

On the Shock-Response-Spectrum Recursive Algorithm of Kelly and Richman

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1 Introduction

The monograph “Principles and Techniques of Shock Data Analysis” written by Kelly and Richman in 1969 has become a seminal reference on the shock response spectrum (SRS) [1]. Because of its clear physical descriptions and mathematical presentation of the SRS, it has been cited in multiple handbooks on the subject [2,3] and research articles [4–10]. Because of continued interest, two additional versions of the monograph have been published: a second edition by Scavuzzo and Pusey in 1996 [11] and a reprint of the original edition in 2008 [12]. The main purpose of this note is to correct several typographical errors in the manuscript’s presentation of a recursive algorithm for SRS calculations. These errors are consistent across all three editions of the monograph. The secondary purpose of this note is to present a Matlab implementation of the corrected algorithm.

2 Continuous-time solution

The shock response spectrum considers the response of a single degree-of-freedom damped mechanical oscillator to a base excitation. The motion of the oscillator is $y(t)$, and the motion of the base is $x(t)$. This system obeys the following equation of motion.

$$\ddot{y} + 2\zeta\omega_n(\dot{y} - \dot{x}) + \omega_n^2(y - x) = 0 \quad (1)$$

Here, ω_n is the natural frequency of the oscillator, and ζ is the damping ratio. Accelerometer data for the base excitation, however, provides knowledge of \ddot{x} not x or \dot{x} . Therefore, it is convenient to consider the relative motion is $\xi(t) = y - x$ and the relative equation of motion.

$$\ddot{\xi} + 2\zeta\omega_n\dot{\xi} + \omega_n^2\xi = -\ddot{x} \quad (2)$$

For arbitrary initial conditions, ξ_0 and $\dot{\xi}_0$, Eq. (2) has the following solution.

$$\begin{aligned} \xi(t) = & e^{-\zeta\omega_n t} \left[\xi_0 \left(\cos \omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin \omega_d t \right) + \frac{\dot{\xi}_0}{\omega_d} \sin \omega_d t \right] \\ & - \frac{1}{\omega_d} \int_0^t \ddot{x}(\tau) e^{-\zeta\omega_n(t-\tau)} \sin \omega_d(t-\tau) d\tau \end{aligned} \quad (3)$$

Here, $\omega_d = \omega_n \sqrt{1 - \zeta^2}$. Differentiation gives the solution for the relative velocity.

$$\begin{aligned} \dot{\xi}(t) = & e^{-\zeta\omega_n t} \left[-\xi_0 \frac{\omega_n}{\sqrt{1 - \zeta^2}} \sin \omega_d t + \dot{\xi}_0 \left(\cos \omega_d t - \frac{\zeta}{\sqrt{1 - \zeta^2}} \sin \omega_d t \right) \right] \\ & - \int_0^t \ddot{x}(\tau) e^{-\zeta\omega_n(t-\tau)} \left[\cos \omega_d(t-\tau) - \frac{\zeta}{\sqrt{1 - \zeta^2}} \sin \omega_d(t-\tau) \right] d\tau \end{aligned} \quad (4)$$

Finally, the absolute acceleration of the oscillator can be reconstructed using the relative position and velocity.

$$\ddot{y}(t) = \ddot{\xi} + \ddot{x} = -2\zeta\omega_n \dot{\xi}(t) - \omega_n^2 \xi(t) \quad (5)$$

Equations (3) and (4) correspond Eq. (4.21) and (4.27), respectively, in Kelly and Richman; however, they reflect several errors in the original source.

