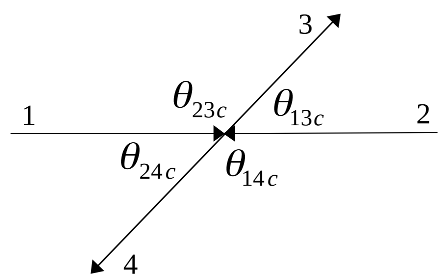

Figure 4: Definition of angles in the *cm* frame. Note that $\theta_{13c} = \theta_{24c}$ and $\theta_{14c} = \theta_{23c}$. The angle between the final state particles is $\theta_{34} = \theta_{13} + \theta_{14}$. In the cm frame, $\theta_{34} = \pi$. Note that the angle θ is defined as the angle between the vectors arranged tail to tail.

All quantities are now to be considered as lab frame quantities. The components are

$$\begin{aligned} \text{x component :} \qquad |\mathbf{p}_1| &= |\mathbf{p}_3|\cos\theta_{13} + |\mathbf{p}_4|\cos\theta_{14} \ , \qquad (138) \\ \text{y component :} \qquad 0 &= |\mathbf{p}_3|\sin\theta_{13} - |\mathbf{p}_4|\sin\theta_{14} \ . \qquad (139) \end{aligned}$$

Now eliminate $|\mathbf{p}_4|$ and θ_{14}. The y component equation gives

$$\sin\theta_{14} = \frac{|\mathbf{p}_3|\sin\theta_{13}}{|\mathbf{p}_4|} \ . \qquad (140)$$

Use $\cos\theta = \pm\sqrt{1-\sin^2\theta}$ and substitute into the x component,

$$|\mathbf{p}_1| = |\mathbf{p}_3|\cos\theta_{13} \pm |\mathbf{p}_4|\sqrt{1 - \frac{|\mathbf{p}_3|^2\sin^2\theta_{13}}{|\mathbf{p}_4^2|}}$$

$$= |\mathbf{p}_3|\cos\theta_{13} \pm \sqrt{|\mathbf{p}_4|^2 - |\mathbf{p}_3|^2\sin^2\theta_{13}} \ . \tag{141}$$

This gives

$$\begin{aligned}(|\mathbf{p}_1| - |\mathbf{p}_3|\cos\theta_{13})^2 &= |\mathbf{p}_4|^2 - |\mathbf{p}_3|^2\sin^2\theta_{13} \\ &= |\mathbf{p}_1|^2 - 2|\mathbf{p}_1||\mathbf{p}_3|\cos\theta_{13} + |\mathbf{p}_3|^2\cos^2\theta_{13} \ , \end{aligned} \tag{142}$$

so that

$$|\mathbf{p}_4|^2 = |\mathbf{p}_1|^2 - 2|\mathbf{p}_1||\mathbf{p}_3|\cos\theta_{13} + |\mathbf{p}_3|^2 = (\mathbf{p}_1 - \mathbf{p}_3)^2 \ . \tag{143}$$

Note that when solving collision problems with the x and y components of momentum conservation, one always ends up with the Mandelstam variable relation anyway, if angles are of interest. Thus, it is good practice to simply start with Mandelstam variables when angles are involved. Squaring the quantity,

$$(\mathbf{p}_1 - \mathbf{p}_3)^2 \equiv t \ , \tag{144}$$

will directly give the angle θ_{13}. Similarly, squaring the quantity,

$$(\mathbf{p}_1 - \mathbf{p}_4)^2 \equiv u \ , \tag{145}$$

will directly give the angle θ_{14}. Now, a very interesting result is that in the lab frame, for reaction (111), the speed or energy of particle 3 is given in terms of the angle at which particle 3 emerges, namely θ_{13}. Similarly, the speed or energy of particle 4 is given in terms of the angle at which particle 4 emerges, namely θ_{14}. In other words, the angles of the final particles determines their energy and vice-versa.

3.2 Relations between momentum and angle of scattered particle

3.2.1 Center of momentum frame

Consider the reaction,

$$1 + 2 \to 3 + 4$$

in the cm frame. As will be shown, the momentum of particle 3 or 4 is *not* determined by the angle at which it emerges from the collision.

3.2.2 Lab frame

In the lab frame, the momentum of particle 3 or 4 *does* depend on the angle at which it emerges from the collision. This is shown below. Again, consider the *elastic* reaction

$$1 + 2 \to 3 + 4 \ ,$$

where the target particle 2 is at rest in the lab and the projectile particle 1 is fired into the stationary target. Assume that we know the value of the momentum of particle 1 to be $\mathbf{p}_{1l}$. One can prove the following useful results.

(a) The momentum of particles 3 and 4 can be expressed in terms of their scattering angles θ_{13l} and θ_{14l}, where θ_{1il} is the angle between the momentum of particle i and the incident beam direction of particle 1, as measured in the lab frame. One can show that

$$|\mathbf{p}_{3l}| = |\mathbf{p}_{1l}| \, \frac{m_3 \left(\cos\theta_{13l} \pm \sqrt{\frac{m_2 m_4}{m_1 m_3} - \sin^2\theta_{13l}} \right)}{m_3 + m_4} \tag{146}$$

and

$$|\mathbf{p}_{4l}| = |\mathbf{p}_{1l}| \, \frac{m_4 \left(\cos\theta_{14l} \pm \sqrt{\frac{m_2 m_3}{m_1 m_4} - \sin^2\theta_{14l}} \right)}{m_3 + m_4} \; . \tag{147}$$

Note that $|\mathbf{p}_{4l}|$ is the same as $|\mathbf{p}_{3l}|$, except for the interchange of the particle labels $3 \leftrightarrow 4$.

(b) We will evaluate $|\mathbf{p}_{3l}|$ and $|\mathbf{p}_{4l}|$ for an elastic billiard ball collision when $m_1 = m_3$ & $m_2 = m_4$ and also when $m_1 = m_4$ & $m_2 = m_3$.

(c) Notice that the solutions for $|\mathbf{p}_{3l}|$ and $|\mathbf{p}_{4l}|$ are double valued. For the case of an elastic billiard ball collision when $m_1 = m_3$ & $m_2 = m_4$, we will interpret the two solutions.

(d) We shall evaluate $|\mathbf{p}_{3l}(\theta_{13l} = 0, \pi, \frac{\pi}{2})|$ and $|\mathbf{p}_{4l}(\theta_{14l} = 0, \pi, \frac{\pi}{2})|$. We shall also work out these expressions when $m_1 = m_3$ & $m_2 = m_4$.

(e) We shall interpret the solutions $|\mathbf{p}_{3l}(\theta_{13l} = \pi/2)|$ and $|\mathbf{p}_{4l}(\theta_{14l} = \pi/2)|$ when $m_1 = m_3$ & $m_2 = m_4$.

(f) Consider a 1-dimensional collision where $\theta_{13l} = 0$ or π and $\theta_{14l} = 0$ or π. We have already evaluated $|\mathbf{p}_{3l}(\theta_{13l} = 0, \pi)|$, and $|\mathbf{p}_{4l}(\theta_{14l} = 0, \pi)|$ for an elastic billiard ball collision when $m_1 = m_3$ & $m_2 = m_4$. We will interpret the results for the three special cases i) $m_1 >> m_2$, ii) $m_1 = m_2$, and iii) $m_1 << m_2$.

We shall now prove the statements given above. All quantities are assumed to be lab frame quantities.

(a) Conservation of momentum, mass and energy is

$$\mathbf{p}_1 + \mathbf{p}_2 = \mathbf{p}_3 + \mathbf{p}_4 \; , \tag{148}$$
$$m_1 + m_2 = m_3 + m_4 \; , \tag{149}$$
$$T_1 + T_2 = T_3 + T_4 \; . \tag{150}$$

We use the Mandelstam variable,

$$t \equiv (\mathbf{p}_1 - \mathbf{p}_3)^2 = (\mathbf{p}_4 - \mathbf{p}_2)^2$$

$$
\begin{aligned}
&= |\mathbf{p}_1|^2 + |\mathbf{p}_3|^2 - 2|\mathbf{p}_1||\mathbf{p}_3|\cos\theta_{13} \\
&= |\mathbf{p}_4|^2, \qquad \text{in the lab frame where } \mathbf{p}_2 = 0 \ .
\end{aligned} \tag{151}
$$

Note that

$$
|\mathbf{p}|^2 \equiv \mathbf{p}^2 \ . \tag{152}
$$

Conservation of kinetic energy can be written (with $T_2 = 0$),

$$
\frac{|\mathbf{p}_1|^2}{2m_1} = \frac{|\mathbf{p}_3|^2}{2m_3} + \frac{|\mathbf{p}_4|^2}{2m_4} \ , \tag{153}
$$

which gives

$$
\mathbf{p}_4^2 = m_4 \left(\frac{|\mathbf{p}_1|^2}{m_1} - \frac{|\mathbf{p}_3|^2}{m_3} \right) \ , \tag{154}
$$

so that the Mandelstam variable t becomes

$$
|\mathbf{p}_1|^2 + |\mathbf{p}_3|^2 - 2|\mathbf{p}_1||\mathbf{p}_3|\cos\theta_{13} = \frac{m_4}{m_1}\mathbf{p}_1^2 - \frac{m_4}{m_3}\mathbf{p}_3^2 \ , \tag{155}
$$

from which we get the quadratic equation for $|\mathbf{p}_3|$,

$$
\mathbf{p}_3^2 \left(1 + \frac{m_4}{m_3}\right) - 2|\mathbf{p}_1||\mathbf{p}_3|\cos\theta_{13} + |\mathbf{p}_1|^2 \left(1 - \frac{m_4}{m_1}\right) = 0 \ . \tag{156}
$$

A general quadratic equation can be written

$$
ax^2 + bx + c = 0 \ , \tag{157}
$$

with solution

$$
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \ . \tag{158}
$$

For our equation, we have

$$
\begin{aligned}
b^2 - 4ac &= 4|\mathbf{p}_1|^2 \cos^2\theta_{13} - 4|\mathbf{p}_1|^2 \left(1 - \frac{m_4}{m_1}\right)\left(1 + \frac{m_4}{m_3}\right) \\
&= 4|\mathbf{p}_1|^2 \left[\cos^2\theta_{13} - \left(1 - \frac{m_4}{m_1}\right)\left(1 + \frac{m_4}{m_3}\right)\right] \\
&= 4|\mathbf{p}_1|^2 \left[\cos^2\theta_{13} - 1 + \frac{m_4}{m_1} - \frac{m_4}{m_3} + \frac{m_4^2}{m_1 m_3}\right] \\
&= 4|\mathbf{p}_1|^2 \left[-\sin^2\theta_{13} + \frac{1}{m_1 m_3}(m_3 m_4 - m_1 m_4 + m_4^2)\right]
\end{aligned}
$$

$$
\begin{aligned}
&= 4|\mathbf{p}_1|^2 \left[\frac{m_4}{m_1 m_3}(m_3 - m_1 + m_4) - \sin^2\theta_{13} \right] \\
&= 4|\mathbf{p}_1|^2 \left(\frac{m_2 m_4}{m_1 m_3} - \sin^2\theta_{13} \right) , \qquad (159)
\end{aligned}
$$

where we have used conservation of mass, $m_3 - m_1 + m_4 = m_2$, in the last equation. Thus, the solution to the quadratic equation is

$$
\begin{aligned}
|\mathbf{p}_3| &= \frac{2|\mathbf{p}_1|\cos\theta_{13} \pm 2|\mathbf{p}_1|\sqrt{\frac{m_2 m_4}{m_1 m_3} - \sin^2\theta_{13}}}{2(1 + \frac{m_4}{m_3})} \\
&= |\mathbf{p}_1| \, \frac{m_3 \left(\cos\theta_{13} \pm \sqrt{\frac{m_2 m_4}{m_1 m_3} - \sin^2\theta_{13}} \right)}{m_3 + m_4} , \qquad (160)
\end{aligned}
$$

which is the desired result. We could work everything out again to determine $|\mathbf{p}_4|$. However, rather than going through all the algebra again, it should be obvious that our answer is the same as that obtained previously except that subscript 3 is replaced by 4. Thus, we immediately deduce that

$$
|\mathbf{p}_4| = |\mathbf{p}_1| \, \frac{m_4 \left(\cos\theta_{14} \pm \sqrt{\frac{m_2 m_3}{m_1 m_4} - \sin^2\theta_{14}} \right)}{m_3 + m_4} . \qquad (161)
$$

The Mandelstam variable that will appear is

$$
u \equiv (\mathbf{p}_1 - \mathbf{p}_4)^2 = |\mathbf{p}_3|^2 , \qquad \text{in the lab frame where } \mathbf{p}_2 = 0. \qquad (162)
$$

(b) If $m_1 = m_3$ & $m_2 = m_4$ (elastic billiard ball collision), the above results reduce to

$$
\begin{aligned}
|\mathbf{p}_3| &= |\mathbf{p}_1| \, \frac{m_1 \left(\cos\theta_{13} \pm \sqrt{\frac{m_2^2}{m_1^2} - \sin^2\theta_{13}} \right)}{m_1 + m_2} , \\
|\mathbf{p}_4| &= |\mathbf{p}_1| \, \frac{m_2(\cos\theta_{14} \pm \cos\theta_{14})}{m_1 + m_2} \\
&= |\mathbf{p}_1| \, \frac{2m_2\cos\theta_{14}}{m_1 + m_2} \qquad \text{or} \qquad 0. \qquad (163)
\end{aligned}
$$

If $m_1 = m_4$ & $m_2 = m_3$, we get

$$
\begin{aligned}
|\mathbf{p}_3| &= |\mathbf{p}_1| \, \frac{m_2\,(\cos\theta_{13} \pm \cos\theta_{13}\,)}{m_1 + m_2} \\
&= |\mathbf{p}_1| \, \frac{2m_2\cos\theta_{13}}{m_1 + m_2} \qquad \text{or} \qquad 0, \qquad (164)
\end{aligned}
$$

and

$$|\mathbf{p}_4| \quad = \quad |\mathbf{p}_1| \, \frac{m_1 \left(\cos\theta_{14} \pm \sqrt{\frac{m_2^2}{m_1^2} - \sin^2\theta_{14}} \right)}{m_1 + m_2} \; . \tag{165}$$

(c) For $m_1 = m_3$ & $m_2 = m_4$, we had

$$|\mathbf{p}_3| \quad = \quad |\mathbf{p}_1| \, \frac{m_1 \left(\cos\theta_{13} \pm \sqrt{\frac{m_2^2}{m_1^2} - \sin^2\theta_{13}} \right)}{m_1 + m_2} \; , \tag{166}$$

and

$$|\mathbf{p}_4| \quad = \quad |\mathbf{p}_1| \, \frac{2m_2 \cos\theta_{14}}{m_1 + m_2} \qquad \text{or} \qquad 0. \tag{167}$$

The identification, $m_1 = m_3$ & $m_2 = m_4$, means that particle 3 is the scattered projectile and particle 4 is the recoil target. First, we interpret the $|\mathbf{p}_4| = 0$ solution. This implies that $\mathbf{p}_4 = 0$. We already have $\mathbf{p}_2 = 0$ in the lab frame. Thus, conservation of momentum $\mathbf{p}_1 + \mathbf{p}_2 = \mathbf{p}_3 + \mathbf{p}_4$ implies that

$$\mathbf{p}_1 = \mathbf{p}_3 \; , \tag{168}$$

and together with

$$\mathbf{p}_2 = \mathbf{p}_4 = 0 \; , \tag{169}$$

this means that the projectile *misses* the target. Thus, the target remains undisturbed ($\mathbf{p}_4 = 0$) and the projectile continues on with its original momentum ($\mathbf{p}_3 = \mathbf{p}_1$) as if nothing happened. Obviously, the other solution $|\mathbf{p}_4| = |\mathbf{p}_1| \, \frac{2m_2 \cos\theta_{14}}{m_1+m_2}$ signifies a real *collision*, not a miss. Now we interpret the $|\mathbf{p}_3|$ solutions. Firstly, we consider whether there is a *miss* solution as above. The *miss* solution corresponded to $\mathbf{p}_1 = \mathbf{p}_3$ which implies that $\theta_{13} = 0$. Thus, if $\theta_{13} \neq 0$, a real collision has occurred and the momentum $|\mathbf{p}_3|$ is simply a double valued function of the scattering angle θ_{13}, and no further interpretation is warranted. We look at the $\theta_{13} = 0$ solution. This gives

$$|\mathbf{p}_3| \quad = \quad |\mathbf{p}_1| \tag{170}$$

or

$$|\mathbf{p}_3| \quad = \quad |\mathbf{p}_1| \, \frac{m_1 - m_2}{m_1 + m_2} \; . \tag{171}$$

Clearly $|\mathbf{p}_3| = |\mathbf{p}_1|$ corresponds to the *miss* solution and $|\mathbf{p}_3| = |\mathbf{p}_1| \, \frac{m_1 - m_2}{m_1+m_2}$ corresponds to the real collision.

(d) Using equations (146) and (147), we obtain

$$
\begin{aligned}
|\mathbf{p}_3(\theta_{13}=0)| &= |\mathbf{p}_1|\,\frac{m_3}{m_3+m_4}\left(1\pm\sqrt{\frac{m_2m_4}{m_1m_3}}\right)\,, \\
&= |\mathbf{p}_1|\,\frac{m_1\pm m_2}{m_1+m_2}\,, \qquad \text{when } m_1=m_3\ \ \&\ \ m_2=m_4\,,
\end{aligned}
\tag{172}
$$

$$
\begin{aligned}
|\mathbf{p}_3(\theta_{13}=\pi)| &= -|\mathbf{p}_1|\,\frac{m_3}{m_3+m_4}\left(1\mp\sqrt{\frac{m_2m_4}{m_1m_3}}\right)\,, \\
&= -|\mathbf{p}_1|\,\frac{m_1\mp m_2}{m_1+m_2}\,, \qquad \text{when } m_1=m_3\ \ \&\ \ m_2=m_4\,,
\end{aligned}
\tag{173}
$$

$$
\begin{aligned}
|\mathbf{p}_3(\theta_{13}=\pi/2)| &= \pm|\mathbf{p}_1|\,\frac{m_3}{m_3+m_4}\sqrt{\frac{m_2m_4}{m_1m_3}-1}\,, \\
&= \pm|\mathbf{p}_1|\,\sqrt{\frac{m_2-m_1}{m_1+m_2}}\,, \qquad \text{when } m_1=m_3\ \ \&\ \ m_2=m_4\,,
\end{aligned}
\tag{174}
$$

$$
\begin{aligned}
|\mathbf{p}_4(\theta_{14}=0)| &= |\mathbf{p}_1|\,\frac{m_4}{m_3+m_4}\left(1\pm\sqrt{\frac{m_2m_3}{m_1m_4}}\right)\,, \\
&= |\mathbf{p}_1|\,\frac{2m_2}{m_1+m_2}\ \text{ or }\ 0\,, \qquad \text{when } m_1=m_3\ \ \&\ \ m_2=m_4\,,
\end{aligned}
\tag{175}
$$

$$
\begin{aligned}
|\mathbf{p}_4(\theta_{14}=\pi)| &= -|\mathbf{p}_1|\,\frac{m_4}{m_3+m_4}\left(1\mp\sqrt{\frac{m_2m_3}{m_1m_4}}\right)\,, \\
&= -|\mathbf{p}_1|\,\frac{2m_2}{m_1+m_2}\ \text{ or }\ 0\,, \qquad \text{when } m_1=m_3\ \ \&\ \ m_2=m_4\,,
\end{aligned}
\tag{176}
$$

and

$$
\begin{aligned}
|\mathbf{p}_4(\theta_{14}=\pi/2)| &= \pm|\mathbf{p}_1|\,\frac{m_4}{m_3+m_4}\sqrt{\frac{m_2m_3}{m_1m_4}-1}\,, \\
&= 0\,, \qquad \text{when } m_1=m_3\ \ \&\ \ m_2=m_4\,.
\end{aligned}
\tag{177}
$$

Note that solutions which give negative $|\mathbf{p}|$ are unphysical and must be discarded.

(e) When $m_1=m_3$ and $m_2=m_4$, we found the physical solutions were

$$
|\mathbf{p}_3(\theta_{13}=\pi/2)| = |\mathbf{p}_1|\,\sqrt{\frac{m_2-m_1}{m_1+m_2}}\,,
\tag{178}
$$

and

$$|\mathbf{p}_4(\theta_{14} = \pi/2)| \quad = \quad 0. \tag{179}$$

This says that the target can *never* be scattered into $\theta_{14} = \pi/2$. The projectile can only be scattered into $\theta_{13} = \pi/2$ if $m_2 > m_1$. This makes sense. It is saying that the projectile can only be scattered into $\theta_{13} = \pi/2$ if it is lighter than the target.

