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AN OPTIMAL INITIAL GUESS GENERATOR FOR ENTRY INTERFACE TARGETERS

Juan S. Senent*

If a pure numerical iterative approach is used, targeting entry interface (EI) conditions for nominal and abort return trajectories or for correction maneuvers can be computationally expensive. This paper describes an algorithm to obtain an optimal impulsive maneuver that generates a trajectory satisfying a set of EI targets: inequality constraints on longitude, latitude and azimuth and a fixed flight-path angle. Most of the calculations require no iterations, making it suitable for real-time applications or large trade studies. This algorithm has been used to generate initial guesses for abort trajectories during Earth-Moon transfers.

INTRODUCTION

Calculating a return trajectory that targets a very specific set of (EI) conditions can be computationally expensive if a pure numerical iterative approach is considered. The number of constraints and the nature of them, as we will see later, can make the problem very difficult for a numerical optimizer since multiple local optima and feasibility problems will appear. As an intermediate solution we can calculate an initial guess that can be used later by the numerical optimizer. This paper describes an algorithm to generate such initial guess with the following features:

- It is analytical, in order to compute the optimal maneuver no numerical iteration is needed. The only exception is the calculation of the velocity magnitude from the time of flight. Since we are only generating an initial guess, it is beneficial to do it in a very fast way. The numerical optimizer will obtain the final trajectory taking into account more realistic gravity models.
- It finds the global optimal solution. Depending on the inequality constraints, this optimization problem might present several local optima. This algorithm searches for solutions in all the feasible regions guaranteeing a global optimal solution. Algorithms using a numerical optimizer might run into a local optimum.
- It can handle inequality constraints in some of the EI targets. Sometimes the EI targets are defined as a large database of points that create an area. We can check individually each point, which can be computationally very expensive, or create one or more rectangular areas that include them. In this way we will reduce the computational time dramatically.
- It can provide information about the feasibility of the problem. One the disadvantages of using a pure numerical approach (e.g. a numerical optimizer) is that we do not have an insight of the problem. If the optimizer fails to converge, we will not know the reason. It might be due to our problem formulation, or an error in the program that runs the simulation or it might be

*Research Engineer, Odyssey Space Research, Aeroscience and Flight Mechanics Division/EG5, Johnson Space Center, 2101 Nasa Parkway, Houston, Texas, 77058

that the problem is not feasible. This algorithm can generate information about the feasibility of the problem since it can provide which inequality constraints could not be satisfied. We can use the information to redesign the return trajectory or create a new EI area.

We can specify the EI targets in different ways: longitude, latitude, azimuth or cross-range and down-range, etc. The type of targets and constraints that the algorithm can handle are described in Table 1. In the first part of the paper, we will describe the solution to the problem when the inequality constraints are not active. In this way, given the state of the spacecraft at a given time, a time of flight between the burn and EI, and a given velocity magnitude, we will obtain an optimal impulsive maneuver that targets: altitude and flight-path angle. We can formulate an equivalent problem by using the velocity magnitude instead of the time of flight. Once the basic optimal problem is solved then we will add one by one the latitude, longitude and azimuth inequality constraints in the next sections of the paper. Adding these inequality constraints will generate disjoint EI regions that can be reached and therefore each region will have its own local optimum that has to be checked.

Table 1. EI variables and constraints

EI variable	Description
Altitude (h)	Fixed
Longitude (λ)	Inequality constraint
Latitude (L)	Inequality constraint
Velocity magnitude (v)	Fixed (or calculated from time of flight)
Azimuth (Az)	Inequality constraint
Flight-path angle (γ)	Fixed

The applications of this algorithm as an initial guess generator include: a nominal or an abort return trajectory from the Moon that targets a specific geographical area at EI and that takes into account a safe module disposal (avoiding some regions, e.g. some islands) or computing correction maneuvers before EI to target a very specify EI point. This algorithm has been extensively used and validated in the generation of abort return trajectories generated after the translunar injection maneuver.

BASIC ALGORITHM

In this section we will examine the basic EI targeting problem with no inequality constraints. The statement of the problem is the folowing:

Given the state of the spacecraft: position and velocity $\mathbf{r}_a$, $\mathbf{v}_a$, calculate: a minimum impulsive maneuver that targets the entry interface position magnitude r_e, the entry interface velocity magnitude v_e and the entry interface flight-path angle γ_e. Once we solve this problem we can formulate a similar one where the target is the time of flight instead of v_e.

Part I

In order to solve the problem, the algorithm will take several steps, the first part will consist of calculating the basic parameters of the return trajectory:

- Calculate the semi-major axis and eccentricity of the return trajectory (also the parameter of the orbit)

$$a = \frac{r_e}{2 - \frac{r_e v_e^2}{\mu}} \qquad e = \sqrt{1 - \frac{(r_e v_e \cos\gamma_e)^2}{a\mu}} \qquad p = a(1-e^2) = \frac{(r_e v_e \cos\gamma_e)^2}{\mu}$$

- Calculate the velocity mangnitude after the return manevuer

$$v_a = \sqrt{v_e^2 + 2\mu\left(\frac{1}{r_a} - \frac{1}{r_e}\right)} \rightarrow v_{e_{min}} = \sqrt{2\mu\left(\frac{1}{r_e} - \frac{1}{r_a}\right)} \tag{1}$$

where $v_{e_{min}}$ is the minimum v_e that is necessary to obtain a return trajectory

- Calculate the true anomaly at EI

$$\theta_e = \arctan\left(\frac{\tan\gamma_e}{1 - \frac{r_e}{p}}\right)$$

- Calculate flight-path angle after the return maneuver γ_a. There are two solutions:

$$\gamma_{a_{1,2}} = \pm\arccos\left(\frac{r_e v_e \cos\gamma_e}{r_a v_a}\right) \tag{2}$$

Additionally, the following condition should be satisfied:

$$\left|\frac{r_e v_e \cos\gamma_e}{r_a v_a}\right| \leq 1$$

- Calculate the true anomaly after the return maneuver θ_a for the previous two solutions

$$\theta_{a_{1,2}} = \arctan\left(\frac{\tan\gamma_{a_{1,2}}}{1 - \frac{r_a}{p}}\right)$$

- Calculate the TOF from the abort maneuver to EI. First calculate the eccentric anomaly at EI:

$$E_e = \pm\arccos\left(\frac{a - r_e}{ae}\right)$$

Although there are two solutions to the above equation, the EI is before periapse so only the negative one will be used. The same equation should be applied to the eccentric anomaly after the abort maneuver:

$$E_{a_{1,2}} = \pm\arccos\left(\frac{a - r_a}{ae}\right)$$

Again, there are solutions to this equation only if

$$\left|\frac{a - r_a}{ae}\right| \leq 1 \Leftrightarrow a(1-e) \leq r_a \leq a(1+e)$$

Finally, the TOF can be calculated:

$$TOF_{1,2} = t_e - t_{a_{1,2}} = \sqrt{\frac{a^3}{\mu}}\left[E_e - E_{a_{1,2}} - e\left(\sin E_e - \sin E_{a_{1,2}}\right)\right] \tag{3}$$

Note that the TOF calculation is only valid if $e < 1$. Therefore this version of the algorithm is only applicable for elliptical return trajectories.

Two families of solutions have been found. Although these families will satisfy the constraints they will have two different TOFs.

Part II

In the second part of the algorithm we will calculate the relationship between the position a velocity vectors at EI: $\mathbf{r}_e, \mathbf{v}_e$. If we use the f and g functions in terms of the transfer angle ($\Delta\theta$) (see for example ****BOND****)

$$\begin{aligned} \mathbf{r}_a &= \mathbf{r}_e f + \mathbf{v}_e g &(4)\\ \mathbf{v}_a &= \mathbf{r}_e \dot{f} + \mathbf{v}_e \dot{g} &(5) \end{aligned}$$

where

$$\begin{aligned} f &= 1 - \frac{r_a}{p}(1 - \cos\Delta\theta)\\ g &= \frac{r_a r_e}{\sqrt{\mu p}}\sin\Delta\theta\\ \dot{f} &= \sqrt{\frac{\mu}{p}}\left(\frac{1-\cos\Delta\theta}{\sin\Delta\theta}\right)\left[\frac{1}{p}(1-\cos\Delta\theta) - \frac{1}{r_e} - \frac{1}{r_a}\right]\\ \dot{g} &= 1 - \frac{r_e}{p}(1-\cos\Delta\theta) \end{aligned}$$

where $\Delta\theta = \theta_a - \theta_e$. We can now compute the angle α between the $\mathbf{r}_a$ and $\mathbf{v}_e$:

$$\mathbf{r}_a^{\mathsf{T}}\mathbf{v}_e = \mathbf{r}_e^{\mathsf{T}}\mathbf{v}_e f + v_e^2 g = r_e v_e \sin\gamma_e f + v_e^2 g = r_a v_e \cos\alpha$$

$$\alpha = \arccos\left(\frac{r_e \sin\gamma_e f + v_e g}{r_a}\right)$$

or alternatively,

$$|\mathbf{r}_a \times \mathbf{v}_e| = |(\mathbf{r}_e f + \mathbf{v}_e g)\times \mathbf{v}_e| = |f\mathbf{r}_e \times \mathbf{v}_e| = r_e v_e \cos\gamma_e |f| = r_a v_e \sin\alpha$$

$$\tan\alpha = \frac{r_e v_e \cos\gamma_e |f|}{r_e v_e \sin\gamma_e f + v_e^2 g} = \frac{r_e \cos\gamma_e |f|}{r_e \sin\gamma_e f + v_e g}$$

We can see that the angle α is constant and therefore all the possible $\mathbf{v}_e$ are contained in a cone around $\mathbf{r}_a$ (see Figure 1) whose equation can be defined as:

$$\mathbf{v}_e = C\hat{\mathbf{r}}_a + R\cos A\hat{\mathbf{u}} + R\sin A\hat{\mathbf{v}} \tag{6}$$

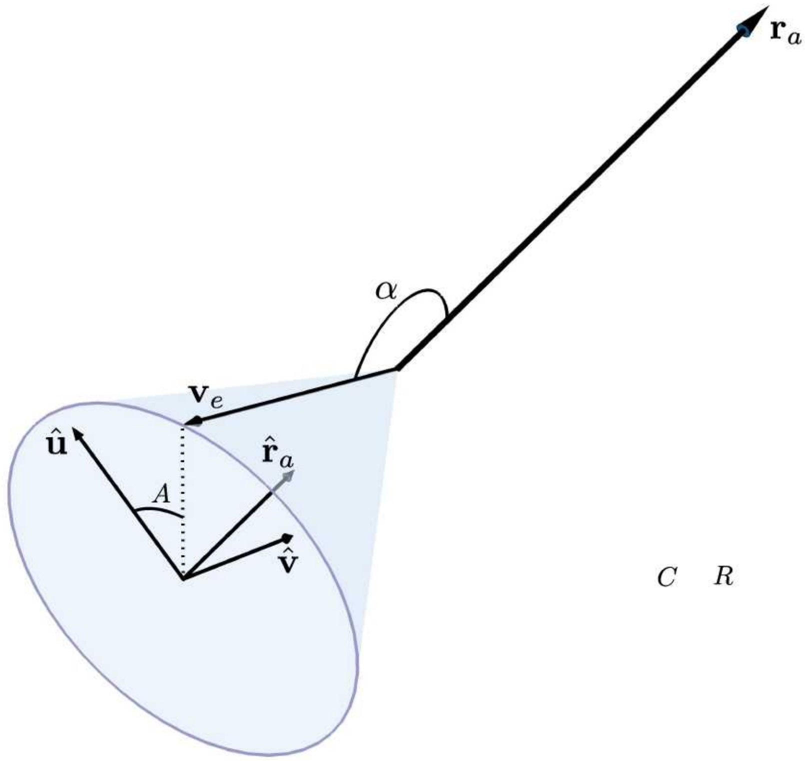


Figure 1. Relationship between the $\mathbf{r}_a$ and $\mathbf{v}_e$ vectors.