3 Discrete-data approximation

In practice, evaluating Eqs. (3) and (4) requires several considerations. First, the input accelerations are often only available at discrete time instants, and the output accelerations may only be desired at the same instants. The data points are here considered to be separated in time by a regular interval Δt . Next, to avoid evaluating the integrals in Equations (3) and (4) over $\tau = 0 \rightarrow t$ for each instant in time, a recursive algorithm is desired. The solutions for $\xi_k \equiv \xi(t_k)$ and $\dot{\xi}_k \equiv \dot{\xi}(t_k)$ are treated as initial conditions in calculating $\xi_{k+1} \equiv \xi(t_{k+1})$ and $\dot{\xi}_{k+1} \equiv \dot{\xi}(t_{k+1})$, and the integrals only need to be evaluated over $\tau = 0 \rightarrow \Delta t$.

$$\begin{aligned} \xi_{k+1} = & e^{-\zeta\omega_n \Delta t} \left[\xi_k \left(\cos \omega_d \Delta t + \frac{\zeta}{\sqrt{1 - \zeta^2}} \sin \omega_d \Delta t \right) + \frac{\dot{\xi}_k}{\omega_d} \sin \omega_d \Delta t \right] \\ & - \frac{1}{\omega_d} \int_0^{\Delta t} \ddot{x}(t_k + \tau) e^{-\zeta\omega_n(\Delta t - \tau)} \sin \omega_d(\Delta t - \tau) d\tau \end{aligned} \quad (6)$$

$$\begin{aligned} \dot{\xi}_{k+1} = & e^{-\zeta\omega_n \Delta t} \left[-\xi_k \frac{\omega_n}{\sqrt{1 - \zeta^2}} \sin \omega_d \Delta t + \dot{\xi}_k \left(\cos \omega_d \Delta t - \frac{\zeta}{\sqrt{1 - \zeta^2}} \sin \omega_d \Delta t \right) \right] \\ & - \int_0^{\Delta t} \ddot{x}(t_k + \tau) e^{-\zeta\omega_n(\Delta t - \tau)} \left[\cos \omega_d(\Delta t - \tau) - \frac{\zeta}{\sqrt{1 - \zeta^2}} \sin \omega_d(\Delta t - \tau) \right] d\tau \end{aligned} \quad (7)$$

In order to evaluate the integral terms in Eqs. (6) and (7) a continuous approximation of the discrete data needs to be formed. Kelly and Richman considered a parabolic approximation using the $(k-1)$, k , and

$(k + 1)$ data points.

$$\ddot{x}(t_k + \tau) = \ddot{x}_k + S_k \frac{\tau}{\Delta t} + \frac{S_{k-1}^2}{2} \left(\frac{\tau^2}{\Delta t^2} - \frac{\tau}{\Delta t} \right) \quad (8)$$

$$S_k = \ddot{x}_{k+1} - \ddot{x}_k$$

$$S_{k-1}^2 = \ddot{x}_{k+1} - 2\ddot{x}_k + \ddot{x}_{k-1}$$

Given a discrete time-history of accelerometer data, the resulting oscillator acceleration at any instant in time can now be calculated by substituting Eq. (8) into Eqs. (6) and (7), and using the resulting values in Eq. (5). Evaluating Eqs. (6) and (7) requires the calculation of the following Duhamel integrals.

$$\begin{aligned} I_1 &\equiv \int_0^{\Delta t} e^{-\zeta\omega_n(\Delta t - \tau)} \sin \omega_d(\Delta t - \tau) d\tau \\ &= \frac{\sqrt{1 - \zeta^2}}{\omega_n} \left[1 - e^{-\zeta\omega_n\Delta t} \left(\cos \omega_d\Delta t + \frac{\zeta}{\sqrt{1 - \zeta^2}} \sin \omega_d\Delta t \right) \right] \end{aligned} \quad (9)$$

$$\begin{aligned} I_2 &\equiv \int_0^{\Delta t} \tau e^{-\zeta\omega_n(\Delta t - \tau)} \sin \omega_d(\Delta t - \tau) d\tau \\ &= \frac{1}{\omega_n^2} \left\{ \omega_d\Delta t - 2\zeta\sqrt{1 - \zeta^2} + e^{-\zeta\omega_n\Delta t} \left[2\zeta\sqrt{1 - \zeta^2} \cos \omega_d\Delta t - (1 - 2\zeta^2) \sin \omega_d\Delta t \right] \right\} \end{aligned} \quad (10)$$