(f) When $m_1 = m_3$ and $m_2 = m_4$, we obtained

$$|\mathbf{p}_3(\theta_{13} = 0)| \quad = \quad |\mathbf{p}_1| \qquad \text{or} \qquad |\mathbf{p}_1| \, \frac{m_1 - m_2}{m_1 + m_2} \, , \tag{180}$$

$$|\mathbf{p}_3(\theta_{13} = \pi)| \quad = \quad -|\mathbf{p}_1| \qquad \text{or} \qquad |\mathbf{p}_1| \, \frac{m_2 - m_1}{m_1 + m_2} \, , \tag{181}$$

$$|\mathbf{p}_4(\theta_{14} = 0)| \quad = \quad |\mathbf{p}_1| \, \frac{2m_2}{m_1 + m_2} \qquad \text{or} \qquad 0, \tag{182}$$

and

$$|\mathbf{p}_4(\theta_{14} = \pi)| \quad = \quad -|\mathbf{p}_1| \, \frac{2m_2}{m_1 + m_2} \qquad \text{or} \qquad 0. \tag{183}$$

We can never have the magnitude of a vector being a negative quantity, and so these solutions must be dismissed as unphysical. Also, we have already decided above that the cases $|\mathbf{p}_3(\theta_{13} = 0)| = |\mathbf{p}_1|$ and $|\mathbf{p}_4| = 0$ correspond to the *miss* solution. Discarding the unphysical and *miss* solutions, we are left with

$$|\mathbf{p}_3(\theta_{13} = 0)| \quad = \quad |\mathbf{p}_1| \, \frac{m_1 - m_2}{m_1 + m_2} \, , \qquad \text{for } m_1 \geq m_2 \, , \tag{184}$$

$$|\mathbf{p}_3(\theta_{13} = \pi)| \quad = \quad |\mathbf{p}_1| \, \frac{m_2 - m_1}{m_1 + m_2} \, , \qquad \text{for } m_2 \geq m_1 \, , \tag{185}$$

and

$$|\mathbf{p}_4(\theta_{14} = 0)| \quad = \quad |\mathbf{p}_1| \, \frac{2m_2}{m_1 + m_2} \, . \tag{186}$$

where we have indicated in parentheses the allowed mass values for physically acceptable solutions. Writing $|\mathbf{p}_i| = m_i |\mathbf{v}_i|$ gives (with $m_1 = m_3$ and $m_2 = m_4$)

$$|\mathbf{v}_3(\theta_{13} = 0)| \quad = \quad |\mathbf{v}_1| \, \frac{m_1 - m_2}{m_1 + m_2} \, , \qquad \text{for } m_1 \geq m_2 \, , \tag{187}$$

$$|\mathbf{v}_3(\theta_{13} = \pi)| \quad = \quad |\mathbf{v}_1| \, \frac{m_2 - m_1}{m_1 + m_2} \, , \qquad \text{for } m_2 \geq m_1 \, , \tag{188}$$

$$|\mathbf{v}_4(\theta_{14} = 0)| \quad = \quad |\mathbf{v}_1| \, \frac{2m_1}{m_1 + m_2} \, . \tag{189}$$

For $m_1 >> m_2$, we have

$$
\begin{aligned}
|\mathbf{v}_3(\theta_{13} = 0)| &\approx |\mathbf{v}_1| \,, && (190)\\
|\mathbf{v}_4(\theta_{14} = 0)| &\approx 2|\mathbf{v}_1| \,. && (191)
\end{aligned}
$$

That is, the projectile continues on at the same speed and the target moves off at twice the speed of the projectile, both moving in the same direction as the original projectile.

For $m_1 = m_2$, we have

$$
\begin{aligned}
|\mathbf{v}_3(\theta_{13} = 0 \text{ or } \pi)| &\approx 0 \,, && (192)\\
|\mathbf{v}_4(\theta_{14} = 0)| &\approx |\mathbf{v}_1| \,. && (193)
\end{aligned}
$$

That is, the projectile stops and the target moves off at the speed of the projectile, in the same direction as the projectile.

For $m_1 << m_2$, we have

$$
\begin{aligned}
|\mathbf{v}_3(\theta_{13} = \pi)| &\approx |\mathbf{v}_1| \,, && (194)\\
|\mathbf{v}_4(\theta_{14} = 0)| &\approx 0 \,. && (195)
\end{aligned}
$$

That is, the projectile bounces off from the target with the same speed but in the opposite direction, and the massive target remains stationary.

3.2.3 Qualitative understanding of the $|\mathbf{p}_{3l}|$ solutions

We now develop a qualitative understanding of the two solutions, namely for a fixed angle θ_{13l}, there are two solutions for $|\mathbf{p}_{3l}|$. Consider the case of a billiard ball collision where a projectile billiard ball scatters off a target ball. In the above example, this is the case where $m_1 = m_3$ and $m_2 = m_4$ (elastic billiard ball collision), which was part (b) of the example. For that case, the momentum of particle 3 reduced to

$$
|\mathbf{p}_{3l}| = |\mathbf{p}_{1l}| \frac{m_1 \left(\cos\theta_{13l} \pm \sqrt{\frac{m_2^2}{m_1^2} - \sin^2\theta_{13l}} \right)}{m_1 + m_2} \,. \tag{196}
$$

Write this as

$$
\begin{aligned}
|\mathbf{p}_{3l}^{+}| &= |\mathbf{p}_{1l}| \frac{m_1 \left(\cos\theta_{13l} + \sqrt{\frac{m_2^2}{m_1^2} - \sin^2\theta_{13l}} \right)}{m_1 + m_2} \,, && (197)\\
|\mathbf{p}_{3l}^{-}| &= |\mathbf{p}_{1l}| \frac{m_1 \left(\cos\theta_{13l} - \sqrt{\frac{m_2^2}{m_1^2} - \sin^2\theta_{13l}} \right)}{m_1 + m_2} \,. && (198)
\end{aligned}
$$

The solution $|\mathbf{p}_{3l}^-|$ allows for a negative value. This solution is obviously unphysical and may be discarded. For $|\mathbf{p}_{3l}^-|$ to be positive, we must have

$$\begin{aligned} \cos\theta_{13l} &> \sqrt{\frac{m_2^2}{m_1^2} - \sin^2\theta_{13l}} \;, \\ \Rightarrow \quad \cos^2\theta_{13l} &> \frac{m_2^2}{m_1^2} - \sin^2\theta_{13l} = \frac{m_2^2}{m_1^2} + \cos^2\theta_{13l} - 1 \;, \\ \Rightarrow \quad 1 &> \frac{m_2^2}{m_1^2} \;, \\ \Rightarrow \quad m_1 &> m_2 \;. \end{aligned}$$

Only if this is true, do we get both solutions $|\mathbf{p}_{3l}^+|$ and $|\mathbf{p}_{3l}^-|$. Otherwise, we only get the one solution $|\mathbf{p}_{3l}^+|$. So given that $m_1 > m_2$, the question remains as to why we get two solutions $|\mathbf{p}_{3l}^\pm|$ at the *same* angle θ_{13l} for a *fixed incident energy.* Note that the two solutions are distinguished by their outgoing energy, or speed, or momentum magnitude. This is illustrated in figure 5, where a more massive projectile scatters off a less massive target. It can clearly be seen that the same scattering angle can occur for *different impact parameters.* If the projectile hits the target at small impact parameter then it will impart a relatively large amount of its momentum to the target and emerge with a relatively small speed at angle θ.

On the other hand, if the projectile hits the target at a large impact parameter then it will impart a relatively small amount of its momentum to the target and emerge with a relatively higher speed at the *same* angle θ. *Thus, the reason as to why there are two different momenta associated with the same scattering angle is because there can be two different impact parameters associated with that angle.* Obviously, if we do an experiment and actually roll one billiard ball into another then there will be only *one* emerging momentum depending on the collision impact parameter. However, when we calculate a cross section, we must *sum over all impact parameters.* Thus, a cross section calculation will be the result of many billiard ball collisions at a variety of integrated impact parameters. The fact that the same lab angle gives rise to two different lab energies is also related to transformation from the cm frame to the lab frame. Later, we shall find that two different angles in the cm frame can give rise to the same angle in the lab frame. From figure 5, it is clear that the two different impact parameters will correspond to two different angles in the cm frame.

3.3 Relations between initial and final momenta in lab and cm frames

In this section, we develop some further useful results, involving relations between the initial and final momenta in the lab and cm frames as well as relations between angles.

3.3.1 Lab frame

Consider the *elastic* reaction

$$1 + 2 \to 3 + 4 \;,$$

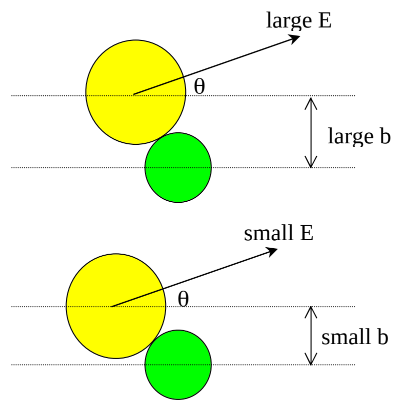

Figure 5: Elastic billiard ball collision where the mass of the projectile is greater than the mass of the target. b is the impact parameter. The balls are imagined to have the same density and so the more massive ball is drawn larger. (If the balls are of different density and have the same size, the dynamics shown in this figure are not altered.

where the target particle 2 is at rest in the lab and the projectile particle 1 is fired into the stationary target. We will derive a formula relating the momenta of particles 3 and 4 with the relative angle θ_{34} between particles 3 and 4. The formula will also contain the masses. This is much easier to work out by squaring the conservation of momentum equation, rather than breaking it up into components. After deriving this formula, we will show that it predicts $\theta_{34} = 90^\circ$ when all the particle masses are equal.

We proceed as follows. Conservation of momentum is

$$\mathbf{p}_1 + \mathbf{p}_2 = \mathbf{p}_3 + \mathbf{p}_4 \ . \tag{199}$$

Squaring gives the non-relativistic Mandelstam variable,

$$\begin{aligned} s &\equiv (\mathbf{p}_1 + \mathbf{p}_2)^2 = (\mathbf{p}_3 + \mathbf{p}_4)^2 \\ &= |\mathbf{p}_3|^2 + |\mathbf{p}_4|^2 + 2|\mathbf{p}_3||\mathbf{p}_4|\cos\theta_{34} \\ &= |\mathbf{p}_{1l}|^2 \ , \qquad \text{in the lab frame where } \mathbf{p}_{2l} = 0 \ . \end{aligned} \tag{200}$$

Conservation of kinetic energy can be written (with $T_{2l} = 0$),

$$\frac{|\mathbf{p}_{1l}|^2}{2m_1} = \frac{|\mathbf{p}_3|^2}{2m_3} + \frac{|\mathbf{p}_4|^2}{2m_4} \ , \tag{201}$$

so that the Mandelstam variable becomes

$$m_1\left(\frac{|\mathbf{p}_3|^2}{m_3} + \frac{|\mathbf{p}_4|^2}{m_4}\right) = |\mathbf{p}_3|^2 + |\mathbf{p}_4|^2 + 2|\mathbf{p}_3||\mathbf{p}_4|\cos\theta_{34} \ , \tag{202}$$

which is the desired formula. When $m_1 = m_2 = m_3 = m_4$, this becomes

$$0 = \cos\theta_{34} \qquad \Rightarrow \qquad \theta_{34} = 90^\circ \ . \tag{203}$$

3.3.2 Center of momentum frame

Again, consider the *elastic* reaction

$$1 + 2 \to 3 + 4$$

but now we will analyze the reaction in the cm frame. Assume that we know the value of the momentum of particle 1 in the cm frame to be $\mathbf{p}_{1c}$. We can obtain some useful results.

(a) We will work out a formula for $|\mathbf{p}_{3c}|$ as a function of $|\mathbf{p}_{1c}|$ and show that it does not depend on any angle.

(b) We will do the same for $|\mathbf{p}_{4c}|$.

(c) We will show that the scattering angles are related by $\theta_{13c} = \theta_{24c}$.

We proceed as follows.

(a) The non-relativistic Mandelstam variables are

$$s \equiv (\mathbf{p}_1 + \mathbf{p}_2)^2 = (\mathbf{p}_3 + \mathbf{p}_4)^2 \ , \tag{204}$$
$$t \equiv (\mathbf{p}_1 - \mathbf{p}_3)^2 = (\mathbf{p}_4 - \mathbf{p}_2)^2 \ , \tag{205}$$
$$u \equiv (\mathbf{p}_1 - \mathbf{p}_4)^2 = (\mathbf{p}_3 - \mathbf{p}_2)^2 \ , \tag{206}$$

and conservation of kinetic energy is

$$\frac{\mathbf{p}_1^2}{m_1} + \frac{\mathbf{p}_2^2}{m_2} = \frac{\mathbf{p}_3^2}{m_3} + \frac{\mathbf{p}_4^2}{m_4} \ . \tag{207}$$

Now, specialize to the cm frame, where

$$\mathbf{p}_{1c} + \mathbf{p}_{2c} = \quad 0 \quad = \mathbf{p}_{3c} + \mathbf{p}_{4c} \ , \tag{208}$$
$$\Rightarrow \quad |\mathbf{p}_{1c}| = |\mathbf{p}_{2c}| \equiv |\mathbf{p}_{ic}| \qquad \text{and} \qquad |\mathbf{p}_{3c}| = |\mathbf{p}_{4c}| \equiv |\mathbf{p}_{fc}| \ . \tag{209}$$

Eliminate $|\mathbf{p}_{2c}|$ and $|\mathbf{p}_{4c}|$ from the conservation of energy equation, giving

$$\mathbf{p}_{1c}^2\left(\frac{1}{m_1} + \frac{1}{m_2}\right) = \mathbf{p}_{3c}^2\left(\frac{1}{m_3} + \frac{1}{m_4}\right) \ , \tag{210}$$

$$\mathbf{p}_{1c}^2\left(\frac{m_1 + m_2}{m_1 m_2}\right) = \mathbf{p}_{3c}^2\left(\frac{m_3 + m_4}{m_3 m_4}\right) \ , \tag{211}$$

$$\Rightarrow \quad \mathbf{p}_{3c}^2 = \mathbf{p}_{1c}^2 \frac{m_3 m_4 (m_1 + m_2)}{m_1 m_2 (m_3 + m_4)} \ . \tag{212}$$

Define

$$\mu^2 \equiv \frac{m_3 m_4 (m_1 + m_2)}{m_1 m_2 (m_3 + m_4)} \;, \tag{213}$$

which gives

$$|\mathbf{p}_{3c}| \;=\; \mu |\mathbf{p}_{1c}| \;, \tag{214}$$

or

$$|\mathbf{p}_{fc}| \;=\; \mu |\mathbf{p}_{ic}| \;, \tag{215}$$

which is the desired result, giving the relation between the momentum of particle 3 in terms of the incident momentum. No angle appears in these formulas.

(b) In the cm frame, $|\mathbf{p}_{1c}| = |\mathbf{p}_{2c}| \equiv |\mathbf{p}_{ic}|$ and $|\mathbf{p}_{3c}| = |\mathbf{p}_{4c}| \equiv |\mathbf{p}_{fc}|$ so that

$$\begin{aligned} |\mathbf{p}_{4c}| &= \mu |\mathbf{p}_{2c}| \\ &= \mu |\mathbf{p}_{1c}| \;. \end{aligned} \tag{216}$$

(c) The non-relativistic Mandelstam t variable is

$$\begin{aligned} t &\equiv (\mathbf{p}_1 - \mathbf{p}_3)^2 = (\mathbf{p}_4 - \mathbf{p}_2)^2 \\ &= \mathbf{p}_1^2 + \mathbf{p}_3^2 - 2|\mathbf{p}_1||\mathbf{p}_3| \cos\theta_{13} \\ &= \mathbf{p}_2^2 + \mathbf{p}_4^2 - 2|\mathbf{p}_2||\mathbf{p}_4| \cos\theta_{24} \;. \end{aligned} \tag{217}$$

In the cm frame, this becomes

$$\begin{aligned} t &= \mathbf{p}_i^2 + \mathbf{p}_f^2 - 2|\mathbf{p}_i||\mathbf{p}_f| \cos\theta_{13} \\ &= \mathbf{p}_i^2 + \mathbf{p}_f^2 - 2|\mathbf{p}_i||\mathbf{p}_f| \cos\theta_{24} \;, \\ \Rightarrow \quad \cos\theta_{13c} &= \cos\theta_{24c} \;, \\ \Rightarrow \quad \theta_{13c} &= \theta_{24c} \;. \end{aligned}$$

4 Relativistic two body kinematics

If the projectile or center of momentum frame moves fast enough, then two different angles in the center of momentum or projectile frames get boosted to only a single angle in the lab frame. In the lab, these two boosts will be distinguished by different lab energies or momenta. This is very important for 3 - dimensional space radiation transport codes which require cross sections in the lab (spacecraft) frame, which must be added incoherently from the two center of momentum or projectile frame angles. For a 2 - body final state, the formula which gives this double - valued lab momentum in terms of the lab angle, is given in equation (272). Other results, such as relating Mandelstam variables to energy and momentum variables, are also derived.

4.1 Introduction

In this section, we wish to consider reactions of the type

$$1 + 2 \to 3 + 4 + 5 + \cdots ,$$

where the numbers denote the particles and the initial state always consists of just two bodies, whereas the final state can contain any number of particles. A 2 - body final state reaction is

$$1 + 2 \to 3 + 4 , \tag{218}$$

whereas a 3 - body final state reaction is

$$1 + 2 \to 3 + 4 + 5 . \tag{219}$$

The reaction kinematics are the same as shown in the previous figures 3 and 4. We are interested in calculating angular and spectral distributions for any of the final state particles. Thus, our primary interest is in the energy or angle of the jth final state particle,

$$E_j \quad \text{or} \quad \theta_j ,$$

which give us the cross sections of primary interest,

$$\frac{d\sigma}{dE_j} \quad \text{or} \quad \frac{d\sigma}{d\theta_j} .$$

If E_j and θ_j are dependent on each other, then $\frac{d\sigma}{dE_j}$ and $\frac{d\sigma}{d\theta_j}$ are also dependent on each other and the doubly differential cross section, $\frac{d^2\sigma}{dE_j\theta_j}$ has no meaning. In this case, the relationship between E_j and θ_j is used to convert between $\frac{d\sigma}{dE_j}$ and $\frac{d\sigma}{d\theta_j}$.

4.1.1 3 - body final state

When there are 3 or more particles in the final state, then E_j and θ_j are independent of each other. One can confirm this by working out the kinematics. In that case, one can form the differential cross sections $\frac{d\sigma}{dE_j}$, $\frac{d\sigma}{d\theta_j}$ and $\frac{d^2\sigma}{dE_j\theta_j}$.

4.1.2 2 - body final state

This situation is more complicated. In the cm frame, E_j and θ_j are independent, but still one cannot form $\frac{d\sigma}{dE_j}$, because in the cm frame, E_j is fixed by the initial energy and the masses; see eqs. 235 and 236. However, θ_j is undetermined and so one can form $\frac{d\sigma}{d\theta_j}$ in the cm frame. In the lab frame, E_j and θ_j are functions of each other and so $\frac{d\sigma}{dE_j}$ and $\frac{d\sigma}{d\theta_j}$ are equivalent to each other. The proof of these statements is the main topic of this section.

4.1.3 Elastic and inelastic scattering

Elastic scattering is defined as that in which *kinetic energy and mass are conserved* [Griffiths 92]. For the reaction with a 2 - body final state, as in equation (218), this will mean that

$$T_1 + T_2 = T_3 + T_4 \ , \tag{220}$$

$$m_1 + m_2 = m_3 + m_4 \ . \tag{221}$$

If particles 3 and 4 are *different* from particles 1 and 2, then it will be *very* unlikely to find particles with just the right masses so that $m_3 + m_4 = m_1 + m_2$. Thus, elastic scattering will almost always mean that ([19] (p. 31), and [20] (p. 403)

$$m_1 = m_3, \qquad m_2 = m_4, \qquad |\mathbf{p}_{ic}| = |\mathbf{p}_{fc}| = |\mathbf{p}| \tag{222}$$

Thus, for all intents and purposes, we have the following statement.

Elastic scattering occurs when the particle identites do not change.

Examples are

$$N + N \rightarrow N + N \ ,$$

and

$$\pi + N \rightarrow \pi + N \ .$$

An example of an inelastic reaction is

$$N + N \rightarrow N + \Delta \ .$$

This can be described with a similar formalism, simply by allowing particle masses to change [7] (p. 100).

