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By J. Cassens

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# METHOD OF DETERMINING THE WEIGHTS OF THE MOST IMPORTANT SIMPLE GIRDERS*

By J. Cassens

This paper presents a series of tables for the simple and more common types of girders, similar to the tables given in handbooks under the heading "Strength of Materials," for determining the moments, deflections, etc., of simple beams. Instead of the uniform cross section there assumed, the formulas given here apply only to girders of "uniform strength," i.e., it is assumed that a girder is so dimensioned that a given load subjects it to a uniform stress throughout its whole length. This principle is particularly applicable to very strong structures. Girders of uniform strength are the lightest girders conceivable, because any girder, all of whose members are stressed to the limit, can not be surpassed by a lighter girder, if the two girders have the same form. The weight $G$ of a member of length $l$, cross section $F$ and specific gravity $\gamma$ is:

$$G = Fl\gamma \tag{1}$$

Instead of this it is also possible to write

$$G = Sl\frac{\gamma}{\sigma_e} \tag{2}$$

if the member carries the load $S$ and is dimensioned according to the stress $\sigma_e$. With a given static arrangement and a given load, $S$, $l$ and $\gamma$ can be very accurately determined for any member. The attainable stress $\sigma_e$ in tension members is easily determined. However, when stability problems arise, e.g., buckling, and tilting phenomena (and they can be avoided in hardly any static arrangements), it is often difficult to obtain a sufficiently accurate value of $\sigma_e$ for estimating the weight. In a simple framework, for which the assumption of pin joints is permissible, there are only tension and compression members. For the latter it is then necessary to determine only the value of $\sigma_e$. This is best accomplished by ascertaining the values of the forces occurring

*"Gewichtsermittlung der wichtigsten einfachen Träger." Zeitschrift für Flugtechnik und Motorluftschiffahrt (published by R. Oldenbourg, Munich and Berlin), August 14, 1931, pp. 456-463.

in the members. In the table, therefore, general data are given regarding the magnitudes of the forces before every weight. It would be impracticable to determine the force and dimensions of every compression member and therefrom to calculate the value of $\sigma_e$. It fully suffices to test a few samples of each type of compression members and from them find the mean value of $\sigma_e$ for each type. Where such determinations must be made often, it is very practical, for certain sections (e.g., tubes with a fixed wall-thickness ratio), to plot the attainable stress $\sigma_e$ against $\frac{\sqrt{P}}{l}$ *. Such graphs will save much work.

There is the same relation between the general formula for the force in a member and the corresponding weight formula as between a differential and an integral. The expression for the force in the member is multiplied by $\frac{\gamma}{\sigma_e}$ and $l$ according to equation (2), which gives the weight of any member in the framework. Instead of the infinitely small quantity dx, we work, in cases 1 to 8 with a finite quantity, namely the length of the member, i.e., with $\underline{a}$ for the chords, with $\underline{d}$ for the diagonals, and with $\underline{h}$ for the vertical members. Hence we use no integral formulas, but derive the sums of series. In case 1, the series of the upper-chord weights reads:

$$P \frac{a}{h} \frac{\gamma}{\sigma_e} a \left(1 + 2 + 3 + \ldots\ldots + (n - 1)\right)$$

The sum of the series is $\frac{(n - 1)n}{2}$. The series of the lower-chord weights reads:

$$P \frac{a}{h} \frac{\gamma}{\sigma_e} a \left(1 + 2 + 3 + \ldots\ldots + n\right)$$

The sum of this series is $\frac{n(n + 1)}{2}$. This yields

$$P \frac{l}{h} \frac{\gamma}{\sigma_e} a \frac{n - 1 + n + 1}{2} = P\, l \frac{\gamma}{\sigma_e} \frac{l}{h}$$

In cases 2 and 3 the summation is very similarly made with respect to the details. In case 4 (chord weight) summations of the following series have to be made:

$$\frac{Q}{2} \frac{a}{h} \frac{\gamma}{\sigma_e} \frac{a}{n} \left[1^2 + 2^2 + 3^2 + \ldots\ldots + (n - 1)^2\right]$$

* H. Wagner: Remarks on Airplane Struts and Girders under Compressive and Bending Stresses. Index Valves. Z.F.M., June 14, 1928. T.M. No. 500, N.A.C.A., 1929.

and

$$\frac{Q}{2}\ \frac{a}{h}\ \frac{\gamma}{\sigma_e}\ \frac{a}{n}\left[1^2 + 2^2 + 3^2 + \ldots. + n^2\right]$$

The sums of the series are:

$$\frac{(n-1)\ n(2n-1)}{6} \text{ and } \frac{n(n+1)\ (2n+1)}{6}$$

The given weight formula is obtained by slight changes. It is not necessary to give the series for the web members. It needs only to be noted that their order is always 1 less than that of the chords. Case 5 yields the following series:

$$\frac{Q}{3}\ \frac{a}{h}\ \frac{\gamma}{\sigma_e}\ \frac{a}{n^2}\left[1^3 + 2^3 + 3^3 + \ldots. + (n-1)^3\right]$$

and

$$\frac{Q}{3}\ \frac{a}{h}\ \frac{\gamma}{\sigma_e}\ \frac{a}{n^2}\left[1^3 + 2^3 + 3^3 + \ldots. + n^3\right]$$

Their sums are:

$$\left[\frac{(n-1)\ n}{2}\right]^2 \text{and } \left[\frac{n(n+1)}{2}\right]^2$$

In case 8 the series for the upper-chord weights is

$$\begin{aligned} G^{(OG)} &= P\ \frac{l}{\Delta h}\ \frac{\gamma}{\sigma_e}\ a_1\ (B^0 - 1 + B^1 - 1 + \\ &\qquad + B^2 - 1 + B^3 - 1 + \ldots. + B^{n-1} - 1) \\ &= P\ \frac{l}{\Delta h}\ \frac{\gamma}{\sigma_e}\ a_1\left[\frac{B^n - 1}{B - 1} - n\right] \end{aligned}$$

We now write:

$$B - 1 = \frac{\tan\beta}{\tan\varphi} = \frac{\Delta h}{h_o}\ \frac{a_1}{l}$$

When it is remembered that

$$l = \Sigma a_k = a_1\ \frac{B^n - 1}{B - 1}$$

from

$$B - 1 = \frac{\Delta h}{h_o}\ \frac{a_1}{l}$$

we obtain

$$B - 1 = \frac{\Delta h}{h_o}\ \frac{B - 1}{B^n - 1}$$

Hence

$$B^n - 1 = \frac{h_n - h_o}{h_o} = \frac{h_n}{h_o} - 1$$

Consequently

$$B^n = \frac{h_n}{h_o} \text{ and } n = \frac{\log \frac{h_n}{h_o}}{\log B}$$

These simplifications yield

$$G^{(OG)} = P \frac{l}{\Delta h} \frac{\gamma}{\sigma_e} l \left( 1 - n \frac{a_1}{l} \right)$$

The weight of the lower chord is similarly obtained.

