Error Control With Perfectly Matched Layer or Damping Layer Treatments for Computational Aeroacoustics With Jet Flows

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1   Abstract

In this paper we show by means of numerical experiments that the error intruduced in a numerical domain because of a Perfectly Matched Layer or Damping Layer boundary treatment can be controlled. These experimental demonstrations are for acoustic propagation with the Linearized Euler Equations with both uniform and steady jet flows. The propagating signal is driven by a time harmonic pressure source. Combinations of Perfectly Matched and Damping Layers are used with different damping profiles. These layer and profile combinations allow the relative error introduced by a layer to be kept as small as desired, in principle. Tradeoffs between error and cost are explored.

2   Introduction

Aeroacoustics research is conducted with theoretical analysis, experimental observations, and computation. Computation is being used more often because of improvements in numerical methods and computer resources, and is increasingly valuable as a source of insight and as a tool for design [3, 18, 23, 28]. Artificial boundary treatments are a critical element for Computational AeroAcoustics (CAA), since they are intended to be used to restrict a numerical domain whithout causing unacceptable error in the numerical solution. A poor boundary treatment can produce errors in the numerical simulation that grow to the size of the solution itself, or larger, making the numerical solution of only qualitative or aesthetic interest. A poor boundary treatment that might produce small errors could be very costly, making an accurate numerical solution practically unobtainable. A good boundary treatment is particularly important for simulating time dependant phenomena with detailed physics, where experimants may be difficult because of cost or impossible because of scale and detail, experiments such as for investigating jet noise. The recent reviews of the extensive work done to date on this problem include [2, 5, 6, 13, 15, 16, 24, 27]. In this paper we are concerned with Perfectly Matched Layer (PML) and Damping Layer (DL) treatments for acoustic propagation by means of Linearized Euler Equations (LEE) in two space dimensions with uniform and steady jet flows. A PML boundary treatment is designed for this system by ensuring that the plane wave solutions for the LEE in the numerical domain perfectly match the plane wave solutions for the adjacent PML domain along the interface between them.

Control of the error introduced back into the numerical domain by the PML is attempted by a combination of numerical domain filtering and interface conditions, and the PML domain damping terms, damping rates and profiles, layer width, and outer boundary treatment.

In [11] we compared three PML treatments for this problem. Hagstrom [14] has used transform methods to analytically formulate a PML boundary treatment (PML A) by operator approximation for this system, which has also been further analysed by Motamed [19] and Fröling [4]. Since PML A has been developed with linear analysis for constant coefficient problems, it is not readily apparent how it can be adapted to problems with nonuniform base flows, or to fully nonlinear problems. Karni [17] has used eigensystem analysis to develop a PML treatment (PML B) that use directional damping in an artificial boundary layer. Karni [17] has observed that the damping rate need not be the same for the different eigenfunctions of the hyperbolic system, but that the damping terms should not alter the eigenvectors of the system with respect to the direction of damping. In [11] we showed that a Thompson like characteristic analysis [25, 26] can be used to easily obtain a general damping layer formulation that does not alter the eigenvectors of the Hyperbolic system. The eigensystem analysis that Karni has used could be applied locally, and is not restricted to constant coefficient linear problems. The familiar DL or sponge zone treatment does not alter the eigenvectors of the system, and the damping profile can be chosen to impose a smooth interface between the numerical domain and the sponge zone or damping layer. A DL boundary treatment has been recently discussed in [21], where the damping term $\sigma\vec{U}$ is called a relaxation source term, and the damping coefficient $\sigma = 1/\tau$, where τ is a relaxation time. Bodony [1] has also recently given an analysis of sponge zones for computational fluid dynamics. We have called this familiar, simple, and easily generalized treatment PML C, even though it is not properly a PML.