$$\begin{aligned} I_3 &\equiv \int_0^{\Delta t} \tau^2 e^{-\zeta\omega_n(\Delta t - \tau)} \sin \omega_d(\Delta t - \tau) d\tau \\ &= -\frac{1}{\omega_n^3} \left\{ 4\zeta\omega_d\Delta t + \sqrt{1 - \zeta^2} (2 - 8\zeta^2 - \omega_n^2\Delta t^2) \right. \\ &\quad \left. + e^{-\zeta\omega_n\Delta t} \left[(8\zeta^2 - 2) \sqrt{1 - \zeta^2} \cos \omega_d\Delta t + (8\zeta^2 - 6) \zeta \sin \omega_d\Delta t \right] \right\} \end{aligned} \quad (11)$$

$$\begin{aligned} I_4 &\equiv \int_0^{\Delta t} e^{-\zeta\omega_n(\Delta t - \tau)} \cos \omega_d(\Delta t - \tau) d\tau \\ &= \frac{1}{\omega_n} \left[\zeta - e^{-\zeta\omega_n\Delta t} \left(\zeta \cos \omega_d\Delta t - \sqrt{1 - \zeta^2} \sin \omega_d\Delta t \right) \right] \end{aligned} \quad (12)$$

$$\begin{aligned} I_5 &\equiv \int_0^{\Delta t} \tau e^{-\zeta\omega_n(\Delta t - \tau)} \cos \omega_d(\Delta t - \tau) d\tau \\ &= \frac{1}{\omega_n^2} \left\{ 1 - 2\zeta^2 + \zeta\omega_n\Delta t - e^{-\zeta\omega_n\Delta t} \left[(1 - 2\zeta^2) \cos \omega_d\Delta t + 2\zeta\sqrt{1 - \zeta^2} \sin \omega_d\Delta t \right] \right\} \end{aligned} \quad (13)$$

$$\begin{aligned}
I_6 &\equiv \int_0^{\Delta t} \tau^2 e^{-\zeta \omega_n (\Delta t - \tau)} \cos \omega_d (\Delta t - \tau) d\tau \\
&= \frac{1}{\omega_n^3} \left\{ 2\zeta (4\zeta^2 - 3) + 2(1 - 2\zeta^2) \omega_n \Delta t + \zeta \omega_n^2 \Delta t^2 \right. \\
&\quad \left. - e^{-\zeta \omega_n \Delta t} \left[2\zeta (4\zeta^2 - 3) \cos \omega_d \Delta t + 2\sqrt{1 - \zeta^2} (1 - 4\zeta^2) \sin \omega_d \Delta t \right] \right\}
\end{aligned} \tag{14}$$

Using Eqs. (9-14), the solutions for ξ_{k+1} and $\dot{\xi}_{k+1}$ can now be rewritten as shown.

$$\xi_{k+1} = B_1 \xi_k + B_2 \dot{\xi}_k + B_3 \ddot{\xi}_k + B_4 S_k + B_5 S_{k-1}^2 \tag{15}$$

$$\frac{\dot{\xi}_{k+1}}{\omega_n} = B_6 \xi_k + B_7 \dot{\xi}_k + B_8 \ddot{\xi}_k + B_9 S_k + B_{10} S_{k-1}^2 \tag{16}$$

Here, the B coefficients are defined using Eqs. (6) and (7) and the definitions of I_1 through I_6 .

$$B_1 = e^{-\zeta \omega_n \Delta t} \left(\cos \omega_d \Delta t + \frac{\zeta}{\sqrt{1 - \zeta^2}} \sin \omega_d \Delta t \right) \tag{17}$$

$$B_2 = \frac{e^{-\zeta \omega_n \Delta t}}{\omega_d} \sin \omega_d \Delta t \tag{18}$$

$$B_3 = -\frac{I_1}{\omega_d} \tag{19}$$

$$B_4 = -\frac{I_2}{\omega_d \Delta t} \tag{20}$$

$$B_5 = -\frac{1}{2\omega_d} \left(\frac{I_3}{\Delta t^2} - \frac{I_2}{\Delta t} \right) \tag{21}$$