4.2 Mandelstam variables

For the reaction

$$1 + 2 \rightarrow 3 + 4 \ ,$$

the Mandelstam variables are defined as

$$s \equiv (p_1 + p_2)^2 = (p_3 + p_4)^2 \ , \tag{223}$$

$$t \equiv (p_1 - p_3)^2 = (p_2 - p_4)^2 \ , \tag{224}$$

$$u \equiv (p_1 - p_4)^2 = (p_2 - p_3)^2 \ . \tag{225}$$

Note that these are Lorentz invariant. An important result for the Mandelstam variables is that the sum of the variables is the sum of the masses squared,

$$s + t + u = m_1^2 + m_2^2 + m_3^2 + m_4^2 \ . \tag{226}$$

4.2.1 Use of the s variable

In the cm frame, $\sqrt{s}$ is the total energy, and

$$\sqrt{s} = E_{1c} + E_{2c} \ . \tag{227}$$

The magnitude of the cm momenta can be written as

$$|\mathbf{p}_{ic}| \equiv |\mathbf{p}_{1c}| = |\mathbf{p}_{2c}| \ , \tag{228}$$

$$|\mathbf{p}_{fc}| \equiv |\mathbf{p}_{3c}| = |\mathbf{p}_{4c}| \ . \tag{229}$$

Defining

$$\lambda_{jk} \equiv \lambda(s, m_j^2, m_k^2) = (s - m_j^2 - m_k^2)^2 - 4m_j^2 m_k^2 \ , \tag{230}$$

one can then prove the following important results.

$$|\mathbf{p}_{ic}| = \sqrt{\frac{\lambda_{12}}{4s}} \ , \tag{231}$$

$$|\mathbf{p}_{fc}| = \sqrt{\frac{\lambda_{34}}{4s}} \ , \tag{232}$$

$$E_{1c} = \frac{s + m_1^2 - m_2^2}{\sqrt{4s}} \ , \tag{233}$$

$$E_{2c} = \frac{s + m_2^2 - m_1^2}{\sqrt{4s}} \ , \tag{234}$$

$$E_{3c} = \frac{s + m_3^2 - m_4^2}{\sqrt{4s}} \ , \tag{235}$$

$$E_{4c} = \frac{s + m_4^2 - m_3^2}{\sqrt{4s}} \ . \tag{236}$$

Note that the energies and momenta of the final state particles, 3 and 4, in the cm frame are fixed. That is, they depend only on the masses and incident energy $\sqrt{s}$. They do not depend on *any angles. This means that one cannot form the spectral distribution $\frac{d\sigma}{dE_j}$ in the cm frame. (j is the label for particles 3 or 4.) Conversely, therefore, the scattering angles θ_{13} and θ_{24} cannot depend on the momenta or energies of the final state particles.*

4.2.2 Use of the t variable

Conservation of 4-momentum is

$$p_1 + p_2 = p_3 + p_4 \ , \tag{237}$$

and so the t variable can be defined and expanded as

$$\begin{aligned} t &\equiv (p_1 - p_3)^2 = (p_2 - p_4)^2 \\ &= m_1^2 + m_3^2 - 2E_1E_3 + 2|\mathbf{p}_1||\mathbf{p}_3|\cos\theta_{13} \\ &= m_2^2 + m_4^2 - 2E_2E_4 + 2|\mathbf{p}_2||\mathbf{p}_4|\cos\theta_{24} \ . \end{aligned} \tag{238}$$

Therefore, a t distribution, $\frac{d\sigma}{dt}$ is equivalent to an angular distribution, $\frac{d\sigma}{d\theta}$. Because the angles θ_{13} and θ_{24} appear directly in t, it is useful in obtaining the relationship between E_j and θ_j as mentioned above. This is discussed below.

4.2.3 Equal mass particles

Now, consider the case when all the particle masses are equal, i.e. $m_1 = m_2 = m_3 = m_4 \equiv m$. One can show that $|\mathbf{p}_{ic}| = |\mathbf{p}_{fc}|$. Label

$$k \equiv |\mathbf{p}_{ic}| = |\mathbf{p}_{fc}| \ , \tag{239}$$

and let the cm scattering angle between particles 1 and 3 be

$$\theta_{13c} \equiv \theta \ . \tag{240}$$

The following results can be shown ([2] (p. 102).

$$s = 4(k^2 + m^2) \ , \tag{241}$$

$$t = -2k^2(1 - \cos\theta) = -4k^2\sin^2(\theta/2) \ , \tag{242}$$

$$u = -2k^2(1 + \cos\theta) = -4k^2\cos^2(\theta/2) \ . \tag{243}$$

4.3 Relations between E_j and θ_j

4.3.1 Center of momentum frame

In the cm frame, we have

$$\mathbf{p}_1 + \mathbf{p}_2 \equiv 0 \ , \quad \text{and} \quad \mathbf{p}_3 + \mathbf{p}_4 \equiv 0 \ , \tag{244}$$

showing that the relative angle θ_{34} between the final state particles 3 and 4 is $\theta_{34} = 180°$. Thus, from figure 4, we see that

$$\theta_{13c} = \theta_{24c} \equiv \theta \ . \tag{245}$$

Also,

$$|\mathbf{p}_1| = |\mathbf{p}_2| \equiv |\mathbf{p}_i| \quad \text{and} \quad |\mathbf{p}_3| = |\mathbf{p}_4| \equiv |\mathbf{p}_f| \ . \tag{246}$$

We can use the t variable to show that

$$E_{3c} = \frac{s + m_3^2 - m_4^2}{\sqrt{4s}} \ . \tag{247}$$

This can be proved as follows. Use of equation (238) gives

$$\begin{aligned} t &\equiv (p_1 - p_3)^2 = (p_2 - p_4)^2 \\ &= m_1^2 + m_3^2 - 2E_1E_3 + 2|\mathbf{p}_i||\mathbf{p}_f| \cos\theta \\ &= m_2^2 + m_4^2 - 2E_2E_4 + 2|\mathbf{p}_i||\mathbf{p}_f| \cos\theta \ , \end{aligned} \tag{248}$$

and the $\cos\theta$ term cancels out on both sides, so that

$$m_1^2 + m_3^2 - 2E_1E_3 = m_2^2 + m_4^2 - 2E_2E_4 \ . \tag{249}$$

Using the results

$$E_1 + E_2 = \sqrt{4s} \ , \tag{250}$$

and

$$s + m_3^2 - m_4^2 = m_1^2 + m_3^2 - m_2^2 - m_4^2 + 2E_2(E_1 + E_2) \ , \tag{251}$$

which gives equation (247). Thus, we have used the t variable to evaluate the relation between E_j and θ_j in the cm frame. We have found that the relation is trivial, in that E_3 does not depend on θ_{13}. We use this same technique below to find the (non-trivial) relation between E_j and θ_j in the lab frame.

4.3.2 Lab frame

Conservation of 4-momentum gives

$$p_1 + p_2 = p_3 + p_4 \ , \tag{252}$$

$$t = (p_1 - p_3)^2 = (p_2 - p_4)^2 \ , \tag{253}$$

$$t = m_1^2 + m_3^2 - 2E_1E_3 + 2|\mathbf{p}_1||\mathbf{p}_3| \cos\theta_{13} = m_2^2 + m_4^2 - 2m_2E_{4l} \ , \tag{254}$$

or

$$t = m_1^2 + m_3^2 - 2E_1E_3 + 2\sqrt{E_1^2 - m_1^2}\sqrt{E_3^2 - m_3^2}\ \cos\theta_{13}\ . \tag{255}$$

Eliminating E_{4l}, using conservation of energy, $E_{4l} = E_1 + m_2 - E_3$ in the lab frame gives equation(254) as [10] (p. 102)

$$\begin{aligned} E_{3l}(E_{1l} + m_2) - |\mathbf{p}_{1l}||\mathbf{p}_{3l}|\cos\theta_{13l} &= E_{1l}m_2 + \frac{1}{2}(m_1^2 + m_2^2 + m_3^2 - m_4^2) \\ &= \frac{1}{2}(s + m_3^2 - m_4^2)\ , \qquad \text{(lab frame)}, \end{aligned} \tag{256}$$

with $|\mathbf{p}_{1l}| \equiv \sqrt{E_{1l}^2 - m_1^2}$ and $|\mathbf{p}_{3l}| \equiv \sqrt{E_{3l}^2 - m_3^2}$. We have also used $s = m_1^2 + m_2^2 + 2m_2E_{1l}$. This is in agreement with references [10] (equation 4.65) and [7] (p. 127). Recall that when we did this analysis for the cm case, the $\cos\theta$ term canceled out on both sides. However, here for the lab case, no such cancelation occurs so that $\cos\theta$ remains, and this will give rise to the non-trivial relation between E_j and θ_j. We now solve this important quadratic equation. Defining

$$a \equiv \frac{1}{2}(m_1^2 + m_2^2 + m_3^2 - m_4^2 + 2m_2E_{1l}) = \frac{1}{2}(s + m_3^2 - m_4^2)\ , \tag{257}$$

$$b \equiv E_{1l} + m_2\ , \tag{258}$$

$$c \equiv |\mathbf{p}_{1l}|\cos\theta_{13l} = \sqrt{E_{1l}^2 - m_1^2}\ \cos\theta_{13l}\ , \tag{259}$$

enables equation (256) to be re-written as

$$bE_{3l} - c|\mathbf{p}_{3l}| = a\ , \tag{260}$$

which is written either in terms of E_{3l} as

$$bE_{3l} - c\sqrt{E_{3l}^2 - m_3^2} = a\ , \tag{261}$$

or in terms of $|\mathbf{p}_{3l}|$ as

$$b\sqrt{|\mathbf{p}_{3l}|^2 + m_3^2} - c|\mathbf{p}_{3l}| = a\ . \tag{262}$$

Firstly, we solve equation (261). Squaring gives

$$(b^2 - c^2)E_{3l}^2 - 2abE_{3l} + (a^2 + c^2m_3^2) = 0\ , \tag{263}$$

which has the solution

$$E_{3l} = \frac{ab \pm c\sqrt{a^2 - m_3^2(b^2 - c^2)}}{b^2 - c^2}\ , \tag{264}$$

or

$$E_{3l} = \frac{ab \pm |\mathbf{p}_{1l}| \cos\theta_{13l} \sqrt{a^2 - m_3^2 \, (b^2 - \mathbf{p}_{1l}^2 \cos^2\theta_{13l})}}{b^2 - \mathbf{p}_{1l}^2 \cos^2\theta_{13l}} \ , \tag{265}$$

with $|\mathbf{p}_{1l}| \equiv \sqrt{E_{1l}^2 - m_1^2}$. This equation agrees with Byckling [14] (equation 74) and Jackson [20] (equation 12.53). E_{3l} has maximum and minimum values. Based on energy conservation alone, upper and lower bounds are given by

$$E_{3l}^{\text{lower}} = m_3 \ , \tag{266}$$

and from energy conservation

$$\begin{aligned} E_3 &= E_1 + E_2 - E_4 \ , & (267) \\ E_{3l} &= E_{1l} + m_2 - E_{4l} \ , & (268) \end{aligned}$$

which implies that

$$E_{3l}^{\text{upper}} = E_{1l} + m_2 - E_{4l} = E_{1l} + m_2 - m_4 \ . \tag{269}$$

Note that if $m_2 = m_4$, then $E_{3l}^{\text{max}} = E_{1l}$, which makes sense. The actual maximum and minimum values of energy are determined by conservation of both energy and momentum, and are found be setting $\theta_{13l} = 0$ in equation 265. Equations (256 - 265) express the fundamental relation connecting E_{3l} to θ_{13l}. Given the fact that such a relation exists clearly shows that the angle is determined by the energy and vice versa. Thus, one *cannot* form $\frac{d\sigma}{dT}$ and $\frac{d\sigma}{d\Omega}$ *independently.* $\frac{d\sigma}{dT}$ and $\frac{d\sigma}{d\Omega}$ are *equivalent* to each other. Note, however, that we were unable to form $\frac{d\sigma}{dT}$ in the cm frame because the final momenta are constant and are not related to the angle. Secondly, we solve equation (262). Squaring gives

$$(c^2 - b^2)|\mathbf{p}_{3l}|^2 + 2ac|\mathbf{p}_{3l}| + (a^2 - b^2 m_3^2) = 0 \ , \tag{270}$$

which has the solution

$$|\mathbf{p}_{3l}| = \frac{-ac \pm b\sqrt{a^2 + m_3^2(c^2 - b^2)}}{c^2 - b^2} \ , \tag{271}$$

or [21] (p. 52)

$$|\mathbf{p}_{3l}| = \frac{-a|\mathbf{p}_{1l}| \cos\theta_{13l} \pm b\sqrt{a^2 + m_3^2 \, [\, |\mathbf{p}_{1l}|^2 \cos^2\theta_{13l} - (E_{1l} + m_2)^2 \,]}}{|\mathbf{p}_{1l}|^2 \cos^2\theta_{13l} - (E_{1l} + m_2)^2} \ . \tag{272}$$

Physical solutions are found more readily using the $|\mathbf{p}_{3l}|$ *solutions in (272) rather than the* E_{3l} *solutions in (265).* The reason is as follows. The quantity

$$\alpha_{3c} \equiv \frac{\beta_{cl}}{\beta_{3c}} , \tag{273}$$

is the speed of the cm frame with respect to the lab divided by the speed of particle 3 in the cm frame. As pointed out by Jackson [20] (p. 403), two roots are allowed in equation (265) for $\alpha_{3c} > 1$ but only one root for $\alpha_{3c} < 1$. Also, a physical constraint is that the solutions must obey $E_{3l} > m_3$. However, for certain cases one obtains *both* roots no matter what the value of α_{3c}, even though in both cases $E_{3l} > m_3$. Clearly, $E_{3l} > m_3$ *is not enough to constrain the physical solutions* [10]. The physical constraint on the magnitude of momentum is $|\mathbf{p}_{3l}| > 0$. In the cases cited above, when one obtains two E_{3l} roots for $\alpha_{3c} < 1$, one finds that equation (272) gives one of the two $|\mathbf{p}_{3l}|$ solutions as negative. (Of course one obtains the E_{3l} solutions from $|\mathbf{p}_{3l}|$ by substituting into $E_{3l} = \sqrt{|\mathbf{p}_{3l}|^2 + m_3^2}$, using both the positive and negative values of $|\mathbf{p}_{3l}|$.) The negative solution is clearly unphysical and therefore one only has *one* physical solution when $\alpha_{3c} < 1$. In other words, the use of the $|\mathbf{p}_{3l}|$ solutions in (272) rather than the E_{3l} solutions in (265) enables us to immediately eliminate the unphysical solutions and we then see that two roots are allowed for $\alpha_{3c} > 1$ but only one root for $\alpha_{3c} < 1$.

An example of an unphysical case discussed above occurs with the input parameters $m_1 = m_3 = 1, m_2 = m_4 = 2, E_{1l} = 2m_1, \theta_{13l} = \pi/7$, resulting in $\alpha_{3c} = .625, E_{3l}^+ = 1.86344, E_{3l}^- = 1.08538, |\mathbf{p}_{3l}^+| = 1.57238, |\mathbf{p}_{3l}^-| = -0.42196$. As a final point, note that equation (270) can be written as

$$A|\mathbf{p}_{3l}|^2 - 2B|\mathbf{p}_{3l}| - C = 0 , \tag{274}$$

which has the solution

$$|\mathbf{p}_{3l}| = \frac{B \pm \sqrt{B^2 + AC}}{A} \equiv \frac{B \pm D}{A} , \tag{275}$$

where

$$\begin{aligned}
D &\equiv \sqrt{B^2 + AC} , && (276)\\
E &\equiv E_{1l} + m_2 , && (277)\\
K &\equiv \frac{1}{2}(m_1^2 + m_2^2 + m_3^2 - m_4^2 + 2m_2E_{1l}) = \frac{1}{2}(s + m_3^2 - m_4^2) , && (278)\\
A &\equiv E^2 - |\mathbf{p}_{1l}|^2 \cos^2\theta_{13l} = s + |\mathbf{p}_{1l}|^2 \sin^2\theta_{13l} , && (279)\\
B &\equiv K|\mathbf{p}_{1l}|\cos\theta_{13l} , && (280)\\
C &\equiv K^2 - m_3^2E^2 . && (281)
\end{aligned}$$

C can be re-written as

$$4C = (s + m_3^2 - m_4^2)^2 - 4m_3^2(E_{1l} + m_2)^2$$

$$= \quad (s+m_3^2-m_4^2)^2 - \frac{m_3^2}{m_2^2}(s+m_2^2-m_1^2)^2 \; , \tag{282}$$

or

$$4Cm_2^2 \quad = \quad m_2^2(s+m_3^2-m_4^2)^2 - m_3^2(s+m_2^2-m_1^2)^2 \; . \tag{283}$$

Now, it turns out that

$$\alpha_{3c} > 1 \qquad \text{iff} \qquad C < 0 \; , \qquad \text{=> 2 solutions,} \tag{284}$$

$$\alpha_{3c} \leq 1 \qquad \text{iff} \qquad C > 0 \; , \qquad \text{=> 1 solution.} \tag{285}$$

Here, the mathematical symbol iff means "if and only if". α_{3c} is given in equation (445) as

$$\alpha_{3c} \equiv \frac{\beta_c}{\beta_{3c}} \quad = \quad \frac{s+m_3^2-m_4^2}{s-m_1^2+m_2^2}\sqrt{\frac{\lambda_{12}}{\lambda_{34}}} \; . \tag{286}$$

Note that $C(s) = 0$ has the solutions

$$s_1 \quad = \quad \frac{m_2(m_4^2-m_3^2)+m_3(m_1^2-m_2^2)}{m_2+m_3} \; , \tag{287}$$

$$s_2 \quad = \quad \frac{m_2(m_4^2-m_3^2)-m_3(m_1^2-m_2^2)}{m_2-m_3} \; , \tag{288}$$

and $\alpha_{3c}(s) = 1$ has the same solutions, showing that $C = 0$ iff $\alpha_{3c} = 1$, which is consistent with (285). One can now vary s and study the behavior of C and α_{3c}. One can check (285) numerically by doing a parametric plot of $C(s)$ and $\alpha_{3c}(s)$, letting s vary.

4.3.3 Proof of equation (285)

First, note the following.

$$A \quad > \quad 0 \;\; \text{always,} \tag{289}$$

$$B^2 \quad > \quad 0 \;\; \text{always,} \tag{290}$$

$$K \quad > \quad 0 \;\; \text{always.} \tag{291}$$

Equation (291) can be seen by noting that the minimum value of $K = \frac{1}{2}(s+m_3^2-m_4^2)$ occurs at the minimum value of s, which is at threshold where $s_{\text{threshold}} = (m_3+m_4)^2$. (Note that s is Lorentz invariant so that this relation is true in any frame.) Thus, the minimum value of K is $K_{\text{threshold}} = \frac{1}{2}[(m_3+m_4)^2+m_3^2-m_4^2] = m_3(m_3+m_4)$, which is always > 0. Given $D \equiv \sqrt{B^2+AC}$, it follows that (with $|B| = \sqrt{B^2}$)

$$D > |B| \quad \text{iff} \quad C > 0, \tag{292}$$

$$D < |B| \quad \text{iff} \quad C < 0. \tag{293}$$

Secondly, note that we need to consider forward and backward scattering separately because of the following.