In [11] we showed that PML A and PML B were reasonably similar in performance, with both producing disturbances that are one to three orders of magnitude smaller than those produced by PML C for similar levels of damping. In this paper we consider both PML B and C treatments by themselves, and we introduce and test a combination of PML B and PML C that is intended to combine the best features of both. This combined PML treatment will be called PML BC. PML BC has PML B terms predominate near the interface, and uses a smooth transition away from the interface to an outer damping treatment that is purely PML C. PML BC is essentially the same as a PML B with different damping profiles for each directional eigenvector, and is intended to incorporate the best features of both separate PML treatments, the accurate and nondistorting interface of PML B with the nonreflecting omnidirectional damping of PML C near the outer PML boundary. Comparisons will be made of the relative errors for combinations of damping profiles and layer widths.

3 PML/DL for the LEE with Uniform Base Flows

The nondimensionalised Linearized Euler Equations (LEE) in two space dimensions with a uniform base flow $(u_b, v_b) = (M_x, M_y)$ and a source can be written for the perturbation

pressure and velocity $\vec{U} = (p, u, v)^T$ as

$$\frac{\partial \vec{U}}{\partial t} + A_c \frac{\partial \vec{U}}{\partial x} + B_c \frac{\partial \vec{U}}{\partial y} = \vec{G},$$

where A_c and B_c are the constant matrices

$$A_c = \begin{pmatrix} M_x & 1 & 0 \\ 1 & M_x & 0 \\ 0 & 0 & M_x \end{pmatrix}, \quad \text{and} \quad B_c = \begin{pmatrix} M_y & 0 & 1 \\ 0 & M_y & 0 \\ 1 & 0 & M_y \end{pmatrix}.$$

In our numerical experiments we always use the pressure source $\vec{G} = (g, 0, 0)^T$, with

$$g(x, y, t) = 0.01 \sin[2\pi t] \exp[-36(x^2 + y^2)].$$

This compact radiating source propagates a time harmonic pressure disturbance with maximum values that are $O[10^{-4}]$. These equations apply in the numerical domain

$$\Omega_N = \{(x, y) \in [x_L, x_R] \times [y_B, y_T]\},$$

where we are concerned with obtaining accurate results. The numerical domain is surrounded with a combination of PML and DL domains, which we take to be Cartesian with linear boundaries. The total combined domain is

$$\Omega = \{(x, y) \in [x_L - w_L, x_R + w_R] \times [y_B - w_B, y_T + w_T]\},$$

where w_L and w_R are the PML/DL domain widths on the left and right, and w_B and w_T are the widths on the bottom and top. The PML/DL or damping domain is just

$$\Omega_D = \Omega - \Omega_N.$$

We will use the homogeneous LEE in the PML/DL domain Ω_D, since the source term g is virtually zero outside of the numerical domain Ω_N. A source in Ω_N near the interface with Ω_D could be accomodated merely by including the source in the damping domain equations.

For this constant coefficient case, a PML B treatment [17, 11] perpendicular to the x axis is

$$\frac{\partial \vec{U}}{\partial t} + A_c \frac{\partial \vec{U}}{\partial x} + B_c \frac{\partial \vec{U}}{\partial y} + \delta\sigma(x) A_c \vec{U} = 0,$$

where $\sigma(x)$ is the damping profile, δ the damping scale, and A_c the damping coefficient matrix. Similarly, a PML B treatment perpendicular to the y axis is

$$\frac{\partial \vec{U}}{\partial t} + A_c \frac{\partial \vec{U}}{\partial x} + B_c \frac{\partial \vec{U}}{\partial y} + \delta\sigma(y) B_c \vec{U} = 0.$$

Note that since PML B is derived from a directional eigenvector analysis [17, 11], δ should be positive for a layer on the right, and negative for a layer on the left, and similarly for up and down. A PML C treatment can be written simply as

$$\frac{\partial \vec{U}}{\partial t} + A_c \frac{\partial \vec{U}}{\partial x} + B_c \frac{\partial \vec{U}}{\partial y} + \delta\sigma \vec{U} = 0,$$

where σ would depend on the variable across the layer in either coordinate direction, with $\delta \geq 0$. In any of the four corners, the PML/DL treatments from the two adjacent sides can be combined by simply adding them. For PML B, this additive corner treatment is just

$$\frac{\partial \vec{U}}{\partial t} + A_c \frac{\partial \vec{U}}{\partial x} + B_c \frac{\partial \vec{U}}{\partial y} + \delta \left(\sigma(x) A_c + \sigma(y) B_c \right) \vec{U} = 0,$$

and for PML C it is

$$\frac{\partial \vec{U}}{\partial t} + A_c \frac{\partial \vec{U}}{\partial x} + B_c \frac{\partial \vec{U}}{\partial y} + \delta \left(\sigma(x) + \sigma(y) \right) \vec{U} = 0.$$