The weight of the diagonal members is constant, as in case 1. It can be easily demonstrated that the length of the diagonal members increases in the same ratio (from d, to $d_n$) as their load decreases. The same is true of the vertical members.

The great advantage of integration over summation of the series is illustrated by cases 9, 10 and 11.

For the upper-chord weight in case 9 we have the expression:

$$G^{(OG)} = Pl \frac{\gamma}{\sigma_e} \int_0^1 \frac{x}{l} \frac{dx}{h_o + \Delta h \frac{x}{l}}$$

Division of the integrand yields:

$$\frac{x}{l} \frac{dx}{h_o + \Delta h \frac{x}{l}} = \frac{1}{h_o \, l} \frac{x \, dx}{\frac{\Delta h}{h_o} \frac{x}{l} + 1} =$$

$$= \frac{dx}{\Delta h} \left[ 1 - \frac{l \, h_o}{\Delta h} \frac{1}{x + \frac{l \, h_o}{\Delta h}} \right]$$

Then

$$G^{(OG)} = P \frac{l}{\Delta h} \frac{\gamma}{\sigma_e} \left[ \int_0^l dx - \frac{l\, h_o}{\Delta h} \int_0^l \frac{dx}{x + \frac{l\, h_o}{\Delta h}} \right]$$

$$= P \frac{l}{\Delta h} \frac{\gamma}{\sigma_e} \left\{ l - \frac{l\, h_o}{\Delta h} \left[ \ln \left( l + \frac{l\, h_o}{\Delta h} \right) - \ln \frac{l\, h_o}{\Delta h} \right] \right\}$$

$$= Pl \frac{\gamma}{\sigma_e} \frac{l}{\Delta h} \left[ 1 - \frac{h_o}{\Delta h} \ln \frac{h_n}{h_o} \right]$$

The weight of the lower chord is similarly obtained.

If it were now desired to determine whether this formula with the assumption that $h_n = h_o$, i.e., with uniform depth of girder, would yield the same value as the formula in case 1, we would arrive at the indeterminate value o/o. Likewise the last formula in case 9, for only a slight difference in depth, i.e., when $\Delta h$ is small with respect to $h_n$, would yield a numerical value which could not be accurately calculated with the slide rule. This difficulty is overcome by another investigation of the integrand, which is developed in an infinite series:

$$\frac{x}{l} \quad \frac{dx}{h_o + \Delta h \frac{x}{l}} =$$

$$= \frac{x \cdot dx}{h_o\, l} \left( 1 - \frac{x}{C} + \left(\frac{x}{C}\right)^2 - \left(\frac{x}{C}\right)^3 + \left(\frac{x}{C}\right)^4 - + \ldots \right)$$

in which $C = \frac{l\, h_o}{\Delta h}$

The integration of the separate terms, in which x occurs only in the numerator, offers no further difficulty and, with slight modifications, leads to the second formula of case 9. A test shows that this formula for a uniform depth $h_n = h_o$ agrees with the formula in case 1.

It is unnecessary to carry the derivations of cases 10 and 11 further, since they follow the same course as those already given. Moreover, all the best mathematical textbooks used by engineers give detailed examples of such problems.

To enable a better understanding of cases 12 and 13, I am adding Figures 1 to 3 which refer particularly to case 13. It only remains to explain the overhanging end of the beam. (Case 4.) Figure 2 represents a special case where the force with upper-chord $O_x$ does not change its sign in the inner panel. If $O_x$ intersects the zero axis, it does so at two points, which can be calculated with the aid of $x_2$ and $x_3$. (See second column of tables. "The integration limits for $G_o$ and $G_u$".) If, however, $x_2$ and $x_3$ yield imaginary, negative or otherwise useless values, $O_x$ retains its sign between A and B. There are then only two integration limits: the lower, $x_1 = l\ (1 - \epsilon)$; the upper, $x = l$.

The lower-chord force $U_x$ can change its sign only once, namely at $x_4$. If this value is negative or greater than $l$, $U_x$ retains the sign between A and B. The integration limits are then the same as for $O_x$.

For the special case, where both the lower and the upper chord weights retain their signs in the inner panel, the whole chord weight, including the overhang is given in the column "Remarks" for case 13. The bracketed expression is plotted against $\epsilon$ in Figure 4.

The solid-line portions of the curves are strictly correct. The portions to the left of the line C-C appear quite different because the lower-chord force intersects the zero axis. The portions to the right of the line D-D are given quite a different course by the intersection of the lower-chord force.

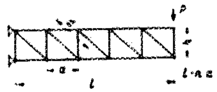
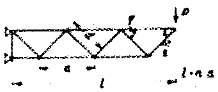
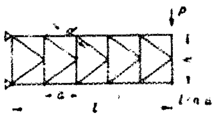
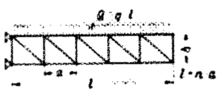
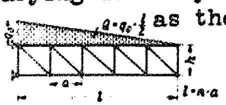
Case 12 may be considered as a special variation of case 13, in which the depth of the strut foot under A, namely a, becomes infinity ($\alpha = \infty$). In practice this means that, for h/a = 0.1 or less, the considerably more troublesome computation work of case 13 can be saved and the values of case 12 can be used instead. The bracketed chord values of case 12 are plotted in Figure 5 against $\epsilon = l_1/l$.

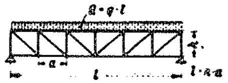
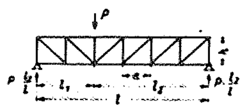
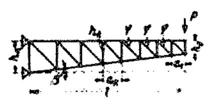
For shearing-force weights, it does not matter whether case 12 or 13 is used. The coefficients are plotted against $\epsilon = l_1/l$ in Figure 6.

Lastly the nondimensional or absolute coefficients of the strut weight $G^{(S)}$ might be plotted similarly to Figures 4 to 6. It is omitted here, because the strut is generally subjected to compressive forces and the reader, in selecting the best supporting point B, must consider the variability of $\sigma_e$, which is often hardly possible. It can only be remarked that the smallest coefficient is 2 at a strut inclination of 45°, when $\epsilon = \eta$.