Forward scattering:

$$-\frac{\pi}{2} < \theta_{13l} < \frac{\pi}{2} \quad => \quad \cos\theta_{13l} > 0 \;=>\; B > 0.$$

Backward scattering:

$$\theta_{13l} < -\frac{\pi}{2} \;\text{ or }\; \theta_{13l} > \frac{\pi}{2} \quad => \quad \cos\theta_{13l} < 0 \;=>\; B < 0.$$

Now, write the solutions as

$$|\mathbf{p}_{3l}^{+}| = \frac{B+D}{A}\,, \tag{294}$$

$$|\mathbf{p}_{3l}^{-}| = \frac{B-D}{A}\,, \tag{295}$$

and note that

$$D > 0 \quad \text{always.}$$

i) First, consider the case of forward scattering, i.e. $B > 0$.

$$B > 0,\; A > 0 \text{ always},\; D > 0 \text{ always}, \;\; => |\mathbf{p}_{3l}^{+}| > 0,$$

and, with $B > 0$, $A > 0$ always, equations (292) and (293) imply that

$$|\mathbf{p}_{3l}^{-}| > 0 \;\; \text{(allowed physically)} \;\; \text{if} \;\; |\mathrm{B}| > \mathrm{D} \quad => \quad C < 0,$$
$$|\mathbf{p}_{3l}^{-}| < 0 \;\; \text{(not allowed physically)} \;\; \text{if} \;\; \mathrm{D} > |\mathrm{B}| \quad => \quad C > 0.$$

Thus,

$$\text{if } \; C < 0 \quad => \quad \text{both roots } |\mathbf{p}_{3l}^{\pm}| \text{ are allowed,}$$
$$\text{if } \; C > 0 \quad => \quad \text{only one root } |\mathbf{p}_{3l}^{+}| \text{ is allowed.}$$

ii) Now, consider the case of backward scattering, i.e. $B < 0$.

$$B < 0,\; A > 0 \text{ always},\; D > 0 \text{ always}, \;\; => |\mathbf{p}_{3l}^{-}| < 0$$

and, with $B < 0$, $A > 0$ always, equations (292) and (293) imply that

$$|\mathbf{p}_{3l}^{+}| > 0 \;\; \text{(allowed physically)} \;\; \text{if} \;\; \mathrm{D} > |\mathrm{B}| \quad => \quad C > 0,$$
$$|\mathbf{p}_{3l}^{+}| < 0 \;\; \text{(not allowed physically)} \;\; \text{if} \;\; \mathrm{D} < |\mathrm{B}| \quad => \quad C < 0.$$

Thus,

$$\begin{array}{lll} \text{If } C < 0 & => & \text{both roots } |\mathbf{p}^{\pm}_{3l}| \text{ are not allowed.} \\ \text{If } C > 0 & => & \text{only one root } |\mathbf{p}^{+}_{3l}| \text{ is allowed.} \end{array}$$

iii) Now, combine the results for both forward and backward scattering and summarize as follows.

$$\begin{array}{lll} \text{If } C < 0 & => & \text{both roots } |\mathbf{p}^{\pm}_{3l}| \text{ are either both allowed(forward scattering),} \\ & & \text{or both not allowed (backward scattering).} \\ \text{If } C > 0 & => & \text{only one root } |\mathbf{p}^{+}_{3l}| \text{ is allowed,} \\ & & \text{(for both forward and backward scattering).} \end{array}$$

4.4 Further results

4.4.1 Momenta in the cm frame

We wish to prove that $|\mathbf{p}_{fc}| = \sqrt{\frac{\lambda_{34}}{4s}}$ and $|\mathbf{p}_{ic}| = \sqrt{\frac{\lambda_{12}}{4s}}$. Note that using

$$F_{any\ frame} = \frac{1}{2}\sqrt{\lambda_{12}}\ , \tag{296}$$

these equations imply that

$$F_c = \sqrt{s}\,|\mathbf{p}_{1c}| = (E_1 + E_2)|\mathbf{p}_{1c}|\ . \tag{297}$$

We proceed with the proofs as follows. Eliminate $\mathbf{p}_4$ in terms of the incident energy. Write the Mandelstam variable

$$\begin{aligned} s &\equiv (p_1 + p_2)^2 = (p_3 + p_4)^2 \\ &= (E_1 + E_2)^2 - (\mathbf{p}_1 + \mathbf{p}_2)^2 = (E_3 + E_4)^2 - (\mathbf{p}_3 + \mathbf{p}_4)^2\ . \end{aligned} \tag{298}$$

In the cm frame, $\mathbf{p}_1 + \mathbf{p}_2 = 0 = \mathbf{p}_3 + \mathbf{p}_4$ and $|\mathbf{p}_1| = |\mathbf{p}_2| \equiv |\mathbf{p}_{ic}|$ and $|\mathbf{p}_3| = |\mathbf{p}_4| \equiv |\mathbf{p}_{fc}|$. Thus,

$$\begin{aligned} s &= (E_1 + E_2)^2 = (E_3 + E_4)^2 = E_3^2 + E_4^2 + 2E_3E_4 \\ &= \mathbf{p}^2_{fc} + m_3^2 + \mathbf{p}^2_{fc} + m_4^2 + 2\sqrt{(\mathbf{p}^2_{fc} + m_3^2)(\mathbf{p}^2_{fc} + m_4^2)}\ , \end{aligned} \tag{299}$$

and

$$s - m_3^2 - m_4^2 - 2\mathbf{p}^2_{fc} = 2\sqrt{(\mathbf{p}^2_{fc} + m_3^2)(\mathbf{p}^2_{fc} + m_4^2)}\ . \tag{300}$$

Squaring both sides gives

$$\begin{aligned} &(s - m_3^2 - m_4^2)^2 - 4\mathbf{p}^2_{fc}(s - m_3^2 - m_4^2) + 4\mathbf{p}^4_{fc} \\ &\qquad = 4[\mathbf{p}^4_{fc} + \mathbf{p}^2_{fc}(m_3^2 + m_4^2) + m_3^2 m_4^2]\ , \end{aligned} \tag{301}$$

and

$$
\begin{aligned}
4\mathbf{p}_{fc}^2 s &= (s - m_3^2 - m_4^2)^2 - 4m_3^2 m_4^2 \\
&= \lambda(s, m_3^2, m_4^2) = \lambda_{34} \ ,
\end{aligned}
\tag{302}
$$

which gives [10] (p. 100)

$$
|\mathbf{p}_{fc}| = \sqrt{\frac{\lambda_{34}}{4s}} \ . \tag{303}
$$

Similarly,

$$
|\mathbf{p}_{ic}| = \sqrt{\frac{\lambda_{12}}{4s}} \ . \tag{304}
$$

4.4.2 Energies in the cm frame

Energies E_{1c}, E_{3c} and E_{4c} may be expressed in terms of the Mandelstam invariant s,

$$
s = (p_1 + p_2)^2 = (E_1 + E_2)^2 - (\mathbf{p}_1 + \mathbf{p}_2)^2 \ . \tag{305}
$$

In the cm frame (with $|\mathbf{p}_1| = |\mathbf{p}_2| \equiv |\mathbf{p}|$),

$$
\sqrt{s} = E_1 + E_2 \quad \textit{with} \quad E_1^2 = \mathbf{p}^2 + m_1^2 \quad \textit{and} \quad E_2^2 = \mathbf{p}^2 + m_2^2 \ . \tag{306}
$$

Thus,

$$
E_2^2 = E_1^2 - m_1^2 + m_2^2 \ , \tag{307}
$$

giving

$$
\begin{aligned}
\sqrt{s} - E_1 &= E_2 \\
&= \sqrt{E_1^2 - m_1^2 + m_2^2}
\end{aligned}
\tag{308}
$$

$$
\begin{aligned}
(\sqrt{s} - E_1)^2 &= E_1^2 - m_1^2 + m_2^2 \\
&= s - \sqrt{4s} E_1 + E_1^2 \ .
\end{aligned}
\tag{309}
$$

Thus,

$$
s - \sqrt{4s} E_1 = -m_1^2 + m_2^2 \ , \tag{310}
$$

giving

$$
E_{1c} = \frac{s + m_1^2 - m_2^2}{\sqrt{4s}} \ . \tag{311}
$$

The result for E_{3c} and E_{4c} are derived similarly [12] (p. 15). One can now calculate

$$|\mathbf{p}_{1c}| = \sqrt{E_{1c}^2 - m_1^2}\ , \tag{312}$$

and the result agrees with the previously derived equation,

$$|\mathbf{p}_{ic}| = \sqrt{\frac{\lambda_{12}}{4s}}\ . \tag{313}$$

4.4.3 Bounds on Mandelstam variable t

We wish to obtain formulas for t_0 and t_π. Specifically, we will prove that

$$\begin{aligned} t_0(t_\pi) &= (E_{1c} - E_{3c})^2 - (|\mathbf{p}_{1c}| \mp |\mathbf{p}_{3c}|)^2 \\ &= \left[\frac{m_1^2 - m_2^2 - m_3^2 + m_4^2}{\sqrt{4s}}\right]^2 - (|\mathbf{p}_{1c}| \mp |\mathbf{p}_{3c}|)^2 \\ &= \frac{1}{4s}[(m_1^2 - m_2^2 - m_3^2 + m_4^2)^2 - (\sqrt{\lambda_{12}} \mp \sqrt{\lambda_{34}})^2]\ . \end{aligned} \tag{314}$$

The notation $\mp$ means that the equation for t_0 has the $-$ sign and the equation for t_π has the $+$ sign. We proceed with the proof as follows. The Mandelstam variable is

$$\begin{aligned} t &= (p_1 - p_3)^2 \\ &= (E_1 - E_3)^2 - (\mathbf{p}_1 - \mathbf{p}_3)^2\ , \end{aligned} \tag{315}$$

and we use

$$\begin{aligned} (\mathbf{p}_1 - \mathbf{p}_3)^2 &= |\mathbf{p}_1|^2 + |\mathbf{p}_3|^2 - 2\mathbf{p}_1.\mathbf{p}_3 \\ &= |\mathbf{p}_1|^2 + |\mathbf{p}_3|^2 - 2|\mathbf{p}_1||\mathbf{p}_3|\cos\theta \\ &= |\mathbf{p}_1|^2 + |\mathbf{p}_3|^2 - 2|\mathbf{p}_1||\mathbf{p}_3|(1 - 2\sin^2\frac{\theta}{2}) \\ &= |\mathbf{p}_1|^2 + |\mathbf{p}_3|^2 - 2|\mathbf{p}_1||\mathbf{p}_3| + 4|\mathbf{p}_1||\mathbf{p}_3|\sin^2\frac{\theta}{2} \\ &= (|\mathbf{p}_1| - |\mathbf{p}_3|)^2 + 4|\mathbf{p}_1||\mathbf{p}_3|\sin^2\frac{\theta}{2}\ . \end{aligned} \tag{316}$$

With the definitions, $t_0 \equiv t(\theta_c = 0)$ and $t_\pi \equiv t(\theta_c = \pi)$, we get

$$t_0(t_\pi) = (E_{1c} - E_{3c})^2 - (|\mathbf{p}_{1c}| \mp |\mathbf{p}_{3c}|)^2 \tag{317}$$

as desired. For the proof of the second equation above, use the previously derived results,

$$E_{1c} = \frac{s + m_1^2 - m_2^2}{\sqrt{4s}} \qquad \text{and} \qquad \mathrm{E}_{3c} = \frac{\mathrm{s} + \mathrm{m}_3^2 - \mathrm{m}_4^2}{\sqrt{4\mathrm{s}}}\ , \tag{318}$$

to give

$$(E_{1c} - E_{3c})^2 = \frac{m_1^2 - m_2^2 - m_3^2 + m_4^2}{\sqrt{4s}} \ , \tag{319}$$

which results in

$$\begin{aligned} t &= (E_{1c} - E_{3c})^2 - (\mathbf{p}_{1c} - \mathbf{p}_{3c})^2 \\ &= (E_{1c} - E_{3c})^2 - (|\mathbf{p}_1| - |\mathbf{p}_3|)^2 - 4|\mathbf{p}_1||\mathbf{p}_3| \sin^2 \frac{\theta_c}{2} \ . \end{aligned} \tag{320}$$

With the definitions $t_0 \equiv t(\theta_c = 0)$ and $t_\pi \equiv t(\theta_c = \pi)$, we obtain equation (314) as desired.

4.4.4 Momentum and energy thresholds

We now obtain lab energy and momentum thresholds. Consider the 2 - body reaction

$$1 + 2 \rightarrow 3 + 4 \ . \tag{321}$$

We wish to prove the following results.

(a) If the lab momentum of particle 1 is $\mathbf{p}_{1l}$, we will derive expressions for $s \equiv (p_1 + p_2)^2 \equiv (p_3 + p_4)^2$ in terms of E_{1l}, T_{1l} and $\mathbf{p}_{1l}$.

(b) We will obtain the threshold value of s needed to produce the final state particles.

(c) We will obtain the threshold value of T_{1l} needed to produce the final state particles.

We now proceed with the proofs as follows.

(a) Start with

$$\begin{aligned} s &\equiv (p_1 + p_2)^2 v \\ &= (E_1 + E_2)^2 - (\mathbf{p}_1 + \mathbf{p}_2)^2 \ . \end{aligned} \tag{322}$$

In the lab frame, $\mathbf{p}_2 \equiv 0$ and therefore $E_{2l} = m_2$. Thus, in the lab frame

$$\begin{aligned} s &= (E_{1l} + m_2)^2 - \mathbf{p}_1^2 \\ &= E_{1l}^2 + 2m_2E_{1l} + m_2^2 - \mathbf{p}_1^2 \\ &= m_1^2 + m_2^2 + 2m_2E_{1l} = m_1^2 + m_2^2 + 2m_2\sqrt{\mathbf{p}_{1l}^2 + m_1^2} \\ &= (m_1 + m_2)^2 + 2m_2T_{1l} \ . \end{aligned} \tag{323}$$

(b) s is invariant. In the cm frame, $s = (E_1 + E_2)^2 = (E_3 + E_4)^2$ and so the threshold value is

$$s_{\text{threshold}} = (m_3 + m_4)^2 \tag{324}$$

when the final state particles are all produced at rest.

(c) Substituting into the equation above, we get

$$T_{1l\,\text{threshold}} = \frac{(m_3+m_4)^2-(m_1+m_2)^2}{2m_2}\ . \tag{325}$$

4.4.5 Relative scattering angle in the cm frame

Now consider the relative angle between the two final state particles. Consider the reaction $1+2 \to 3+4$. We can obtain some more useful results.

(a) We show that in the center of momentum frame, the particles 3 and 4 scatter at a relative angle of $\theta_{34c} = 180°$. We show this is true in both the non-relativistic and relativistic cases.

(a) Now, consider the lab frame. We will show that

$$\mathbf{p}_1^2 = \mathbf{p}_3^2 + \mathbf{p}_4^2 + 2|\mathbf{p}_3||\mathbf{p}_4|\cos\theta_{34l}\ . \tag{326}$$

where θ_{34l} is the relative scattering angle between particles 3 and 4 in lab frame, and $\mathbf{p}$ refers to the particle 3-momenta. We show this equation is true in both the non-relativistic and relativistic cases.

(c) Now, consider part b) for the case of *elastic* scattering, where *all particle masses are equal.* In the *non-relativistic* case, we will show that the lab angle is $\theta_{34l} = 90°$. (Note: In the non-relativistic case, elastic scattering is defined as conservation of kinetic energy.)

(d) Again, consider part b) for the case of *elastic* scattering, where *all particle masses are equal.* This time consider the *relativistic* case. Now, the scattering angle between the final particles is not 90°, but is instead given by

$$\cos\theta_{34l} = \frac{T_3T_4}{|\mathbf{p}_3||\mathbf{p}_4|} \quad = \quad \frac{T_3(T_1-T_3)}{\sqrt{T_3^2+2mT_3}\sqrt{(T_1-T_3)^2+2m(T_1-T_3)}}\ , \tag{327}$$

where T_3 and T_4 are the kinetic energies of particles 3 and 4 and $|\mathbf{p}_3|$ and $|\mathbf{p}_4|$ are the magnitudes of their 3-momenta. We prove this result for the relativistic case. Also, we show how this reduces to the non-relativistic result $\theta_{34l} = 90°$ for small speeds. (Note: In the relativistic case, elastic scattering is defined as conservation of kinetic energy and mass. Small speed is equivalent to mass being much bigger than kinetic energy.)

We now prove these results as follows.

(a) The cm frame, in both the non-relativistic and relativistic cases, is defined by

$$\mathbf{p}_3 + \mathbf{p}_4 = 0\ , \tag{328}$$

giving

$$\mathbf{p}_3 = -\mathbf{p}_4\ , \tag{329}$$

implying that the relative angle in the center of momentum frame is

$$\theta_{34c} = 180^\circ \ . \tag{330}$$

Alternative method:

$$\mathbf{p}_3 = -\mathbf{p}_4 \ . \tag{331}$$

Define

$$\mathbf{p}_3^2 = \mathbf{p}_4^2 \equiv \mathbf{p}_f^2 \ . \tag{332}$$

Thus,

$$\begin{aligned} (\mathbf{p}_3 + \mathbf{p}_4)^2 = 0 &= \mathbf{p}_3^2 + \mathbf{p}_4^2 + 2|\mathbf{p}_3||\mathbf{p}_4| \cos\theta_{34c} \\ &= 2\mathbf{p}_f^2 + 2\mathbf{p}_f^2 \cos\theta_{34c} \ . \end{aligned} \tag{333}$$

Therefore,

$$2\mathbf{p}_f^2 = -2\mathbf{p}_f^2 \cos\theta_{34c} \tag{334}$$

$$\cos\theta_{34c} = -1 \tag{335}$$

$$\theta_{34c} = 180^\circ \ , \quad \text{in the cm frame} \ . \tag{336}$$

This is true in both the non-relativistic and relativistic cases. No assumptions have been made concerning mass or whether or not the scattering is elastic.

(b) The lab frame in both the non-relativistic and relativistic cases is defined by

$$\mathbf{p}_2 = 0 \ . \tag{337}$$

Conservation of 3-momentum becomes

$$\mathbf{p}_1 + \mathbf{p}_2 = \mathbf{p}_3 + \mathbf{p}_4 \tag{338}$$

$$\mathbf{p}_1 = \mathbf{p}_3 + \mathbf{p}_4 \ , \quad \text{lab frame} \ . \tag{339}$$

$$=> \quad \mathbf{p}_1^2 = \mathbf{p}_3^2 + \mathbf{p}_4^2 + 2|\mathbf{p}_3||\mathbf{p}_4| \cos\theta_{34l} \ . \tag{340}$$

(c) In the *non-relativistic* case, elastic scattering is defined by conservation of kinetic energy,

$$T_1 = T_3 + T_4 \ , \qquad \text{lab frame.} \tag{341}$$

$$\frac{\mathbf{p}_1^2}{2m} = \frac{\mathbf{p}_3^2}{2m} + \frac{\mathbf{p}_4^2}{2m} \tag{342}$$

$$=> \quad \mathbf{p}_1^2 = \mathbf{p}_3^2 + \mathbf{p}_4^2 \ . \tag{343}$$

Substituting into equation (340) gives $\cos\theta_{34l} = 0$ or $\theta_{34l} = 90^\circ$.

(d) In the *relativistic* case, total energy is *always* conserved,

$$E_1 + E_2 = E_3 + E_4 \ , \tag{344}$$
$$T_1 + m_1 + T_2 + m_2 = T_3 + m_3 + T_4 + m_4 \ . \tag{345}$$

In relativistic elastic collisions, kinetic energy, rest energy and mass are all conserved. Thus, for relativistic elastic collisions

$$T_1 + T_2 = T_3 + T_4 \ , \tag{346}$$
$$m_1 + m_2 = m_3 + m_4 \ . \tag{347}$$

The second equation is of course true by our assumption of equal mass particles. In the lab frame,

$$T_1 = T_3 + T_4 \qquad \text{lab frame} \tag{348}$$

which is the same as the non-relativistic case. However, now the expressions for kinetic energy are different, namely

$$\sqrt{\mathbf{p}_1^2 + m} - m = \sqrt{\mathbf{p}_3^2 + m} - m + \sqrt{\mathbf{p}_4^2 + m} - m \ , \qquad \text{lab frame} \ , \tag{349}$$

$$\begin{aligned} \sqrt{\mathbf{p}_1^2 + m} + m &= \sqrt{\mathbf{p}_3^2 + m} + \sqrt{\mathbf{p}_4^2 + m} \\ = E_1 + m &= E_3 + E_4 \ , \end{aligned} \tag{350}$$

where we could have obtained this equation right away by setting $E_2 = m$ in equation (344). This all illustrates how the conservation of energy and kinetic energy work out in the relativistic case, and we have written this out for comparison to the non-relativstic results. However, to actually get a result in terms of the angle θ_{34l}, it is easier to work with 4-vectors. Thus,

$$(p_1 + p_2)^2 = (p_3 + p_4)^2 \ , \tag{351}$$
$$m_1^2 + m_2^2 + 2(E_1 E_2 - \mathbf{p}_1 \cdot \mathbf{p}_2) = m_3^2 + m_4^2 + 2(E_3 E_4 - \mathbf{p}_3 \cdot \mathbf{p}_4) \ . \tag{352}$$

Let $m_1 = m_2 = m_3 = m_4 \equiv m$. This implies elastic scattering, because mass is conserved and as shown above, it implies conservation of kinetic energy. This gives

$$E_1 E_2 - \mathbf{p}_1 \cdot \mathbf{p}_2 = E_3 E_4 - \mathbf{p}_3 \cdot \mathbf{p}_4 \ . \tag{353}$$

The lab frame is defined with $\mathbf{p}_2 = 0$, giving

$$mE_1 = E_3 E_4 - |\mathbf{p}_3||\mathbf{p}_4| \cos\theta_{34l} \tag{354}$$
$$\cos\theta_{34l} = \frac{E_3 E_4 - mE_1}{|\mathbf{p}_3||\mathbf{p}_4|} \ . \tag{355}$$

Conservation of energy gives $E_1 + E_2 = E_3 + E_4$ which, in the lab frame, is $E_1 + m = E_3 + E_4$, or $E_1 = E_3 + E_4 - m$. Substitution gives

$$\cos\theta_{34l} \quad = \quad \frac{T_3 T_4}{|\mathbf{p}_3||\mathbf{p}_4|} \ . \tag{356}$$

This is not the final answer, however, because $T_1 = T_3 + T_4$ giving $T_4 = T_1 - T_3$. Using

$$\mathbf{p}^2 = E^2 - m^2 = (T+m)^2 - m^2 = T^2 + 2mT \ , \tag{357}$$

gives

$$\mathbf{p}_3^2 = T_3^2 + 2mT_3 \ , \tag{358}$$

and

$$\mathbf{p}_4^2 = T_4^2 + 2mT_4 = (T_1 - T_3)^2 + 2m(T_1 - T_3) \ . \tag{359}$$

Thus,

$$\cos\theta_{34l} = \frac{T_3 T_4}{|\mathbf{p}_3||\mathbf{p}_4|} \quad = \quad \frac{T_3(T_1 - T_3)}{\sqrt{T_3^2 + 2mT_3}\sqrt{(T_1 - T_3)^2 + 2m(T_1 - T_3)}} \ . \tag{360}$$

In the non-relativistic limit, energy is dominated by rest mass energy. In other words,

$$m >> T \ . \tag{361}$$

Thus,

$$\begin{aligned} \cos\theta_{34l} \quad &\approx \quad \frac{T_3(T_1 - T_3)}{\sqrt{2mT_3}\sqrt{(2m(T_1 - T_3)}} \\ &= \quad \frac{\sqrt{T_3(T_1 - T_3)}}{2m} \\ &\approx \quad 0 \ , \qquad \text{for} \ \ m >> T \ , \end{aligned} \tag{362}$$

giving

$$\theta_{34l} \approx 90^\circ \tag{363}$$

in the non-relativistic limit.