A more general approach to corners for PML C is to simply make the damping profile multidimensional, with

$$\frac{\partial \vec{U}}{\partial t} + A_c \frac{\partial \vec{U}}{\partial x} + B_c \frac{\partial \vec{U}}{\partial y} + \delta \sigma(x,y) \vec{U} = 0.$$

The additive corner treatment is then just a simple example of a multidimensional damping profile. In [11] we presented an approach for damping in a corner by blending two PML B treatments from the sides. In this paper we will use additive corner damping for both PML B and C, and the multidimensional corner damping profile for some cases with PML C. There appears to be little difference in the errors produced by the various corner treatments, and each of these three approaches produces no significant extra error than is produced by the PML/DL treatments on the sides.

A Zero Boundary Data (ZBD) boundary condition is used on all outer boundaries of the damping layers, with all data set equal to zero along the entire outer boundary. This is a combined Dirichlet and Neumann condition since we keep and propagate spatial derivative data at each grid point. Our rationale is that we are primarily concerned with the layer treatment, and that if there were a good boundary condition, then the entire layer treatment industry would be unnecessary. In addition, if any signal crosses a damping zone and reaches an outer boundary with ZBD boundary condition, then the boundary condition will create a reflected signal with approximately the same strength. This worst of all backscattering from the outer boundary will show how well the layer treatment is performing. This is particularly significant for PML B which damps data for outgoing eigenfunctions and amplifies data for incoming eigenfunctions. Note that we do not use any filtering to eliminate this effect, but just the damping in the formulation of the PML/DL layer treatment. The backscattering can be reduced by using a better outer boundary treatment.

There are various ways to implement a PML or DL. One approach is to simply view the damping terms as ordinary terms in the system, and treat them the same as any other. For complex algorithms this could lead to significant extra effort. A simple, inexpensive and general approach is described by Romenski, Titarev, and Toro in [21]. For the vector variable $\vec{U}$, consider the evolutionary system

$$\frac{\partial \vec{U}}{\partial t} + \Pi(\vec{U}) = 0,$$

where Π is the propagator for $\vec{U}$, with the associated damped system

$$\frac{\partial \vec{U}}{\partial t} + \Pi(\vec{U}) + \delta\vec{U} = 0,$$

where δ is the damping rate. This is approximated in [21] by

$$\frac{\vec{U}^{n+1} - \vec{U}^n}{\triangle t} = -\Pi(\vec{U}^n) - \delta\vec{U}^{n+1}, \quad \text{or} \quad (1 + \delta\triangle t)\vec{U}^{n+1} = \vec{U}^n - \triangle t\Pi(\vec{U}^n).$$

Any explicit algorithm for an undamped evolutionary system can be written as

$$\vec{U}^{n+1} = P(\vec{U}^n),$$

where the algorithm could be to any order in time, with any number of time substeps. The damping in [21] can now be adopted as

$$(1 + \delta\triangle t)\vec{U}^{n+1} = P(\vec{U}^n), \quad \text{or} \quad \vec{U}^{n+1} = \frac{P(\vec{U}^n)}{(1 + \delta\triangle t)}.$$

If the damping terms are not simple, say in the form

$$\frac{\partial \vec{U}}{\partial t} + \Pi(\vec{U}) + D\vec{U} = 0,$$

where D is a matrix, such as in PML B, then we can write

$$(1 + \triangle tD)\vec{U}^{n+1} = P(\vec{U}^n), \quad \text{or} \quad \vec{U}^{n+1} = (1 + \triangle tD)^{-1}P(\vec{U}^n).$$

The inversion of the matrix damping term can be done locally at each grid point. These formulations can be implemented as post processing step after each time step, or after each time substep, or set of substeps. The damping terms for PML B and PML C have been implemented with algorithmic treatments that are the same as any other terms with all of the details required for full accurate time evolution, and also as this simple post processing step, and there is little difference in the numerical domain solution, but there can be a large difference in the required effort.