where$\hat{\mathbf{r}}_a$ is a unit vector in the direction of $\mathbf{r}_a$, and $\hat{\mathbf{u}}$, $\hat{\mathbf{v}}$ will complete the right-handed coordinate system. Their definition will change depending on the application. Also the angle A will determine the location of $\mathbf{v}_e$ in the cone ($A \in [0, 2\pi]$). From Figure 1 we obtain (note: $\alpha \in [0, \pi]$):

$$\begin{aligned} C &= v_e \cos\alpha \\ R &= v_e \sin\alpha \geq 0 \end{aligned}$$

We can define a unit vector in the direction of $\mathbf{v}_e$

$$\hat{\mathbf{v}}_e = \cos\alpha\hat{\mathbf{r}}_a + \sin\alpha\cos A\hat{\mathbf{u}} + \sin\alpha\sin A\hat{\mathbf{v}}$$

From eq. 4 we can obtain an expression for $\mathbf{r}_e$:

7

$$\begin{aligned} \mathbf{r}_e &= \frac{1}{f}\left(\mathbf{r}_a - \mathbf{v}_e g\right) && (7) \\ & \quad \text{using eq. } \mathbf{??} \\ &= \frac{\mathbf{r}_a}{f} - \frac{g}{f}\left(C\hat{\mathbf{r}}_a + R\cos A\hat{\mathbf{u}} + R\sin A\hat{\mathbf{v}}\right) && (8) \\ &= \left(\frac{r_a - gC}{f}\right)\hat{\mathbf{r}}_a - \frac{gR}{f}\left(\cos A\hat{\mathbf{u}} + \sin A\hat{\mathbf{v}}\right) && (9) \end{aligned}$$

Again, all the possible values of $\mathbf{r}_e$ are contained in a cone around $\mathbf{r}_a$.

$$\hat{\mathbf{r}}_e = \left(\frac{r_a - gC}{fr_e}\right)\hat{\mathbf{r}}_a - \frac{gR}{fr_e}\left(\cos A\hat{\mathbf{u}} + \sin A\hat{\mathbf{v}}\right)$$

Using eq. 9 we can obtain:

$$\begin{aligned}
|\hat{\mathbf{r}}_a \times \hat{\mathbf{r}}_e| &= \left|-\frac{gR}{fr_e}\left(\cos A\hat{\mathbf{v}} - \sin A\hat{\mathbf{u}}\right)\right| \\
&= \sqrt{\left(\frac{gR}{fr_e}\right)^2\left(\cos^2 A + \sin^2 A\right)} \\
&= \left|\frac{gR}{fr_e}\right| \\
&= \sin\sigma
\end{aligned}$$

we can express $\hat{\mathbf{r}}_e$ in spherical coordinates as (I don't know if I need this, the second part is much better):

$$\hat{\mathbf{r}}_e = \cos\rho\hat{\mathbf{r}}_a + \sin\rho\cos A\hat{\mathbf{u}} + \sin\rho\sin\mathbf{A}\hat{\mathbf{v}}$$

where:

$$\begin{aligned}
\cos\rho &= \frac{r_a - gC}{fr_e} \\
\sin\rho &= -\frac{gR}{fr_e}
\end{aligned}$$

Relationship between ρ and $\Delta\theta$:

$$\hat{\mathbf{r}}_e = \left(\frac{r_a - gC}{fr_e}\right)\hat{\mathbf{r}}_a - \frac{gR}{fr_e}\left(\cos A\hat{\mathbf{u}} + \sin A\hat{\mathbf{v}}\right)$$

$$\hat{\mathbf{r}}_e^{\mathsf{T}}\hat{\mathbf{r}}_a = \cos\Delta\theta \rightarrow \cos\Delta\theta = \cos\rho$$

$$\begin{aligned}
-\frac{gR}{fr_e} &= -\frac{\frac{r_a r_e}{\sqrt{\mu p}}\sin\Delta\theta v_e\sin\alpha}{fr_e} \\
&= -\sin\Delta\theta\frac{r_e\cos\gamma_e|f|}{\sqrt{\mu p}f} \\
&= -\sin\Delta\theta\frac{|f|}{f}
\end{aligned}$$

Therefore:

$$\begin{array}{lll} \sin\rho = \sin\Delta\theta & \text{if } f \leq 0 \\ \sin\rho = -\sin\Delta\theta & \text{if } f > 0 \end{array} \rightarrow \begin{array}{ll} \rho = \Delta\theta & \text{if } f \leq 0 \\ \rho = -\Delta\theta & \text{if } f > 0 \end{array}$$

$$\text{if } f \leq 0 \rightarrow \hat{\mathbf{r}}_e = \cos\Delta\theta\hat{\mathbf{r}}_a + \sin\Delta\theta\cos A\hat{\mathbf{u}} + \sin\boldsymbol{\Delta}\theta\sin\mathbf{A}\hat{\mathbf{v}} \tag{10}$$

$$\text{if } f > 0 \rightarrow \hat{\mathbf{r}}_e = \cos\Delta\theta\hat{\mathbf{r}}_a - \sin\Delta\theta\cos A\hat{\mathbf{u}} - \sin\boldsymbol{\Delta}\theta\sin\mathbf{A}\hat{\mathbf{v}} \tag{11}$$

If we define $f_s = -\frac{|f|}{f}$, ($f_s = \pm 1$) then

$$-\frac{gR}{fr_e} = f_s \sin\Delta\theta$$

and $\hat{\mathbf{r}}_e$ can be defined by:

$$\hat{\mathbf{r}}_e = \cos\Delta\theta\hat{\mathbf{r}}_a + f_s\left(\sin\Delta\theta\cos A\hat{\mathbf{u}} + \sin\boldsymbol{\Delta}\theta\sin\mathbf{A}\hat{\mathbf{v}}\right) \tag{12}$$

and therefore,

$$\begin{aligned} \hat{\mathbf{r}}_e^{\mathsf{T}}\hat{\mathbf{v}}_e &= \cos\Delta\theta\cos\alpha + f_s\sin\Delta\theta\sin\alpha\cos^2 A + f_s\sin\Delta\theta\sin\alpha\sin^2 A \\ &= \cos\Delta\theta\cos\alpha + f_s\sin\Delta\theta\sin\alpha = \cos\left(\Delta\theta - f_s\alpha\right) = \sin\gamma_e \end{aligned}$$

$$\cos\left(\Delta\theta - f_s\alpha\right) = \sin\gamma_e \rightarrow \sin\left(\frac{\pi}{2} - \Delta\theta + f_s\alpha\right) = \sin\gamma_e$$

$$\gamma_e = \frac{\pi}{2} - \Delta\theta + f_s\alpha$$

$$\Delta\theta - f_s\alpha = \frac{\pi}{2} - \gamma_e \rightarrow \sin\left(\Delta\theta - f_s\alpha\right) = \sin\left(\frac{\pi}{2} - \gamma_e\right) = \cos\gamma_e \geq 0, \quad \text{since} \ \ \gamma_e \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

therefore

$$\sin\left(\Delta\theta - f_s\alpha\right) \geq 0 \tag{13}$$

Also, if $f \leq 0$, we can calculate the angular momentum of the abort transfer orbit as:

$$\hat{\mathbf{r}}_e \times \hat{\mathbf{v}}_e = \begin{vmatrix} \hat{\mathbf{r}}_a & \hat{\mathbf{u}} & \hat{\mathbf{v}} \\ \cos\Delta\theta & f_s\sin\Delta\theta\cos A & f_s\sin\Delta\theta\sin A \\ \cos\alpha & \sin\alpha\cos A & \sin\alpha\sin A \end{vmatrix}$$

$$
\begin{aligned}
\hat{\mathbf{r}}_e \times \hat{\mathbf{v}}_e &= \hat{\mathbf{r}}_a \left(f_s \sin\Delta\theta \cos A \sin\alpha \sin A - \sin\alpha \cos A f_s \sin\Delta\theta \sin A \right) \\
&\quad - \hat{\mathbf{u}} \left(\cos\Delta\theta \sin\alpha \sin A - \cos\alpha f_s \sin\Delta\theta \sin A \right) \\
&\quad + \hat{\mathbf{v}} \left(\cos\Delta\theta \sin\alpha \cos A - \cos\alpha f_s \sin\Delta\theta \cos A \right) \\
&= \sin A \left(\sin\Delta\theta \cos\alpha f_s - \cos\Delta\theta \sin\alpha \right) \hat{\mathbf{u}} - \cos A \left(\sin\Delta\theta \cos\alpha f_s - \cos\Delta\theta \sin\alpha \right) \hat{\mathbf{v}} \\
&= f_s \sin\left(\Delta\theta - f_s\alpha \right) \left(\sin A \hat{\mathbf{u}} - \cos A \hat{\mathbf{v}} \right)
\end{aligned}
$$

$$
|\hat{\mathbf{r}}_e \times \hat{\mathbf{v}}_e| = \sin\left(\Delta\theta - f_s\alpha \right) = \cos\gamma_e
$$

$$
\hat{\mathbf{h}} = \frac{\hat{\mathbf{r}}_\mathbf{e} \times \hat{\mathbf{v}}_\mathbf{e}}{\sin\left(\boldsymbol{\Delta\theta} - \mathbf{f_s}\alpha \right)} = f_s \left(\sin A \hat{\mathbf{u}} - \cos A \hat{\mathbf{v}} \right) \tag{14}
$$

The angular momentum vector remains in the $\hat{\mathbf{u}} - \hat{\mathbf{v}}$ plane, that is, $\hat{\mathbf{r}}_a^\mathsf{T} \hat{\mathbf{h}} = 0$ (esto era obvio pues $\hat{\mathbf{r}}_a$ is on the transfer plane).