4.4.6 Equivalence of $|\mathbf{p}_{ic}|$ and $|\mathbf{p}_{fc}|$

Consider the reaction

$$1 + 2 \rightarrow 3 + 4.$$

Consider the case when all the particle masses are equal, i.e. $m_1 = m_2 = m_3 = m_4 \equiv m$. One can show that $|\mathbf{p}_{ic}| = |\mathbf{p}_{fc}|$. (See the previous results). Label

$$k \equiv |\mathbf{p}_{ic}| = |\mathbf{p}_f| \ , \tag{364}$$

and let the cm scattering angle between particles 1 and 3 be

$$\theta_{13c} \equiv \theta \ . \tag{365}$$

We will show the following results.

$$\begin{aligned}
\text{(a)} \quad s &= 4(k^2 + m^2) \ , && (366)\\
\text{(b)} \quad t &= -2k^2(1 - \cos\theta) = -4k^2 \sin^2(\theta/2) \ , && (367)\\
\text{(c)} \quad u &= -2k^2(1 + \cos\theta) = -4k^2 \cos^2(\theta/2) \ . && (368)
\end{aligned}$$

To show these results, we proceed as follows.

(a) Begin with

$$\begin{aligned}
s &= (p_1 + p_2)^2 \\
&= p_1^2 + p_2^2 + 2p_1.p_2 \\
&= m_1^2 + m_2^2 + 2(E_1E_2 - \mathbf{p}_1.\mathbf{p}_2) \ .
\end{aligned} \tag{369}$$

In the cm frame, $\mathbf{p}_1 + \mathbf{p}_2 = 0$ giving $\mathbf{p}_1.\mathbf{p}_2 = -|\mathbf{p}_1|^2 \equiv -\mathbf{p}_{ic}^2 = -k^2$. Thus

$$\begin{aligned}
s &= m_1^2 + m_2^2 + 2[\sqrt{(k^2 + m_1^2)(k^2 + m_2^2)} + k^2] \\
&= 2m^2 + 2(k^2 + m^2 + k^2) \ \text{ with } \ m_1 = m_2 \equiv m \\
&= 4(k^2 + m^2) \ .
\end{aligned} \tag{370}$$

(b) The Mandelstam variable is

$$\begin{aligned}
t &= (p_1 - p_3)^2 \\
&= p_1^2 + p_3^2 - 2p_1.p_3 \\
&= m_1^2 + m_3^2 - 2(E_1E_2 - \mathbf{p}_1.\mathbf{p}_3) \\
&= m_1^2 + m_3^2 - 2\sqrt{(\mathbf{p}_1^2 + m_1^2)(\mathbf{p}_3^2 + m_3^2)} + 2|\mathbf{p}_1||\mathbf{p}_3|\cos\theta_{13} \ .
\end{aligned} \tag{371}$$

In the cm frame, when $m_1 = m_2 = m_3 = m_4 \equiv m$, we have $|\mathbf{p}_{1c}| = |\mathbf{p}_{3c}| \equiv k$, which gives

$$\begin{aligned}
t &= 2m^2 - 2(k^2 + m^2) + 2k^2 \cos\theta \\
&= -2k^2(1 - \cos\theta) = -4k^2 \sin^2(\theta/2) \ .
\end{aligned} \tag{372}$$

(c) Changing $3 \to 4$ in the above gives

$$\begin{aligned} u &= (p_1 - p_4)^2 \\ &= -2k^2(1 - \cos\theta_{14}) \\ &= -2k^2[1 - \cos(180 - \theta_{13})] \\ &= -2k^2(1 + \cos\theta_{13}) \\ &= -2k^2(1 + \cos\theta) = -4k^2\cos^2(\theta/2) \quad \text{where } \theta \equiv \theta_{13}\ . \end{aligned} \tag{373}$$

The relation $\theta_{14} = 180 - \theta_{13}$ can be seen in figure 6.

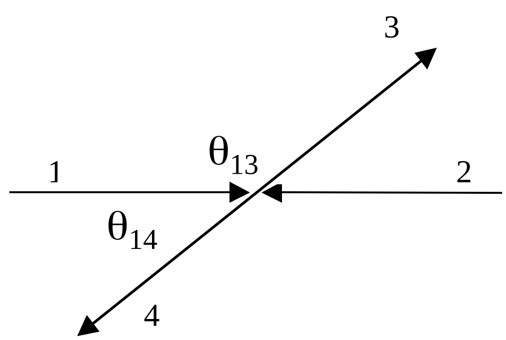

Figure 6: The reaction, $1 + 2 \to 3 + 4$, viewed in the cm frame.

4.4.7 Relationship of s and $\mathbf{p}_{1l}$

For the reaction,

$$1 + 2 \to \text{anything}\ ,$$

we now show that

$$\begin{aligned} s &= m_1^2 + m_2^2 + 2m_2E_{1l} \\ &= m_1^2 + m_2^2 + 2m_2\sqrt{\mathbf{p}_{1l}^2 + m_1^2}\ , \end{aligned} \tag{374}$$

where $\mathbf{p}_{1l}$ is the momentum of particle 1 in the lab frame. This is proved as follows. The Mandelstam variables is

$$\begin{aligned} s &= (p_1 + p_2)^2 = p_1^2 + p_2^2 + 2p_1.p_2 \\ &= m_1^2 + m_2^2 + 2(E_1E_2 - \mathbf{p}_1.\mathbf{p}_2)\ . \end{aligned} \tag{375}$$

The lab (or target) frame is defined as the frame in which the target is at rest, namely $\mathbf{p}_{2l} = 0$. Thus, equation (374) is obtained.

4.4.8 Threshold of s

For the reaction,

$$1 + 2 \to 3 + 4 + 5 \ ,$$

we now derive an expression for the threshold value of s in terms of m_3, m_4, m_5 only. We proceed as follows. Our formula for the threshold kinetic energy was given in equation (325) and generalizes to 3 final state particles as

$$T_{1l} = \frac{(m_3 + m_4 + m_5)^2 - (m_1 + m_2)^2}{2m_2} \ . \tag{376}$$

In the previous result, we showed that

$$\begin{aligned} s &= m_1^2 + m_2^2 + 2m_2 E_{1l} = m_1^2 + m_2^2 + 2m_2(T_{1l} + m_1) \\ &= (m_1 + m_2)^2 + 2m_2 T_{1l} = (m_1 + m_2)^2 + (m_3 + m_4 + m_5)^2 - (m_1 + m_2)^2 \\ &= (m_3 + m_4 + m_5)^2 \ . \end{aligned} \tag{377}$$

An alternative derivation is

$$\begin{aligned} s &= (p_1 + p_2)^2 \\ &= (p_3 + p_4 + p_5)^2 \ , \qquad \text{(due to 4 − momentum conservation)} \\ &= (m_3 + m_4 + m_5)^2 \ , \end{aligned} \tag{378}$$

because all final state particles are produced at rest in the cm frame at threshold.

5 Lorentz transformations

Cross sections are usually calculated in the center of momentum or projectile frames, but space radiation transport codes require cross section formulas in the lab (spacecraft) frame. One therefore must transform cross sections, angle, energies and momenta into the lab frame, by using Lorentz transformations. The general formulas for doing this are derived in this section.

Suppose a frame S' is moving at velocity $\mathbf{v}$ relative to a stationary frame S, as shown in figure 7. We will always choose the longitudinal z direction to be in the direction of motion. The transverse directions x and y will always be perpendicular to the velocity. The Lorentz tranformations for transforming spacetime coordinates from one frame to another as shown in figure 7 are

$$\begin{pmatrix} t' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} t \\ z \end{pmatrix} \ , \tag{379}$$

and

$$x' = x \qquad y' = y \ , \tag{380}$$

where

$$\beta \equiv v \ . \tag{381}$$

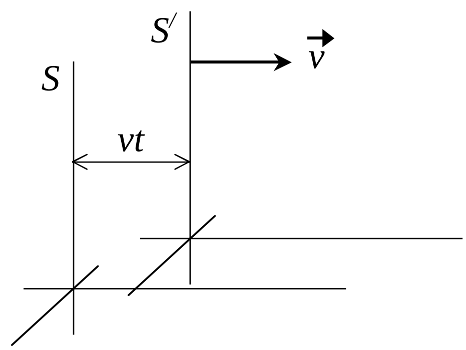

Figure 7: Reference frame S' moves at speed $\mathbf{v}$ relative to S.

Frame S moves relative to S' at speed $-v$, as shown in figure 8. The inverse transformation is

$$\begin{pmatrix} t \\ z \end{pmatrix} = \begin{pmatrix} \gamma & \gamma\beta \\ \gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} t' \\ z' \end{pmatrix} . \tag{382}$$

Thus, the inverse transformations are obtained by *swapping primes and unprimes and changing the sign of* v.

5.1 Lorentz transformations of energy and momentum

Consider a frame S' moving at speed v relative to frame S. Now, imagine that a particle is moving at velocity $\mathbf{u}$ in frame S'. (See figure 2-4 from Tipler [22].) If the particle has energy and momentum E' and $\mathbf{p}'$ as measured in frame S', the Lorentz transformation gives the energy and momentum E and $\mathbf{p}$ measured in frame S.

The coordinates t and $\mathbf{x}$ form a 4-vector $(t, \mathbf{x})$ with an invariant squared length given by $s^2 = t^2 - \mathbf{x}^2$. Similarly, for the energy-momentum 4-vector $(E, \mathbf{p})$, the invariant squared "length" is $E^2 - \mathbf{p}^2 = m^2$. The E and $\mathbf{p}$ obey Lorentz transformations identical to t and $\mathbf{x}$:

$$\begin{pmatrix} E' \\ p'_z \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} E \\ p_z \end{pmatrix} , \tag{383}$$

$$p'_x = p_x \ , \qquad p'_y = p_y \ . \tag{384}$$

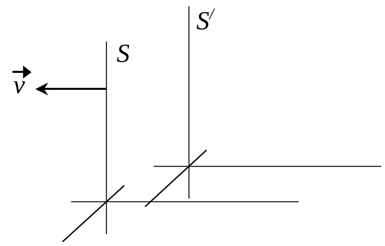

Figure 8: Frame S viewed from S'.

It is often very useful to write this in terms of *parallel* momenta $p_{||} \equiv p_z$ and *transverse* momenta $p_{Tx} \equiv p_x$ or $p_{Ty} \equiv p_y$. We often just write p_T to denote either p_{Tx} or p_{Ty}. The direction parallel or transverse is defined in relation to the velocity of the frame S'. Thus, we write

$$\begin{pmatrix} E' \\ p'_{||} \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} E \\ p_{||} \end{pmatrix}, \qquad p'_T = p_T \ . \tag{385}$$

5.2 Transformation between cm or projectile frame and lab (target) frame

Suppose we have quantities in the cm or projectile frames and we wish to transform to the lab frame. *The cm frame moves at speed β_{cl} relative to the lab frame.* The corresponding γ factor is labelled as γ_{cl}. *The projectile frame moves at speed β_{pl} relative to the lab frame.* The corresponding γ factor is labelled as γ_{pl}. The Lorentz transformations are

$$\begin{pmatrix} E_* \\ p_{||*} \end{pmatrix} = \begin{pmatrix} \gamma_{*l} & -\gamma_{*l}\beta_{*l} \\ -\gamma_{*l}\beta_{*l} & \gamma_{*l} \end{pmatrix} \begin{pmatrix} E_l \\ p_{||l} \end{pmatrix}, \qquad p_{T*} = p_{Tl} \ , \tag{386}$$

and inverse transformations are

$$\begin{pmatrix} E_l \\ p_{||l} \end{pmatrix} = \begin{pmatrix} \gamma_{*l} & \gamma_{*l}\beta_{*l} \\ \gamma_{*l}\beta_{*l} & \gamma_{*l} \end{pmatrix} \begin{pmatrix} E_* \\ p_{||*} \end{pmatrix}, \qquad p_{Tl} = p_{T*} \ , \tag{387}$$

where

$$p_{||} \equiv p_z = |\mathbf{p}| \cos\theta \ , \tag{388}$$

$$p_T = |\mathbf{p}| \sin\theta \ . \tag{389}$$

With the above notation, both cm and projectile frames are included. The notation is as follows. A quantity x_* is the value of the quantity x *evaluated in that particluar frame* with

$$* = c \quad \text{or} \quad * = p \ , \tag{390}$$

and β_{*l} is the speed of that frame *with respect to the lab* with

$$\beta_{*l} = \beta_{cl} \quad \text{or} \quad \beta_{*l} = \beta_{pl} \ . \tag{391}$$

5.2.1 Evaluation of β_{cl} and γ_{cl}

Consider the non-relativistic case. For 2 particles, the position of the center of mass frame, relative to an arbitrary origin, is defined via

$$\mathbf{R} \equiv \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2} \ , \tag{392}$$

giving the velocity of the cm frame as

$$\mathbf{V} \equiv \frac{m_1\mathbf{v}_1 + m_1\mathbf{v}_2}{m_1 + m_2} = \frac{\mathbf{p}_1 + \mathbf{p}_2}{m_1 + m_2} \ . \tag{393}$$

The velocity of the cm frame relative to the lab frame is obtained by setting $\mathbf{p}_2 = 0$ to give

$$\mathbf{V}_{cl} = \frac{\mathbf{p}_{1l}}{m_1 + m_2} \ , \tag{394}$$

where $\mathbf{p}_{1l}$ is the momentum of particle 1 (the projectile) in the lab frame. The relativistic case follows similarly. For a single particle of mass m, $\mathbf{p} = \gamma m\mathbf{v}$ and $E = \gamma m$ giving

$$\beta \equiv \mathbf{v} = \frac{\mathbf{p}}{E} \ , \tag{395}$$

$$\gamma = \frac{E}{m} \ , \tag{396}$$

$$\gamma\mathbf{v} = \frac{\mathbf{p}}{m} \ . \tag{397}$$

For a *system* of particles (Byckling [14] (p. 21) of *total* energy $E = E_1 + E_2 + \cdots$ and *total* momentum $\mathbf{p} = \mathbf{p}_1 + \mathbf{p}_2 + \cdots$, the invariant mass is

$$M \equiv \sqrt{E^2 - \mathbf{p}^2} = \sqrt{s} \ , \tag{398}$$

which, for 2 particles, is [1]

$$M \equiv \sqrt{(E_1 + E_2)^2 - (\mathbf{p}_1 + \mathbf{p}_2)^2} = \sqrt{s} \ . \tag{399}$$

The velocity of the cm frame is

$$\beta \equiv \mathbf{v} = \frac{\mathbf{p}}{E} = \frac{\mathbf{p}_1 + \mathbf{p}_2 + \cdots}{E_1 + E_2 + \cdots} \ , \tag{400}$$

$$\gamma = \frac{E}{M} = \frac{E_1 + E_2 + \cdots}{\sqrt{s}} \ , \tag{401}$$

$$\gamma\beta = \frac{\mathbf{p}}{M} = \frac{\mathbf{p}_1 + \mathbf{p}_2 + \cdots}{\sqrt{s}} \ . \tag{402}$$

For a system of 2 particles, the velocity of the cm frame relative to the lab frame is obtained by setting $\mathbf{p}_2 = 0$, to give [1]

$$\beta_{cl} = \frac{\mathbf{p}_{1l}}{E_{1l} + m_2} , \tag{403}$$

$$\gamma_{cl} = \frac{E_{1l} + m_2}{\sqrt{s}} , \tag{404}$$

$$\gamma_{cl}\beta_{cl} = \frac{\mathbf{p}_{1l}}{\sqrt{s}} , \tag{405}$$

with

$$\sqrt{s} = \sqrt{m_1^2 + m_2^2 + 2m_2 E_{1l}} , \tag{406}$$

where $\mathbf{p}_{1l}$ is the momentum of particle 1 (the projectile) in the lab frame. These results can also be written in terms of invariants. It is straightforward to show that

$$\sqrt{\lambda_{12}^{lab}} = 2m_2 \left|\mathbf{p}_{1l}\right| , \tag{407}$$

and because λ is invariant, it does not need a frame label, i.e.

$$\left|\mathbf{p}_{1l}\right| = \frac{\sqrt{\lambda_{12}}}{2m_2} . \tag{408}$$

Substituting into the above expressions gives ([14] pps. 26 and 73)

$$\beta_{cl} = \frac{\sqrt{\lambda_{12}}}{s - m_1^2 + m_2^2} , \tag{409}$$

$$\gamma_{cl} = \frac{s - m_1^2 + m_2^2}{2m_2\sqrt{s}} , \tag{410}$$

$$\gamma_{cl}\beta_{cl} = \frac{1}{m_2}\sqrt{\frac{\lambda_{12}}{4s}} . \tag{411}$$

5.2.2 Evaluation of β_{pl} and γ_{pl}

Clearly,

$$\beta_{pl} = \frac{\mathbf{p}_{1l}}{E_{1l}} , \tag{412}$$

$$\gamma_{pl} = \frac{E_{1l}}{m_1} = 1 + \frac{T_{1l}}{m_1} , \tag{413}$$

$$\gamma_{pl}\beta_{pl} = \frac{\mathbf{p}_{1l}}{m_1} , \tag{414}$$

where T is the kinetic energy. Again, these results can also be written in terms of invariants. It is straightforward to show that

$$\beta_{pl} = \frac{\sqrt{\lambda_{12}}}{s - m_1^2 - m_2^2} , \tag{415}$$

$$\gamma_{pl} = \frac{s - m_1^2 - m_2^2}{2m_1 m_2} , \tag{416}$$

$$\gamma_{pl}\beta_{pl} = \frac{\sqrt{\lambda_{12}}}{2m_1 m_2} . \tag{417}$$

5.3 Energy transformations

The Lorentz transformation for energy from the lab (l) frame to the starred ($*$) frame is

$$\begin{aligned} E_* &= \gamma_{*l}(E_l - \beta_{*l} p_{||l}) \\ &= \gamma_{*l}(E_l - \beta_{*l}|\mathbf{p}_l| \cos\theta_l) \\ &= \gamma_{*l}\left(E_l - \beta_{*l}\sqrt{E_l^2 - m^2}\cos\theta_l\right) . \end{aligned} \tag{418}$$

The derivative dE_*/dE_l is found by expressing $cos\theta_l$ in (254) in terms of E_l, and substituting for $cos\theta_l$ above to eliminate θ_l and have E_* solely as a function of E_l. The inverse transformation is

$$E_l = \gamma_{*l}\left(E_* + \beta_{*l}\sqrt{E_*^2 - m^2}\cos\theta_*\right) . \tag{419}$$