We have generally used four damping profiles. Let $\xi = x$ or y, depending on whether or not a damping term is for a layer perpendicular to the x or y axis, and let ξ_I be the location in x or y of the interface between the numerical domain Ω_N and the damping domain Ω_D. We assume that all profiles are set to zero inside the numerical domain. The first damping profile was used for the constant coefficient cases in [11], with

$$\sigma_0(\xi, w) = \frac{(\xi - \xi_I)^{2n}}{(w^2 + (\xi - \xi_I)^2)^n},$$

for all ξ inside the damping domain, where w is a scaling factor. Note that the order of the zero of σ_0 at ξ_I and the smoothness of the interface can be controlled by the choice of n. Note also that σ_0 asymptotes to one, instead of growing indefinitely like typical polynomial

damping profiles. The damping profile is multiplied by the maximum damping rate or amplitude δ. The damping amplitude δ combines with the layer width scale w to control the total damping in the layer. The second damping profile is

$$\sigma_1(\xi, w) = \left(\frac{\xi - \xi_I}{w}\right)^{2n} \left(\frac{2w - (\xi - \xi_I)}{w}\right)^{2n}, \quad \text{for} \ \ |\xi_I| \leq |\xi| \leq |\xi_I| + w,$$

and

$$\sigma_1(\xi, w) = 1, \quad \text{for} \ \ |\xi_I| + w \leq |\xi|.$$

Note that $\sigma_1(\xi_I, w) = 0$, and $\sigma_1(\xi_I + w, w) = 1$, and that the first $2n - 1$ derivatives of σ_1 are 0 at both points. The third damping profile is similar to the second, with

$$\sigma_2(\xi, w) = \left(\frac{\xi - \xi_I}{w}\right)^{2n} \left(\frac{2w - (\xi - \xi_I)}{w}\right)^{2n}, \quad \text{for} \ \ |\xi_I| \leq |\xi| \leq |\xi_I| + 2w,$$

and

$$\sigma_2(\xi, w) = 0, \quad \text{otherwise.}$$

Note that $\sigma_2(\xi_I, w) = 0 = \sigma_2(\xi_I + 2w, w)$, and $\sigma_2(\xi_I + w, w) = 1$, and that the first $2n - 1$ derivatives of σ_2 are 0 at all three points. Damping profile σ_2 is used for PML B, and σ_1 for PML C, with $n = 3$ and additive corner treatments for both profiles. The fourth damping profile is

$$\sigma_3(\xi, w) = \exp[-w^2/\xi^2],$$

where $\xi = |x - x_I|$ on the right or left, $\xi = |y - y_I|$ on the top or bottom, and in the four corners $\xi = \sqrt{(x - x_I)^2 + (y - y_I)^2}$. The multidimensional corner treatment is used with σ_3 by design. Note that σ_3 is infinitely smooth at the interface between the numerical domain and the damping layers.

PML BC is a blend of PML B and PML C using a combination of terms with damping profiles σ_1 and σ_2. For a PML BC perpendicular to the x axis on the right, the damping terms are

$$D(x)\vec{U} = \delta_B \sigma_2(x, w_B) A_c \vec{U} + \delta_C \sigma_1(x, w_C) I \vec{U},$$

where I is the identity matrix. The first term is just PML B for $x_I \leq x \leq x_I + 2w$, and the second term is PML C. The use of PML B near the interface is intended to ensure that lower errors are introduced by the PML, and the final persisting use of PML C is intended to provide omnidirectional damping with no amplification of errors back towards the numerical domain. A slightly modified form of PML BC delays the introduction of the third term for a distance w_g from the interface, and can be written as

$$D(x)\vec{U} = \delta_B \sigma_2(x, w_B) A_c \vec{U} + \delta_C \sigma_1(x - w_g, w_C) \vec{U}.$$

This is the form that we prefer, with appropriate modifications for right and left, and up and down PMLs. Note that there are five parameters, the two damping scales, δ_B and δ_C, the two width scales w_B and w_C, and the "gap" scaling w_g before the third pure PML C term becomes active. This PML BC uses a PML B in the layer or zone near the interface for accuracy, with greater accuracy from a smaller damping coefficient δ_B, and PML C away from the interface to prevent back scattering from the outer boundary, with greater damping from a larger damping coefficient δ_C. The interplay of these five parameters will begin to be explored in the numerical experiments reported below.