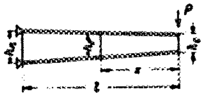
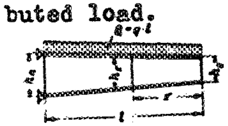
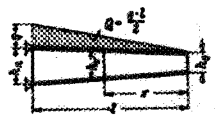
In determining the most favorable a, the fact was disregarded, that, with increasing a, the attainable stress $\sigma_e$ in the compression chord might be greatly diminished. If necessary, this point should be especially investigated.

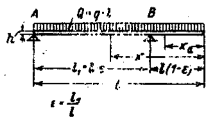
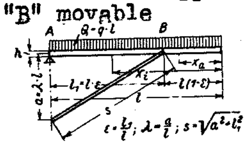
Translation by Dwight M. Miner,
National Advisory Committee
for Aeronautics.

| Case | Load case<br>Support<br>Notation | Both chords<br>Forces: Up'r chord $O_k : O_x$<br>Lower chord: $U_k : U_x$<br>Weights: $G^{(G)}$ | Diagonals<br>Forces: $D_k$<br>Weights: $G^{(D)}$ | Vert'l members<br>Forces: $V_k$<br>Weights: $G^{(V)}$ | | Whole girder<br>Weights: $G$ | Remarks |
|---|---|---|---|---|---|---|---|
| 1 | Cantilever truss — Single load<br> | $O_k = +P\cdot\frac{a}{h}\cdot(k-1)$<br>$U_k = -P\cdot\frac{a}{h}\cdot k$<br>$G^{(G)} = P\cdot l\cdot\frac{\gamma}{\sigma_e}\cdot\frac{l}{h}$ | $D_k = +P\cdot\frac{d}{h}$<br>$G^{(D)} = P\cdot l\,\frac{\gamma}{\sigma_e}\left(\frac{a}{h}+\frac{h}{a}\right)$ | $V_k = -P$<br>$G^{(V)} = P\cdot l\,\frac{\gamma}{\sigma_e}\cdot\frac{h}{a}$ | — | $G = P\cdot l\cdot\frac{\gamma}{\sigma_0}\cdot\left[\frac{1}{\eta_g}\cdot\frac{l}{h}+\frac{1}{\eta_d}\cdot\left(\frac{h}{a}+\frac{a}{h}\right)+\frac{1}{\eta_v}\cdot\frac{h}{a}\right]$<br>$\eta_g = \frac{\sigma_g}{\sigma_0}\cdot\frac{\gamma}{\gamma_g};\ \eta_d = \frac{\sigma_d}{\sigma_0}\frac{\gamma}{\gamma_d};\ \eta_v = \frac{\sigma_v}{\sigma_0}\frac{\gamma}{\gamma_v}$ | The most favorable value of "a"<br>$a = h\cdot\sqrt{1+\frac{\sigma_d}{\sigma_v}\cdot\frac{\gamma_v}{\gamma_d}}$<br>is obtained for: |
| 2 | <br>$d = \sqrt{h^2+\frac{a^2}{4}}$ | $O_k = +P\cdot\frac{a}{h}\cdot\left(k-\frac{1}{2}\right)$<br>$U_k = -P\cdot\frac{a}{h}\cdot k$<br>$G^{(G)} = P\cdot l\cdot\frac{\gamma}{\sigma_e}\cdot\frac{l}{h}$ | $D_k = \pm P\cdot\frac{d}{h}$<br>$G^{(D)} = P\cdot l\cdot\frac{\gamma}{\sigma_v}\left(\frac{a}{2h}+\frac{2h}{a}\right)$ | — | — | $G = P\cdot l\,\frac{\gamma}{\sigma_0}\cdot\left[\frac{1}{\eta_g}\cdot\frac{l}{h}+\frac{1}{\eta_d}\cdot\left(\frac{a}{2h}+\frac{2h}{a}\right)\right]$ | The most favorable "a" lies at:<br>$a = 2\cdot h$; d. h. $\varphi = 45°$ |
| 3 | K-Truss<br><br>$d = \sqrt{a^2+\frac{h^2}{4}}$ | $O_k = +P\cdot\frac{a}{h}\cdot(k-1)$<br>$U_k = -P\cdot\frac{a}{h}\cdot(k-1)$<br>$G^{(G)} = P\cdot l\cdot\frac{\gamma}{\sigma_e}\cdot\frac{l}{h}\cdot\left(1-\frac{1}{n}\right)$ | $D_k = \pm P\cdot\frac{d}{h}$<br>$G^{(D)} = P\cdot l\cdot\frac{\gamma}{\sigma_e}\cdot\left(\frac{2a}{h}+\frac{h}{2a}\right)$ | $V_k = \pm\frac{P}{2}$<br>$G^{(V)} = P\cdot l\cdot\frac{\gamma}{\sigma_e}\cdot\frac{h}{2a}$ | — | $G = P\cdot l\cdot\frac{\gamma}{\sigma_0}\cdot\left[\frac{1}{\eta_g}\cdot\frac{l}{h}\cdot\left(1-\frac{1}{n}\right)+\frac{1}{\eta_d}\cdot\left(\frac{2a}{h}+\frac{h}{2a}\right)+\frac{1}{\eta_v}\cdot\frac{h}{2a}\right]$ | The most favorable "a" lies at:<br>$a = \frac{h}{2}\cdot\sqrt{1+\frac{\sigma_d\,\gamma_v}{\sigma_v\,\gamma_d}}$ |
| 4 | Uniformly distributed load<br> | $O_k = +\frac{Q}{2}\cdot\frac{a}{h}\cdot\frac{(k-1)^2}{n}$<br>$U_k = -\frac{Q}{2}\cdot\frac{a}{h}\cdot\frac{k^2}{n}$<br>$G^{(G)} = \frac{Q\cdot l}{2}\cdot\frac{\gamma}{\sigma_e}\cdot\frac{2}{3}\cdot\frac{l}{h}\left(1+\frac{1}{2n^2}\right)$ | $D_k = \pm Q\cdot\frac{d}{h}\cdot\frac{k-\frac{1}{2}}{n}$<br>$G^{(D)} = \frac{Q\cdot l}{2}\cdot\frac{\gamma}{\sigma_e}\left(\frac{h}{a}+\frac{a}{h}\right)$ | $V_k = -Q\cdot\frac{k+\frac{1}{2}}{n}$<br>$G^{(V)} = \frac{Q\cdot l}{2}\cdot\frac{\gamma}{\sigma_e}\cdot\frac{h}{a}$ | — | $G = \frac{Q\cdot l}{2}\cdot\frac{\gamma}{\sigma_0}\left[\frac{1}{\eta_g}\cdot\frac{2}{3}\cdot\frac{l}{h}\cdot\left(1+\frac{1}{2n^2}\right)+\frac{1}{\eta_d}\cdot\left(\frac{h}{a}+\frac{a}{h}\right)+\frac{1}{\eta_v}\cdot\frac{h}{a}\right]$ | At large enough "n" the most favorable "a" is the same as in case 1. |
| 5 | Distributed load varying directly as the distance from one end<br> | $O_k = +\frac{Q}{3}\cdot\frac{a}{h}\cdot\frac{(k-1)^3}{n^2}$<br>$U_k = -\frac{Q}{3}\cdot\frac{a}{h}\cdot\frac{k^3}{n^2}$<br>$G^{(G)} = \frac{Q\cdot l}{3}\cdot\frac{\gamma}{\sigma_e}\cdot\frac{1}{2}\cdot\frac{l}{h}\cdot\left(1+\frac{1}{n^2}\right)$ | $D_k = +Q\cdot\frac{d}{h}\cdot\frac{k^2-k+\frac{1}{3}}{n^2}$<br>$G^{(D)} = \frac{Q\cdot l}{3}\cdot\frac{\gamma}{\sigma_e}\cdot\left(\frac{h}{a}+\frac{a}{h}\right)$ | $V_k = -Q\cdot\frac{k^2+k+\frac{1}{3}}{n^2}$<br>$G^{(V)} = \frac{Q\cdot l}{3}\cdot\frac{\gamma}{\sigma_e}\cdot\frac{h}{a}$ | — | $G = \frac{Q\cdot l}{3}\cdot\frac{\gamma}{\sigma_0}\cdot\left[\frac{1}{\eta_g}\cdot\frac{1}{2}\cdot\frac{l}{h}\cdot\left(1+\frac{1}{n^2}\right)+\frac{1}{\eta_d}\cdot\left(\frac{h}{a}+\frac{a}{h}\right)+\frac{1}{\eta_v}\cdot\frac{h}{a}\right]$ | At large enough "n" the most favorable "a" is the same as in case 1. |

