CONDITIONING OF THE STABLE, DISCRETE-TIME LYAPUNOV OPERATOR*

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Abstract. The Schatten p-norm condition of the discrete-time Lyapunov operator $\mathcal{L}_A$ defined on matrices $P \in \mathbb{R}^{n\times n}$ by $\mathcal{L}_A P \equiv P - APA^T$ is studied for stable matrices $A \in \mathbb{R}^{n\times n}$. Bounds are obtained for the norm of $\mathcal{L}_A$ and its inverse that depend on the spectrum, singular values and radius of stability of A. Since the solution P of the the discrete-time algebraic Lyapunov equation (DALE) $\mathcal{L}_A P = Q$ can be ill-conditioned only when either $\mathcal{L}_A$ or Q is ill-conditioned, these bounds are useful in determining whether P admits a low-rank approximation, which is important in the numerical solution of the DALE for large n.

Key words. Lyapunov matrix equation, condition estimates, large-scale systems, radius of stability.

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1. Introduction. Properties of the solution P of the discrete algebraic Lyapunov equation (DALE), $P = APA^T + Q$, are closely related to the stability properties of A. For instance, the DALE has a unique solution $P = P^T > 0$ for any $Q = Q^T > 0$ if A is stable [11], a fact also true in infinite-dimensional Hilbert spaces [18]. In the setting treated here with A, Q, $P \in \mathbb{R}^{n\times n}$, A is stable if its eigenvalues $\lambda_i(A)$, $i = 1, \ldots, n$, lie inside the unit circle; the eigenvalues are ordered so that $|\lambda_1(A)| \geq |\lambda_2(A)| \geq \cdots \geq |\lambda_n(A)|$. Here A is always assumed to be stable.

In applications where the dimension n is very large, direct solution of the DALE or even storage of P is impractical or impossible. For instance, in numerical weather prediction applications A is the matrix that evolves atmospheric state perturbations. The DALE and its continuous-time analogs can be solved directly for simplified atmospheric models [6, 23], but in realistic models n is about $10^6 - 10^7$ and even the storage of P is impossible. Krylov subspace [5] and Monte Carlo [9] methods have been used to find low-rank approximations of the right-hand side of the DALE and of the solution of the DALE [10].

The solution P of the DALE can be well approximated by a rank-deficient matrix if P has some small singular values. Therefore, it is useful to identify properties of A or Q that lead to P being ill-conditioned. If A is normal then

$$\frac{\lambda_1(P)}{\lambda_n(P)} \leq \frac{\lambda_1(Q)}{\lambda_n(Q)} \frac{1-|\lambda_n(A)|^2}{1-|\lambda_1(A)|^2}\,; \tag{1.1}$$

the conditioning of P is controlled by that of Q and by the spectrum of A. In the general case, the conditioning of Q and of the discrete-time Lyapunov operator $\mathcal{L}_A$ defined by $\mathcal{L}_A P \equiv P - APA^T$ determine when P may be ill-conditioned.

THEOREM 1.1. *Let A be a stable matrix and suppose that $\mathcal{L}_A P = Q$ for $Q = Q^T > 0$. Then*

$$\|P\|_p \, \|P^{-1}\|_p \leq \|\mathcal{L}_A\|_p \, \|\mathcal{L}_A^{-1}\|_p \, \|Q\|_p \, \|Q^{-1}\|_p\,, \qquad p = \infty\,, \tag{1.2}$$

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where $\|\cdot\|_p$ *is the Schatten p-norm (see Eq. 2.2).*

Theorem 1.1 (see proof in Appendix) follows from $\mathcal{L}_A^{-1}$ and its adjoint being positive operators. Therefore the same connection between rank-deficient approximate solutions and operator conditioning exists for matrix equations such as the continuous algebraic Lyapunov equation. We note that Theorem 1.1 also holds for $1 \le p < \infty$ if either A is singular or $\sigma_1^2(A) \ge 2$; $\sigma_1(A)$ is the largest singular value of A.

Here we characterize the Schatten p-norm condition of $\mathcal{L}_A$. The main results are the following. Theorem 3.1 bounds $\|\mathcal{L}_A\|_p$ in terms of the singular values of A. A lower bound for $\|\mathcal{L}_A^{-1}\|_p$ depending on $\lambda_1(A)$ is presented in Theorem 4.1, generalizing results of [7]. Theorem 4.2 gives lower bounds for $\|\mathcal{L}_A^{-1}\|_1$ and $\|\mathcal{L}_A^{-1}\|_\infty$ in terms of the singular values of A. Theorem 4.6 gives an upper bound for $\|\mathcal{L}_A^{-1}\|_p$ depending on the radius of stability of A and generalizes results in [20]. Three examples illustrating the results are included. The issue of whether $\mathcal{L}_A$ and $\mathcal{L}_A^{-1}$ achieve their norms on symmetric, positive definite matrices is addressed in the concluding remarks.