The trajectories can be classified into two families:

1. The ones that cross the antipode (-$\hat{\mathbf{r}}_a$) and then the entry interface point ($\hat{\mathbf{r}}_e$)
2. The ones that cross the entry interface point ($\hat{\mathbf{r}}_e$) and then the antipode (-$\hat{\mathbf{r}}_a$)

We can find a criteria to distinguish between these two families based on the angular momentum of the transfer orbit. We first get $-\hat{\mathbf{r}}_a \times \hat{\mathbf{r}}_e$

$$
-\hat{\mathbf{r}}_a \times \hat{\mathbf{r}}_e = \begin{vmatrix} \hat{\mathbf{r}}_a & \hat{\mathbf{u}} & \hat{\mathbf{v}} \\ -1 & 0 & 0 \\ \cos\Delta\theta & f_s \sin\Delta\theta \cos A & f_s \sin\Delta\theta \sin A \end{vmatrix}
$$

$$
\begin{aligned}
-\hat{\mathbf{r}}_a \times \hat{\mathbf{r}}_e &= -\hat{\mathbf{u}} \left(-f_s \sin\Delta\theta \sin A \right) + \hat{\mathbf{v}} \left(-f_s \sin\Delta\theta \cos A \right) \\
&= f_s \sin\Delta\theta \left(\sin A \hat{\mathbf{u}} - \cos A \hat{\mathbf{v}} \right) \\
&= \sin\Delta\theta \hat{\mathbf{h}}
\end{aligned}
$$

In this way, we can classify the orbits according to this criteria as follows:

$$
\begin{cases} \text{Family 1} & \text{if} \quad \sin\Delta\theta > 0 \\ \text{Family 2} & \text{if} \quad \sin\Delta\theta < 0 \end{cases}
$$

Also, since

$$
\frac{\hat{\mathbf{r}}_a \times \hat{\mathbf{r}}_e}{|\hat{\mathbf{r}}_a \times \hat{\mathbf{r}}_e|} = -\frac{\sin\Delta\theta}{|\sin\Delta\theta|} \hat{\mathbf{h}} = \pm \hat{\mathbf{h}}
$$

The angular momentum vector is parallel (or antiparallel) to the intersection of the planes defined by $\hat{\mathbf{r}}_a$ and $\hat{\mathbf{r}}_e$ (cosa que ya sabiamos pues los dos vectores pertenecen al plano de transferencia)

We can now calculate the maximum and minumum inclination of the abort trajectory. If the coordinate system is defined:

$\hat{\mathbf{u}} = \frac{proj_{\hat{\mathbf{r}}_a}\mathbf{z}}{|proj_{\hat{\mathbf{r}}_a}\mathbf{z}|} = \frac{proj_{\hat{\mathbf{r}}_a}\mathbf{z}}{\cos DEC_{\mathbf{r}_a}}$ Projection of the $\mathbf{z}$-axis on the plane defined by $\mathbf{r}_a$. Where $proj_{\hat{\mathbf{r}}_a}\mathbf{z} = \mathbf{z} - \left(\hat{\mathbf{r}}_a^{\mathsf{T}}\mathbf{z}\right)\hat{\mathbf{r}}_a =$

$\hat{\mathbf{v}} = \hat{\mathbf{r}}_a \times \hat{\mathbf{u}} = \frac{\hat{\mathbf{r}}_a \times \mathbf{z}}{\cos DEC_{\mathbf{r}_a}}$ To complete the right-handed coordinate system

$\hat{\mathbf{r}}_a$ In the direction of $\mathbf{r}_a$

Note: $|proj_{\hat{\mathbf{r}}_a}\mathbf{z}| = \cos DEC_{\mathbf{r}_a}$

Esto esta MAAAAAALLLLLL. Rehacer con lo que tengo en mis notas: 19/01/09

Since the inclination is the angle between $\mathbf{z}$ and $\hat{\mathbf{h}}$, the minimum inclination will occur when $\hat{\mathbf{h}}$ is parallel to $\hat{\mathbf{u}}$ and the maximum inclination of the abort trajectory will occur when $\hat{\mathbf{h}}$ is antiparallel to $\hat{\mathbf{u}}$. We can calculate the angles A associated with these two situations:

From eq. 14 we can get:

$$INC_{min} = DEC_{\mathbf{r}_a} \quad \leftarrow \quad A = \frac{\pi}{2}$$

$$INC_{max} = \pi - INC_{min} = \pi - DEC_{\mathbf{r}_a} \quad \leftarrow \quad A = -\frac{\pi}{2}$$

where $DEC_{\mathbf{r}_a}$is the declination of $\hat{\mathbf{r}}_a$ in the $\mathbf{x} - \mathbf{y} - \mathbf{z}$ coordinate system (intertial system)

See fig. dibujar figura!!!!

Some properties that can be obtained:

$$\left(\hat{\mathbf{r}}_a \times \hat{\mathbf{r}}_e\right)^{\mathsf{T}} \hat{\mathbf{v}}_e = \hat{\mathbf{r}}_a^{\mathsf{T}} \left(\hat{\mathbf{r}}_e \times \hat{\mathbf{v}}_e\right) = \cos\gamma_e \hat{\mathbf{r}}_a^{\mathsf{T}} \hat{\mathbf{h}} = 0$$

$$\left(\hat{\mathbf{r}}_a \times \hat{\mathbf{r}}_e\right)^{\mathsf{T}} proj_{\hat{\mathbf{r}}_e} \hat{\mathbf{v}}_e = \left(\hat{\mathbf{r}}_a \times \hat{\mathbf{r}}_e\right)^{\mathsf{T}} \left[\hat{\mathbf{v}}_\mathbf{e} - \left(\hat{\mathbf{r}}_e^{\mathsf{T}} \hat{\mathbf{v}}_\mathbf{e}\right) \hat{\mathbf{r}}_e\right] = 0$$

$$\left(\hat{\mathbf{r}}_a \times \hat{\mathbf{r}}_e\right)^{\mathsf{T}} proj_{\hat{\mathbf{r}}_a} \hat{\mathbf{v}}_e = \left(\hat{\mathbf{r}}_a \times \hat{\mathbf{r}}_e\right)^{\mathsf{T}} \left[\hat{\mathbf{v}}_\mathbf{e} - \left(\hat{\mathbf{r}}_a^{\mathsf{T}} \hat{\mathbf{v}}_\mathbf{e}\right) \hat{\mathbf{r}}_a\right] = 0$$

Therefore $\hat{\mathbf{v}}_e$, $proj_{\hat{\mathbf{r}}_e}\hat{\mathbf{v}}_e$ and $proj_{\hat{\mathbf{r}}_a}\hat{\mathbf{v}}_e$lie on the same plane (the one defined by $\hat{\mathbf{r}}_a \times \hat{\mathbf{r}}_e$)

Using eq. 14 we can prove that the amount of plane change only depends on the increment on A.

If we obtain the abort trajectories defined by A_1 and A_2, the angle between the two angular momentum vectors is:

$$\hat{\mathbf{h}}_1 = \sin A_1 \hat{\mathbf{u}} - \cos A_1 \hat{\mathbf{v}}$$

$$\hat{\mathbf{h}}_2 = \sin A_2 \hat{\mathbf{u}} - \cos A_2 \hat{\mathbf{v}}$$

$$\hat{\mathbf{h}}_1^{\mathsf{T}} \hat{\mathbf{h}}_2 = \sin A_1 \sin A_2 + \cos A_1 \cos A_2 = \cos \Delta A = \cos\eta \rightarrow \Delta A = \eta$$

therefore the amount of plane change is defined by the increment in A

Using eq. 14 we can prove that $\hat{\mathbf{r}}_a$ and $-\hat{\mathbf{r}}_a$ are contained in the plane of the orbit:

$$\pm\hat{\mathbf{r}}_a^{\mathsf{T}}\hat{\mathbf{h}} = \pm\hat{\mathbf{r}}_a^{\mathsf{T}}\left(\sin A\hat{\mathbf{u}} - \cos A\hat{\mathbf{v}}\right) = 0$$

We can now calculate $\mathbf{v}_a$ velocity after the abort maneuver. Using eqs. 15, 7 and 6:

$$\begin{aligned}
\mathbf{v}_a &= \mathbf{r}_e\dot{f} + \mathbf{v}_e\dot{g} = \frac{\dot{f}}{f}\left(\mathbf{r}_a - \mathbf{v}_e g\right) + \mathbf{v}_e\dot{g} = \frac{\dot{f}}{f}\mathbf{r}_a + \left(\dot{g} - \frac{g\dot{f}}{f}\right)\mathbf{v}_e = \frac{1}{f}\left(\dot{f}\mathbf{r}_a + \mathbf{v}_e\right) \\
&= \left(\frac{\dot{f}}{f}r_a + C\right)\hat{\mathbf{r}}_a + \frac{R}{f}\left(\cos A\hat{\mathbf{u}} + \sin A\hat{\mathbf{v}}\right) && (15) \\
& && (16)
\end{aligned}$$

Once again, $\mathbf{v}_a$ lies on a cone defined by $\hat{\mathbf{r}}_a$

If $\mathbf{v}_a^-$, velocity vector before the abort maneuver, is known, we can compute the optimal velocity after the abort manuver$\mathbf{v}_a^*$. The optimal velocity is just the projection of $\mathbf{v}_a^-$on the cone defined by eq. 15. We need to calculate the optimal angle A^* that will produce $\mathbf{v}_a^*$.

$$\mathbf{v}_a^* = \left(\frac{\dot{f}}{f}r_a + C\right)\hat{\mathbf{r}}_a + \frac{R}{f}\frac{proj_{\mathbf{r}_a}\mathbf{v}_a^-}{\left|proj_{\mathbf{r}_a}\mathbf{v}_a^-\right|}$$

$$\mathbf{v}_a^* - \mathbf{v}_a = \frac{R}{f}\left[\frac{proj_{\mathbf{r}_a}\mathbf{v}_a^-}{\left|proj_{\mathbf{r}_a}\mathbf{v}_a^-\right|} - \left(\cos A\hat{\mathbf{u}} + \sin A\hat{\mathbf{v}}\right)\right]$$

$$\hat{\mathbf{u}}^{\mathsf{T}}\left(\mathbf{v}_a^* - \mathbf{v}_a\right) = \frac{R}{f}\left[\frac{\hat{\mathbf{u}}^{\mathsf{T}}proj_{\mathbf{r}_a}\mathbf{v}_a^-}{\left|proj_{\mathbf{r}_a}\mathbf{v}_a^-\right|} - \cos A\right] \rightarrow \cos A^* = \frac{\hat{\mathbf{u}}^{\mathsf{T}}proj_{\mathbf{r}_a}\mathbf{v}_a^-}{\left|proj_{\mathbf{r}_a}\mathbf{v}_a^-\right|}$$

$$\hat{\mathbf{v}}^{\mathsf{T}}\left(\mathbf{v}_a^* - \mathbf{v}_a\right) = \frac{R}{f}\left[\frac{\hat{\mathbf{v}}^{\mathsf{T}}proj_{\mathbf{r}_a}\mathbf{v}_a^-}{\left|proj_{\mathbf{r}_a}\mathbf{v}_a^-\right|} - \sin A\right] \rightarrow \sin A^* = \frac{\hat{\mathbf{v}}^{\mathsf{T}}proj_{\mathbf{r}_a}\mathbf{v}_a^-}{\left|proj_{\mathbf{r}_a}\mathbf{v}_a^-\right|}$$

$$\tan A^* = \frac{\hat{\mathbf{v}}^{\mathsf{T}}proj_{\mathbf{r}_a}\mathbf{v}_a^-}{\hat{\mathbf{u}}^{\mathsf{T}}proj_{\mathbf{r}_a}\mathbf{v}_a^-} \tag{17}$$

