A Semiclassical Derivation
of the QCD Coupling

David Batchelor[*]

Radiation Effects & Analysis Group

NASA's Goddard Space Flight Center

Mail Code 561.4

Greenbelt, MD 20771

(Dated: March 17, 2009)

Abstract

The measured value of the QCD coupling α_s at the energy M_{Z^0}, the variation of α_s as a function of energy in QCD, and classical relativistic dynamics are used to investigate virtual pairs of quarks and antiquarks in vacuum fluctuations. For virtual pairs of bottom quarks and antiquarks, the pair lifetime in the classical model agrees with the lifetime from quantum mechanics to good approximation, and the action integral in the classical model agrees as well with the action that follows from the Uncertainty Principle. This suggests that the particles might have small de Broglie wavelengths and behave with well-localized pointlike dynamics. It also permits α_s at the mass energy twice the bottom quark mass to be expressed as a simple fraction: 3/16. This is accurate to approximately 10%. The model in this paper predicts the measured value of $\alpha_s(M_{Z^0})$ to be 0.121, which is in agreement with recent measurements within statistical uncertainties.

PACS numbers: PACS: 03.65.-w, 03.65.Sq, 12.38.Aw, 12.39.-x, 12.39.Pn, 14.65.Dw, 14.65.Fy

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[*]David.A.Batchelor@nasa.gov

I. INTRODUCTION

Recently 't Hooft and others have investigated the possibility that physics at small scales might be governed by "basic dynamical Laws" of a deterministic underlying theory [1]. These laws would not invalidate quantum mechanics, but would yield quantum mechanics when subjected to statistical analysis. In view of that possibility, this paper considers a classical dynamical model that might underlie quantum chromodynamics (QCD [2], [3], [4]) at small scales. The model becomes semiclassical in the usual sense in a natural way.

If a quark and its antiquark are positioned at rest in their center of mass reference frame and are released, then in general their mutual attraction will draw them to a collision at the origin where they will annihilate. Photons or other particles would result, given sufficient energy. However, the quark and antiquark experience a potential energy $U(R)$ [5] as a function of their separation distance that could reduce the mass-energy of the system, if the separation R between particles is small enough. The energy conservation relation

$$\varepsilon = 2\gamma\, m_q c^2 + U(R) = 0 \tag{1}$$

is possible to satisfy, where the first term is the total relativistic energy of the particles, kinetic plus rest mass. ($R = 2r$ with r the radius of either particle from the center of mass.) In classical physics, the collision would not yield any energy and so no photons or particles could be emitted.

If the time-reversed trajectory occurred, with the vacuum spontaneously creating a quark-antiquark pair obeying Eq. (1), then the particles only could move apart in one-dimensional motion to reach turning points separated by R_{max}. Continuing this trajectory so that the particles fall from the turning points back to the origin, they would disappear back into the vacuum, like a virtual quark-antiquark pair (VQAP; see Fig. 1 for the Feynman diagram). This is similar to virtual electron-positron pairs discussed by Greiner (p. 3 of ref. [6]) and Sakurai (p. 139 of ref. [7]). Such virtual pairs of antiparticles are believed to appear and disappear spontaneously in vacuum fluctuations. Considering 't Hooft's exploration of deterministic physics that might underlie quantum mechanics, it would make sense to compare the classical treatment of this problem with the quantum-mechanical VQAP.

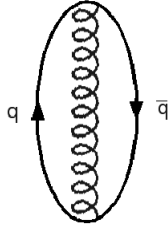Quantum theory implies that this two-particle system of quarks would obey the time-

FIG. 1: Feynman diagram of a virtual quark-antiquark pair (VQAP). Time increases from bottom to top of figure. The curling segment represents gluon interaction.

energy uncertainty relationship for the energy fluctuation $\Delta\varepsilon$ (p. 139 of ref. [7])

$$\Delta\varepsilon\Delta t = \frac{1}{2}\hbar = 5.273 \times 10^{-28}\text{erg s.} \qquad (2)$$

The energy fluctuation is $\Delta\varepsilon = 2m_q c^2$, since the mass-energy of each quark contributes $m_q c^2$. The quantum-mechanical lifetime of the fluctuation is Δt. Since $\hbar$ is the quantum of action, Eq. (2) establishes an action integral that characterizes a VQAP that has $\Delta\varepsilon = 2m_q c^2$.

The first purpose of this paper is to present classical computations of Δt and the action integral for the trajectory described above, which turn out to give results that satisfy Eq. (2) remarkably well, provided that the QCD interaction between the particles is well-described by the potential energy function.