$$B_6 = -\frac{e^{-\zeta \omega_n \Delta t}}{\sqrt{1 - \zeta^2}} \sin \omega_d \Delta t \tag{22}$$

$$B_7 = \frac{e^{-\zeta \omega_n \Delta t}}{\omega_n} \left(\cos \omega_d \Delta t - \frac{\zeta}{\sqrt{1 - \zeta^2}} \sin \omega_d \Delta t \right) \tag{23}$$

$$B_8 = \frac{\zeta I_1}{\omega_d} - \frac{I_4}{\omega_n} \tag{24}$$

$$B_9 = \frac{\zeta I_2}{\omega_d \Delta t} - \frac{I_5}{\omega_n \Delta t} \tag{25}$$

$$B_{10} = \frac{\zeta}{2\omega_d} \left(\frac{I_3}{\Delta t^2} - \frac{I_2}{\Delta t} \right) - \frac{1}{2\omega_n} \left(\frac{I_6}{\Delta t^2} - \frac{I_5}{\Delta t} \right) \tag{26}$$

Finally, the expressions for these coefficients can be simplified using Eqs. (9-14).

$$B_1 = e^{-\zeta \omega_n \Delta t} \left(\cos \omega_d \Delta t + \frac{\zeta}{\sqrt{1 - \zeta^2}} \sin \omega_d \Delta t \right) \tag{27}$$

$$B_2 = \frac{e^{-\zeta \omega_n \Delta t}}{\omega_d} \sin \omega_d \Delta t \tag{28}$$

$$B_3 = -\frac{1}{\omega_n^2} (1 - B_1) \quad (29)$$

$$B_4 = -\frac{1}{\omega_n^2} \left[1 - \frac{2\zeta}{\omega_n \Delta t} (1 - e^{-\zeta \omega_n \Delta t} \cos \omega_d \Delta t) - \frac{(1 - 2\zeta^2) e^{-\zeta \omega_n \Delta t} \sin \omega_d \Delta t}{\omega_d \Delta t} \right] \quad (30)$$

$$B_5 = -\frac{1}{2\omega_n^2} \left\{ -\frac{4\zeta}{\omega_n \Delta t} - \left[\frac{2(1 - 4\zeta^2)}{\omega_n^2 \Delta t^2} - \frac{2\zeta}{\omega_n \Delta t} \right] (1 - e^{-\zeta \omega_n \Delta t} \cos \omega_d \Delta t) + \left(\frac{1 - 2\zeta^2}{\omega_n \Delta t} + \frac{2\zeta(3 - 4\zeta^2)}{\omega_n^2 \Delta t^2} \right) \frac{e^{-\zeta \omega_n \Delta t} \sin \omega_d \Delta t}{\sqrt{1 - \zeta^2}} \right\} \quad (31)$$

$$B_6 = -\omega_n B_2 \quad (32)$$

$$B_7 = \frac{e^{-\zeta \omega_n \Delta t}}{\omega_n} \left(\cos \omega_d \Delta t - \frac{\zeta}{\sqrt{1 - \zeta^2}} \sin \omega_d \Delta t \right) \quad (33)$$

$$B_8 = -\frac{B_2}{\omega_n} \quad (34)$$

$$B_9 = \frac{B_1 - 1}{\omega_n^3 \Delta t} \quad (35)$$

$$B_{10} = -\frac{1}{2\omega_n^2} \left\{ \frac{2}{\omega_n \Delta t} - \left(\frac{1}{\omega_n \Delta t} + \frac{4\zeta}{\omega_n^2 \Delta t^2} \right) (1 - e^{-\zeta \omega_n \Delta t} \cos \omega_d \Delta t) - \left[\frac{2(1 - 2\zeta^2)}{\omega_n^2 \Delta t^2} - \frac{\zeta}{\omega_n \Delta t} \right] \frac{e^{-\zeta \omega_n \Delta t} \sin \omega_d \Delta t}{\sqrt{1 - \zeta^2}} \right\} \quad (36)$$

Equations (27-36) correspond with Eqs. (6.58-67) in Kelly and Richman and Eqs. (7.38-47) in Scavuzzo and Pusey; however, they reflect negative-sign errors in B_3 , B_4 , and B_{10} in the original source. The corrected SRS algorithm for a sequence of input accelerations and a desired range of natural frequencies can be evaluated using Eqs. (5), (15), (16), and (27-36).