Note: if $f < 0$ then from eq. 15 the radius of the cone $\frac{R}{f} < 0$ and therefore the equivalent cone will be such as the original but where the $\hat{\mathbf{u}}$and $\hat{\mathbf{v}}$ are rotated by π. In that case:

$$A^* = \arctan\left(\frac{\hat{\mathbf{v}}^{\mathsf{T}}proj_{\mathbf{r}_a}\mathbf{v}_a^-}{\hat{\mathbf{u}}^{\mathsf{T}}proj_{\mathbf{r}_a}\mathbf{v}_a^-}\right) + \pi$$

Note: If the minimum distance between $\mathbf{v}_a^-$ and $\mathbf{v}_a$ (minimum Δv) is obtained when the angle A^* is such that the unit vector $\cos A^*\hat{\mathbf{u}} + \sin A^*\hat{\mathbf{v}}$ points in the direction of $proj_{\mathbf{r}_a}\mathbf{v}_a^-$, deviations from A^*will increase the distance between $\mathbf{v}_a^-$ and $\mathbf{v}_a$ (increase in Δv). The maximum distance (maximum Δv) will be found for a $\mathbf{v}_a$ generated by $A^* + \pi$, that corresponds to a vector that points to $-proj_{\mathbf{r_a}}\mathbf{v}_a^-$. Therefore the Δv will monotonically increase when A goes from A^* to $A^* + \pi$ and will monotonically decrease when A goes from $A^* + \pi$ to A^*. Therefore, if constraints are impose on A, $[A_{min}, A_{max}]$:

$$A^* = \begin{cases} A^* & \text{if} \quad A^* \in [A_{min}, A_{max}] \\ A_{min} & \text{if} \quad \Delta v_{A_{min}} < \Delta v_{A_{max}} \\ A_{max} & \text{if} \quad \Delta v_{A_{max}} < \Delta v_{A_{min}} \end{cases}$$

Note: From eq. 17 we can see that if the coordinate system is defined as in page 9 , A^* depends only on $\mathbf{v}_a^-$ and $\mathbf{r}_a$. That is, v_e does not affect A^*, therefore from eq. 8 we can say that all the optimal abort trajectories will be on the same plane, since angular momentum vector $\hat{\mathbf{h}}$ is the same. The transfer plane of the optimal abort trajectories will not depend on the v_e.

We can now consider constraints in the latitude at entry interface.

From fig. la figura de la minima inclinacion, we can express $\mathbf{z}$ in the coordinate frame defined in 9 as:

$$\mathbf{z} = \sin\left(\frac{\pi}{2} - DEC_{\mathbf{r}_a}\right)\hat{\mathbf{u}} + \cos\left(\frac{\pi}{2} - DEC_{\mathbf{r}_a}\right)\hat{\mathbf{r}}_\mathbf{a} = \cos DEC_{\mathbf{r}_a}\hat{\mathbf{u}} + \sin DEC_{\mathbf{r}_a}\hat{\mathbf{r}}_\mathbf{a} \tag{18}$$

$$\mathbf{z}^\mathsf{T}\hat{\mathbf{r}}_\mathbf{e} = \sin\left(DEC_{EI}\right)$$

$$\mathbf{z}^\mathsf{T}\hat{\mathbf{r}}_\mathbf{e} = \cos DEC_{\mathbf{r}_a}\sin\Delta\theta\cos A + \cos\Delta\theta\sin DEC_{\mathbf{r}_a}$$

$$\cos A = \frac{\sin DEC_{EI} - \cos\Delta\theta\sin DEC_{\mathbf{r}_a}}{\cos DEC_{\mathbf{r}_a}\sin\Delta\theta} \tag{19}$$

So if we consider that at entry interface (EI) $DEC_{EI} \approx LAT_{EI}$ we can calculate the angle A that will produce an abort trajectory with a determined LAT_{EI}.

Note: for each LAT_{EI} there are two solutions to eq. 11: $\pm A_{LAT_{EI}}$. The positive one will generate positive EI azimuths (direct return, $INC \in \left[0, \frac{\pi}{2}\right]$) and the negative one will produce negative EI azimuths (retrograde return, $INC \in \left[\frac{\pi}{2}, \pi\right]$). We can see this if we calculate the INC of the abort trajectory:

$$\cos INC = \mathbf{z}^\mathsf{T}\hat{\mathbf{h}} = \sin A\cos DEC_{\mathbf{r}_a} \tag{20}$$

If $A \in [0, \pi]$ then we obtain a direct return:

$$\sin A \geq 0, \quad \text{since} \quad \cos DEC_{\mathbf{r}_a} \geq 0 \rightarrow \cos INC \geq 0 \rightarrow INC \in \left[0, \frac{\pi}{2}\right] \rightarrow Az_{EI} \in [0, \pi]$$

If $A \in [-\pi, 0]$ then we obtain a retrograde return:

$$\sin A \leq 0, \quad \text{since} \quad \cos DEC_{\mathbf{r}_a} \geq 0 \rightarrow \cos INC \leq 0 \rightarrow INC \in \left[\frac{\pi}{2}, \pi\right] \rightarrow Az_{EI} \in [\pi, 2\pi]$$

Note: the relationship between the angle A and Az_{EI} is given by:

$$\cos INC = \cos LAT_{EI} \sin AZ_{EI} = \sin A \cos DEC_{\mathbf{r}_a}$$

$$\sin AZ_{EI} = \frac{\sin A \cos DEC_{\mathbf{r}_a}}{\cos LAT_{EI}} \tag{21}$$

We can compute AZ_{EI} as a function of A.

$$\cos AZ_{EI} = \hat{\mathbf{h}}^{\mathsf{T}} \frac{(\hat{\mathbf{r}}_{\mathbf{e}} \times \mathbf{z})}{|\hat{\mathbf{r}}_{\mathbf{e}} \times \mathbf{z}|}$$

$\hat{\mathbf{r}}_{\mathbf{e}} \times \mathbf{z}$ defines a plane with constant longitude that contains $\hat{\mathbf{r}}_{\mathbf{e}}$

From their definition we can compute $\hat{\mathbf{r}}_{\mathbf{e}} \times \mathbf{z}$:

$$\hat{\mathbf{r}}_{\mathbf{e}} \times \mathbf{z} = -\cos DEC_{\mathbf{r}_a} \sin \Delta\theta \cos A \hat{\mathbf{r}}_{\mathbf{a}} + \sin DEC_{\mathbf{r}_a} \sin \Delta\theta \sin A \hat{\mathbf{u}} + (\cos DEC_{\mathbf{r}_a} \cos \Delta\theta - \sin DEC_{\mathbf{r}_a} \sin \Delta\theta \cos A) \hat{\mathbf{v}}$$

also

$$|\hat{\mathbf{r}}_{\mathbf{e}} \times \mathbf{z}| = \sin\left(\frac{\pi}{2} - LAT_{EI}\right) = \cos LAT_{EI}$$

so now we can get

$$\begin{aligned}
\cos AZ_{EI} &= \hat{\mathbf{h}}^{\mathsf{T}} \frac{(\hat{\mathbf{r}}_{\mathbf{e}} \times \mathbf{z})}{|\hat{\mathbf{r}}_{\mathbf{e}} \times \mathbf{z}|} \\
&= \frac{\sin^2 A \sin DEC_{\mathbf{r}_a} \sin \Delta\theta + \cos^2 A \sin DEC_{\mathbf{r}_a} \sin \Delta\theta - \cos A \cos DEC_{\mathbf{r}_a} \cos \Delta\theta}{\cos LAT_{EI}} \\
&= \frac{\sin DEC_{\mathbf{r}_a} \sin \Delta\theta - \cos A \cos DEC_{\mathbf{r}_a} \cos \Delta\theta}{\cos LAT_{EI}}
\end{aligned}$$

Using the above eq. and eq. 21 we can compute $\tan AZ_{EI}$

$$\tan AZ_{EI} = \frac{\sin A \cos DEC_{\mathbf{r}_a}}{\sin DEC_{\mathbf{r}_a} \sin \Delta\theta - \cos A \cos DEC_{\mathbf{r}_a} \cos \Delta\theta} = \frac{\sin A}{\tan DEC_{\mathbf{r}_a} \sin \Delta\theta - \cos A \cos \Delta\theta}$$

If a specific AZ_{EI} is required, we can now solve for A

$$\begin{aligned}
\sin A + \cos A \cos \Delta\theta \tan AZ_{EI} &= \tan AZ_{EI} \tan DEC_{\mathbf{r}_a} \sin \Delta\theta \\
\sin (A + \epsilon) \sqrt{1 + \cos^2 \Delta\theta \tan^2 AZ_{EI}} &= \tan AZ_{EI} \tan DEC_{\mathbf{r}_a} \sin \Delta\theta
\end{aligned}$$

where

$$\epsilon = \arctan \left(\frac{\cos \Delta\theta \tan AZ_{EI}}{1} \right)$$

Therefore we can have two solutions for A

$$A_1 = \arcsin \left(\frac{\tan AZ_{EI} \tan DEC_{\mathbf{r}_a} \sin \Delta\theta}{\sqrt{1 + \cos^2 \Delta\theta \tan^2 AZ_{EI}}} \right) - \epsilon$$

$$A_2 = \pi - \arcsin \left(\frac{\tan AZ_{EI} \tan DEC_{\mathbf{r}_a} \sin \Delta\theta}{\sqrt{1 + \cos^2 \Delta\theta \tan^2 AZ_{EI}}} \right) - \epsilon$$

As an alternative:

$$\sin (A_1 + \epsilon) = \sin (\pi - A_1 - \epsilon) = \sin (A_2 + \epsilon)$$

$$\pi - A_1 - \epsilon = A_2 + \epsilon \rightarrow A_2 = \pi - A_1 - 2\epsilon$$

AZ_{EI} can be also directly computed as follows:

If we define an E-N-R coordinate system associated with $\hat{\mathbf{r}}_e$:

$$\begin{aligned}
\mathbf{R} &:= \hat{\mathbf{r}}_e \\
\mathbf{N} &:= \frac{proj_{\hat{\mathbf{r}}_e} \mathbf{z}}{|proj_{\hat{\mathbf{r}}_e} \mathbf{z}|} \\
\mathbf{E} &:= \mathbf{N} \times \mathbf{R} = \frac{\mathbf{z} \times \hat{\mathbf{r}}_e}{|proj_{\hat{\mathbf{r}}_e} \mathbf{z}|}
\end{aligned}$$

$$\begin{aligned}
AZ_{EI} &= \arctan \left[\frac{\left(proj_{\hat{\mathbf{r}}_e} \mathbf{v}_e\right)^{\mathsf{T}} (\mathbf{z} \times \hat{\mathbf{r}}_e)}{\left(proj_{\hat{\mathbf{r}}_e} \mathbf{v}_e\right)^{\mathsf{T}} proj_{\hat{\mathbf{r}}_e} \mathbf{z}} \right] = \arctan \left[\frac{\left(\mathbf{v}_e - \hat{\mathbf{r}}_e^{\mathsf{T}} \mathbf{v}_e \hat{\mathbf{r}}_e\right)^{\mathsf{T}} (\mathbf{z} \times \hat{\mathbf{r}}_e)}{\left(\mathbf{v}_e - \hat{\mathbf{r}}_e^{\mathsf{T}} \mathbf{v}_e \hat{\mathbf{r}}_e\right)^{\mathsf{T}} (\mathbf{z} - \hat{\mathbf{r}}_e^{\mathsf{T}} \mathbf{z} \hat{\mathbf{r}}_e)} \right] \\
&= \arctan \left[\frac{\mathbf{v}_e^{\mathsf{T}} (\mathbf{z} \times \hat{\mathbf{r}}_e)}{\mathbf{z}^{\mathsf{T}} (\mathbf{v}_e - \hat{\mathbf{r}}_e^{\mathsf{T}} \mathbf{v}_e \hat{\mathbf{r}}_e)} \right] = \arctan \left[\frac{\mathbf{v}_e^{\mathsf{T}} (\mathbf{z} \times \hat{\mathbf{r}}_e)}{\mathbf{v}_e^{\mathsf{T}} proj_{\hat{\mathbf{r}}_e} \mathbf{z}} \right] = \arctan \left[\frac{\mathbf{z}^{\mathsf{T}} (\hat{\mathbf{r}}_e \times \mathbf{v}_e)}{\mathbf{z}^{\mathsf{T}} proj_{\hat{\mathbf{r}}_e} \mathbf{v}_e} \right]
\end{aligned}$$

Since once the LAT_{EI} is specified, A is determined using eq. 11, the above eq. is only useful to obtain the value of the abort trajectory AZ_{EI} but not to specify it. That is, we cannot specify

LAT_{EI} and AZ_{EI} of the abort trajectory at the same time. From the two solutions of the equation only one is valid?? averiguar.

From eq. 12 we can see that the sign of A and $AZ_{EI}$are the same.
Calculating the optimal abort trajectory when latitude constraints $[LAT_{min}, LAT_{max}]$ at EI are imposed. In the rest of the document we will assume that $DEC_{EI} \approx LAT_{EI}$

We can rewrite eq. 11:

$$\cos A = \frac{1}{\cos DEC_{\mathbf{r}_a} \sin \Delta\theta} \sin\left(LAT_{EI}\right) - \frac{\cos \Delta\theta \sin DEC_{\mathbf{r}_a}}{\cos DEC_{\mathbf{r}_a} \sin \Delta\theta} = K_1 \sin\left(LAT_{EI}\right) + K_2 \quad (22)$$

Given the latitude constraints above, a range for the angle A, $[A_{min}, A_{max}]$, can be calculated:

First we will calculate the range of feasible latitudes for this problem. The range of feasible latitudes $[LAT_{fmin}, LAT_{fmax}]$will be such that that eq. 14 has a solution and also it has to be within the range $[LAT_{min}, LAT_{max}]$

$$L_{min} = K_1 \sin LAT_{min} + K_2$$

$$L_{max} = K_1 \sin LAT_{min} + K_2$$

$[LAT_{fmin}, LAT_{fmax}]$ can be calculated using the following table:

	$L_{min} > 1$	$-1 \leq L_{min} \leq 1$	$L_{min} < -1$
$L_{max} > 1$	No solution	$[LAT_{min}, LAT_p]$	$[LAT_m, LAT_p]$
$-1 \leq L_{max} \leq 1$	$[LAT_p, LAT_{max}]$	$[LAT_{min}, LAT_{max}]$	$[LAT_m, LAT_{max}]$
$L_{max} < -1$	$[LAT_p, LAT_m]$	$[LAT_{min}, LAT_m]$	No solution

where LAT_p and LAT_m are defined as:

$$K_1 \sin LAT_{EI} + K_2 = 1 \rightarrow LAT_p = \arcsin\left(\frac{1-K_2}{K_1}\right)$$

$$K_1 \sin LAT_{EI} + K_2 = 1 \rightarrow LAT_m = \arcsin\left(\frac{-1-K_2}{K_1}\right)$$

Once we have the range of feasible latitudes that the abort trajectories can achieve, we have to calculate the range $[A_{min}, A_{max}]$:

Since $LAT_{EI} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, hacer grafica de mis notas 7/14/08

$$K_1 > 0 \rightarrow K_1 \sin\left(LAT_{EI}\right) + K_2 \text{ increases monotonically} \rightarrow \begin{cases} A_{min} = \arccos\left(K_1 \sin\left(LAT_{fmin}\right) + K_2\right) \\ A_{max} = \arccos\left(K_1 \sin\left(LAT_{fmax}\right) + K_2\right) \end{cases}$$

$$K_1 < 0 \rightarrow K_1 \sin\left(LAT_{EI}\right) + K_2 \text{ decreases monotonically} \rightarrow \begin{cases} A_{min} = \arccos\left(K_1 \sin\left(LAT_{fmax}\right) + K_2\right) \\ A_{max} = \arccos\left(K_1 \sin\left(LAT_{fmin}\right) + K_2\right) \end{cases}$$

Note: if the abort trajectory is retrograde then the range would be $[-A_{max}, -A_{min}]$.

Algortihm to compute the optimum abort trajectory when latitude constraints are imposed

1. Specify type of solution 1 or 2 (from eq.3)
2. Specify direct or posigrade
3. Compute the optimum A^* using eq. 10
4. If $A^* \in [A_{min}, A_{max}]$ then END
5. Compute $\Delta v_{A_{min}}$ and $\Delta v_{A_{max}}$ and return the minimum

We can also find the relationship between A a required longitude at entry interface:

We can calculate first the difference between the longitude of the antipode of $\mathbf{r}_a$ and the longitude at any entry interface point $\mathbf{r}_e$

$$\left(proj_{\mathbf{z}}\hat{\mathbf{r}}_e\right)^{\mathsf{T}} \left(proj_{\mathbf{z}} - \hat{\mathbf{r}}_a\right) = \left|proj_{\mathbf{z}}\hat{\mathbf{r}}_e\right| \left|proj_{\mathbf{z}} - \hat{\mathbf{r}}_a\right| \cos\Delta\lambda$$

$$\left|proj_{\mathbf{z}}\hat{\mathbf{r}}_e\right| = \sqrt{1 - \left(\mathbf{z}^{\mathsf{T}}\hat{\mathbf{r}}_e\right)^2} = \sqrt{1 - \sin^2\left(DEC_{EI}\right)} = \cos DEC_{EI}$$

$$\left|proj_{\mathbf{z}} - \hat{\mathbf{r}}_a\right| = \cos DEC_{\mathbf{r}_a}$$

$$\left(proj_{\mathbf{z}}\hat{\mathbf{r}}_e\right)^{\mathsf{T}} \left(proj_{\mathbf{z}} - \hat{\mathbf{r}}_a\right) = -\hat{\mathbf{r}}_e^{\mathsf{T}} proj_{\mathbf{z}}\hat{\mathbf{r}}_a$$

$$\begin{aligned}
\left(proj_{\mathbf{z}}\hat{\mathbf{r}}_e\right)^{\mathsf{T}} \left(proj_{\mathbf{z}} - \hat{\mathbf{r}}_a\right) &= -\hat{\mathbf{r}}_e^{\mathsf{T}} proj_{\mathbf{z}}\hat{\mathbf{r}}_a \\
&= -\hat{\mathbf{r}}_e^{\mathsf{T}} \left[\hat{\mathbf{r}}_a - \left(\mathbf{z}^{\mathsf{T}}\hat{\mathbf{r}}_a\right)\mathbf{z}\right] = -\hat{\mathbf{r}}_e^{\mathsf{T}}\hat{\mathbf{r}}_a + \left(\mathbf{z}^{\mathsf{T}}\hat{\mathbf{r}}_e\right)\left(\mathbf{z}^{\mathsf{T}}\hat{\mathbf{r}}_a\right) \\
&= -\cos\Delta\theta + \sin DEC_{EI} \sin DEC_{\mathbf{r}_a}
\end{aligned}$$

$$\cos\Delta\lambda = \frac{-\cos\Delta\theta + \sin DEC_{EI}\sin DEC_{\mathbf{r}_a}}{\cos DEC_{EI}\cos DEC_{\mathbf{r}_a}}$$

So if a specific $\Delta\lambda$ is required, we can compute DEC_{EI} associated with it:

$$\begin{aligned}
\sin DEC_{EI}\sin DEC_{\mathbf{r}_a} - \cos DEC_{EI}\cos DEC_{\mathbf{r}_a}\cos\Delta\lambda &= \cos\Delta\theta \\
\sin(DEC_{EI} + \epsilon)\sqrt{\sin^2 DEC_{\mathbf{r}_a} + \cos^2 DEC_{\mathbf{r}_a}\cos^2\Delta\lambda} &= \cos\Delta\theta
\end{aligned}$$

$$\epsilon = \arctan\left(\frac{-\cos DEC_{\mathbf{r}_a}\cos\Delta\lambda}{\sin DEC_{\mathbf{r}_a}}\right) = \arctan\left(\frac{-\cos\Delta\lambda}{\tan DEC_{\mathbf{r}_a}}\right)$$

Now we can obtain the two solutions for DEC_{EI}:

$$DEC_{EI_1} = \arcsin\left(\frac{\cos\Delta\theta}{\sqrt{\sin^2 DEC_{\mathbf{r}_a} + \cos^2 DEC_{\mathbf{r}_a}\cos^2\Delta\lambda}}\right) - \epsilon$$

$$DEC_{EI_2} = \pi - \arcsin\left(\frac{\cos\Delta\theta}{\sqrt{\sin^2 DEC_{\mathbf{r}_a} + \cos^2 DEC_{\mathbf{r}_a}\cos^2\Delta\lambda}}\right) - \epsilon$$

$$\sin\left(DEC_{EI_1} + \epsilon\right) = \sin\left(\pi - DEC_{EI_1} - \epsilon\right) = \sin\left(DEC_{EI_2} + \epsilon\right)$$

$$\pi - DEC_{EI_1} - \epsilon = DEC_{EI_2} + \epsilon \rightarrow DEC_{EI_2} = \pi - DEC_{EI_1} - 2\epsilon$$

Alternative formulation!!!!!!!!!!!!