5.4 Angle transformations

The angle is obtained from

$$\tan\theta = \frac{p_T}{p_z} . \tag{420}$$

Thus, the angle of particle j is

$$\begin{aligned} \tan\theta_{jl} = \frac{p_{T\,jl}}{p_{z\,jl}} &= \frac{p_{T\,j*}}{\gamma_{*l}\beta_{*l}E_{j*} + \gamma_{*l}p_{z\,j*}} \\ &= \frac{|\mathbf{p}_{j*}|\sin\theta_{j*}}{\gamma_{*l}\left(\beta_{*l}E_{j*} + |\mathbf{p}_{j*}|\cos\theta_{j*}\right)} . \end{aligned} \tag{421}$$

Defining α_{j*} as the speed of the (cm or projectile) frame relative to the lab divided by the speed of particle j in the (cm or projectile) frame

$$\alpha_{j*} \equiv \frac{\beta_{*l}}{\beta_{j*}} , \tag{422}$$

and using

$$\beta_{j*} = \frac{|\mathbf{p}_{j*}|}{E_{j*}} \ , \tag{423}$$

we obtain

$$\tan\theta_{jl} = \frac{\sin\theta_{j*}}{\gamma_{*l}\left(\cos\theta_{j*} + \alpha_{j*}\right)} \ , \tag{424}$$

See references [20] (p. 402), [14] (p. 42), [12] (p. 17), [19] (p. 26). The above equation is a complicated function of θ because in general

$$\alpha_{j*} = \alpha_{j*}(E_{j*}) = \alpha_{j*}(\theta_{j*}) \ , \tag{425}$$

To make this explicit, the equation is written

$$\tan\theta_{jl} = \frac{\sin\theta_{j*}}{\gamma_{*l}\left[\cos\theta_{j*} + \alpha_{j*}(\theta_{j*})\right]} \ . \tag{426}$$

That is, in general, α_{j*} is a function of θ_{j*}, making $\tan\theta_{jl}$ *a complicated function of θ_{j*}. However, for the cm frame E_{jc} is not a function of θ_{jc}, meaning that α_{jc} is not a function of θ_{jc} [14] (pp. 42, 58).* Also, for a 3 - body final state E_{jc} is not a function of θ_{jc} meaning that α_{jc} is not a function of θ_{jc}. For $* = c$, this is plotted in references [14] (p. 43), [20] (p. 403), [23]. For $\alpha_{j*} > 1$, the function is double valued. That is, two different angles in the (cm or projectile) frame can give rise to the same angle in the lab frame for $\alpha_{j*} > 1$. However, the two particles can be distinguished by their energies, labelled for the cm frame as $E^{\pm}_{jc}$ [20] (p. 402). Later we will need to evaluate $\frac{d\cos\theta_l}{d\cos\theta_*}$. Using $\sec^2\theta - \tan^2\theta = 1$ allows equation (426) to be re-written as

$$\cos\theta_{jl} = \frac{\gamma_{*l}[\cos\theta_{j*} + \alpha_{j*}(\theta_{j*})]}{\sqrt{\gamma^2_{*l}[\cos\theta_{j*} + \alpha_{j*}(\theta_{j*})]^2 + \sin^2\theta_{j*}}} \ , \tag{427}$$

which is written in terms of θ_{j*}, to be contrasted later with (433) written in terms of θ_{jl}. In evaluating $\frac{d\cos\theta_l}{d\cos\theta_*}$, one needs to be very careful, because, as stated above, α_{j*} is a function of θ_{j*}. *That is, in general, α_{j*} is a function of θ_{j*}, making the evaluation of $\frac{d\cos\theta_l}{d\cos\theta_*}$ difficult. However, for the cm frame, E_{jc} is not a function of θ_{jc}, meaning that α_{jc} is not a function of θ_{jc} ([14] pps. 42 and 58).* This is also true for a 3 - body final state. Thus, for the cm frame we have [14] (p. 59)

$$\frac{d\cos\theta_{jl}}{d\cos\theta_{jc}} = \frac{\gamma_{*l}(1 + \alpha_{jc}\cos\theta_{jc})}{[\gamma^2_{*l}(\alpha_{jc} + \cos\theta_{jc})^2 + \sin^2\theta_{jc}]^{3/2}} \ , \tag{428}$$

which is written in terms of θ_{jc}, to be contrasted later with [14] (p. 59) written in terms of θ_{jl}. Using exactly the same technique that was used to derive equation (426), we may also derive the following [14] (p. 42)

$$\tan\theta_{j*}^{\pm} = \frac{\sin\theta_{jl}}{\gamma_{*l}[\cos\theta_{jl} - \alpha_{jl}^{\pm}(\theta_{jl})]} , \tag{429}$$

where

$$\beta_{jl}^{\pm} = \frac{|\mathbf{p}_{jl}^{\pm}|}{E_{jl}^{\pm}} , \tag{430}$$

$$\alpha_{jl}^{\pm} \equiv \frac{\beta_{*l}}{\beta_{jl}^{\pm}} . \tag{431}$$

Again, we have been careful to note that $\alpha_{jl}^{\pm}$ is a function of θ_{jl}, i.e.

$$\alpha_{jl}^{\pm} = \alpha_{jl}^{\pm}(\theta_{jl}) . \tag{432}$$

This time the lab momentum and energy are double valued functions of the lab angle, as discussed previously. This is not the case in the cm frame. There, the energy and momentum are independent of the angle. Now, equation (429) is a complicated function of θ_{jl} because in general $\alpha_{jl} = \alpha_{jl}(\theta_{jl})$. An alternative to equation (429) is to invert equation (426) directly which leads to [14] (p. 43, equation 8.32)

$$\cos\theta_{j*} = \frac{-\alpha_{j*}(\theta_{j*})\gamma_{*l}^2\tan^2\theta_{jl} \pm \sqrt{D}}{1+\gamma_{*l}^2\tan^2\theta_{jl}} , \tag{433}$$

which is written in terms of θ_{jl}, in contrast with equation (427) written in terms of θ_{j*}. The term D is given by

$$D = 1 + \gamma_{*l}^2\tan^2\theta_{jl}[1-\alpha_{j*}^2(\theta_{j*})] , \tag{434}$$

and using [14] (p. 43)

$$1+\gamma_{*l}^2\tan^2\theta_{jl} = \frac{\gamma_{*l}^2}{\cos^2\theta_{jl}}(1-\beta_{*l}^2\cos^2\theta_{jl}) , \tag{435}$$

gives

$$\cos\theta_{j*} = \frac{-\alpha_{j*}(\theta_{j*})\gamma_{*l}^2[1-\cos^2\theta_{jl}] \pm \cos^2\theta_{jl}\sqrt{D}}{\gamma_{*l}^2(1-\beta_{*l}^2\cos^2\theta_{jl})} . \tag{436}$$

This is still a complicated function of θ_{j*} through $\alpha_{j*} = \alpha_{j*}(\theta_{j*})$. However, it has a big advantage for the cm frame because, as discussed above, α_{jc} is *not* a function of θ_{jc}. The derivative is [14] (p. 59)

$$\frac{d\cos\theta_{jc}}{d\cos\theta_{jl}} = \frac{\cos\theta_{jl}}{\gamma_{cl}^2(1-\beta_{cl}^2\cos^2\theta_{jl})^2}\frac{(\alpha_{jc}\pm\sqrt{D})^2}{(\pm\sqrt{D})}$$

$$
\begin{aligned}
&= \left(\frac{|\mathbf{p}_{jl}|}{|\mathbf{p}_{jc}|}\right)^2 \frac{1}{(\pm\sqrt{D}\cos\theta_{jl})} \\
&= \frac{\pm|\mathbf{p}_{jl}|^2}{\gamma_{cl}|\mathbf{p}_{jc}|(|\mathbf{p}_{jl}| - E_{jl}\beta_{cl}\cos\theta_{jl})} \,, \qquad (437)
\end{aligned}
$$

which is written in terms of θ_{jl}, in contrast with equation (428) written in terms of θ_{jc}. Here, $|\mathbf{p}_{jl}|$ given by (272) and $E_{jl} = \sqrt{|\mathbf{p}_{jl}|^2 + m_i^2}$ where

$$
D = 1 + \gamma_{cl}^2 \tan^2\theta_{jl}(1 - \alpha_{jc}^2) \,. \qquad (438)
$$

5.4.1 Evaluation of α_{3c} and α_{4c}

We have already obtained β_{cl} in equation (409) above. Now,

$$
\beta_{jc} = \frac{|\mathbf{p}_{jc}|}{E_{jc}} \,. \qquad (439)
$$

Using the results from section 4.2.1,

$$
\begin{aligned}
|\mathbf{p}_{3c}| = |\mathbf{p}_{4c}| = |\mathbf{p}_{fc}| &= \sqrt{\frac{\lambda_{34}}{4s}} \,, && (440) \\
E_{3c} &= \frac{s + m_3^2 - m_4^2}{\sqrt{4s}} \,, && (441) \\
E_{4c} &= \frac{s + m_4^2 - m_3^2}{\sqrt{4s}} \,, && (442)
\end{aligned}
$$

which gives

$$
\begin{aligned}
\beta_{3c} &= \frac{\sqrt{\lambda_{34}}}{s + m_3^2 - m_4^2} \,, && (443) \\
\beta_{4c} &= \frac{\sqrt{\lambda_{34}}}{s + m_4^2 - m_3^2} \,, && (444)
\end{aligned}
$$

which results in [14] (p. 73)

$$
\begin{aligned}
\alpha_{3c} \equiv \frac{\beta_{cl}}{\beta_{3c}} &= \frac{s + m_3^2 - m_4^2}{s - m_1^2 + m_2^2}\sqrt{\frac{\lambda_{12}}{\lambda_{34}}} && (445) \\
\alpha_{4c} \equiv \frac{\beta_{cl}}{\beta_{4c}} &= \frac{s + m_4^2 - m_3^2}{s - m_1^2 + m_2^2}\sqrt{\frac{\lambda_{12}}{\lambda_{34}}} \,. && (446)
\end{aligned}
$$

We see that α_{3c} is the same as α_{4c} except for the interchange $3 \leftrightarrow 4$. This gives exactly the same result as α in equation (12.51) of reference [20] and τ in reference [19].

5.5 Transformation of angular distributions

Suppose we have $\frac{d\sigma}{d\Omega_{j*}}$ in the cm or projectile frames and we wish to obtain $\frac{d\sigma}{d\Omega_{jl}}$ in the lab frame. Clearly,

$$\frac{d\sigma}{d\cos\theta_{jl}} = \frac{d\sigma}{d\cos\theta_{j*}}\frac{d\cos\theta_{j*}}{d\cos\theta_{jl}} \ , \tag{447}$$

and using $d\Omega = 2\pi d(\cos\theta)$ gives

$$\frac{d\sigma}{d\Omega_{jl}} = \frac{d\cos\theta_{j*}}{d\cos\theta_{jl}}\frac{d\sigma}{d\Omega_{j*}} \ . \tag{448}$$

Using equation (428) gives

$$\frac{d\sigma}{d\Omega_{jl}} = \frac{[\gamma_{*l}^2(\alpha_{jc}+\cos\theta_{jc})^2+\sin^2\theta_{jc}]^{3/2}}{\gamma_{*l}|1+\alpha_{jc}\cos\theta_{jc}|}\frac{d\sigma}{d\Omega_{jc}} \ , \tag{449}$$

which agrees with Joachain [19] (equation 2.104). This is written in terms of θ_{jc}, in contrast to (450) written in terms of θ_{jl}. *Equation (449) is one of our most important results. It tells us how to transform an angular distribution from the cm frame to the lab frame. The way to use it is as follows. Suppose you want the lab frame angular distribution* $\frac{d\sigma}{d\Omega_{jl}}$ *evaluated at a particular lab angle, say* $\theta_{jl} = 45°$. *Then, evaluate the corresponding cm angle* θ_{jc} *by substituting* $\theta_{jl} = 45°$ *into equation (433). This cm angle is simply substituted into the right hand side of equation (449), and the number obtained is the value of* $\frac{d\sigma}{d\Omega_{jl}}$. *Now of course, because of the double valued nature of (433), you might get two angles, say* θ_{jc}^+ *and* θ_{jc}^-. *If this occurs, it simply means that there are two terms on the right hand side of (449) which must be added incoherently (i.e. ordinary simple addition with no interference terms).* However, equation (449) is inconvenient in one sense. If we want the angular distribution $\frac{d\sigma}{d\Omega_{jl}}$ in the lab frame, then it would be very nice to have it *only* as a function of the lab frame angle θ_{jl}. But equation (449) is written in terms of the cm angle θ_{jc}. Using reference [14] (p. 59) gives

$$\begin{aligned}
\frac{d\sigma}{d\Omega_{jl}} &= \frac{\cos\theta_{jl}}{\gamma_{cl}^2(1-\beta_{cl}^2\cos\theta_{jl}^2)^2}\frac{(\alpha_{3c}\pm\sqrt{D})^2}{(\pm\sqrt{D})}\frac{d\sigma}{d\Omega_{jc}} \\
&= \left(\frac{|\mathbf{p}_{jl}|}{|\mathbf{p}_{jc}|}\right)^2\frac{1}{(\pm\sqrt{D}\cos\theta_{jl})}\frac{d\sigma}{d\Omega_{jc}} \\
&= \frac{|\mathbf{p}_{jl}|^2}{\gamma_{cl}|\mathbf{p}_{jc}|(|\mathbf{p}_{jl}|-E_{jl}\beta_{cl}\cos\theta_{jl})}\frac{d\sigma}{d\Omega_{jc}} \ .
\end{aligned} \tag{450}$$

This is written in terms of θ_{jl}, in contrast to equation (449) written in terms of θ_{jc}. However, the right hand side still contains $\frac{d\sigma}{d\Omega_{jc}}$ which is a function of θ_{jc}. To eliminate this dependance, substitute equation (436) wherever $\cos\theta_{jc}$ appears in $\frac{d\sigma}{d\Omega_{jc}}$. This will result in the right hand side of equation (450) being only a function of θ_{jl}. Of course, it will be double valued sometimes.

5.6 Double differential cross sections

In order to transform an area element such as $dxdy$ or a volume element such as $dxdydz$, we need to use the Jacobian defined as

$$\frac{\partial(x,y)}{\partial(u,v)} \equiv \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \equiv \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v} \ , \tag{451}$$

so that under a change of coordinates, an integral becomes

$$\int\int f(x,y)dxdy = \int\int f[x(u,v),y(u,v)]\frac{\partial(x,y)}{\partial(u,v)}dudv \ , \tag{452}$$

or, in other words,

$$dxdy = \frac{\partial(x,y)}{\partial(u,v)}dudv \ . \tag{453}$$

The proof of this is usually given in advanced calculus books. If several transformations need to be carried out, we can use the *chain rule* properties of the Jacobian, namely

$$\frac{\partial(x_1\cdots x_n)}{\partial(z_1\cdots z_n)} = \frac{\partial(x_1\cdots x_n)}{\partial(y_1\cdots y_n)}\frac{\partial(y_1\cdots y_n)}{\partial(z_1\cdots z_n)} \ . \tag{454}$$

We are interested in relating the double differential cross section between the cm or projectile frames and the lab frame, as in

$$\frac{d^2\sigma}{dE_{jl}d\Omega_{jl}} = \frac{d^2\sigma}{dE_{j*}d\Omega_{j*}}\frac{\partial(E_{j*},\Omega_{j*})}{\partial(E_{jl},\Omega_{jl})} \ , \tag{455}$$

which is the analog of equation (447). *In the following analysis, it is essential that E_l and θ_l are independent. This is true for 3 particles in the final state, but, as we saw previously, it is not true for only 2 particles in the final state.* Re-write the Lorentz transformations (387) as (leaving off the index i)

$$\begin{aligned} E_* &= \gamma_{*l}(E_l - \beta_{*l}p_{||l}) \ , && (456) \\ p_{||*} &= \gamma_{*l}(p_{||l} - \beta_{*l}E_l) \ , && (457) \\ P_{T*} &= p_{Tl} \ , && (458) \end{aligned}$$

or

$$\begin{aligned} E_* &= \gamma_{*l}(E_l - \beta_{*l}|\mathbf{p}_l|\cos\theta_l) \ , && (459) \\ |\mathbf{p}_*|\cos\theta_* &= \gamma_{*l}(|\mathbf{p}_l|\cos\theta_l - \beta_{*l}E_l) \ , && (460) \\ |\mathbf{p}_*|\sin\theta_* &= |\mathbf{p}_l|\sin\theta_l \ . && (461) \end{aligned}$$

Now our two independent variables are E and θ (or Ω). For the case of 3 or more particles in the final state, E is no longer a function of θ. This makes calculation of the derivatives simpler than for the case of only 2 final state particles. The above equation can be written as

$$\begin{aligned} E_* &= \gamma_{*l}\left(E_l - \beta_{*l}\sqrt{E_l^2 - m^2}\cos\theta_l\right) , && (462) \\ \cos\theta_* &= \frac{\gamma_{*l}}{|\mathbf{p}_*|}\left(\sqrt{E_l^2 - m^2}\cos\theta_l - \beta_{*l}E_l\right) , && (463) \\ \frac{|\mathbf{p}_*|}{|\mathbf{p}_l|} &= \frac{\sin\theta_l}{\sin\theta_*} . && (464) \end{aligned}$$

The Jacobian is now easily evaluated as

$$\frac{\partial(E_*, \Omega_*)}{\partial(E_l, \Omega_l)} = \frac{|\mathbf{p}_l|}{|\mathbf{p}_*|} = \frac{\sin\theta_*}{\sin\theta_l} , \tag{465}$$

to give

$$\frac{d^2\sigma}{dE_{jl}d\Omega_{jl}} = \frac{\sin\theta_{j*}}{\sin\theta_{jl}}\,\frac{d^2\sigma}{dE_{j*}d\Omega_{j*}} , \tag{466}$$

in agreement with Joachain [19] (p. 30), Dedrick [23], and Byckling [14] (p. 60). This may be expressed purely in terms of θ_{jl} by using 433 to give θ_{j*} in terms of θ_{jl}.