| Case | Load case<br>Support<br>Notation | Both chords<br>Forces: Up'r chord<br>$O_k : O_x$<br>Lower chord: $U_k : U_x$<br>Weights: $G^{(G)}$ | Diagonals<br>Forces: $D_k$<br>Weights: $G^{(D)}$ | Vert'l members<br>Forces: $V_K$<br>Weights: $G^{(V)}$ | | Whole girder<br>Weights: G | Remarks |
|---|---|---|---|---|---|---|---|
| 6 | Truss on two supports. Uniformly distributed load<br><br>»n« eine gerade Zahl<br>"n" is an even number | $O_k = -\frac{Q}{2}\cdot\frac{a}{h}\cdot\left(k-\frac{k^2}{n}\right)$<br>$U_k = +\frac{Q}{2}\cdot\frac{a}{h}\cdot\left[(k-1)-\frac{(k-1)^2}{n}\right]$<br>$G^{(G)} = \frac{Q\cdot l}{4}\cdot\frac{\gamma}{\sigma_e}\cdot\frac{2}{3}\cdot\frac{l}{h}\cdot\left(1-\frac{1}{n^2}\right)$ | $D_k = +Q\,\frac{d}{h}\left(\frac{1}{2}-\frac{k-\frac{1}{2}}{n}\right)$<br>$G^{(D)} = \frac{Q\cdot l}{4}\cdot\frac{\gamma}{\sigma_e}\cdot\left(\frac{h}{a}+\frac{a}{h}\right)$ | $V_k = -Q\cdot\left(\frac{1}{2}-\frac{k+\frac{1}{2}}{n}\right)$<br>$G^{(V)} = \frac{Q\cdot l}{4}\cdot\frac{\gamma}{\sigma_e}\cdot\frac{h}{a}$ | — | $G = \frac{Q\cdot l}{4}\cdot\frac{\gamma}{\sigma_0}\cdot\left[\frac{1}{\eta_g}\cdot\frac{2}{3}\cdot\frac{l}{h}\cdot\left(1-\frac{1}{n^2}\right)+\right.$<br>$\left.+\frac{1}{\eta_d}\cdot\left(\frac{h}{a}+\frac{a}{h}\right)+\frac{1}{\eta_v}\,\frac{h}{a}\right]$ | At large enough "n" the most favorable "a" is the same as in case 1. |
| 7 | Truss on two supports. Single load.<br><br>$l_1 = m\cdot a \quad l_2 = n\cdot a$ | left<br>links: $O_j = -P\cdot\frac{l_2}{l}\cdot\frac{a}{h}\cdot j$<br>$U_j = +P\cdot\frac{l_2}{l}\cdot\frac{a}{h}\,(j-1)$<br>right<br>rechts: $O_k = -P\cdot\frac{l_1}{l}\cdot\frac{a}{h}\cdot k$<br>$U_k = +P\cdot\frac{l_1}{l}\cdot\frac{a}{h}\,(k-1)$<br>$G^{(G)} = P\,\frac{l_1\cdot l_2}{l}\cdot\frac{\gamma}{\sigma_e}\cdot\frac{l}{h}$ | left<br>links: $D_j = +P\cdot\frac{l_2}{l}\cdot\frac{d}{h}$<br>right<br>rechts: $D_k = +P\,\frac{l_1}{l}\cdot\frac{d}{h}$<br>$G^{(D)} = P\cdot\frac{l_1\cdot l_2}{l}\cdot\frac{\gamma}{\sigma_e}\,2\left(\frac{a}{h}+\frac{h}{a}\right)$ | left<br>links: $V_j = -P\cdot\frac{l_2}{l}$<br>right<br>rechts: $V_k = -P\cdot\frac{l_1}{l}$<br>$G^{(V)} = P\,\frac{l_1\cdot l_2}{l}\cdot\frac{\gamma}{\sigma}\cdot 2\,\frac{h}{a}$ | — | $G = P\cdot\frac{l_1\cdot l_2}{l}\,\frac{\gamma}{\sigma_0}\cdot\left[\frac{1}{\eta_g}\cdot\frac{l}{h}+\frac{2}{\eta_d}\left(\frac{h}{a}+\right.\right.$<br>$\left.\left.+\frac{a}{h}\right)+\frac{2}{\eta_v}\cdot\frac{h}{a}\right]$ | The most favorable "a" is the same as in case 1. |
| 8 | Inclined chords. Single load.<br><br>$\varphi = $ konst. $\Delta h = h_n - h_0$;<br>$\operatorname{tg}\beta = \frac{\Delta h}{l}$; $h_k = h_0\cdot B^k$;<br>$a_k = a_1\cdot B^{k-1}$; $B = 1+\frac{\operatorname{tg}\beta}{\operatorname{tg}\varphi}$<br>$= 1+\frac{a_1}{l}\cdot\frac{\Delta h}{h_0}$; $l' = \frac{l}{\operatorname{tg}\beta}$;<br>$n = \frac{\log\frac{h_n}{h_0}}{\log B}$ | $O_k = +P\cdot\frac{l}{\Delta h}\cdot\left(1-\frac{1}{B^{k-1}}\right)$<br>$O_k = +P\cdot\frac{l}{h_0}\cdot\frac{k-1}{\frac{l}{a_1}+k\,\frac{\Delta h}{h_0}}$<br>$U_k = -P\,\frac{l'}{\Delta h}\cdot\left(1-\frac{1}{B^k}\right)$<br>$U_k = -P\,\frac{l'}{h_0}\cdot\frac{k}{\frac{l}{a_1}+(k+1)\,\frac{\Delta h}{h_0}}$<br>$G^{(G)} = P\cdot l\,\frac{\gamma}{\sigma_e}\left[\frac{l}{\Delta h}\cdot\left(1-\frac{n\cdot a_1}{l}\right)+\frac{l'}{l}\cdot\frac{l'}{\Delta h}\left(1-\frac{n\cdot a_1}{l\cdot B}\right)\right]$ | $D_k = +\frac{P}{\sin\varphi}\cdot\frac{1}{B^k}$<br>$G^{(D)} = P\cdot l\,\frac{\gamma}{\sigma_e}\cdot\frac{2}{\sin(2\varphi)}\cdot\frac{n\,h_0}{l\cdot B}$ | $V_k = -P\cdot\frac{1}{B^k}$<br>$G^{(V)} = P\cdot l\,\frac{\gamma}{\sigma_e}\cdot\frac{n\cdot h_0}{l}$ | — | $G = P\cdot l\cdot\frac{\gamma}{\sigma_0}\cdot\left\{\frac{1}{\eta_g}\cdot\left[\frac{l}{\Delta h}\left(1-\frac{n\cdot a_1}{l}\right)+\right.\right.$<br>$\left.+\frac{l'}{l}\cdot\frac{l'}{\Delta h}\left(1-\frac{n\cdot a_1}{l\cdot B}\right)\right]+\frac{1}{\eta_d}\cdot$<br>$\left.\cdot\frac{2}{\sin(2\varphi)}\cdot\frac{n\cdot h_0}{l\cdot B}+\frac{1}{\eta_v}\cdot\frac{n\cdot h_0}{l}\right\}$ | |