We can also find the relationship between A a required longitude at entry interface:

We can calculate first the difference between the longitude of the center of the entry interface points $c_s\mathbf{r}_a$ and the longitude at any entry interface point $\mathbf{r}_e$

$$\left(proj_{\mathbf{z}}\hat{\mathbf{r}}_e\right)^{\mathsf{T}}\left(proj_{\mathbf{z}}c_s\hat{\mathbf{r}}_a\right) = \left|proj_{\mathbf{z}}\hat{\mathbf{r}}_e\right|\left|proj_{\mathbf{z}}c_s\hat{\mathbf{r}}_a\right|\cos\Delta\lambda$$

$$\left|proj_{\mathbf{z}}\hat{\mathbf{r}}_e\right| = \sqrt{1 - \left(\mathbf{z}^{\mathsf{T}}\hat{\mathbf{r}}_e\right)^2} = \sqrt{1 - \sin^2\left(DEC_{EI}\right)} = \cos DEC_{EI}$$

$$\left|proj_{\mathbf{z}}c_s\hat{\mathbf{r}}_a\right| = \cos DEC_{\mathbf{r}_a}$$

$$\left(proj_{\mathbf{z}}\hat{\mathbf{r}}_e\right)^{\mathsf{T}}\left(proj_{\mathbf{z}}c_s\hat{\mathbf{r}}_a\right) = c_s\hat{\mathbf{r}}_e^{\mathsf{T}} proj_{\mathbf{z}}\hat{\mathbf{r}}_a$$

$$\begin{aligned}
\left(proj_{\mathbf{z}}\hat{\mathbf{r}}_e\right)^{\mathsf{T}}\left(proj_{\mathbf{z}}c_s\hat{\mathbf{r}}_a\right) &= c_s\hat{\mathbf{r}}_e^{\mathsf{T}} proj_{\mathbf{z}}\hat{\mathbf{r}}_a \\
&= c_s\hat{\mathbf{r}}_e^{\mathsf{T}}\left[\hat{\mathbf{r}}_a - \left(\mathbf{z}^{\mathsf{T}}\hat{\mathbf{r}}_a\right)\mathbf{z}\right] = c_s\hat{\mathbf{r}}_e^{\mathsf{T}}\hat{\mathbf{r}}_a - c_s\left(\mathbf{z}^{\mathsf{T}}\hat{\mathbf{r}}_e\right)\left(\mathbf{z}^{\mathsf{T}}\hat{\mathbf{r}}_a\right) \\
&= c_s\cos\Delta\theta - c_s\sin DEC_{EI}\sin DEC_{\mathbf{r}_a} = -c_s\left(-\cos\Delta\theta + \sin DEC_{EI}\sin DEC_{\mathbf{r}_a}\right)
\end{aligned}$$

$$\cos\Delta\lambda = \frac{-c_s\left(-\cos\Delta\theta + \sin DEC_{EI}\sin DEC_{\mathbf{r}_a}\right)}{\cos DEC_{EI}\cos DEC_{\mathbf{r}_a}}$$

So if a specific $\Delta\lambda$ is required, we can compute DEC_{EI} associated with it:

$$\cos DEC_{EI} \cos DEC_{\mathbf{r}_a} \cos \Delta\lambda + c_s \sin DEC_{EI} \sin DEC_{\mathbf{r}_a} = c_s \cos \Delta\theta$$

$$c_s \cos DEC_{EI} \cos DEC_{\mathbf{r}_a} \cos \Delta\lambda + \sin DEC_{EI} \sin DEC_{\mathbf{r}_a} = \cos \Delta\theta$$

$$\sin DEC_{EI} \sin DEC_{\mathbf{r}_a} + c_s \cos DEC_{EI} \cos DEC_{\mathbf{r}_a} \cos \Delta\lambda = \cos \Delta\theta$$

$$\sin(DEC_{EI} + \epsilon)\sqrt{\sin^2 DEC_{\mathbf{r}_a} + \cos^2 DEC_{\mathbf{r}_a} \cos^2 \Delta\lambda} = \cos \Delta\theta$$

$$\epsilon = \arctan\left(\frac{c_s \cos DEC_{\mathbf{r}_a} \cos \Delta\lambda}{\sin DEC_{\mathbf{r}_a}}\right) = \arctan\left(\frac{c_s \cos \Delta\lambda}{\tan DEC_{\mathbf{r}_a}}\right)$$

Now we can obtain the two solutions for DEC_{EI}:

$$DEC_{EI_1} = \arcsin\left(\frac{\cos \Delta\theta}{\sqrt{\sin^2 DEC_{\mathbf{r}_a} + \cos^2 DEC_{\mathbf{r}_a} \cos^2 \Delta\lambda}}\right) - \epsilon$$

$$DEC_{EI_2} = \pi - \arcsin\left(\frac{\cos \Delta\theta}{\sqrt{\sin^2 DEC_{\mathbf{r}_a} + \cos^2 DEC_{\mathbf{r}_a} \cos^2 \Delta\lambda}}\right) - \epsilon$$

$$\sin(DEC_{EI_1} + \epsilon) = \sin(\pi - DEC_{EI_1} - \epsilon) = \sin(DEC_{EI_2} + \epsilon)$$

$$\pi - DEC_{EI_1} - \epsilon = DEC_{EI_2} + \epsilon \rightarrow DEC_{EI_2} = \pi - DEC_{EI_1} - 2\epsilon$$

Once we have the DEC_{EI} associated with the required $\Delta\lambda$, we can compute the angle A using eq. 19. Therefore in order to compute A such that the abort trajectory has a specific longitude at entry interface the following procedure should carried out:

1. Compute the longitude associate with the antipode of $\mathbf{r}_a$
2. Compute $\Delta\lambda = \lambda_e - \lambda_{-\mathbf{r}_a}$
3. Compute the latitude(s) DEC_{EI} associated with this value of $\Delta\lambda$
4. Compute the angle(s) A assciated with DEC_{EI}

We can also calculate the angle A that can generates the maximum $\Delta\lambda$. The maximum $\Delta\lambda$ occurs when a meridian is tangent to the projection of the circle generated by $\hat{\mathbf{r}}_a$ (see Fig. HACER LA FIGURA!!!!!!!):

$$\left(proj_{\hat{\mathbf{r}}_e}\mathbf{z}\right)^{\mathsf{T}}\left(proj_{\hat{\mathbf{r}}_a}\hat{\mathbf{r}}_e\right) = 0$$

$$\begin{aligned}
proj_{c_s\hat{\mathbf{r}}_a}\hat{\mathbf{r}}_e &= \\
&= \left[\hat{\mathbf{r}}_e - \left(c_s\hat{\mathbf{r}}_a^{\mathsf{T}}\hat{\mathbf{r}}_e\right)c_s\hat{\mathbf{r}}_a\right] = \left[\hat{\mathbf{r}}_e - \cos\Delta\theta\hat{\mathbf{r}}_a\right] \\
&= f_s\sin\Delta\theta\left(\cos A\hat{\mathbf{u}} + \sin A\hat{\mathbf{v}}\right)
\end{aligned}$$

$$\mathbf{z}^{\mathsf{T}}\hat{\mathbf{r}}_e = \sin DEC_{\mathbf{r}_a}\cos\Delta\theta + f_s\sin\Delta\theta\cos DEC_{\mathbf{r}_a}\cos A$$

$$\begin{aligned}
\left[\mathbf{z} - \left(\mathbf{z}^{\mathsf{T}}\hat{\mathbf{r}}_e\right)\hat{\mathbf{r}}_e\right]^{\mathsf{T}}\left[f_s\sin\Delta\theta\left(\cos A\hat{\mathbf{u}} + \sin A\hat{\mathbf{v}}\right)\right] &= 0 \\
\left[\mathbf{z} - \left(\mathbf{z}^{\mathsf{T}}\hat{\mathbf{r}}_e\right)\hat{\mathbf{r}}_e\right]^{\mathsf{T}}\left(\cos A\hat{\mathbf{u}} + \sin A\hat{\mathbf{v}}\right) &= 0 \text{ since } f_s\sin\Delta\theta \neq 0\ (\Delta\theta \neq \\
\cos DEC_{\mathbf{r}_a}\cos A - \left(\mathbf{z}^{\mathsf{T}}\hat{\mathbf{r}}_e\right)\left(f_s\sin\Delta\theta\cos^2 A + f_s\sin\Delta\theta\sin^2 A\right) &= 0 \\
\cos DEC_{\mathbf{r}_a}\cos A - \left(\sin DEC_{\mathbf{r}_a}\cos\Delta\theta + f_s\sin\Delta\theta\cos DEC_{\mathbf{r}_a}\cos A\right)f_s\sin\Delta\theta &= 0 \\
\cos DEC_{\mathbf{r}_a}\cos A - \sin DEC_{\mathbf{r}_a}\cos\Delta\theta f_s\sin\Delta\theta - \sin^2\Delta\theta\cos DEC_{\mathbf{r}_a}\cos A &= 0 \\
\cos DEC_{\mathbf{r}_a}\cos A\left(1 - \sin^2\Delta\theta\right) - \sin DEC_{\mathbf{r}_a}\cos\Delta\theta f_s\sin\Delta\theta &= 0
\end{aligned}$$

$$\cos A = \frac{f_s\sin DEC_{\mathbf{r}_a}\cos\Delta\theta\sin\Delta\theta}{\cos DEC_{\mathbf{r}_a}\left(1 - \sin^2\Delta\theta\right)} = f_s\tan DEC_{\mathbf{r}_a}\tan\Delta\theta$$

If there are no solutions to the eq. it will mean that the maximum $\Delta\lambda$is reached when $A = A_{min}$ or $A = A_{max}$.

If a range of longitudes is specified, we can calculate the range(s) of the angle A associated with it.

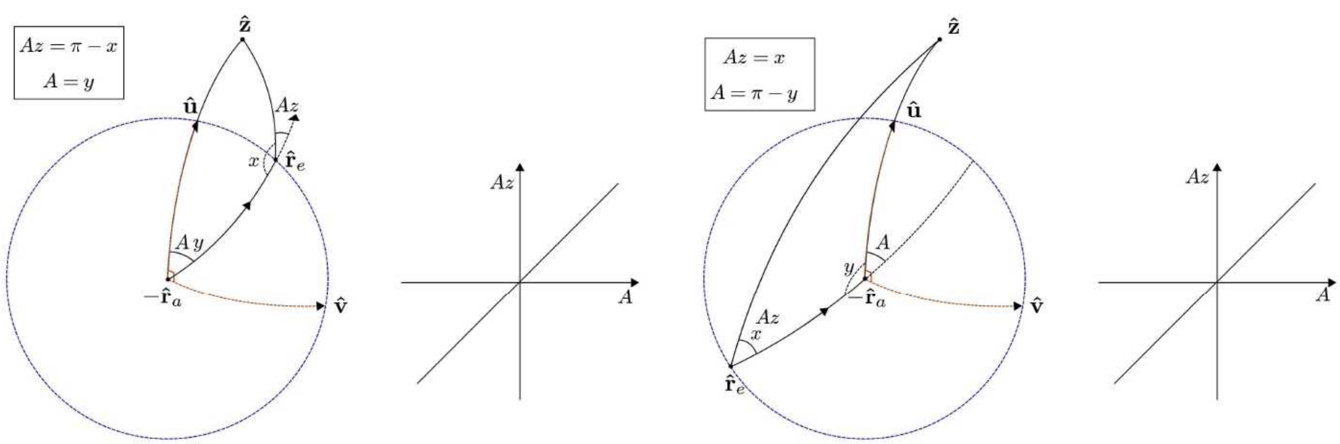
NOTE ON AZIMUTHS

In general, we can classify all the cases according to three criteria: family, EI plane position and orientation of the $\hat{\mathbf{r}}_\mathbf{e}$ in the $(\hat{\mathbf{u}} - \hat{\mathbf{v}})$ plane.