2. Preliminaries. We investigate the condition number $\kappa(\mathcal{L}_A) = \|\mathcal{L}_A\|\,\|\mathcal{L}_A^{-1}\|$, where $\|\cdot\|$ is a norm on $\mathbb{R}^{n^2\times n^2}$ induced by a matrix norm on $\mathbb{R}^{n\times n}$. Specifically, for $\mathcal{M} \in \mathbb{R}^{n^2\times n^2}$ we consider norms defined by

$$\|\mathcal{M}\|_p = \max_{S\neq 0\in\mathbb{R}^{n\times n}} \frac{\|\mathcal{M}S\|_p}{\|S\|_p}, \quad 1 \le p \le \infty, \tag{2.1}$$

where the Schatten matrix p-norm for $S \in \mathbb{R}^{n\times n}$ is defined by

$$\|S\|_p = \left(\sum_{i=1}^{n} (\sigma_i(S))^p\right)^{1/p}; \tag{2.2}$$

$\sigma_i(S)$ are the singular values of S with ordering $\sigma_1(S) \ge \sigma_2(S) \ge \cdots \ge \sigma_n(S) \ge 0$. On $\mathbb{R}^{n\times n}$, $\|\cdot\|_2$ is the Frobenius norm and $\|\cdot\|_\infty = \sigma_1(\cdot)$. If $S = S^T \ge 0$ then $\|S\|_1 = \operatorname{tr} S$. The following lemma about the Schatten p-norms follows from their being unitarily invariant [1, p. 94].

LEMMA 2.1. *For any three matrices X, Y and $Z \in \mathbb{R}^{n\times n}$,*

$$\|XYZ\|_p \le \|X\|_\infty \|Y\|_p \|Z\|_\infty, \qquad 1 \le p \le \infty. \tag{2.3}$$

The $p = 2$ Schatten norm on $\mathbb{R}^{n\times n}$ is equivalently defined as $\|S\|_2^2 = (S,S)$, where $(\cdot,\cdot)$ is the inner product on $\mathbb{R}^{n\times n}$ defined by $(S_1,S_2) = \operatorname{tr} S_1^T S_2$. This norm corresponds to the usual Euclidean norm on $\mathbb{R}^{n^2}$ since $\|S\|_2^2$ is equal to the sum of the squares of the entries of S. As a consequence $\kappa_2(\mathcal{L}_A) = \sigma_1(\mathcal{L}_A)/\sigma_{n^2}(\mathcal{L}_A)$, where $\sigma_1(\mathcal{L}_A)$ and $\sigma_{n^2}(\mathcal{L}_A)$ are respectively the largest and smallest singular values of $\mathcal{L}_A$. The adjoint of $\mathcal{L}_A$ is given by $\mathcal{L}_A^* S = \mathcal{L}_{A^T} S = S - A^T S A$.

We now state some lemmas about mappings $\mathcal{M} \in \mathbb{R}^{n^2\times n^2}$ and about the spectra of $\mathcal{L}_A$ and A.

LEMMA 2.2 ((15) of [2]). $\|\mathcal{M}\|_p \le \|\mathcal{M}\|_1^{1/p}\|\mathcal{M}\|_\infty^{1-1/p}, \quad 1 \le p \le \infty.$

LEMMA 2.3. $\|\mathcal{M}\|_1 = \|\mathcal{M}^*\|_\infty$.

LEMMA 2.4 (See proof of Theorem 1, [2]). *If $\mathcal{M}S > 0$ for all $S \in \mathbb{R}^{n\times n}$ such that $S > 0$, then $\|\mathcal{M}\|_\infty = \|\mathcal{M}I\|_\infty$.*

LEMMA 2.5 ([13, 14]). *The n^2 eigenvalues of $\mathcal{L}_A$ are $1 - \lambda_i(A)\overline{\lambda_j(A)}$, $1 \le i, j \le n$.*

3. The norm of the Lyapunov operator. If A is normal, then $\mathcal{L}_A$ is normal, and its conditioning in the $p = 2$ Schatten norm depends only on its eigenvalues. Therefore when A is normal,

$$\|\mathcal{L}_A^{-1}\|_2 = \frac{1}{\sigma_{n^2}(\mathcal{L}_A)} = \frac{1}{|\lambda_{n^2}(\mathcal{L}_A)|} = \frac{1}{1-|\lambda_1(A)|^2}, \tag{3.1}$$

and

$$\|\mathcal{L}_A\|_2 = \sigma_1(\mathcal{L}_A) = |\lambda_1(\mathcal{L}_A)| = \max_{i,j} |1 - \lambda_i(A)\overline{\lambda_j(A)}|. \tag{3.2}$$

For general A, the following theorem bounds $\|\mathcal{L}_A\|_p$ in terms of the singular values of A.