5.7 Derivation of $d\cos\theta_{jc}/d\cos\theta_{jl}$

Equation (436) gives

$$\cos\theta_{jc} = \frac{-\alpha_{jc}\gamma_c^2(1-\cos^2\theta_{jl}) \pm \cos^2\theta_{jl}\sqrt{D}}{\gamma_c^2(1-\beta_c^2\cos^2\theta_{jl})} , \tag{467}$$

where

$$D = 1 + \gamma_c^2\tan^2\theta_{jl}(1-\alpha_{jc}^2) . \tag{468}$$

We will show that

$$\frac{d\cos\theta_{jc}}{d\cos\theta_{jl}} = \frac{\cos\theta_{jl}}{\gamma_c^2(1-\beta_c^2\cos\theta_{jl}^2)^2}\frac{(\alpha_{jc}\pm\sqrt{D})^2}{(\pm\sqrt{D})} . \tag{469}$$

To simplify the algebra, make the following definitions.

$$\begin{aligned} y &\equiv \cos\theta_{jc} , && (470) \\ x &\equiv \cos\theta_{jl} , && (471) \\ \gamma_c &\equiv \gamma , && (472) \end{aligned}$$

$$
\begin{aligned}
\beta_c &\equiv \beta \,, &(473)\\
\alpha_{jc} &\equiv \alpha \,, &(474)\\
y' &\equiv \frac{dy}{dx} \,. &(475)
\end{aligned}
$$

We have

$$
y \equiv \frac{f}{g} = \frac{-\alpha\gamma^2(1-x^2) \pm x^2\sqrt{D}}{\gamma^2(1-\beta^2x^2)} \,, \tag{476}
$$

where

$$
\begin{aligned}
f &\equiv -\alpha\gamma^2(1-x^2) \pm x^2\sqrt{D} \,, &(477)\\
g &\equiv \gamma^2(1-\beta^2x^2) = \gamma^2 - (\gamma^2-1)x^2 \,, &(478)
\end{aligned}
$$

where we have used the result $\gamma^2\beta^2 = \gamma^2 - 1$. Note that we can also write

$$
D = 1 + \gamma^2(1-\alpha^2)(\frac{1}{x^2} - 1) \,. \tag{479}
$$

Taking derivatives with respect to x, we obtain

$$
\begin{aligned}
D' &= -\frac{2\gamma^2}{x^3}(1-\alpha^2) \,, &(480)\\
f' &= 2\alpha\gamma^2 x \pm \left[2x\sqrt{D} - \frac{\gamma^2(1-\alpha^2)}{x\sqrt{D}}\right] \,, &(481)\\
\Rightarrow \quad \pm x\sqrt{D}f' &= \pm 2\alpha\gamma^2x^2\sqrt{D} + 2x^2D - \gamma^2(1-\alpha^2) \,, &(482)\\
\Rightarrow \quad \pm x\sqrt{D}gf' &= \pm 2\alpha\gamma^4x^2\sqrt{D} + 2x^2\gamma^2D - \gamma^4(1-\alpha^2) &\\
&\mp 2\alpha\gamma^4x^4\sqrt{D} - 2x^4\gamma^2D + \gamma^4(1-\alpha^2)x^2 &\\
&\pm 2\alpha\gamma^2x^4\sqrt{D} + 2x^4D - \gamma^2(1-\alpha^2)x^2 \,, &(483)\\
-(\pm x\sqrt{D}g'f) &= \pm 2\alpha\gamma^2x^2\sqrt{D} \mp 2\alpha\gamma^2x^4\sqrt{D} - 2x^4D &\\
&\mp 2\alpha\gamma^4x^2\sqrt{D} \pm 2\alpha\gamma^4x^4\sqrt{D} + 2x^4\gamma^2D \,. &(484)
\end{aligned}
$$

Putting this together gives

$$
\begin{aligned}
\pm x\sqrt{D}(gf' - g'f) &= 2x^2\gamma^2D + \gamma^4(1-\alpha^2)(x^2-1) &\\
&\quad -\gamma^2(1-\alpha^2)x^2 \pm 2\alpha\gamma^2x^2\sqrt{D} &\\
&= \gamma^2x^2(D \pm 2\alpha\sqrt{D} + \alpha^2) &\\
&= \gamma^2x^2(\alpha \pm \sqrt{D})^2 \,, &(485)
\end{aligned}
$$

finally giving

$$
y' = \frac{gf' - g'f}{g^2} = \frac{1}{\gamma^4(1-\beta^2x^2)^2}\frac{\gamma^2x^2(\alpha \pm \sqrt{D})^2}{(\pm x\sqrt{D})}
$$

$$= \frac{x(\alpha \pm \sqrt{D})^2}{\gamma^2(1-\beta^2 x^2)^2(\pm\sqrt{D})} . \tag{486}$$

Translating back into the original variables gives

$$\frac{d\cos\theta_{jc}}{d\cos\theta_{jl}} = \frac{\cos\theta_{jl}}{\gamma_c^2(1-\beta_c^2\cos\theta_{jl}^2)^2}\frac{(\alpha_{jc}\pm\sqrt{D})^2}{(\pm\sqrt{D})} . \tag{487}$$

6 Two body final state cross sections

Three dimensional radiation transport codes require differential cross sections in the lab (spacecraft) frame. However, calculations are most easily done in the center of momentum or projectile frame. The previous section presented general results for Lorentz transforming cross sections, energies, angle and momenta from one frame to another. Two body final state cross section transformations are much more complicated than three or more bodies, because the energy and angle are *dependent* variables for two body final states. The present section is a culmination of this paper, in that explicit expressions are written down for Lorentz transforming 2 - body final state cross sections. The key equations (558) - (561) tell one exactly how to transform a 2 - body final state angular distribution from the center of momentum to the lab (spacecraft) frame, with all the relevant kinematic factors written in terms of lab variables.

6.1 Differential cross sections

6.1.1 General form of differential cross sections

The differential cross section is

$$d\sigma = \frac{\mathcal{S}}{4F}|\mathcal{M}|^2(2\pi)^4 d\Phi_2(p_1+p_2;p_3,p_4) . \tag{488}$$

Using $e \equiv (2\pi)^3 2E$, the phase space factor is given by

$$\begin{aligned} d\Phi_2(p_1+p_2;p_3,p_4) &= \delta^4(p_1+p_2-p_3-p_4)\frac{d^3p_3}{(2\pi)^3 2E_3}\frac{d^3p_4}{(2\pi)^3 2E_4} \\ &= \delta(E_1+E_2-E_3'-E_4)\frac{d^3p_4}{e_3' e_4} \\ &= \delta(E_1+E_2-E_3'-E_4)\frac{|\mathbf{p}_4|^2}{e_3' e_4}d|\mathbf{p}_4|d\Omega_4 , \end{aligned} \tag{489}$$

where it is now understoood that

$$\mathbf{p}_3 \equiv \mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_4 , \tag{490}$$

so that in the above equation, we have (with $|\mathbf{p}_4|^2 \equiv \mathbf{p}_4^2$)

$$E_4 = \sqrt{\mathbf{p}_4^2 + m_4^2} , \tag{491}$$

$$E_3' \equiv \sqrt{(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_4)^2 + m_3^2} \,. \tag{492}$$

To eliminate the δ function, we could now use the Ho-Kim phase space method discussed previously [10]. Simplify the above expression by writing

$$d|\mathbf{p}_4| = \frac{d|\mathbf{p}_4|}{d(E_3' + E_4)} d(E_3' + E_4) \,, \tag{493}$$

so that $d(E_3' + E_4)$ will kill the $\delta(E_1 + E_2 - E_3' - E_4)$ term. It is actually $\frac{d(E_3'+E_4)}{d|\mathbf{p}_4|}$ that we will evaluate. We use $d(E_3' + E_4)$ rather than just dE_3 because *both* E_3' and E_4 have the term $|\mathbf{p}_4|$ present in them. The phase space factor then becomes

$$d\Phi_2(p_1 + p_2; p_3, p_4) = \frac{|\mathbf{p}_4|^2}{e_3' e_4} \frac{d|\mathbf{p}_4|}{d(E_3' + E_4)} d\Omega_4 \,, \quad \text{(in general)}, \tag{494}$$

which is identical in form to equation (73), because both phase space factors have a 2 - body final state. (The latter equation is obtained from the former via the substitutions $1 \to 4$ and $2 \to 3$.) The Griffiths phase space method, follows exactly as discussed previously and will give the same result. The differential cross section now becomes

$$\frac{d\sigma}{d\Omega_4} = \frac{\mathcal{S}}{64\pi^2 F} \, |\mathcal{M}|^2 \, \frac{|\mathbf{p}_4|^2}{E_3' E_4} \, \frac{d|\mathbf{p}_4|}{d(E_3' + E_4)} \,, \quad \text{(in general)}, \tag{495}$$

which agrees with Ho-Kim [10] (equation 4.57).

6.1.2 Angular distribution in cm frame

We now evaluate this expression by evaluating $\frac{d|\mathbf{p}_4|}{d(E_3'+E_4)}$ in a particular frame. Evaluate the differential cross section in the cm frame, where $\mathbf{p}_1 + \mathbf{p}_2 = 0$ and $E_3' = \sqrt{|\mathbf{p}_4|^2 + m_3^2}$. We obtain

$$\left[\frac{d(E_3' + E_4)}{d|\mathbf{p}_4|}\right]_c = |\mathbf{p}_4| \frac{E_3' + E_4}{E_3' E_4} \,. \tag{496}$$

Also,

$$\begin{aligned} d\Phi_2(p_1 + p_2; p_3, p_4)_c &= \frac{|\mathbf{p}_4|^2}{e_3' e_4} \frac{d|\mathbf{p}_4|}{d(E_3' + E_4)} d\Omega_{4c} \\ &= \frac{|\mathbf{p}_4|}{e_3' e_4} \frac{E_3' E_4}{E_3' + E_4} d\Omega_{4c} \\ &= \frac{|\mathbf{p}_4|}{4(2\pi)^6 \, (E_3' + E_4)} d\Omega_{4c} \,, \end{aligned} \tag{497}$$

which compares to equation (82). The differential cross section is therefore

$$d\sigma = \frac{\mathcal{S}}{4F}|\mathcal{M}|^2(2\pi)^4 d\Phi_2(p_1+p_2;p_3,p_4) = \frac{\mathcal{S}}{64\pi^2 F}\,\frac{|\mathbf{p}_4|}{E_3'+E_4}\,|\mathcal{M}|^2\,d\Omega_{4c}\ , \tag{498}$$

giving

$$\frac{d\sigma}{d\Omega_{4c}} = \frac{\mathcal{S}}{64\pi^2 F}\,\frac{|\mathbf{p}_4|}{E_3'+E_4}\,|\mathcal{M}|^2\ . \tag{499}$$

Now, use (21) for F with $\sqrt{s}=E_3'+E_4$. The angular distribution in the cm frame is, therefore (with $|\mathbf{p}_{4c}|=|\mathbf{p}_{3c}|\equiv|\mathbf{p}_{fc}|$ and $|\mathbf{p}_{2c}|=|\mathbf{p}_{1c}|\equiv|\mathbf{p}_{ic}|$ in the cm frame),

$$\begin{aligned}\frac{d\sigma}{d\Omega_{4c}} &= \frac{\mathcal{S}}{64\pi^2 s}\left|\frac{\mathbf{p}_{fc}}{\mathbf{p}_{ic}}\right||\mathcal{M}|^2\\ &= \frac{\mathcal{S}}{64\pi^2 s}\sqrt{\frac{\lambda_{34}}{\lambda_{12}}}\,|\mathcal{M}|^2\ ,\end{aligned} \tag{500}$$

where we have used the following results,

$$|\mathbf{p}_{fc}| = \sqrt{\frac{\lambda_{34}}{4s}}\ ,\qquad \text{and}\qquad |\mathbf{p}_{\rm ic}| = \sqrt{\frac{\lambda_{12}}{4\rm s}}\ . \tag{501}$$

The latter expression for $\frac{d\sigma}{d\Omega_{4c}}$ is far superior to the former, because the latter is written in terms of the single variable s, whereas the former *appears* to contain two independent variables $\mathbf{p}_{ic}$ and $\mathbf{p}_{fc}$. This agrees with references [7] (p. 100), [14] (p. 80), [10] (p. 100), [2] (p. 200). A similar formula for $\frac{d\sigma}{d\Omega_{3c}}$ is obtained by interchanging $3\leftrightarrow 4$ to give ([14] p. 80)

$$\frac{d\sigma}{d\Omega_{3c}} = \frac{d\sigma}{d\Omega_{4c}}\ . \tag{502}$$

The angle Ω_3 is understood to be Ω_{13}, the angle of particle 3 with respect to particle 1. Written explicitly, the above formula becomes

$$\frac{d\sigma}{d\Omega_{13c}} = \frac{d\sigma}{d\Omega_{24c}}\ . \tag{503}$$

When scalar products are encountered such as $\mathbf{p}_2\cdot\mathbf{p}_4 = |\mathbf{p}_4||\mathbf{p}_2|\cos\theta_{24}$, it becomes necessary to specify relative angles such as θ_{24}.

6.1.3 t distribution

The above results are often written in terms of the Mandelstam invariant variable t defined as

$$\begin{aligned} t &\equiv (p_1-p_3)^2 = (p_2-p_4)^2\\ &= m_2^2 - 2E_2E_4 + 2\mathbf{p}_2\cdot\mathbf{p}_4 + m_4^2\end{aligned}$$

$$= \quad m_4^2 + m_2^2 - 2\sqrt{\mathbf{p}_4^2 + m_4^2}\sqrt{\mathbf{p}_2^2 + m_2^2} + 2|\mathbf{p}_4||\mathbf{p}_2|\cos\theta_{24} \;, \tag{504}$$

giving

$$dt = 2|\mathbf{p}_2||\mathbf{p}_4| d\cos\theta_{24} \;, \tag{505}$$

where θ_{24} is the angle between $\mathbf{p}_2$ and $\mathbf{p}_4$. Because of a minus sign ambiguity to come later, we need now to thoroughly establish our convention for a solid angle. Thus, we now present some elementary considerations. The solid angle is

$$\begin{aligned} d\Omega &= \sin\theta d\theta d\phi \\ &= -d\cos\theta d\phi \\ &= -2\pi d\cos\theta \;. \end{aligned} \tag{506}$$

Note that, when integrated this gives the total solid angle. This works as follows.

$$\int_0^\pi \sin\theta d\theta = [-\cos\theta]_0^\pi = -[\cos\theta]_{\theta=0}^{\theta=\pi} = -(-1-1) = 2 \;, \tag{507}$$

or equivalently,

$$\int_0^\pi \sin\theta d\theta = -\int_1^{-1} d\cos\theta = -[\cos\theta]_{\cos\theta=1}^{\cos\theta=-1} = -(-1-1) = 2 \;. \tag{508}$$

This gives the total solid angle as

$$\begin{aligned} \int d\Omega &= \int d\phi \int sin\theta d\theta = 2\pi \int sin\theta d\theta \\ &= -2\pi \int_1^{-1} d\cos\theta = 4\pi \;. \end{aligned} \tag{509}$$

Using equations (505) and (506) gives $dt = -\frac{1}{\pi}|\mathbf{p}_2||\mathbf{p}_4| d\Omega$. Using

$$\frac{d\sigma}{dt} = \frac{d\sigma}{d\Omega_c}\frac{d\Omega_c}{dt} = -\frac{\pi}{|\mathbf{p}_{2c}||\mathbf{p}_{4c}|}\frac{d\sigma}{d\Omega_c} \;, \tag{510}$$

gives [10] (p. 101), with $|\mathbf{p}_{2c}| = |\mathbf{p}_{1c}| \equiv |\mathbf{p}_{ic}|$, and using equation (501) gives

$$\begin{aligned} \frac{d\sigma}{dt} &= -\frac{\mathcal{S}}{64\pi s}\frac{1}{|\mathbf{p}_{ic}|^2}|\mathcal{M}|^2 \\ &= -\frac{\mathcal{S}}{16\pi\lambda_{12}}|\mathcal{M}|^2 \\ &= -\frac{\mathcal{S}}{64\pi F^2}|\mathcal{M}|^2 \;. \end{aligned} \tag{511}$$

Note that all these equations have a minus sign in front, which does not mean the cross section is negative, because t takes on negative values. However, most authors, such as de Wit [7] (p.

100), Ho-Kim [10] (p. 99), and Byckling [14] (p. 81) do not have a minus sign in such formulas. The reason for this is that most authors write

$$\begin{aligned} d\Omega &= d\cos\theta d\phi \\ &= 2\pi d\cos\theta \;, \end{aligned} \tag{512}$$

instead of equation (506). This is explained as follows. Note that the same answer for equation (508) is obtained with

$$\int_0^\pi \sin\theta d\theta = +\int_{-1}^1 d\cos\theta = [\cos\theta]_{\cos\theta=-1}^{\cos\theta=1} = (1 - -1) = 2 \;. \tag{513}$$

In other words, we can change the definition of solid angle to

$$d\Omega \equiv +2\pi d\cos\theta \;, \tag{514}$$

provided that it is understood that the limits of integration are reversed, as in equation (513). This gives the same solid angle as obtained in equation (509) as

$$\int d\Omega = +2\pi \int_{-1}^1 d(\cos\theta) = 4\pi \;. \tag{515}$$

With this convention, the t integration also changes. Previously, with $d\Omega = -2\pi d\cos\theta$, the t integration would have been $\int_{t_0}^{t_\pi} dt$ where $t_0 \equiv t(\theta = 0)$ and $t_\pi \equiv t(\theta = \pi)$. However, with our new convention, the t integration must be

$$d\Omega = +2\pi d\cos\theta \quad => \quad \int_{t_\pi}^{t_0} dt \;. \tag{516}$$

This will be important when we calculate the total cross section,

$$\sigma = \int_{t_\pi}^{t_0} \frac{d\sigma}{dt} dt \;. \tag{517}$$

However, because $s+t+u = \sum_i m_i^2$, then u has opposite behaviour to t. Thus, if u integrations are being performed, then it must be

$$\sigma = \int_{u_0}^{u_\pi} \frac{d\sigma}{du} du \;. \tag{518}$$

With the above considerations, equations (511) then become

$$\begin{aligned} \frac{d\sigma}{dt} &= \frac{\mathcal{S}}{64\pi s} \frac{1}{|\mathbf{p}_{ic}|^2} |\mathcal{M}|^2 \\ &= \frac{\mathcal{S}}{16\pi\lambda_{12}} |\mathcal{M}|^2 \end{aligned}$$

$$= \quad \frac{\mathcal{S}}{64\pi F^2}|\mathcal{M}|^2 \; , \tag{519}$$

where by (21)

$$\lambda_{12} = (s - m_1^2 - m_2^2)^2 - 4m_1^2 m_2^2 = 4F^2 \; . \tag{520}$$

The above expressions for $\frac{d\sigma}{dt}$ are *valid in any frame because σ and t are invariant.* They agree with references [1] (equation 34.30) and [14] (p. 81) and [10] (equation 4.62). The invariant amplitude often consists of *direct (t-channel)* and *exchange (u-channel)* terms,

$$\mathcal{M} \equiv \mathcal{M}_d + \mathcal{M}_e \; . \tag{521}$$

Cross sections, however, always involve the amplitude squared $|\mathcal{M}|^2$, and a new *interference* term arises which is the cross term in the square. Thus,

$$|\mathcal{M}|^2 \equiv |\mathcal{M}_d|^2 + |\mathcal{M}_e|^2 + |\mathcal{M}_i|^2 \; , \tag{522}$$

where

$$|\mathcal{M}_i|^2 \equiv 2\mathcal{M}_d\mathcal{M}_e \; . \tag{523}$$

Thus, the cross section can be written as a direct, exchange and interference cross section,

$$\frac{d\sigma_d}{dt} \quad = \quad \frac{\mathcal{S}}{16\pi\lambda_{12}}|\mathcal{M}_d|^2 \; , \tag{524}$$

$$\frac{d\sigma_e}{dt} \quad = \quad \frac{\mathcal{S}}{16\pi\lambda_{12}}|\mathcal{M}_e|^2 \; , \tag{525}$$

$$\frac{d\sigma_i}{dt} \quad = \quad \frac{\mathcal{S}}{16\pi\lambda_{12}}|\mathcal{M}_i|^2 = \frac{\mathcal{S}}{16\pi\lambda_{12}}\, 2\mathcal{M}_d\mathcal{M}_e \; . \tag{526}$$

The total cross section will be

$$\begin{aligned} \sigma \quad &= \quad \int_{t_\pi}^{t_0} \frac{d\sigma}{dt} dt \\ = \sigma_d + \sigma_e + \sigma_i \quad &= \quad \int_{t_\pi}^{t_0} \frac{d\sigma_d}{dt} dt + \int_{t_\pi}^{t_0} \frac{d\sigma_e}{dt} dt + \int_{t_\pi}^{t_0} \frac{d\sigma_i}{dt} dt \; . \end{aligned} \tag{527}$$

When performing integrations to obtain total cross sections, one finds the analysis is much simpler if the direct, exchange and interference terms are treated separately. Note that equation (3.18) in reference [15] and equation (6) in reference [13] are incorrect; the factor 4 should not appear in the denominator. In the notations of those references, the correct equation is $\frac{d\sigma}{dt} = \frac{1}{64\pi}|\mathcal{M}|^2 \frac{1}{I^2}$.

6.1.4 Cross section units

We consider equation (519) to specifically show how cross section units can be checked. The units of the formula (519) for $d\sigma/dt$ are verified as follows. On the left hand side, σ has units of mb or GeV^{-2}. The Mandelstam variable, t, has units of GeV^2. Thus, $d\sigma/dt$ has units of mb/GeV^2 or GeV^{-4}. The invariant amplitude will be of the form $\mathcal{M} = \frac{g^2}{t-m^2}$ (for the direct term), where $t \equiv (p_1 - p_3)^2$ and g is the coupling constant, and m is the mass of the exchange particle. On the right hand side of (519), the coupling g^2 has units of GeV^2 (in scalar theory), thus $\mathcal{M}$ is dimensionless. λ has units of GeV^4. Thus, the right hand side has units of GeV^{-4}, matching the left hand side.