3

| Case | Load case<br>Support<br>Notation | Both chords<br>Forces: Up'r chord $O_k : O_x$<br>Lower chord: $U_k : U_x$<br>Weights: $G^{(G)}$ | Diagonals<br>Forces: $D_k$<br>Weights: $G^{(D)}$ | Vert'l members<br>Forces: $V_k$<br>Weights: $G^{(V)}$ | Other shear bracing<br>Forces: $S_x$<br>Weights: $G^{(S)}$ | Whole girder<br>Weights: G | Remarks |
|---|---|---|---|---|---|---|---|
| 9 | Plate girder. Inclined chords. Single load.<br><br>$h_x = h_0 + \Delta h \frac{x}{l};\ \Delta h = h_n - h_0$ | $\begin{matrix} O_x \\ U_x \end{matrix} = \pm P \cdot l \frac{x}{l} \frac{1}{h_x}$<br>$G^{(G)} = P \cdot l \cdot \frac{\gamma}{\sigma_e} 2 \frac{l}{\Delta h} \cdot \left(1 - \frac{h_0}{\Delta h} \cdot \ln \frac{h_n}{h_0}\right)$<br>or<br>oder:<br>$G^{(G)} = P \cdot l \cdot \frac{\gamma}{\sigma_0} \cdot 2 \frac{l}{h_0} \left[\frac{1}{2} - \frac{1}{3} \frac{\Delta h}{h_0} + \frac{1}{4}\left(\frac{\Delta h}{h_0}\right)^2 - \frac{1}{5}\left(\frac{\Delta h}{h_0}\right)^3 + \frac{1}{6}\left(\frac{\Delta h}{h_0}\right)^4 - + \ldots\right]$ | — | — | $S_x = P - P \cdot x \cdot \frac{1}{x + l \frac{h_0}{\Delta h}}$<br>$G^{(S)} = P \cdot l \frac{\gamma}{\tau_e} \frac{h_0}{\Delta h} \ln \frac{h_n}{h_0}$<br>or<br>oder:<br>$G^{(S)} = P \cdot l \cdot \frac{\gamma}{\tau_e} \left[1 - \frac{1}{2} \frac{\Delta h}{h_0} + \frac{1}{3}\left(\frac{\Delta h}{h_0}\right)^2 - \frac{1}{4}\left(\frac{\Delta h}{h_0}\right)^3 + \frac{1}{5}\left(\frac{\Delta h}{h_0}\right)^4 - + \ldots\right]$ | — | The first $G^{(G)}$ holds good for large $\Delta h$ as compared with $h_n$.<br>The second $G^{(G)}$ for smaller $\Delta h$ |
| 10 | Plate girder. Inclined chords. Uniformly distributed load.<br><br>$h_x = h_0 + \Delta h \frac{x}{l};\ \Delta h = h_n - h_0$ | $\begin{matrix} O_x \\ U_x \end{matrix} = + \frac{Q \cdot l}{2} \cdot \left(\frac{x}{l}\right)^2 \cdot \frac{1}{h_x}$<br>$G^{(G)} = \frac{Q \cdot l}{2} \frac{\gamma}{\sigma_e} \cdot 2 \frac{l}{\Delta h} \cdot \left[\frac{1}{2} - \frac{h_0}{\Delta h} + \left(\frac{h_0}{\Delta h}\right)^2 \cdot \ln\left(\frac{h_n}{h_0}\right)\right]$<br>or<br>oder:<br>$G^{(G)} = \frac{Q \cdot l}{2} \cdot \frac{\gamma}{\sigma_e} \cdot 2 \frac{l}{h_0} \cdot \left[\frac{1}{3} - \frac{1}{4} \frac{\Delta h}{h_0} + \frac{1}{5}\left(\frac{\Delta h}{h_0}\right)^2 - \frac{1}{6}\left(\frac{\Delta h}{h_0}\right)^3 + - \ldots\right]$ | — | — | $S_x = Q \frac{x}{l} - \frac{Q \cdot l}{2}\left(\frac{x}{l}\right)^2 \cdot \frac{1}{x + l \frac{h_0}{\Delta h}}$<br>$G^{(S)} = \frac{Q \cdot l}{2} \frac{\gamma}{\tau_e} \cdot \left[\frac{1}{2} + \frac{h_0}{\Delta h} - \left(\frac{h_0}{\Delta h}\right)^2 \ln \frac{h_n}{h_0}\right]$<br>or<br>oder:<br>$G^{(S)} = \frac{Q \cdot l}{2} \cdot \frac{\gamma}{\tau_e} \left[1 - \frac{1}{3} \frac{\Delta h}{h_0} + \frac{1}{4}\left(\frac{\Delta h}{h_0}\right)^2 - \frac{1}{5}\left(\frac{\Delta h}{h_0}\right)^3 + \frac{1}{6}\left(\frac{\Delta h}{h_0}\right)^4 - + \ldots\right]$ | — | The first $G^{(G)}$ holds good for large $\Delta h$ as compared with $h_n$.<br>The second $G^{(G)}$ for smaller $\Delta h$ |
| 11 | Plate girder. Inclined chords. Distributed load. varying directly as the distance from one end<br><br>$h_x = h_0 + \Delta h \cdot \frac{x}{l};\ \Delta h = h_n - h_0$ | $\begin{matrix} O_x \\ U_x \end{matrix} = \pm \frac{Q \cdot l}{3} \cdot \left(\frac{x}{l}\right)^3 \cdot \frac{1}{h_x}$<br>$G^{(G)} = \frac{Q \cdot l}{3} \cdot \frac{\gamma}{\sigma_e} 2 \cdot \frac{l}{\Delta h}\left[\frac{1}{3} - \frac{1}{2} \frac{h_0}{\Delta h} + \left(\frac{h_0}{\Delta h}\right)^2 - \left(\frac{h_0}{\Delta h}\right)^3 \ln\left(\frac{h_n}{h_0}\right)\right]$<br>or<br>oder:<br>$G^{(G)} = \frac{Q \cdot l}{3} \cdot \frac{\gamma}{\sigma_e} \cdot 2 \cdot \frac{l}{h_0} \cdot \left[\frac{1}{4} - \frac{1}{5} \cdot \frac{\Delta h}{h_0} + \frac{1}{6}\left(\frac{\Delta h}{h_0}\right)^2 - \frac{1}{7}\left(\frac{\Delta h}{h_0}\right)^3 + - \ldots\right]$ | | | $S_x = Q\left(\frac{x}{l}\right)^2 - \frac{Ql}{3}\left(\frac{x}{l}\right)^3 \cdot \frac{1}{x + l \frac{h_0}{\Delta h}}$<br>$G^{(S)} = \frac{Ql}{3} \frac{\gamma}{\tau_e} \cdot \left[\frac{2}{3} + \frac{1}{2} \frac{h_0}{\Delta h} - \left(\frac{h_0}{\Delta h}\right)^2 + \left(\frac{h_0}{\Delta h}\right)^3 \ln \frac{h_n}{h_0}\right]$<br>or<br>oder:<br>$G^{(S)} = \frac{Ql}{3} \frac{\gamma}{\tau_e} \left[1 - \frac{1}{4} \frac{\Delta h}{h_0} + \frac{1}{5}\left(\frac{\Delta h}{h_0}\right)^2 - \frac{1}{6}\left(\frac{\Delta h}{h_0}\right)^3 + \frac{1}{7}\left(\frac{\Delta h}{h_0}\right)^4 - + \ldots\right]$ | | The first $G^{(G)}$ holds good for large $\Delta h$ as compared with $h_n$.<br>The second $G^{(G)}$ for smaller $\Delta h$ |