THEOREM 3.1.

$$|1-\sigma_1^2(A)| \le \max_j |1-\sigma_j^2(A)| \le \|\mathcal{L}_A\|_p \le 1+\sigma_1^2(A), \quad 1 \le p \le \infty. \tag{3.3}$$

Proof. Note that $\mathcal{L}_A v_j v_j^T = v_j v_j^T - \sigma_j^2 u_j u_j^T$, where u_j and v_j are respectively the j-th left and right singular vectors of A such that $Av_j = \sigma_j u_j$. The lower bound follows from $\|u_j u_j^T\|_p = \|v_j v_j^T\|_p = 1$ and

$$\|\mathcal{L}_A\|_p \ge \|v_j v_j^T - \sigma_j^2 u_j u_j^T\|_p \ge \left|\|v_j v_j^T\|_p - \|\sigma_j^2 u_j u_j^T\|_p\right| = \left|1-\sigma_j^2\right|. \tag{3.4}$$

The upper bound follows from

$$\|\mathcal{L}_A P\|_p \le \|P\|_p + \|APA^T\|_p \le \|P\|_p + \|A\|_\infty^2 \|P\|_p. \qquad \square \tag{3.5}$$

If A is normal, $\sigma_j(A)$ can be replaced by $|\lambda_j(A)|$ in Theorem 3.1, and $\|\mathcal{L}_A\|_p \le 1 + |\lambda_1(A)|^2$. If A is normal and $(-\overline{\lambda_1(A)})$ is an eigenvalue of A, then $1 + |\lambda_1(A)|^2$ is an eigenvalue of $\mathcal{L}_A$ and $\|\mathcal{L}_A\|_p = 1 + |\lambda_1(A)|^2$.

Theorem 3.1 shows that $\|\mathcal{L}_A\|_p$ is large and contributes to ill-conditioning if and only if $\sigma_1(A)$ is large, a situation that occurs in various applications [3, 22]. If $\sigma_1(A) \gg 1$ and $|\lambda_1(A)| < 1$, A is highly nonnormal [8, p. 314] and as Corollary 4.8 will show, close to an unstable matrix.

4. The norm of the inverse Lyapunov operator. We first show that a sufficient condition for $\|\mathcal{L}_A^{-1}\|_p$ to be large is that $\lambda_1(A)$ be near the unit circle. The condition is necessary when A is normal.

THEOREM 4.1. *Let A be a stable matrix. Then*

$$\|\mathcal{L}_A^{-1}\|_p \ge \frac{1}{1-|\lambda_1(A)|^2}, \qquad 1 \le p \le \infty, \tag{4.1}$$

with equality holding if A is normal.

Proof. To obtain the lower bound, let z_1 be the leading eigenvector of A, $Az_1 = \lambda_1(A)z_1$, and note that $\mathcal{L}_A z_1 z_1^H = (1-|\lambda_1(A)|^2) z_1 z_1^H$ where $(\cdot)^H$ denotes conjugate transpose. Either $\operatorname{Re} z_1 z_1^H \neq 0$ or $\operatorname{Im} z_1 z_1^H \neq 0$ is an eigenvector of $\mathcal{L}_A$, and it follows that $\|\mathcal{L}_A^{-1}\|_p \ge (1-|\lambda_1(A)|^2)^{-1}$. Finally, if A is normal, then

$$\mathcal{L}_{A^T}^{-1} I = \mathcal{L}_A^{-1} I = \sum_{i=1}^{n} \frac{1}{1-|\lambda_i(A)|^2} z_i z_i^H, \tag{4.2}$$

and $\|\mathcal{L}_A^{-1}\|_\infty = \|\mathcal{L}_A^{-1}\|_1 = (1-|\lambda_1(A)|^2)^{-1}$. Using Lemma 2.2 gives $\|\mathcal{L}_A^{-1}\|_p \leq (1-|\lambda_1(A)|^2)^{-1}$ when A is normal, and therefore $\|\mathcal{L}_A^{-1}\|_p = (1-|\lambda_1(A)|^2)^{-1}$. □

When A is nonnormal, $\|\mathcal{L}_A^{-1}\|_p$ can be large without $\lambda_1(A)$ being near the unit circle. For instance, if $\sigma_1(A)$ is large or more generally if $\|A^k\|_\infty$ converges to zero slowly as a function of k, then $\|\mathcal{L}_A^{-1}\|_p$ is large. We show this fact first for $p = 1, \infty$.