- As it is mentioned above, the family (1 or 2) criteria can obtained by using $\sin\Delta\theta$
- EI plane position. For the same velocity vector, the azimuth value will change its sign if the EI plane is on the side of the antipode or not. The criteria we will use is the sign of $\cos\Delta\theta$. If this sign is negative the EI plane is on the side of the antipode (see eq. 12) and viceversa.

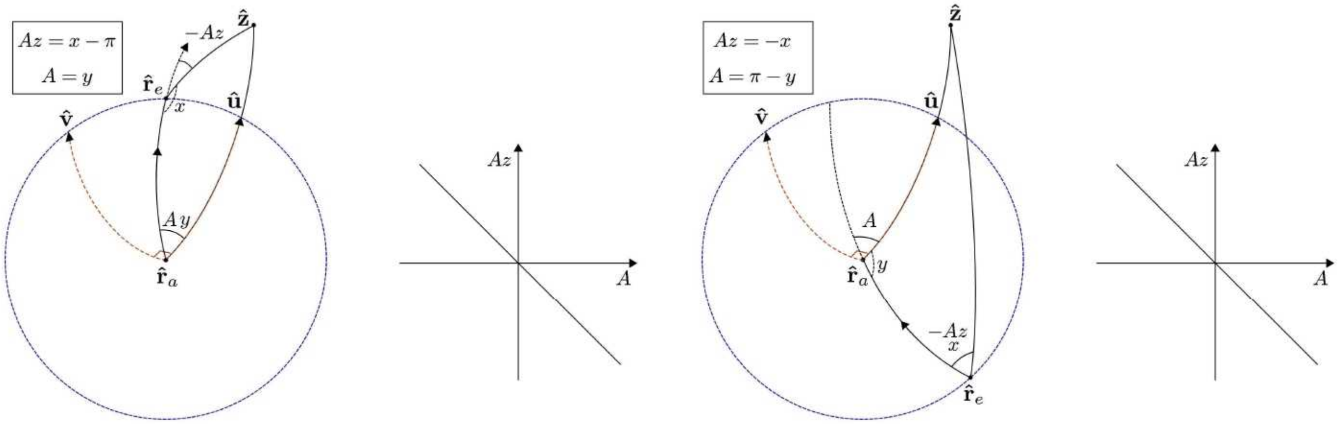
Table 2. Azimuth case classification

Family	EI plane position	$\hat{\mathbf{r}}_\mathbf{e}$ in the $(\hat{\mathbf{u}} - \hat{\mathbf{v}})$ plane		Family	EI plane position		
$\sin \Delta\theta$	$\cos \Delta\theta$	$f_s \sin \Delta\theta$	Case	$\sin \Delta\theta$	$\cos \Delta\theta$	f_s	Case
< 0	< 0	< 0	G	< 0	< 0	> 0	G
< 0	< 0	> 0	C	< 0	< 0	< 0	C
< 0	> 0	< 0	F	< 0	> 0	> 0	F
< 0	> 0	> 0	B	< 0	> 0	< 0	B
> 0	< 0	< 0	E	> 0	< 0	< 0	E
> 0	< 0	> 0	A	> 0	< 0	> 0	A
> 0	> 0	< 0	H	> 0	> 0	< 0	H
> 0	> 0	> 0	D	> 0	> 0	> 0	D



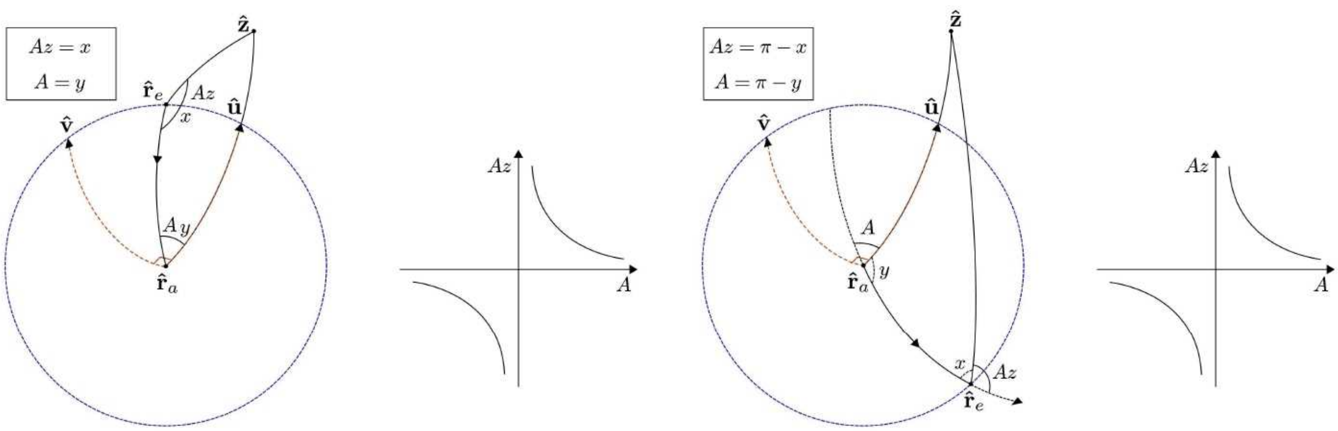
Case A Case G

Figure 2. Azimuth type 1



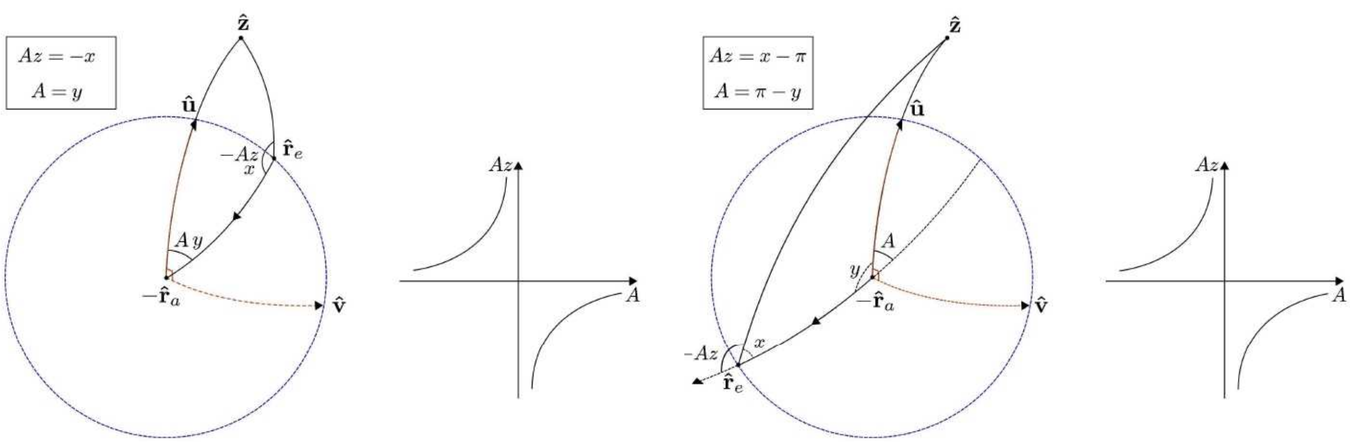
Case B Case H

Figure 3. Azimuth type 2



Case D Case F

Figure 4. Azimuth type 3



Case C Case E

Figure 5. Azimuth type 4

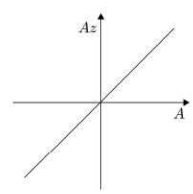
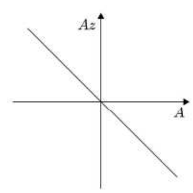
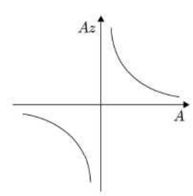
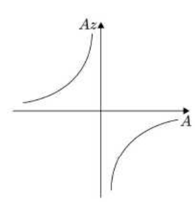
- The orientation of $\hat{\mathbf{r}}_\mathbf{e}$ in the $(\hat{\mathbf{u}}-\hat{\mathbf{v}})$ plane can be easily obtained using $f_s \sin\Delta\theta$. If $f_s \sin\Delta\theta > 0$ ....EXPLAIN!!!!!

Some cases are equivalent, for example: cases A and or B and H (see above figure). In some cases Az changes continuously when the angle A is varied, in some other cases there is a discontinuity when $A = 0$. Also, the signs of A and Az can be the same or opposite. In this way,

$$\begin{array}{lll} \text{if } f_s > 0 & \rightarrow & sign(Az) = sign(A) \\ \text{if } f_s < 0 & \rightarrow & sign(Az) \neq sign(A) \end{array} \tag{23}$$

$$\begin{array}{lll} \text{if } f_s \cos\theta > 0 & \rightarrow & \text{discontinuity at } A = 0 \\ \text{if } f_s \cos\theta < 0 & \rightarrow & \text{continuity at } A = 0 \end{array}$$

Finally, we can classify all of the azimuth types according to the following table:

f_s	$\cos\Delta\theta$	Azimuth type	
> 0	< 0	1	
< 0	> 0	2	
> 0	> 0	3	
< 0	< 0	4	

Note on the extremes of the $\arccos$ function

$$y = \arccos\left(f(a)\right)$$

$$y' = \frac{-f'(a)}{\sqrt{1-f(a)^2}}$$

$$y'' = -\frac{f''(a)\left(1-f(a)^2\right)^{1/2} - \frac{1}{2}\left(1-f(a)^2\right)^{-1/2}\left(-2f(a)f'(a)\right)}{1-f(a)^2}$$

if $y' = 0 \Leftrightarrow f'(a) = 0$, therefore

$$y''\,\big|_{y'=0} = -\,f''(a)\left(1-f(a)^2\right)^{-1/2}$$

$$\begin{array}{lll} \text{if } f''(a) > 0 & \rightarrow & y'' < 0 \rightarrow \text{maximum} \\ \text{if } f''(a) < 0 & \rightarrow & y'' > 0 \rightarrow \text{minimum} \end{array} \tag{24}$$

Calculating the maximum or minimum azimuth

From the law of cosines we can get:

$$\cos l = \cos a \cos r + \sin a \sin r \cos x$$

$$\cos x = \frac{\cos l - \cos a \cos r}{\sin a \sin r}$$

where:

$r \in [0, \pi]$: angle between the entry interface position and the antipode
$l \in [0, \pi]$: angle between the north (z-axis) and the antipode
$a \in [0, \pi]$: angle between the north (z-axis) and the entry interface position

From the results of the previous section we can now calculate the maximum or minimum x. We only need to calculate the conditions for $\cos x$. Since r and l are constant, $\cos x$ will depend only on a,

$$\frac{d\cos x}{da} = \frac{\sin^2 a \cos r \sin r - \cos a \sin r\left(\cos l - \cos a \cos r\right)}{\left(\sin a \sin r\right)^2} = \frac{\cos r - \cos a \cos l}{\sin^2 a \sin r}$$

therefore

$$\frac{d\cos x}{da} = 0 \rightarrow \cos r - \cos a \cos l = 0 \rightarrow \cos a = \frac{\cos r}{\cos l}$$