The second purpose of the paper follows from the fact that QCD cannot specify the value of α_s at arbitrary energy or 4-momentum scale Q without an established measurement of α_s at some particular energy μ [8]; but once the renormalized coupling $\alpha_s(\mu^2)$ is measured, then QCD precisely gives the variation of α_s as a function of energy (the "running" coupling). The present paper offers a theory based on the action integral that establishes the value of α_s at the energy scale twice the bottom quark mass-energy to good approximation. This enables one to use the QCD running coupling $\alpha_s(Q^2)$ to determine the coupling strength in general in the usual way to good approximation, and suggests that the magnitude of α_s is fundamentally established by this underlying dynamical law.

II. QCD POTENTIAL ENERGY FUNCTION

As discussed in detail by Lucha *et al.* (especially pp. 161-162 of ref. [5]), a potential energy function serves to describe the bound states of heavy quarks (charm, bottom, and top). For light quarks the QCD interaction is not satisfactorily described by a potential energy function and will not be attempted here. Here we apply the standard potential energy treatment to the intermediate-mass charm and bottom quarks. More elaborate treatment of the most massive top quark case is necessary, due to effects of spin-spin interactions between the top quark and antiquark [9].

We use the standard Cornell potential [10], [11]

$$V(R) = -\frac{4}{3}\frac{\alpha_s \hbar c}{R} + aR \ , \tag{3}$$

where α_s is the dimensionless QCD strong coupling strength and $a \approx 0.25$ GeV^2. The second term, aR, is only significant for $R > 10^{-13}$ cm. We will not need to consider the aR term, since the first term with the Coulomb-like dependence turns out to strongly dominate the potential because $R_{max} \ll 10^{-13}$ cm for VQAPs.

The VQAP is a form of bound state. The standard way to account for the variation of $\alpha_s(Q^2)$ in a quark bound state is to let α_s depend on the quark masses and use $Q^2 = (m_1 + m_2)^2$, with the m_i the quark masses (see p. 129 of ref. [12]). To model a VQAP we may then compute α_s to leading order (Eq. (6) of Ref. [8]). So the QCD coupling is then given by

$$\alpha_s(Q^2) = \frac{1}{\beta_0 \ln(Q^2/\Lambda^2)} \ , \tag{4}$$

where β_0 is defined by

$$\beta_0 = \frac{33 - 2n_f}{12\pi} \ , \tag{5}$$

n_f is the number of quark flavors with masses much less than $m_1 + m_2$, and Λ is the QCD scale energy,

$$\Lambda^2 = \frac{\mu^2}{e^{1/(\beta_0 \alpha_s(\mu^2))}} \ . \tag{6}$$

Λ is approximately 0.093 GeV, assuming that $\mu \equiv M_{Z^0} = 91.2$ GeV (the mass-energy of the Z_0 particle), $\alpha_s(M_{Z^0}) = 0.119 \pm 0.002$ (in units defining $c \equiv 1$). This is a typical value of Λ for a one-loop approximation ([8] p. R31).

For the running coupling near the charm quark mass $n_f = 4$, and near the bottom quark mass $n_f = 5$. We now can use $m_1 = m_2 = m_q$ in the expression for Q^2 and compute α_s

from Eq. (4) for a running coupling. Then with a charm quark current mass $m_q = m_c$ of 1.27 GeV/c^2 we have $\alpha_s(2m_c c^2) = 0.228$; with a bottom quark current mass $m_q = m_b$ of 4.2 GeV/c^2 we have $\alpha_s(2m_b c^2) = 0.167$ [13].

III. CLASSICAL TRAJECTORY AND LIFETIME OF VQAP

Let us calculate the trajectory lifetime t_{vq} for a VQAP. If one solves the energy equation (1) for the dynamics using $V(R)$, the nonrelativistic potential energy, then the particle velocities nevertheless exhibit relativistic motion, approaching c asymptotically at the origin $r = 0$. Thus it is necessary to correct the potential energy for relativistic effects. Jackson ([14], p. 185) demonstrates how this is done by using the relativity factor γ defined in Eq. (10) below. For this trajectory of linear motion, the transformation $R \rightarrow \gamma R$ in the expression for $V(R)$ performs the appropriate modification of the potential energy function (since we are considering the center-of-mass reference frame). The potential energy function in Eq. (3) becomes

$$U(R) = -\frac{4}{3}\frac{\alpha_s}{\gamma R}\,\hbar\,c. \tag{7}$$

With this $U(R)$ we can solve Eq. (1) for R_{max} at the turning point (where $\gamma = 1$):

$$2\,m_q\,c^2 \equiv -U(R_{max}) = \frac{4}{3}\frac{\alpha_s}{R_{max}}\hbar\,c \tag{8}$$

$$R_{max} = \frac{2}{3}\frac{\alpha_s \hbar}{m_q\,c}\ . \tag{9}$$

For the charm quark current mass m_c of 1.27 GeV/$c^2 \equiv 2.26 \times 10^{-24}$ g, we find the charm VQAP has an $R_{max} = 2.35 \times 10^{-15}$ cm.