THEOREM 4.2. *Let A be a stable matrix. For all $m \geq 1$,*

$$\|\mathcal{L}_A^{-1}\|_1 = \left\| \sum_{k=0}^{\infty} \left(A^k\right)^{\mathrm{T}} A^k \right\|_\infty \geq \left\| \sum_{k=0}^{m} \left(A^k\right)^{\mathrm{T}} A^k \right\|_\infty + \frac{\sigma_n^{2(m+1)}(A)}{1-\sigma_n^2(A)}, \tag{4.3}$$

$$\|\mathcal{L}_A^{-1}\|_\infty = \left\| \sum_{k=0}^{\infty} A^k \left(A^k\right)^{\mathrm{T}} \right\|_\infty \geq \left\| \sum_{k=0}^{m} A^k \left(A^k\right)^{\mathrm{T}} \right\|_\infty + \frac{\sigma_n^{2(m+1)}(A)}{1-\sigma_n^2(A)}. \tag{4.4}$$

In particular,

$$\|\mathcal{L}_A^{-1}\|_p \geq 1 + \sigma_1^2(A) + \frac{\sigma_n^4(A)}{1-\sigma_n^2(A)}, \qquad p = 1, \infty. \tag{4.5}$$

Proof. The operator $\mathcal{L}_A^{-1}$ applied to $S \in \mathbb{R}^{n\times n}$ can be expressed as [18]

$$\mathcal{L}_A^{-1} S = \sum_{k=0}^{\infty} A^k S \left(A^k\right)^{\mathrm{T}}. \tag{4.6}$$

Applying Lemma 2.4 gives $\|\mathcal{L}_A^{-1}\|_\infty = \|\mathcal{L}_A^{-1} I\|_\infty$, with the inequality in (4.4) being a consequence of

$$\left\| \sum_{k=0}^{\infty} A^k \left(A^k\right)^{\mathrm{T}} \right\|_\infty \geq \left\| \sum_{k=0}^{m} A^k \left(A^k\right)^{\mathrm{T}} \right\|_\infty + \lambda_n \left(\sum_{k=m+1}^{\infty} A^k \left(A^T\right)^k \right), \tag{4.7}$$

and

$$\lambda_n \left(\sum_{k=m+1}^{\infty} A^k \left(A^T\right)^k \right) \geq \sum_{k=m+1}^{\infty} \lambda_n \left(A^k \left(A^T\right)^k \right) \geq \sum_{k=m+1}^{\infty} \sigma_n^{2k}(A) = \frac{\sigma_n^{2(m+1)}(A)}{1-\sigma_n^2(A)}, \tag{4.8}$$

where we have used the facts that for matrices $W, X, Y \in \mathbb{R}^{n\times n}$ with X, Y being symmetric positive semi-definite, $\lambda_i(X+Y) \geq \lambda_i(X) + \lambda_n(Y)$ and $\lambda_i(WXW^T) \geq \sigma_n^2(W)\lambda_i(X)$ [17]. Likewise the $p = 1$ results follow from $\|\mathcal{L}_A^{-1}\|_1 = \|\mathcal{L}_{A^T}^{-1} I\|_\infty$. □

Lower bounds for $1 < p < \infty$ follow trivially, e.g.,

$$\|\mathcal{L}_A^{-1}\|_p \geq \frac{\|\mathcal{L}_A^{-1} I\|_p}{\|I\|_p} = \frac{\|\mathcal{L}_A^{-1} I\|_p}{n^{1/p}} \geq n^{-1/p} \|\mathcal{L}_A^{-1}\|_\infty, \tag{4.9}$$

but give little information when n is large. A lower bound for $1 \leq p \leq \infty$ depending on $\sigma_1(A)$ and independent of n is given in Corollary 4.9.

We now relate $\|\mathcal{L}_A^{-1}\|_p$ to the distance from A to the set of unstable matrices as measured by its *radius of stability* [15].

DEFINITION 4.3. *For any stable matrix $A \in \mathbb{R}^{n\times n}$ define the radius of stability $r(A)$ by*

$$r(A) \equiv \min_{0\leq\theta\leq 2\pi} \|(e^{i\theta} I - A)^{-1}\|_\infty^{-1} = \min_{0\leq\theta\leq 2\pi} \|R(e^{i\theta}, A)\|_\infty^{-1}, \tag{4.10}$$

where the resolvent of A is $R(\lambda, A) = (\lambda I - A)^{-1}$.

If A is normal and stable, then $r(A) = 1 - |\lambda_1(A)|$. However, if A is nonnormal and if its eigenvalues are *sensitive* to perturbations, then $r(A) \ll 1 - |\lambda_1(A)|$. The sensitivity of the eigenvalues of A is most completely described by its *pseudospectrum* [21]. The radius of stability $r(A)$ is the largest value of ϵ such that the ϵ-pseudospectrum of A lies inside the unit circle; $r(A)$ being small indicates that the ϵ-pseudospectrum of A is close to the unit circle for small ϵ. The following theorem shows that when $r(A)$ is small, $\|\mathcal{L}_A^{-1}\|_p$ must be large.