We have a necessary condition for finding the extreme:

if we define $c_s = \frac{\cos \Delta\theta}{|\cos \Delta\theta|}$

$$\begin{aligned}
\cos l &= \mathbf{z}^{\mathsf{T}} \left(c_s \hat{\mathbf{r}}_a\right) = c_s \sin DEC_{\mathbf{r}_a} \\
\sin l &= \left|\mathbf{z} \times \left(c_s \hat{\mathbf{r}}_a\right)\right| = \cos DEC_{\mathbf{r}_a} \\
\cos r &= \hat{\mathbf{r}}_e^{\mathsf{T}} \left(c_s \hat{\mathbf{r}}_a\right) = |\cos \Delta\theta| \\
\sin r &= \left|\hat{\mathbf{r}}_e \times \left(c_s \hat{\mathbf{r}}_a\right)\right| = |\sin \Delta\theta|
\end{aligned}$$

$$\tan l = \frac{\cos DEC_{\mathbf{r}_a}}{c_s \sin DEC_{\mathbf{r}_a}} \rightarrow \begin{cases} \cot DEC_{\mathbf{r}_a} \rightarrow l = \frac{\pi}{2} - DEC_{\mathbf{r}_a} & \text{if } c_s > 0 \\ -\cot DEC_{\mathbf{r}_a} \rightarrow l = \frac{\pi}{2} + DEC_{\mathbf{r}_a} & \text{if } c_s < 0 \end{cases}$$

$$\tan r = \frac{|\sin \Delta\theta|}{|\cos \Delta\theta|}$$

$$\left|\frac{\cos r}{\cos l}\right| \leq 1 \rightarrow \left|\frac{\cos \Delta\theta}{\sin DEC_{\mathbf{r}_a}}\right| \leq 1$$

$$\frac{d^2 \cos x}{da^2} = \frac{\sin a \cos l \sin^2 a \sin r - 2\left(\cos r - \cos a \cos l\right) \sin a \cos a \sin r}{\left(\sin^2 a \sin r\right)^2}$$

$$\left.\frac{d^2 \cos x}{da^2}\right|_{\frac{d \cos x}{da}=0} = \frac{\sin^3 a \cos l \sin r}{\left(\sin^2 a \sin r\right)^2} = \frac{\sin a \cos l}{\sin r}$$

If we take into account:

$$\sin r = |\sin \Delta\theta| \geq 0$$

$$\sin a \geq 0$$

$$\cos l = c_s \sin DEC_{\mathbf{r}_a}$$

From eq. 24 we can obtain a criteria for maximum or minimum azimuth

$$\left.\frac{d^2 \cos x}{da^2}\right|_{\frac{d \cos x}{da}=0} \begin{cases} \geq 0 & \text{if } c_s \sin DEC_{\mathbf{r}_a} \geq 0 \rightarrow x_{max} \\ < 0 & \text{if } c_s \sin DEC_{\mathbf{r}_a} < 0 \rightarrow x_{min} \end{cases}$$

We can find now the maximum or minimum x. First,

$$\sin a > 0 \rightarrow \sqrt{1-\cos^2 a} = \sqrt{1-\left(\frac{\cos r}{\cos l}\right)^2} = \frac{\sqrt{\cos^2 l - \cos^2 r}}{|\cos l|}$$

$$\begin{aligned}
\cos x &= \frac{\cos l - \frac{\cos r}{\cos l}\cos r}{\sin a \sin r} = \frac{|\cos l|\left(\cos^2 l - \cos^2 r\right)}{\sqrt{\cos^2 l - \cos^2 r}\,\sin r \cos l} = \frac{|\cos l|\sqrt{\cos^2 l - \cos^2 r}}{\cos l \sin r} \\
&= \frac{|\sin DEC_{\mathbf{r}_a}|\sqrt{\sin^2 DEC_{\mathbf{r}_a} - \cos^2 \Delta\theta}}{c_s \sin DEC_{\mathbf{r}_a}\,|\sin\Delta\theta|} \\
&= sign\left(\frac{\sqrt{\sin^2 DEC_{\mathbf{r}_a} - \cos^2 \Delta\theta}}{|\sin\Delta\theta|},\ c_s \sin DEC_{\mathbf{r}_a}\right)
\end{aligned}$$

$$x = \arccos\left[sign\left(\frac{\sqrt{\sin^2 DEC_{\mathbf{r}_a} - \cos^2 \Delta\theta}}{|\sin\Delta\theta|},\ c_s \sin DEC_{\mathbf{r}_a}\right)\right]$$

Finally, Azcan be calculated (see diagrams ....explain)

$$Az = \begin{cases} \pi - x & \text{if } c_s \sin\Delta\theta < 0 \\ x & \text{if } c_s \sin\Delta\theta > 0 \end{cases}$$

$$\text{if } c_s \sin\Delta\theta < 0 \rightarrow \begin{cases} x_{min} \rightarrow Az_{max} \\ x_{max} \rightarrow Az_{min} \end{cases}$$

$$\text{if } c_s \sin\Delta\theta > 0 \rightarrow \begin{cases} x_{min} \rightarrow Az_{min} \\ x_{max} \rightarrow Az_{max} \end{cases}$$

Also the angle y associated to Az can be calculated as follows:

$$\cos a = \cos l \cos r + \sin l \sin r \cos y$$

$$\begin{aligned}
\cos y = & \tfrac{\cos a - \cos l \cos r}{\sin l \sin r} \\
= & \frac{\frac{\cos r}{\cos l} - \cos l \cos r}{\sin l \sin r} = \frac{\cos r - \cos^2 l \cos r}{\sin l \sin r \cos l} \\
= & \frac{\cos r \sin l}{\sin r \cos l} = \frac{|\cos\Delta\theta| \cos DEC_{\mathbf{r}_a}}{|\sin\Delta\theta|\, c_s \sin DEC_{\mathbf{r}_a}} = \frac{c_s\,|\cos\Delta\theta|}{|\sin\Delta\theta| \tan DEC_{\mathbf{r}_a}} = \frac{\cos\Delta\theta}{|\sin\Delta\theta| \tan DEC_{\mathbf{r}_a}}
\end{aligned}$$

From eq. 23, therefore we can calculate A as:

$$y = \arccos\left(\frac{\cos\Delta\theta}{|\sin\Delta\theta| \tan DEC_{\mathbf{r}_a}}\right)$$

The magnitude and sign of Acan be calculated (see diagrams ....explain ...the sign Acan be determined by the signs of Az and f_s)

$$A = \begin{cases} sign(\pi - y,\ f_s Az) & \text{if } f_s \sin\Delta\theta < 0 \\ sign(y,\ f_s Az) & \text{if } f_s \sin\Delta\theta > 0 \end{cases}$$

The above derivations calculate only $Az \in [0, \pi]$ if we are interested in retrograde solutions $(Az',\ A')$, all we have to do is:

$$\begin{aligned} Az' &= -Az \\ A' &= -A \end{aligned}$$

$$\begin{aligned} Az_{min} &\rightarrow Az'_{max} \\ Az_{max} &\rightarrow Az'_{min} \end{aligned}$$

CALCULATING V_E GIVEN THE TOF

$$\begin{aligned} e\cos E_e &= 1 - \frac{r_e}{a} \\ e\sin E_e &= \frac{\mathbf{r}_e\mathbf{v}_e}{\sqrt{\mu a}} = \frac{r_e v_e \sin\gamma_e}{\sqrt{\mu a}} \end{aligned}$$

Note:

$$\frac{a r_e v_e}{\sqrt{\mu a}} = \frac{a r_e\sqrt{\frac{2\mu}{r_e} - \frac{\mu}{a}}}{\sqrt{\mu a}} = \frac{\sqrt{2\mu a^2 r_e - \mu a r_e^2}}{\sqrt{\mu a}} = \sqrt{2ar_e - r_e^2} = \sqrt{r_e\,(2a - r_e)}$$

$$e = \frac{p}{r_p} - 1 = \frac{\frac{r_p(2a-r_p)}{a}}{r_p} - 1 = \frac{(2a - r_p)}{a} - 1 = \frac{a - r_p}{a}$$

$$\sin E_e = \frac{r_e v_e \sin\gamma_e}{e\sqrt{\mu a}} = \frac{r_e v_e \sin\gamma_e}{\frac{a-r_p}{a}\sqrt{\mu a}} = \frac{\sin\gamma_e\sqrt{r_e\,(2a - r_e)}}{a - r_p}$$

$$\cos E_e = \frac{a - r_e}{ea} = \frac{a - r_e}{\frac{a-r_p}{a}a} = \frac{a - r_e}{a - r_p}$$

$$\tan E_e = \frac{\frac{r_e v_e \sin\gamma_e}{\sqrt{\mu a}}}{1 - \frac{r_e}{a}} = \frac{\frac{a r_e v_e \sin\gamma_e}{\sqrt{\mu a}}}{a - r_e} = \frac{\sin\gamma_e\sqrt{r_e\,(2a - r_e)}}{a - r_e}$$

$$\cos E_a = \frac{a - r_a}{ea} = \frac{a - r_a}{a - r_p}$$

$$L = r_e - a$$

$$M = -\sin\gamma_e \sqrt{r_e\,(2a - r_e)}$$

$$K = r_a - a$$

$$\sin E_e = \frac{-M}{a - r_p}$$

$$\cos E_e = \frac{-L}{a - r_p} \rightarrow (a - r_p)^2 = \frac{L^2}{1 - \sin^2 E_e} = \frac{L^2}{1 - \frac{M^2}{(a-r_p)^2}} = \frac{(a - r_p)^2 L^2}{(a - r_p)^2 - M^2}$$

$$(a - r_p)^2 = \frac{(a - r_p)^2 L^2}{(a - r_p)^2 - M^2} \rightarrow 1 = \frac{L^2}{(a - r_p)^2 - M^2} \rightarrow (a - r_p)^2 = L^2 + M^2$$

$$\cos^2 E_a = \left(\frac{a - r_a}{a - r_p}\right)^2 = \frac{K^2}{L^2 + M^2}$$

$$\sin E_a = \pm\sqrt{1 - \frac{K^2}{L^2 + M^2}}$$

The sign of $\sin E_a$ will depend on the type of solution: 1 or 2

$$\text{if } E_a \in (-\pi, 0)\ E_a < 0 \rightarrow \sin E_a < 0\,\text{Type1}$$

$$\text{if } E_a \in (0, \pi)\ E_a > 0 \rightarrow \sin E_a > 0\,\text{Type2}$$

$$\cos E_a = \frac{a - r_a}{a - r_p} = \frac{-K}{\sqrt{L^2 + M^2}}$$

Note:

$$a - r_p \geq 0$$

that's why we get the positive solution of $\sqrt{L^2 + M^2}$

$$\tan E_a = \frac{\pm\sqrt{1 - \frac{K^2}{L^2+M^2}}}{\frac{-K}{\sqrt{L^2+M^2}}} = \pm\frac{\sqrt{L^2 + M^2 - K^2}}{-K}$$