| Case | Load case<br>Support<br>Notation | Both chords<br>Forces: Up'r ch'd $O_k$: $O_x$<br>Lower chord: $U_k$: $U_x$<br>Weights: $G^{(G)}$ | Diagonals<br>Forces: $D_k$<br>Weights: $G^{(D)}$ | Vert'l members<br>Forces: $V_k$<br>Weights: $G^{(V)}$ | Other shear bracing<br>Forces: $S_x$<br>Weights: $G^{(S)}$ | Whole girder<br>Weights: G | Remarks |
|---|---|---|---|---|---|---|---|
| 12 | Cantilever on two supports. Uniformly distributed load.<br><br>Support "B" movable | Overhang<br>$\frac{O_x}{U_x} = \pm \frac{Q \cdot l}{2} \left(\frac{x}{l}\right)^2 \frac{1}{h}$<br>Inner panel<br>$\frac{O_x}{U_x} = \pm \frac{Q \cdot l}{2} \cdot \left[\left(\frac{x}{l}\right)^2 - \frac{1}{\varepsilon}\frac{x}{l} + \frac{1-\varepsilon}{\varepsilon}\right] \cdot \frac{1}{h}$<br>For $\varepsilon > 0{,}5$:<br>$G^{(G)} = \frac{Q \cdot l}{2} \cdot \frac{\gamma}{\sigma_e} \cdot 2 \frac{l}{h} \left(3 + \frac{2}{\varepsilon^2} - \frac{\varepsilon}{2} - \frac{4}{\varepsilon} - \frac{1}{3\varepsilon^3}\right)$<br>$f'(\varepsilon) = 2\left(3 + \frac{2}{\varepsilon^2} - \frac{\varepsilon}{2} - \frac{4}{\varepsilon} - \frac{1}{3\varepsilon^3}\right)$<br>For $\varepsilon < 0{,}5$:<br>$G^{(G)} = \frac{Q \cdot l}{2} \cdot \frac{\gamma}{\sigma_e} \cdot 2 \cdot \frac{l}{h} \left(\frac{1}{3} - \frac{\varepsilon}{2}\right)$<br>$f''(\varepsilon) = 2\left(\frac{1}{3} - \frac{\varepsilon}{2}\right)$ | For a trussed girder as in case 1. Over hang<br>$D_k = \pm \frac{Q}{n}\frac{d}{h}\left(k - \frac{1}{2}\right)$<br>Inner panel<br>$D_k = \pm \frac{Q}{n} \cdot \frac{d}{h} \cdot \left[\frac{n}{\varepsilon}\frac{1}{2} - \left(k - \frac{1}{2}\right)\right]$<br>For $\varepsilon > 0{,}5$:<br>$G^{(D)} = \frac{Q \cdot l}{2} \cdot \frac{\gamma}{\sigma_e} \left(\frac{a}{h} + \frac{h}{a}\right) F'(\varepsilon)$<br>$F'(\varepsilon) = 2\left[(1-\varepsilon)^2 + \left(1 - \frac{0{,}5}{\varepsilon}\right)^2\right]$<br>For $\varepsilon < 0{,}5$:<br>$G^{(D)} = \frac{Q \cdot l}{2} \frac{\gamma}{\sigma_e} \cdot \left(\frac{a}{h} + \frac{h}{a}\right) F''(\varepsilon)$<br>$F''(\varepsilon) = 2 \cdot (1-\varepsilon)^2$ | $V_k = \pm \frac{Q}{n}\left(k + \frac{1}{2}\right)$<br>Inner panel<br>$V_k = \pm \frac{Q}{n} \cdot \left[\frac{n}{\varepsilon}\frac{1}{2} - \left(k - \frac{1}{2}\right)\right]$<br>For $\varepsilon > 0{,}5$:<br>$G^{(V)} = \frac{Q \cdot l}{2} \cdot \frac{\gamma}{\sigma_e} \frac{h}{a} \cdot F'(\varepsilon)$<br>For $\varepsilon < 0{,}5$:<br>$G^{(V)} = \frac{Q \cdot l}{2} \cdot \frac{\gamma}{\sigma_e} \frac{h}{a} \cdot F''(\varepsilon)$ | Overhang<br>$S_x = Q \frac{x}{l}$<br>Inner panel<br>$S_x = Q\left(\frac{0{,}5}{\varepsilon} - \frac{x}{l}\right)$<br>For $\varepsilon > 0{,}5$:<br>$G^{(S)} = \frac{Q \cdot l}{2} \frac{\gamma}{\sigma_e} \cdot F'(\varepsilon)$<br>$F'(\varepsilon) = 2\left[(1-\varepsilon)^2 + \left(1 - \frac{0{,}5}{\varepsilon}\right)^2\right]$<br>For $\varepsilon < 0{,}5$:<br>$G^{(S)} = \frac{Q \cdot l}{2} \cdot \frac{\gamma}{\sigma_e} \cdot F''(\varepsilon)$<br>$F''(\varepsilon) = 2 \cdot (1-\varepsilon)^2$ | For $\varepsilon > 0{,}5$:<br>$G = \frac{Ql}{2} \cdot \frac{\gamma}{\sigma_e} \cdot \left[\frac{1}{\eta_g}\frac{l}{h} \cdot