6.1.5 Relation between $\frac{d\sigma}{d\Omega_c}$ and $\frac{d\sigma}{dt}$

The differential cross section $\frac{d\sigma}{dt}$ can be thought of as an *angular distribution* because t is directly related to the scattering angle $\cos\theta$. Re-writing equation (504) gives

$$\frac{t - m_2^2 - m_4^2 + 2E_2E_4}{2|\mathbf{p}_2||\mathbf{p}_4|} = \cos\theta_{24} \; . \tag{528}$$

In the cm frame, $\mathbf{p}_1 + \mathbf{p}_2 = 0 = \mathbf{p}_3 + \mathbf{p}_4$ and $|\mathbf{p}_1| = |\mathbf{p}_2| \equiv |\mathbf{p}_{ic}|$ and $|\mathbf{p}_3| = |\mathbf{p}_4| \equiv |\mathbf{p}_{fc}|$, which gives equation (504) as

$$\begin{aligned} t &\equiv (p_4 - p_2)^2 \\ &= m_4^2 + m_2^2 - 2\sqrt{\mathbf{p}_{fc}^2 + m_4^2}\sqrt{\mathbf{p}_{ic}^2 + m_2^2} + 2|\mathbf{p}_{fc}||\mathbf{p}_{ic}|\cos\theta_{24c} \; , \end{aligned} \tag{529}$$

or

$$t = m_2^2 + m_4^2 + \frac{1}{2s}\left[-\sqrt{(\lambda_{12} + 4sm_2^2)(\lambda_{34} + 4sm_4^2)} + \sqrt{\lambda_{12}\lambda_{34}}\cos\theta_{24c}\right] \; , \tag{530}$$

using equation (501). Notice that when $m_1 = m_2 = m_3 = m_4$, this reduces to $|\mathbf{p}_{fc}| = |\mathbf{p}_{ic}|$ which is used to obtain equation (6.51) in Griffiths [2]. This can also be written [16] (p. 8), [19] (p. 31), [14] (pp. 79, 80),

$$\cos\theta_{13c} = \frac{s(t-u) + (m_1^2 - m_2^2)(m_3^2 - m_4^2)}{\sqrt{\lambda_{12}\lambda_{34}}} \; . \tag{531}$$

Byckling [14] (p. 80) and Leon [12] (p. 16) also write equivalent expressions for the lab frame angle. Byckling explains how to convert these expressions for the s, t, u channels. Using (514) and (530), we obtain

$$dt = \frac{\sqrt{\lambda_{12}\lambda_{34}}}{2s}\, d\cos\theta = \frac{\sqrt{\lambda_{12}\lambda_{34}}}{4\pi s}\, d\Omega \; . \tag{532}$$

$\frac{d\sigma}{d\Omega_c}$ is written directly in terms of the angular distribution $\frac{d\sigma}{dt}$. Using equations (530) and (532), we obtain [19] (p. 40)

$$\frac{d\sigma}{d\Omega_{24c}} = \frac{\sqrt{\lambda_{12}\lambda_{34}}}{4\pi s}\frac{d\sigma}{dt} , \tag{533}$$

with t given by equation (530), where $\theta_{24} \equiv \theta_4$. Since $\theta_{13} = \theta_{24}$ in the cm frame, $d\theta_{24}/d\theta_{13} = 1$, therefore

$$\frac{d\sigma}{d\Omega_{13}} = \frac{d\sigma}{d\Omega_{24}}\frac{d\Omega_{24}}{d\Omega_{13}} = \frac{d\sigma}{d\Omega_{24}} , \tag{534}$$

and

$$\frac{d\sigma}{d\Omega_{13c}} = \frac{\sqrt{\lambda_{12}\lambda_{34}}}{4\pi s}\frac{d\sigma}{dt} . \tag{535}$$

Now, t can also be written as

$$\begin{aligned} t &\equiv (p_3 - p_1)^2 \\ &= m_3^2 + m_1^2 - 2\sqrt{\mathbf{p}_{fc}^2 + m_3^2}\sqrt{\mathbf{p}_{ic}^2 + m_1^2} + 2|\mathbf{p}_{fc}||\mathbf{p}_{ic}|\cos\theta_{13c} \\ &= m_1^2 + m_3^2 + \frac{1}{2s}\left[-\sqrt{(\lambda_{12} + 4sm_1^2)(\lambda_{34} + 4sm_3^2)} + \sqrt{\lambda_{12}\lambda_{34}}\cos\theta_{13c}\right] . \end{aligned} \tag{536}$$

Just as θ has a minimum and maximum value (i.e. 0 and π), so too does t. The minimum and maximum values are denoted

$$t_0 \equiv t(\theta_c = 0) , \tag{537}$$

and

$$t_\pi \equiv t(\theta_c = \pi) . \tag{538}$$

These are given by [1] (p. 188), [10] (p. 101)

$$\begin{aligned} t_0(t_\pi) &= (E_{1c} - E_{3c})^2 - (|\mathbf{p}_{1c}| \mp |\mathbf{p}_{3c}|)^2 \\ &= \left[\frac{m_1^2 - m_2^2 - m_3^2 + m_4^2}{\sqrt{4s}}\right]^2 - (|\mathbf{p}_{1c}| \mp |\mathbf{p}_{3c}|)^2 \\ &= \frac{1}{4s}\left[(m_1^2 - m_2^2 - m_3^2 + m_4^2)^2 - (\sqrt{\lambda_{12}} \mp \sqrt{\lambda_{34}})^2\right] , \end{aligned} \tag{539}$$

where the $\mp$ notation means that t_0 has the $-$ sign and t_π has the $+$ sign, i.e. $t_0 = (E_{1c} - E_{3c})^2 - (|\mathbf{p}_{1c}| - |\mathbf{p}_{3c}|)^2$ and $t_\pi = (E_{1c} - E_{3c})^2 - (|\mathbf{p}_{1c}| + |\mathbf{p}_{3c}|)^2$. Equation (539) is obtained by substituting 0 and π into equation (536). It is also derived in section 4.4. In plotting an angular distribution, $\cos\theta$ must not exceed the maximum and minimum values. Similarly, when plotting the cross section $\frac{d\sigma}{dt}$, with t on the horizontal axis, then t must not exceed t_0 or t_π.

6.1.6 Lab frame spectral distribution

In the following derivation, T_{4l} is the kinetic energy of particle number 4. The Mandelstam invariant t is

$$\begin{aligned} t &\equiv (p_1 - p_3)^2 = (p_2 - p_4)^2 \,, \\ &= m_1^2 - 2E_1E_3 + 2\mathbf{p}_1 \cdot \mathbf{p}_3 + m_3^2 \,, \\ &= m_2^2 - 2E_2E_4 + 2\mathbf{p}_2 \cdot \mathbf{p}_4 + m_4^2 \,. \end{aligned} \tag{540}$$

Previously, we used the $1, 3$ variables, but now it proves convenient to use the $2, 4$ variables because the lab (target) frame is defined by $\mathbf{p}_2 \equiv 0$ giving $E_{2l} = m_2$ and we obtain

$$t = m_2^2 - 2m_2E_{4l} + m_4^2 \,, \tag{541}$$

or, since $E_4 = T_4 + m_4$,

$$t = (m_2 - m_4)^2 - 2m_2T_{4l} \,, \tag{542}$$

which gives

$$dt = -2m_2 \, dT_{4l} \,. \tag{543}$$

de Wit [7] (p. 101) chooses particle number 4 as the target *recoil* particle. Notice that there is no θ_{24l} appearing in the above two expressions. This makes sense because in the lab frame particle 2 is at rest and so it is impossible to have an angle defined relative to particle 2. Using equations (542) and (543), we obtain the lab frame spectral distribution of particle 4 ([7] (p. 101)

$$\frac{d\sigma}{dT_{4l}} = \frac{d\sigma}{dE_{4l}} = -2m_2\frac{d\sigma}{dt} \,, \tag{544}$$

with t given by equation (542). We also have by (533)

$$\begin{aligned} \frac{d\sigma}{dT_{4l}} &= -2m_2\frac{d\sigma}{dt} \\ &= \frac{8\pi m_2 s}{\sqrt{\lambda_{12}\lambda_{34}}}\frac{d\sigma}{d\Omega_{4c}} \\ &= \frac{2\pi m_2}{|\mathbf{p}_{ic}||\mathbf{p}_{fc}|}\frac{d\sigma}{d\Omega_{4c}} \,. \end{aligned} \tag{545}$$

The spectral distribution of particle 3 is now easily obtained. Conservation of energy gives

$$\begin{aligned} & E_{1l} + E_{2l} = E_{3l} + E_{4l} \\ = \; & \text{constant.} \end{aligned} \tag{546}$$

Thus,

$$dE_{3l} = -dE_{4l} , \tag{547}$$
$$dT_{3l} = -dT_{4l} . \tag{548}$$

Alternatively, we could have substituted energy conservation in the lab frame, $E_4 = E_1 + m_2 - E_3$ into equation (542), to obtain

$$t = m_4^2 - m_2^2 + 2m_2(T_{3l} + m_3 - E_{1l}) , \tag{549}$$

giving

$$dt = 2m_2 dT_{3l} . \tag{550}$$

Thus,

$$\frac{d\sigma}{dT_{3l}} = \frac{d\sigma}{dE_{3l}} = +2m_2\frac{d\sigma}{dt} , \tag{551}$$

with t given by equation (549).

6.1.7 Lab frame angular distribution

To obtain the angular distribution in terms of $\frac{d\sigma}{dt}$, solve (256) for $\cos\theta_{13l}$ and differentiate, as in

$$\cos\theta_{13l} = \frac{E_{3l}(E_{1l}+m_2) - \frac{1}{2}(m_1^2+m_2^2+2m_2E_{1l}+m_3^2-m_4^2)}{|\mathbf{p}_{1l}||\mathbf{p}_{3l}|} \tag{552}$$

$$\frac{d(\cos\theta_{13l})}{dE_{3l}} = \frac{d(\cos\theta_{13l})}{dT_{3l}} = \frac{E_{3l}(m_1^2+m_2^2+m_3^2-m_4^2+2m_2E_{1l}) - 2m_3^2(E_{1l}+m_2)}{2\sqrt{E_{1l}^2-m_1^2}\,(E_{3l}^2-m_3^2)^{3/2}} \quad \text{(lab frame)} , \tag{553}$$

which, upon substitution into equation (551), and with the implied integral over $d\phi$, $d\Omega = 2\pi\, d(\cos\theta)$, gives

$$\begin{aligned}
\frac{d\sigma}{d\Omega_{3l}} &= \frac{m_2}{\pi}\left|\frac{|\mathbf{p}_{1l}||\mathbf{p}_{3l}|}{E_{1l}+m_2-\frac{|\mathbf{p}_{1l}|}{|\mathbf{p}_{3l}|}E_{3l}\cos\theta_{13l}}\right|\frac{d\sigma}{dt} \\
&= \frac{m_2}{\pi}\left|\frac{2\sqrt{E_{1l}^2-m_1^2}\,(E_{3l}^2-m_3^2)^{3/2}}{E_{3l}(m_1^2+m_2^2+m_3^2-m_4^2+2E_{1l}m_2)-2m_3^2(E_{1l}+m_2)}\right|\frac{d\sigma}{dt} ,
\end{aligned} \tag{554}$$

with t given by equation (255). In equation (554), the tall absolute value brackets are necessary. *E_{3l} is a function of θ*, as shown in equation (265), where in general E_{3l} is a double - valued function of θ, and $dE_{3l}/d(\cos\theta_{3l})$ can have *both positive and negative* values. However, $d\sigma/d\Omega_{3l}$ is positive definite, therefore, must depend only on the absolute value of $d(\cos\theta_{13l})/dE_{3l}$. By substituting for E_{3l} from (265), (554) is expressed entirely in terms of θ_{3l}. Equation (554) agrees with Ho-Kim [10] (p. 101) , Byckling [14] (p. 80), and deWit [7] (p. 101). Note that if equation (553) is substituted into the left hand side of equation (554), a large cancellation of terms on the left and right hand sides takes place and one is left back with equation (551), again showing the equivalence of $\frac{d\sigma}{dT}$ and $\frac{d\sigma}{d\Omega}$ in the lab frame.

We now present an alternative derivation of the above results. We evaluate equation (495) in the lab frame, defined as $\mathbf{p}_{2l} \equiv 0$. This means evaluating $\frac{d(E'_{3l}+E_{4l})}{d|\mathbf{p}_{4l}|}$ in the lab frame. E'_{3l} becomes

$$E'_{3l} = \sqrt{(\mathbf{p}_{1l} - \mathbf{p}_{4l})^2 + m_3^2} = \sqrt{|\mathbf{p}_{1l}|^2 + |\mathbf{p}_{4l}|^2 - 2|\mathbf{p}_{1l}||\mathbf{p}_{4l}|\cos\theta + m_3^2}\ , \tag{555}$$

with $\theta \equiv \theta_{4l}$. This gives

$$\frac{d(E'_{3l} + E_{4l})}{d|\mathbf{p}_{4l}|} = |\mathbf{p}_{4l}|\frac{E'_{3l} + E_{4l}(1 - \alpha\cos\theta)}{E'_{3l}E_{4l}}\ , \tag{556}$$

where

$$\alpha \equiv \frac{|\mathbf{p}_{1l}|}{|\mathbf{p}_{4l}|}\ . \tag{557}$$

Thus

$$\frac{d\sigma}{d\Omega_{4l}} = \frac{\mathcal{S}}{64\pi^2 F}\ \frac{|\mathbf{p}_{4l}|}{E'_{3l} + E_{4l}(1 - \alpha\cos\theta)}\ |\mathcal{M}|^2\ . \tag{558}$$

Using equation (25) for $F_l = m_2|\mathbf{p}_{1l}|$ and $E_1 + E_2 = E_{1l} + m_2 = E_{3l} + E_{4l}$ gives

$$\frac{d\sigma}{d\Omega_{4l}} = \frac{\mathcal{S}}{64\pi^2 m_2}\left|\frac{\mathbf{p}_{4l}}{\mathbf{p}_{1l}}\right|\frac{1}{E_{1l} + m_2 - \left|\frac{\mathbf{p}_{1l}}{\mathbf{p}_{4l}}\right| E_{4l}\cos\theta_{4l}}\ |\mathcal{M}|^2\ , \tag{559}$$

which agrees with Byckling [14] (p. 80) and Ho-Kim [10] (equation 4.64). A similar formula for $\frac{d\sigma}{d\Omega_{3l}}$ is obtained by interchanging $3 \leftrightarrow 4$ to give Byckling [14] (p. 80)

$$\frac{d\sigma}{d\Omega_{3l}} = \frac{\mathcal{S}}{64\pi^2 m_2}\left|\frac{\mathbf{p}_{3l}}{\mathbf{p}_{1l}}\right|\frac{1}{E_{1l} + m_2 - \left|\frac{\mathbf{p}_{1l}}{\mathbf{p}_{3l}}\right| E_{3l}\cos\theta_{3l}}\ |\mathcal{M}|^2\ . \tag{560}$$

Upon combining (535) with (554), one obtains

$$\begin{aligned}\frac{d\sigma}{d\Omega_{3l}} &= \frac{4m_2 s}{\sqrt{\lambda_{12}\lambda_{34}}}\left|\frac{|\mathbf{p}_{1l}||\mathbf{p}_{3l}|}{E_{1l} + m_2 - \frac{|\mathbf{p}_{1l}|}{|\mathbf{p}_{3l}|}E_{3l}\cos\theta_{13l}}\right|\frac{d\sigma}{d\Omega_{13c}} \\ &= \frac{4m_2 s}{\sqrt{\lambda_{12}\lambda_{34}}}\left|\frac{2\sqrt{E_{1l}^2 - m_1^2}\ (E_{3l}^2 - m_3^2)^{3/2}}{E_{3l}(m_1^2 + m_2^2 + m_3^2 - m_4^2 + 2E_{1l}m_2) - 2m_3^2(E_{1l} + m_2)}\right|\frac{d\sigma}{d\Omega_{13c}}\ ,\end{aligned} \tag{561}$$

which provides a way of transforming a cm frame angular distribution to a lab frame distribution. This provides an alternative to equation (450). Performing a numerical evaluation of equations (450) and (561), which must give the same answer, enables one to perform an excellent test of these results.

7 Conclusions

This paper has provided a thorough discussion of the transformation of 3 - dimensional differential cross sections from the projectile or center of momentum frames to the lab frame. This is important because transport codes require all cross sections in the lab frame. Several important issues, such as the use of double valued and infinite functions have been discussed extensively. Such a detailed and comprehensive treatment does not appear in previous literature. Special attention has been paid to writing differential cross sections entirely in terms of lab variables, because these are used in 3 - dimensional radiation transport codes. 2 - body final state cross sections are much more complicated than 3 - body final states, because the energy and angle are dependent variables. Equations (558) - (561) tell one exactly how to transform a 2 - body final state angular distribution from the center of momentum frame to the lab (spacecraft) frame, with all the relevant kinematic factors written in terms of lab variables. These results will be very useful in enhancing space radiation transport codes, such as HZETRN, to be capable of fully 3 - dimensional transport.

References

[1] W. M. Yao et al., J. Phys. G **33**, 1 (2006).

[2] D. Griffiths, *Introduction to elementary particles.* (Wiley, New York, 1987).

[3] M. E. Peskin and D. V. Schroeder, *An introduction to quantum field theory*, (Perseus books, Reading, Massachusetts, 1995).

[4] W. Greiner and J. Reinhardt, *Field quantization*, (Springer, New York, 1996).

[5] W. Greiner and J. Reinhardt, *Quantum electrodynamics*, 2nd ed., (Springer, New York, 1996).

[6] F. Gross, *Relativistic quantum mechanics and field theory*, (Wiley, New York, 1993).

[7] B. de Wit and J. Smith, *Field theory in particle physics.* (North-Holland, Amsterdam, 1986).

[8] M. D. Scadron, *Advanced quantum theory*, (Springer-Verlag, New York, 1979).

[9] J. D. Bjorken and S. D. Drell, *Relativistic quantum mechanics*, (McGraw-Hill, New York, 1964).

[10] Q. Ho-Kim and P. Xuan Yem, *Elementary particles and their interactions*, (Springer, New York, 1998).

[11] I. J. R. Aitchison and A. J. G. Hey, *Gauge theories in particle physics*, 3rd ed., (Institute of Physics, Bristol, 2003).

[12] M. Leon, *Particle physics: an introduction*, (Academic Press, New York, 1973).

[13] V. Dmitriev, O. Sushkov, and C. Gaarde, Nucl. Phys. A **459**, 503 (1986).

[14] E. Byckling and K. Kajantie, *Particle kinematics*, (Wiley, New York, 1973).

[15] F. A. Cucinotta, J. W. Wilson and J. W. Norbury, *Parameterizations of pion energy spectrum in nucleon - nucleon collisions*, NASA Technical memorandum TM - 1998 - 208722.

[16] H. Pilkuhn, *The interactions of hadrons*, (North-Holland, Amsterdam, 1967).

[17] J. W. Norbury, Phys. Rev. D **42**, 3696 (1993).

[18] M. Vidovic, M. Greiner, C. Best and G. Soff, Phys. Rev. C **47**, 2308 (1993).

[19] C. J. Joachain, *Quantum collision theory*, (North-Holland, Amsterdam, 1983).

[20] J. D. Jackson, *Classical electrodynamics*, 1st ed., (Wiley, New York, 1972).

[21] R. Hagedorn, *Relativistic kinematics* (Benjamin, New York, 1963).

[22] P. A. Tipler and R. A. llewellyn, *Modern physics*, 3rd ed., (Freeman, New York, 1999).

[23] K. G. Dedrick, Rev. Mod. Phys. **34**, 429 (1962).

REPORT DOCUMENTATION PAGE	*Form Approved* *OMB No. 0704-0188*

1. REPORT DATE *(DD-MM-YYYY)*: 01- 11 - 2008

2. REPORT TYPE: Technical Publication

3. DATES COVERED *(From - To)*:

4. TITLE AND SUBTITLE: Differential Cross Section Kinematics for 3-Dimensional Transport Codes

5a. CONTRACT NUMBER:

5b. GRANT NUMBER:

5c. PROGRAM ELEMENT NUMBER:

5d. PROJECT NUMBER:

5e. TASK NUMBER:

5f. WORK UNIT NUMBER: 651549.02.07.01

6. AUTHOR(S)
Norbury, John W.; and Dick, Frank

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)
NASA Langley Research Center
Hampton, VA 23681-2199

8. PERFORMING ORGANIZATION REPORT NUMBER: L-19382

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)
National Aeronautics and Space Administration
Washington, DC 20546-0001

10. SPONSOR/MONITOR'S ACRONYM(S): NASA

11. SPONSOR/MONITOR'S REPORT NUMBER(S): NASA/TP-2008-215543

12. DISTRIBUTION/AVAILABILITY STATEMENT
Unclassified - Unlimited
Subject Category 93
Availability: NASA CASI (301) 621-0390

13. SUPPLEMENTARY NOTES

14. ABSTRACT
In support of the development of 3-dimensional transport codes, this paper derives the relevant relativistic particle kinematic theory. Formulas are given for invariant, spectral and angular distributions in both the lab (spacecraft) and center of momentum frames, for collisions involving 2, 3 and n - body final states.

15. SUBJECT TERMS
Cross sections; Relativistic kinematics

16. SECURITY CLASSIFICATION OF: a. REPORT	b. ABSTRACT	c. THIS PAGE	17. LIMITATION OF ABSTRACT	18. NUMBER OF PAGES	19a. NAME OF RESPONSIBLE PERSON
					STI Help Desk (email: help@sti.nasa.gov)
					19b. TELEPHONE NUMBER *(Include area code)*
U	U	U	UU	90	(301) 621-0390

Standard Form 298 (Rev. 8-98)
Prescribed by ANSI Std. Z39.18