4   Numerical Algorithm

All of the experiments that are reported in this paper are conducted with the Hermite/Cauchy-Kovalevsky/Taylor method (see [7], [8], [9], [10], [12], [11]). This method uses two staggered uniform grids, offset by half a grid spacing in both x and y. The first or basic grid is used for initial data and for the numerical solution at each full time step. The second or offset grid is used for the numerical solution at each half time step. The time evolution through a half time step from one grid to the next starts with multidimensional Hermite spatial interpolation of the local data on the current grid stencil, with a tensor interpolant for a rectangular grid. The half time step then proceeds by using the governing equations for Cauchy-Kovalevsky recursion to produce all of the required time derivatives from the spatial derivatives produced by the interpolant. The half time step ends by propagating through a half time step with a Taylor series. This approach is not simply an algorithm, but is actually a method for developing or specifying numerical algorithms. The recursion routines are dependant upon the governing equations, but they can be written independantly from the order of the method. Various methods can be realized for a particular system by simply swapping the interpolation routines, using different stencils and data. This approach has had various names, but in this paper, we will call it the Hermite/Cauchy-Kovalevsky/Taylor method, or the HCKT method. In this paper we use the c2o1 method, which is a HCKT method that uses a four point 2×2 grid cell, and interpolates with a bicubic Hermite interpolant at the cell center, with data for $\{f, f_x, f_y, f_{xy}\}$ at each grid point for any evolving variable f. This is a tensor interpolant which simultaneously and consistently calculates all derivatives in the local bicubic spatial expansion. The spatial interpolant is used as a local initial data surface, and all required time derivatives are computed from the local data surface by means of a Cauchy-Kovalevsky recursion. The new solution data is obtained at the cell center with a Taylor series in time. The time expansion is to fourth order terms, but because of the bicubic tensor interpolant, the overall order is third in space and time, since a fourth order time expansion requires fourth order spatial data which the cubic interpolant does not provide. Note that the local time evolution is exact up to the order of the method. For two dimensional hyperbolic systems, the local data is correctly propagated along characteristic surfaces up to the order of the method. In this sense, the HCKT methods can be said to correctly extend the method of characteristics to multidimensional hyperbolic systems. By using the correct Cauchy-Kovalevsky recursion, this has been done for nonlinear problems as well, in particular for convectively dominated compressible Navier-Stokes equations.

5   Results for the LEE with a Uniform Flow

The numerical experiments in this section are for the LEE with $(p_b, u_b, v_b) = (1, 0.4, 0)$ as the base flow, and with the pressure source

$$\frac{\partial \vec{U}}{\partial t} + A_c \frac{\partial \vec{U}}{\partial x} + B_c \frac{\partial \vec{U}}{\partial y} = \vec{G},$$

where $\vec{G} = (g, 0, 0)^T = (0.01 \sin[2\pi t] \exp[-36(x^2 + y^2)], 0, 0)^T$. The propagated variables are all initially set to zero. All of the results are with the c2o1 HCKT algorithm, which is third

order accurate in space and time. The algorithm is run with a Taylor series time expansion that has fourth order terms, but the third order Hermite interpolant restricts the overall algorithm to third order. The data that we report is

$$\|p - p_R\|_{\infty,\Omega_N} = \max\{|p(x, y, t_F) - p_R(x, y, t_F)| : (x, y) \in \Omega_N\},$$

where t_F is the final time for the simulation being considered, and where p_R is a reference solution produced with no damping of any kind on a Big Enough Domain (BED) so that the solution in Ω_N is undisturbed at t_F by any reflection back from the domain boundary.