f'(\varepsilon) + \frac{1}{\eta_d}\left(\frac{a}{h} + \frac{h}{a}\right) \cdot F'(\varepsilon) + \frac{1}{\eta_v}\frac{h}{a} \cdot F'(\varepsilon)\right]$<br>For $\varepsilon < 0{,}5$:<br>$G = \frac{Q \cdot l}{2} \cdot \frac{\gamma}{\sigma_e} \cdot \left[\frac{1}{\eta_g}\frac{l}{h} \cdot f''(\varepsilon) + \frac{1}{\eta_d}\left(\frac{a}{h} + \frac{h}{a}\right) \cdot F''(\varepsilon) + \frac{1}{\eta_v}\frac{h}{a} F''(\varepsilon)\right]$ | $O_x, U_x$ become $= 0$ for<br>$x = l\left(\frac{l}{\varepsilon} - l\right)$<br>$S_x = 0$ for<br>$x = \frac{l}{2\varepsilon}$ and an optimum for $f'(\varepsilon)$ lies at<br>$\varepsilon = \frac{2}{3}$<br>An optimum for $F'(\varepsilon)$ lies at<br>$\varepsilon = 0.7$ |
| 13 | Braced cantilever. Uniformly distributed load. Support "B" movable<br><br>Integration limits for $G_o$ and $G_u$<br>$x_1 = l(1-\varepsilon)$;<br>$x_2 = l\left[\frac{0{,}5}{\varepsilon} - \sqrt{\left(1 - \frac{0{,}5}{\varepsilon}\right)^2 - \frac{h}{2a}}\right]$<br>$x_3 = l\left[\frac{0{,}5}{\varepsilon} + \sqrt{\left(1 - \frac{0{,}5}{\varepsilon}\right)^2 - \frac{h}{2a}}\right]$<br>$x_4 = l\left[\frac{0{,}5}{\varepsilon} \pm \sqrt{\left(1 - \frac{0{,}5}{\varepsilon}\right)^2 + \frac{h}{2a}}\right]$ | Over hang<br>$\frac{O_x}{U_x} = \pm \frac{Q \cdot l}{2} \cdot \left(\frac{x}{l}\right)^2 \cdot \frac{1}{h}$<br>Inner panel<br>$Q_x = + \frac{Q \cdot l}{2}\left[\frac{h}{2a} + C\right]\frac{1}{h}$<br>$U_x = + \frac{Q \cdot l}{2}\left[\frac{h}{2a} - C\right]\frac{1}{h}$<br>$C = \left[\left(\frac{x}{l}\right)^2 - \frac{1}{\varepsilon}\frac{x}{l} + \frac{1-\varepsilon}{\varepsilon}\right]$<br>Weight of overhang<br>$G^{(G)}_{(kr)} = \frac{Q \cdot l}{2} \cdot \frac{\gamma}{\sigma_e} \cdot \frac{2}{3} \cdot \frac{l}{h}(1-\varepsilon)^3$<br>Weight of upper chord<br>$G^{(G)}_O = \frac{Q\,l}{2}\frac{\gamma}{\sigma_e}\frac{l}{h}\left\{M_{x_1}^{x_2} + M_{x_3}^{x_2} + M_{x_3}^{x=l}\right\}$<br>$M = \left[\frac{h}{2a} \cdot \frac{x}{l} + \frac{1}{3}\left(\frac{x}{l}\right)^3 - \frac{1}{2\varepsilon}\left(\frac{x}{l}\right)^2 + \frac{1-\varepsilon}{\varepsilon}\frac{x}{l}\right]$<br>Weight of lower chord<br>$G^{(G)}_U = \frac{Q \cdot l}{2} \cdot \frac{\gamma}{\sigma} \cdot \frac{l}{h}\left\{N_{x_1}^{x_4} + N_{x=l}^{x_4}\right\}$<br>$N = \left[\frac{h}{2a} \cdot \frac{x}{l} - \frac{1}{3}\left(\frac{x}{l}\right)^3 + \frac{1}{2\varepsilon}\left(\frac{x}{l}\right)^2 - \frac{1-\varepsilon}{\varepsilon}\frac{x}{l}\right]$ | For a trussed girder the figures under 12 apply.<br>Force on strut<br>$T = \frac{Q \cdot l}{2}\sqrt{1 + \left(\frac{\lambda}{\varepsilon}\right)^2 \cdot \frac{1}{a}}$<br>Strut weight<br>$G^{(T)} = \frac{Q \cdot l}{2}\frac{\gamma}{\sigma_e}\left(\frac{\varepsilon}{\lambda} + \frac{\lambda}{\varepsilon}\right)$ | Inner panel | Same as in case 12. | | If h/a=0.1 or less, the chord weight differs but slightly from that given under case 12. If $x_2, x_3$ and $x_4$ yield only useless values, the weight of both chords is,<br>$G^{(G)} = \frac{Ql}{2} \cdot \frac{\gamma}{\sigma_e}\frac{l}{h} \cdot \left[\varepsilon \cdot \frac{h}{a} + \frac{2}{3}(1-\varepsilon)^3\right]$ |