4 Comparison and Results

The corrections described in the preceding paragraphs were verified by comparing the corrected algorithm and the original algorithm to an independent SRS code. The independent code used a piecewise-linear approximation for the base acceleration, as described by Paz [13]. The various algorithms were applied to accelerometer data from the ignition environment of live-fire testing of the Space Shuttle Reusable Solid Rocket Motor (RSRM). Specifically, data was evaluated from the radial channel at station 1479.5 on Technical Evaluation Motor 13. The data was sampled at 10,000 Hz. The acceleration time history is shown in Fig. (1).

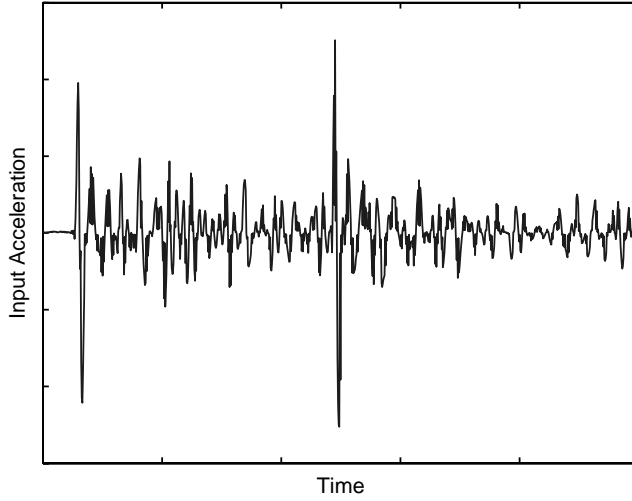


Fig. 1: Time history of sample input acceleration.

The SRS of this acceleration data is shown in Fig. (2) as calculated using three different algorithms. The corrected algorithm of Eqs. (5), (15), (16), and (27-36) are compared with the uncorrected equations from Kelly and Richman as well as the independent code. For each algorithm, a damping ratio of $\zeta = 0.05$ was used, and the peak response was calculated for a range of natural frequencies at one-third octaves up to the Nyquist frequency. The corrected algorithm and the independent code show strong agreement with each other; however, the uncorrected algorithm displays large differences in the high-frequency regime. The MATLAB implementation of the corrected algorithm is available for download at <http://www.eng.auburn.edu/users/sinclaj/SRS/>.

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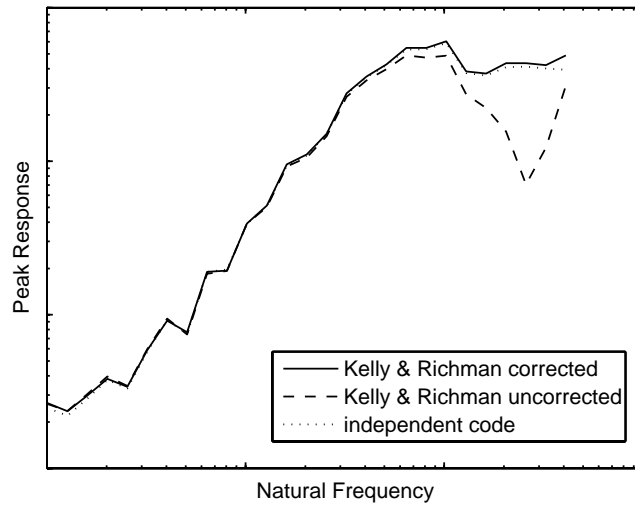


Fig. 2: SRS of input acceleration for $\zeta = 0.05$ using three different algorithms.

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