Checking the terms in Eq. (3) for $V(R_{max})$ shows that the first term is about -100 times the second term. This confirms that the quarks are so deep in the potential well that the aR term of the Cornell potential can be neglected in solving the problem.

Let us define the time from appearance of a quark at the origin $r = 0$ to the time that the quark stops at the turning point $r = \frac{1}{2}R_{max}$ as $\frac{1}{2}t_{vq}$. The t_{vq} is the classical equivalent of the quantum-mechanical Δt that we seek. We note that

$$\gamma^2 \equiv \frac{1}{1 - \beta^2} \qquad \beta = \frac{1}{c}\frac{dr}{dt} \tag{10}$$

and solve for dt, which we shall integrate. We rewrite the energy equation (1) with $\zeta \equiv R/R_{max}$ as

$$\gamma^2 = \zeta^{-1} \Rightarrow dt = \frac{dr}{c\sqrt{1-\zeta}} .$$ (11)

The time for the particle to fall from $r = R_{max}/2$ back to $r = 0$ is also $\frac{1}{2}t_{vq}$, so we have

$$t_{vq} = 2\int_0^{R_{max}/2} \frac{dr}{c\sqrt{1-\zeta}} = \frac{R_{max}}{c}\int_0^1 \frac{d\zeta}{\sqrt{1-\zeta}} = \frac{R_{max}}{c}\frac{\sqrt{\pi}\,\Gamma(1)}{\Gamma(\frac{3}{2})} = \frac{2R_{max}}{c} = \frac{4}{3}\frac{\alpha_s\hbar}{m_q c^2}$$ (12)

The integral is given in ref. ([15], p. 974). The value of t_{vq} is the total time for either quark to travel from $r = 0$ to its turning point and back to $r = 0$.

For the charm quark, with $m_q = m_c \approx 2.26 \times 10^{-24}$ g, we find the trajectory lifetime of the VQAP to be $t_{vq} \approx 1.57 \times 10^{-25}$ s. In comparison, the standard lifetime of the charm VQAP, given by the Uncertainty Principle expressed in Eq. (2), is $\Delta t = \hbar/(4m_c c^2) \approx 1.29 \times 10^{-25}$ s. So t_{vq} from the classical computation is approximately 22% larger than Δt. This is remarkably close agreement of the classical lifetime with the quantum mechanical lifetime.

For the bottom quark, with $m_q = m_b \approx 7.48 \times 10^{-24}$ g, we find the trajectory lifetime of the VQAP to be $t_{vq} \approx 3.47 \times 10^{-26}$ s. In comparison, the standard lifetime of the bottom VQAP, given by the Uncertainty Principle expressed in Eq. (2), is $\Delta t = \hbar/(4m_b c^2) \approx 3.90 \times 10^{-26}$ s. So t_{vq} from the classical computation is approximately 11% smaller than Δt. This also is remarkably close agreement of the classical lifetime with the quantum mechanical lifetime.

IV. ACTION INTEGRAL FOR THE TRAJECTORY

A key step in quantizing a classical model to make it a semiclassical model of a quantum system is computation of the action integral. In the present model, that is done as follows. The expression for the integral of action associated with a potential function U acting on a particle, in the relativistic case, is given by Lanczos ([16], p. 321):

$$\mathcal{A} = -\int_{t_1}^{t_2} U\frac{ds}{c}$$ (13)

where $ds = c\,dt/\gamma$. Considering the integrated action of the potential energy field in a VQAP, we compute the field action integrated over t_{vq}:

$$\mathcal{A} = -2\int_0^{t_{vq}/2}\left(-\frac{4\alpha_s\hbar c}{3\gamma R}\right)\frac{dt}{\gamma} = \frac{8\alpha_s\hbar}{3}\int_0^{R_{max}/2}\frac{dr}{\gamma^2 R\sqrt{1-\zeta}}$$ (14)

$$\mathcal{A} = \frac{4\alpha_s\hbar}{3}\int_0^1 \frac{d\zeta}{\zeta^{-1}\zeta\sqrt{1-\zeta}} = \frac{4\alpha_s\hbar}{3}\int_0^1 \frac{d\zeta}{\sqrt{1-\zeta}} = \frac{8\alpha_s}{3}\hbar$$ (15)

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For the charm VQAP, $\alpha_s(2m_c c^2) = 0.228$ and therefore $\mathcal{A} = 0.61\,\hbar$. This action integral is only 22% larger than the exact VQAP quantum fluctuation action in Eq. (2), $\frac{1}{2}\hbar$.