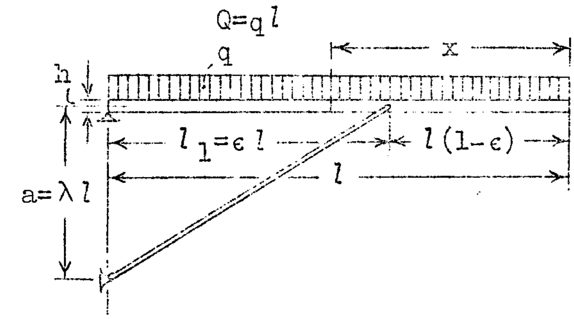



Fig. 1  System and load diagram

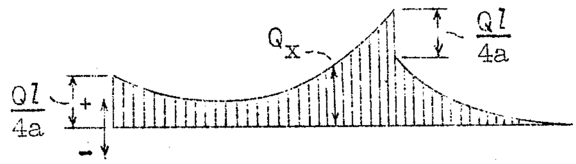



Fig. 2  Force distribution in upper flange

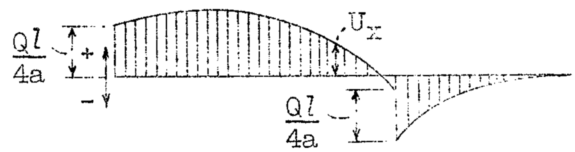
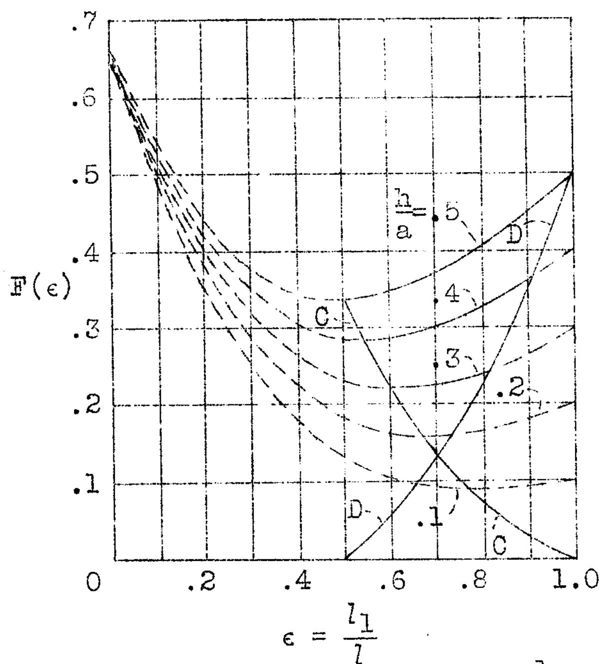



Fig. 3  Force distribution in lower flange




$$F(\epsilon) = \left[\epsilon \frac{h}{a} + \frac{2}{3}(1-\epsilon)^3\right]$$

Fig. 4  To be noted under Case 13

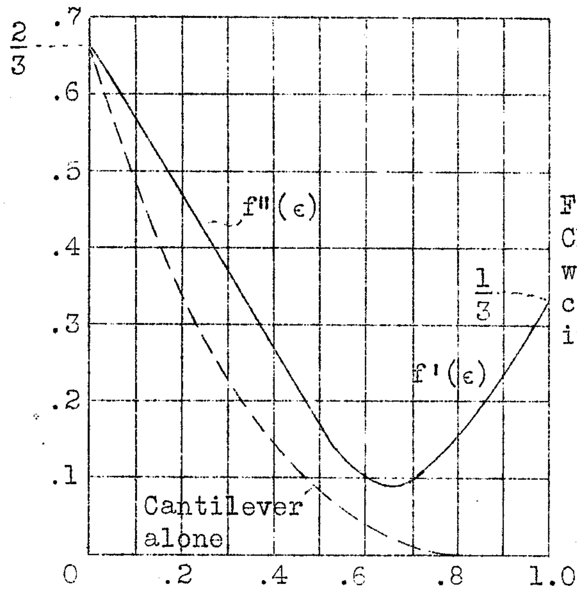



Fig. 5
Chord-weight coefficient in Case 12

$$\epsilon = \frac{l_1}{l}$$

$$f'' \ (\epsilon) = 2 \left(\frac{1}{3} - \frac{\epsilon}{2}\right)$$

$$f' \ (\epsilon) = 2 \left(3 + \frac{2}{\epsilon^2} - \frac{\epsilon}{2} - \frac{4}{\epsilon} - \frac{1}{3\epsilon^3}\right)$$

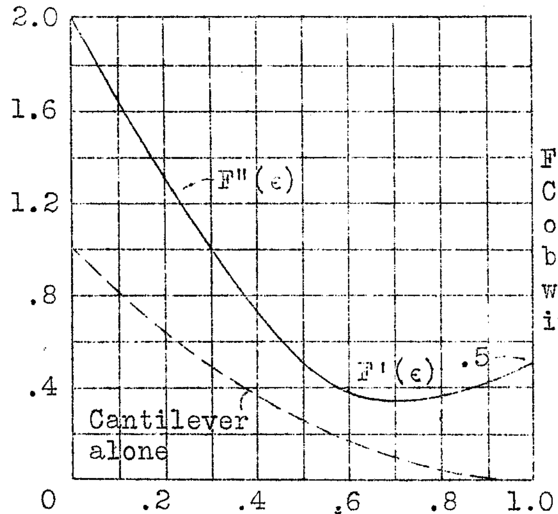



Fig. 6
Coefficients of shear bracing weights in Case 12

$$\epsilon = \frac{l_1}{l}$$

$$F'' \ (\epsilon) = 2 \ (1-\epsilon)^2$$

$$F' \ (\epsilon) = 2 \left[(1-\epsilon)^2 + \left(1 - \frac{0.5}{\epsilon}\right)\right]$$