THEOREM 4.4 (Proven for $p = \infty$ in [7]). *Let A be a stable matrix. Then*

$$\|\mathcal{L}_A^{-1}\|_p \geq \frac{1}{2r(A) + r^2(A)}, \qquad 1 \leq p \leq \infty. \tag{4.11}$$

Proof. There exists a matrix $E \in \mathbb{R}^{n \times n}$ with $|\lambda_1(A + E)| = 1$ and $\|E\|_\infty = r(A)$. Therefore there exists a vector x with $x^H x = 1$ such that $(A + E)x = e^{i\theta} x$ for some $0 \leq \theta \leq 2\pi$. Using $\|xx^H\|_p = 1$ and Lemma 2.1 gives

$$\begin{aligned} \|\mathcal{L}_A xx^H\|_p &= \| - Exx^H E^T + e^{i\theta} xx^H E^T + e^{-i\theta} Exx^H \| \\ &\leq \|Exx^H E^T\|_p + \|xx^H E^T\|_p + \|Exx^H\|_p \\ &\leq \|E\|_\infty^2 + 2\|E\|_\infty = r^2(A) + 2r(A), \end{aligned} \tag{4.12}$$

and we have

$$\|\mathcal{L}_A^{-1}\|_p \geq \frac{\|\mathcal{L}_A^{-1}\mathcal{L}_A xx^H\|_p}{\|\mathcal{L}_A xx^H\|_p} = \frac{1}{\|\mathcal{L}_A xx^H\|_p} \geq \frac{1}{2r(A) + r^2(A)}. \qquad \square \tag{4.13}$$

A consequence of Theorem 4.4 is the following lower bound for $r(A)$ in terms of $\|\mathcal{L}_A^{-1}\|_p$.

COROLLARY 4.5. *Let A be a stable matrix. Then*

$$r(A) \geq \frac{\|\mathcal{L}_A^{-1}\|_p^{-1}}{1 + \sqrt{1 + \|\mathcal{L}_A^{-1}\|_p^{-1}}}, \qquad 1 \leq p \leq \infty. \tag{4.14}$$

Bounds for $r(A)$ are useful in robust stability [12] and in the study of perturbations of the discrete algebraic Riccati equation (DARE) [19]. In [19, Lemma 2.2] the bound

$$r(A) \geq \frac{\|\mathcal{L}_A^{-1}\|_\infty^{-1}}{\sigma_1(A) + \sqrt{\sigma_1^2(A) + \|\mathcal{L}_A^{-1}\|_\infty^{-1}}}, \tag{4.15}$$

was used to formulate conditions under which a perturbed DARE has a unique, symmetric, positive definite solution. Since the lower bound in (4.14) with $p = \infty$ is sharper than that in (4.15) when $\sigma_1(A) > 1$, it can be used to show existence of a unique, symmetric, positive definite solution of the perturbed DARE for a larger class of perturbations [19, Theorem 4.1].

We generalize to Schatten p-norms the conjecture of [7] proven in [20] for the Frobenius norm.

THEOREM 4.6. *Let A be a stable matrix. Then*

$$\|\mathcal{L}_A^{-1}\|_p \leq \frac{1}{r^2(A)}, \quad 1 \leq p \leq \infty. \tag{4.16}$$

Proof. $\mathcal{L}_A^{-1} I$ can be expressed as [20, 13],

$$\mathcal{L}_A^{-1} I = \frac{1}{2\pi} \int_0^{2\pi} R(e^{i\theta}, A) R(e^{i\theta}, A)^H \, d\theta \,. \tag{4.17}$$

Therefore, from Lemma 2.4,

$$||\mathcal{L}_A^{-1}||_\infty = ||\mathcal{L}_A^{-1} I||_\infty \leq \frac{1}{2\pi} \int_0^{2\pi} ||R(e^{i\theta}, A)||_\infty^2 \, d\theta \leq \frac{1}{r^2(A)} \,. \tag{4.18}$$

The inequality (4.16) for $p = 1$ follows from $||\mathcal{L}_A^{-1}||_1 = ||\mathcal{L}_{A^T}^{-1} I||_\infty$ and $r(A) = r(A^T)$. The theorem follows from Lemma 2.2. □

As a consequence, any solution of the DALE can be used to obtain an upper bound for $r(A)$.

COROLLARY 4.7. *Let A be a stable matrix and let $\mathcal{L}_A P = Q$. Then*

$$r^2(A) \leq \frac{||Q||_p}{||P||_p} \,, \qquad 1 \leq p \leq \infty \,. \tag{4.19}$$

Theorem 4.6 can be combined with any lower bound for $||\mathcal{L}_A^{-1}||_p$ to obtain an upper bound for $r(A)$. For instance, from Theorem 4.2 we get the following upper bound.

COROLLARY 4.8. *Let A be a stable matrix. Then*

$$r^2(A) \leq \frac{1}{1 + \sigma_1^2(A)} \,. \tag{4.20}$$

Combining Corollary 4.8 and Theorem 4.4 gives a lower bound for $||\mathcal{L}_A^{-1}||_p$.