In the case of the bottom quark, the action integral for the model of the VQAP is found by substituting $\alpha_s(2m_b c^2) = 0.167$ into Eq. (15), and we find $\mathcal{A} = 0.45\,\hbar$. This is 10% lower than the quantum mechanical action for a VQAP.

This means that the model's representation of the bottom VQAP inherently is approximately quantized – a remarkable agreement between a quantum-mechanical characteristic of a dynamical system and the classical description of it. In comparison, semiclassical models for mesons, which achieve excellent agreement with measurements of meson masses [17], [18] need to be formulated with additional quantization conditions that introduce the factor $\hbar$. We have not imposed any quantization conditions upon the trajectory in this dynamical model. The model herein achieves approximate quantization at $\alpha_s(2m_b c^2)$ based upon only the measured value of $\alpha_s(M_{Z^0})$, the QCD theoretical energy dependence of $\alpha_s(Q^2)$, and relativistic dynamical theory (Eqs. (1) and (7)).

The discrepancy between t_{vq} and Δt in the case of the charm quark may be accounted for in part, as the author will show elsewhere [9]: for the charm quark and top quark, spin-spin interactions which have been ignored here become important and increase t_{vq} and $\mathcal{A}$ in such a way as to bring into closer agreement the classical and quantum results.

V. CONCLUSIONS

This good agreement between the classical trajectory lifetime and the quantum uncertainty lifetime at the key mass-energy of the bottom quark is surprising, but it may have a simple physical explanation: if the vacuum creates these particles in motion at $v \approx c$, then their de Broglie wavelengths $\lambda = h/p$ should be small, so that the quarks are pointlike. Dynamics of point masses would then be applicable. It is interesting that the length scale of any VAP is usually characterized in standard literature by assuming that $v \approx c$ [7].

The salient logic in this paper's result is the following. The measurement of $\alpha_s(M_{Z^0})$ and the one-loop Λ obtained from the so-called 'modified minimal subtraction scheme' of renormalization theory predict $\alpha_s(2m_b c^2)$ [8]. From this we may use the classical trajectory lifetime of the VQAP to compute the t_{vq} and action $\mathcal{A}$, obtaining $\mathcal{A} \approx \frac{1}{2}\hbar$ in approximate agreement with quantum mechanics.

Since $\hbar$ is a more universal and fundamental parameter than α_s, $\hbar$ intuitively would seem to be the governing parameter in Eq. (15).

If $\alpha_s(2m_b c^2)$ equalled 3/16 then $\mathcal{A}$ would exactly equal $\frac{1}{2}\hbar$. Setting Eq. (4) equal to 3/16, Q^2 equal to $(2m_b c^2)^2$, and solving for Λ yields $\Lambda = 0.106$ GeV instead of the standard 0.093 GeV. With this value of Λ, Eq. (4) gives $\alpha_s(91.2 \text{ GeV} \equiv M_{Z^0}) = 0.121$. This is only 2% different from the measured value upon which the accuracy of QCD depends, and is within the statistical uncertainty in $\alpha_s(M_{Z^0})$ quoted in Ref. [8].

We now have a logical link between the measured $\alpha_s(M_{Z^0})$ and the action integral from the Uncertainty Principle, $\frac{1}{2}\hbar$. It is reasonable to reverse this logical sequence and infer that the action integral $\frac{1}{2}\hbar$ is what governs the value of $\alpha_s(M_{Z^0})$. This reverse argument from $\mathcal{A} = \frac{1}{2}\hbar$ through Eqs. (15), (12), and (4) to $\alpha_s(M_{Z^0})$ is the derivation mentioned in this paper's title. In this way 't Hooft's vision of an underlying deterministic Law (classical relativistic dynamics) enables QCD and renormalization theory to account for the magnitude of α_s, not just the variation of $\alpha_s(Q^2)$ with energy scale.

Acknowledgments

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