COROLLARY 4.9. *Let A be a stable matrix. Then*

$$||\mathcal{L}_A^{-1}||_p \geq \frac{1 + \sigma_1^2(A)}{1 + 2\sqrt{1 + \sigma_1^2(A)}} \,, \qquad 1 \leq p \leq \infty \,. \tag{4.21}$$

5. Examples. We present three examples that illustrate how ill-conditioning of $\mathcal{L}_A$ leads to low-rank approximate solutions of the DALE.

EXAMPLE 1. *Almost unit eigenvalues.* Take $A = \lambda z z^T$ where λ and z are real, $0 < \lambda < 1$ and $z^T z = 1$. The matrix A is symmetric and $\mathcal{L}_A$ is self-adjoint. The eigenvalues of A are $(\lambda, 0, \ldots, 0)$. The operator $\mathcal{L}_A$ has singular values (and eigenvalues) $(1, \ldots, 1, 1 - \lambda^2)$. Therefore $||\mathcal{L}_A||_2 = 1$ and $1 \leq ||\mathcal{L}_A||_p \leq 1 + \lambda^2$ from Theorem 3.1. The norm of the inverse Lyapunov operator is

$$||\mathcal{L}_A^{-1}||_p = \frac{1}{1 - \lambda^2} \,, \qquad 1 \leq p \leq \infty \,, \tag{5.1}$$

according to Theorem 4.1. As the eigenvalue λ approaches the unit circle, $\mathcal{L}_A$ is increasingly poorly conditioned. The solution of the DALE for this choice of A is:

$$P = \frac{\lambda^2}{1 - \lambda^2} \left(z^T Q z\right) z z^T + Q \,. \tag{5.2}$$

A "natural" rank-1 approximation $\tilde{P}$ of P is $\tilde{P} = \lambda^2 (1 - \lambda^2)^{-1} (z^T Q z) z z^T$. As the eigenvalue λ approaches the unit circle, if $(z^T Q z)$ is nonzero, P is increasingly well-approximated by $\tilde{P}$ in the sense that $||P - \tilde{P}||_p / ||P||_p$ approaches zero.

EXAMPLE 2. *Large singular values.* Take $A = \sigma y z^T$ where $\sigma > 0$ and y and z are real unit n-vectors. The matrix A has at most one nonzero eigenvalue, namely $\lambda = \sigma(y^T z)$, taken to be less than one in absolute value. The sensitivity s of the eigenvalue λ is the cosine of the angle between y and z, i.e., $s = \lambda/\sigma$ for $\lambda \neq 0$, indicating that λ is sensitive to perturbations to A when σ is large [8].

Theorem 3.1 gives that $1 + \sigma^2 \geq \|\mathcal{L}_A\|_p \geq |1 - \sigma^2|$, showing that $\|\mathcal{L}_A\|_p$ is large when σ is large. From Lemmas 2.3 and 2.4,

$$\|\mathcal{L}_A^{-1}\|_1 = \|\mathcal{L}_A^{-1}\|_\infty = 1 + \frac{\sigma^2}{1-\lambda^2}, \tag{5.3}$$

and it follows from Lemma 2.2 that $\|\mathcal{L}_A^{-1}\|_p \leq 1 + \sigma^2/(1-\lambda^2)$. A lower bound for the $p = 2$ norm is

$$\|\mathcal{L}_A^{-1}\|_2 \geq \|\mathcal{L}_A^{-1} z z^T\|_2 = \sqrt{1 + 2\frac{\lambda^2}{1-\lambda^2} + \frac{\sigma^4}{(1-\lambda^2)^2}}. \tag{5.4}$$

The matrix A is near an unstable matrix when either $|\lambda|$ is near unity or when σ is large since

$$\left\|\left(e^{i\theta} I - \sigma y z^T\right)^{-1}\right\|_\infty = \left\|e^{-i\theta} I + \frac{\sigma e^{-2i\theta}}{1 - \lambda e^{-i\theta}} y z^T\right\|_\infty \geq 1 + \frac{2|\lambda|}{1-|\lambda|} + \frac{\sigma^2}{(1-|\lambda|)^2}. \tag{5.5}$$

Therefore $r(A) \leq (1 - |\lambda|)/\sigma$ and a lower bound on $\|\mathcal{L}_A^{-1}\|_p$ follows from Theorem 4.4. When either $|\lambda|$ is close to unity or when σ is large, $r(A)$ is small and $\kappa_p(\mathcal{L}_A)$ is large.

The solution of the DALE is

$$P = \frac{\sigma^2}{1-\lambda^2}\left(z^T Q z\right) y y^T + Q. \tag{5.6}$$

When $\mathcal{L}_A$ is ill-conditioned and $\left(z^T Q z\right) \neq 0$, the rank-1 matrix $\tilde{P} = \sigma^2(1-\lambda^2)^{-1}(z^T Q z) y y^T$ is a good approximation of P in the sense that $\|P - \tilde{P}\|_p / \|P\|_p$ is small.

EXAMPLE 3. *Sensitive eigenvalues.* Consider the dynamics arising from the one-dimensional advection equation, $w_t + w_x = 0$ for $0 \leq x \leq n$, with boundary condition $w(0,t) = 0$. The matrix A that advances the n-vector $w(x = 1, 2, \ldots, n, t = t_0)$ to $w(x = 1, 2, \ldots, n, t = t_0 + 1)$ is the $n \times n$ matrix with ones on the sub-diagonal and zero elsewhere, i.e., the transpose of an $n \times n$ Jordan block with zero eigenvalue. Adding stochastic forcing with covariance Q at unit time intervals leads to the DALE, $\mathcal{L}_A P = Q$, where P is the steady-state covariance of w.

Since $\sigma_1(A) = 1$, Theorem 3.1 yields $1 \leq \|\mathcal{L}_A\|_p \leq 2$. Further, since $\|\mathcal{L}_A\|_1 \geq \|\mathcal{L}_A e_1 e_1^T\|_1 = \|e_1 e_1^T - e_2 e_2^T\|_1 = 2$, where e_j is the j-th column of the identity matrix, $\|\mathcal{L}_A\|_1 = 2$. A similar argument with $\mathcal{L}_{A^T}$ gives $\|\mathcal{L}_A\|_\infty = 2$. Calculating $\mathcal{L}_A^{-1} I$ and $\mathcal{L}_{A^T}^{-1} I$ gives $\|\mathcal{L}_A^{-1}\|_\infty = \|\mathcal{L}_A^{-1}\|_1 = n$. Therefore, using Lemma 2.2, $\|\mathcal{L}_A^{-1}\|_p \leq n$. Also,

$$\|\mathcal{L}_A^{-1}\|_2 \geq \frac{\|\mathcal{L}_A^{-1} e_1 e_1^T\|_2}{\|e_1 e_1^T\|_2} = \sqrt{n}. \tag{5.7}$$

A direct calculation shows that

$$\|(e^{i\theta} I - A)^{-1}\|_2^2 = \left\|\sum_{k=0}^{n-1} A^k e^{-i(k+1)\theta}\right\|_2^2 = \frac{n(n+1)}{2}, \tag{5.8}$$

for any real θ. Since $\sqrt{n}\|(e^{i\theta}I - A)^{-1}\|_\infty \geq \|(e^{i\theta}I - A)^{-1}\|_2$, we have $r^2(A) \leq 2/(n+1)$. Theorem 4.4 then gives a lower bound for $\|\mathcal{L}_A^{-1}\|_p$, $1 \leq p \leq \infty$. Thus as n becomes large, that is, as the domain becomes large with respect to the advection length scale, $\mathcal{L}_A$ is increasingly ill-conditioned.

The elements P_{ij} of the solution P of the DALE are

$$P_{ij} = e_i^T P e_j = \sum_{k=0}^{n-1} e_i^T A^k Q (A^T)^k e_j = \sum_{k=0}^{\min(i-1,j-1)} Q_{i-k,j-k}\,. \tag{5.9}$$

Therefore if $Q = Q^T > 0$, a "natural" rank-m approximation of P is the matrix $\tilde{P}$ defined by

$$\tilde{P}_{i,j} = \begin{cases} P_{i,j}\,, & n-m < i,j \leq n \\ 0 & \text{otherwise.} \end{cases} \tag{5.10}$$

When Q is diagonal, P is also diagonal and

$$P_{ii} = \sum_{k=1}^{i} Q_{kk}\,. \tag{5.11}$$

In this case, each $Q_{kk} > 0$ and $\tilde{P}$ is the best rank-m approximation of P in the sense of minimizing $\|P - \tilde{P}\|_p$. We note that $\tilde{P}$ is associated with the left-most part of the domain $0 \leq x \leq n$.

6. Concluding Remarks. Results about $\|\mathcal{L}_A^{-1}\|_p$ translate into bounds for solutions of the DALE. For instance, the solution P of the DALE for $Q = Q^T \geq 0$ satisfies

$$\operatorname{tr} P \leq \|\mathcal{L}_A^{-1}\|_1 \operatorname{tr} Q\,, \tag{6.1}$$

and the upper bound is achieved for $Q = w_1 w_1^T$, where w_1 is the leading eigenvector of $\mathcal{L}_{A^T}^{-1} I$. In the $p = \infty$ norm, $\mathcal{L}_A^{-1}$ achieves its norm on the identity. In the $p = 2$ norm, $\mathcal{L}_A^{-1}$ does not in general achieve its norm on the identity, and the question arises whether it achieves its norm on any symmetric, positive semi-definite matrix. The forward operator $\mathcal{L}_A$ does not in general assume its norm on a symmetric, positive semi-definite matrix. The following theorem states that $\mathcal{L}_A^{-1}$ does achieve its $p = 2$ norm on a symmetric, positive semi-definite matrix.

THEOREM 6.1. *There exists a matrix $S = S^T \geq 0$ such that* $\|\mathcal{L}_A^{-1} S\|_2 / \|S\|_2 = \|\mathcal{L}_A^{-1}\|_2$.

Proof. Theorem 8 of [4] states that the inverse of the stable, continuous-time Lyapunov operator achieves its $p = 2$ norm on a symmetric matrix. The proof is easily adapted to give that $\mathcal{L}_A^{-1}$ achieves its $p = 2$ norm on a symmetric matrix. We now show that if $\mathcal{L}_A^{-1}$ achieves its $p = 2$ norm on a symmetric matrix, it does so on a symmetric, positive semi-definite matrix. Suppose that $\|\mathcal{L}_A^{-1} S\|_2 / \|S\|_2 = \|\mathcal{L}_A^{-1}\|_2$ and S is symmetric with Schur decomposition $S = UDU^T$. Define the symmetric, positive semi-definite matrix $S^+ = U|D|U^T$. Then $\|S\|_2 = \|S^+\|_2$ and $-S^+ \leq S \leq S^+$. The positiveness of the stable, discrete-time inverse Lyapunov operator mapping implies that $-\mathcal{L}_A^{-1} S^+ \leq \mathcal{L}_A^{-1} S \leq \mathcal{L}_A^{-1} S^+$, which implies that $\|\mathcal{L}_A^{-1} S\|_2 \leq \|\mathcal{L}_A^{-1} S^+\|_2$. Therefore

$$\frac{\|\mathcal{L}_A^{-1} S\|_2}{\|S\|_2} = \frac{\|\mathcal{L}_A^{-1} S\|_2}{\|S^+\|_2} \leq \frac{\|\mathcal{L}_A^{-1} S^+\|_2}{\|S^+\|_2}\,. \qquad \square \tag{6.2}$$

Additional information about the leading singular vectors of $\mathcal{L}_A^{-1}$ could be useful for determining low-rank approximations of P. The power method can be applied to $\mathcal{L}_{A^T}^{-1}\mathcal{L}_A^{-1}$ to calculate the leading right singular vector and singular value of $\mathcal{L}_A^{-1}$ [7]. However, this approach requires solving two DALEs at each iteration, which may be impractical for large n. If it is practical to store P and to apply $\mathcal{L}_A$ and $\mathcal{L}_{A^T}$, a Lanczos method could be used to compute the trailing eigenvectors of $\mathcal{L}_A\mathcal{L}_{A^T}$ while avoiding the cost of solving any DALEs.

Appendix. *Proof of Theorem 1.1.* By definition, $\|P\|_p \le \|\mathcal{L}_A^{-1}\|_p\, \|Q\|_p$, and it remains to show that $\|P^{-1}\|_\infty \le \|\mathcal{L}_A\|_\infty\, \|Q^{-1}\|_\infty$. Since $P = P^T > 0$, there is a nonzero $x \in \mathbb{R}^n$ such that

$$\|P^{-1}\|_\infty = \frac{1}{\lambda_n(P)} = \frac{x^T x}{x^T\left(\mathcal{L}_A^{-1}Q\right)x} = \frac{\operatorname{tr} xx^T}{\operatorname{tr}\left(\mathcal{L}_A^{-1}Q\right)xx^T} = \frac{\operatorname{tr} xx^T}{\operatorname{tr}\left(\left(\mathcal{L}_{A^T}\right)^{-1} xx^T\right)Q}. \quad \text{(A.1)}$$

Let $B = \mathcal{L}_{A^T}^{-1}(xx^T)$ and note $B = B^T \ge 0$. Then using Lemma 2.3 and $\operatorname{tr} BQ \ge \lambda_n(Q)\operatorname{tr} B$ gives

$$\|P^{-1}\|_\infty = \frac{\operatorname{tr}\mathcal{L}_{A^T}B}{\operatorname{tr} BQ} \le \frac{\operatorname{tr}\mathcal{L}_{A^T}B}{\operatorname{tr} B}\frac{1}{\lambda_n(Q)} \le \|\mathcal{L}_{A^T}\|_1\|Q^{-1}\|_\infty = \|\mathcal{L}_A\|_\infty\|Q^{-1}\|_\infty. \quad \Box \quad \text{(A.2)}$$

Theorem 1.1 holds for $1 \le p \le \infty$ given some restrictions on A. From [16], $\lambda_i(P) \ge \lambda_i(Q) + \sigma_n^2(A)\lambda_n(P)$, and it follows that $\|P^{-1}\|_p \le \|Q^{-1}\|_p$ for $1 \le p \le \infty$. From Theorem 3.1, $\|\mathcal{L}_A\|_p \ge 1$ if either A is singular or $\sigma_1^2(A) \ge 2$. Therefore if either A is singular or $\sigma_1^2(A) \ge 2$,

$$\|P^{-1}\|_p \le \|\mathcal{L}_A\|_p\|Q^{-1}\|_p\,, \qquad 1 \le p \le \infty. \quad \text{(A.3)}$$

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