Table 1: $\|p - p_R\|_{\infty,\Omega_N}$ at $t = 15$, from PML B with σ_1.

δ_B	$w_B = 1$	$w_B = 3$	$w_B = 5$	$w_B = 7$
10.000	4.2233×10^{-7}	1.5443×10^{-7}	3.7913×10^{-8}	8.7781×10^{-9}
1.000	8.8739×10^{-9}	2.6081×10^{-9}	4.5782×10^{-10}	1.5450×10^{-10}
0.100	7.2401×10^{-10}	1.0766×10^{-10}	2.1359×10^{-11}	8.5276×10^{-12}
0.010	7.3263×10^{-11}	9.2124×10^{-12}	2.4369×10^{-12}	7.8338×10^{-13}
0.001	7.3414×10^{-12}	9.0569×10^{-13}	2.4700×10^{-13}	7.7644×10^{-14}

Note that $\|p_R\|_{\infty,\Omega_N} = O[10^{-4}]$.

Table 2: $\|p - p_R\|_{\infty,\Omega_N}$ at $t = 15$, for PML C with σ_1.

δ_C	$w_C = 1$	$w_C = 3$	$w_C = 5$	$w_C = 7$	$w_C = 9$
10.000	2.5833×10^{-5}	1.6333×10^{-6}	8.2610×10^{-7}	6.9061×10^{-7}	2.9978×10^{-7}
1.000	2.9492×10^{-6}	5.1488×10^{-7}	3.5265×10^{-7}	1.2282×10^{-7}	6.4461×10^{-8}
0.100	2.4761×10^{-7}	1.3017×10^{-7}	4.9640×10^{-8}	2.1429×10^{-8}	8.7373×10^{-9}
0.010	3.3743×10^{-8}	1.3779×10^{-8}	5.7895×10^{-9}	2.3194×10^{-9}	9.0230×10^{-10}
0.001	3.4936×10^{-9}	1.3810×10^{-9}	5.8907×10^{-10}	2.3381×10^{-10}	9.0522×10^{-11}

Note that $\|p_R\|_{\infty,\Omega_N} = O[10^{-4}]$.

The first series of computations are with PML B and PML C by themselves, and are presented in Tables 1 and 2. These calculations are for the numerical domain

$$\Omega_N = [-3, 7] \times [-5, 5], \quad \text{with} \quad t_F = 15,$$

with algorithm parameters $\triangle x = 1/24 = \triangle y$, and $\triangle t = 1/48$. The damping domain layer widths are $w_L = w_R = w_B = w_T = 10$, the σ_1 damping profile and additive corner treatments are used for both PML B and PML C, and the damping terms are fully propagated. Note that the propagation speeds of the wave fronts are 1.4 in the $+x$ direction, 0.6 in the $-x$ direction, and 1 in the $\pm y$ directions, and consequently, that the propagating signal never reaches the outer boundary of Ω_D by $t_F = 15$. The data in Tables 1 and 2 clearly shows that errors introduced by a PML/DL can be controlled, even with a PML/DL on all sides and corners of the numerical domain. The three controls that effect the results in Tables 1 and 2 are the type of layer that is used, the damping amplitude d, and the damping width scale w. For every combination of d and w, PML B produces errors that are between two and three

orders of magnitude smaller than those produced by PML C. The effect of both the damping amplitude and width is linear for both PML B and PML C. The solution scale is $O[10^{-4}]$, so that the relative error is from $O[10^{-1}]$, or ten percent, to $O[10^{-10}]$. Any error that is relatively less than $O[10^{-3}]$ is invisible at the scale of the solution, and this can be readily achieved. It should be noted that if the damping layer widths w_B or w_C are greater than 5, then the effect of the maximum damping profile amplitude is not returned to the numerical domain Ω_N where $\|p - p_R\|_{\infty,\Omega_N}$ is computed. The damping profile can cause an error by its initial slope, with the layer scale and profile type both playing a role, by the maximum damping amplitude, and possibly by higher derivatives from the profile shape. The smallest errors in Tables 1 and 2 are produced by combinations of small damping amplitude and large damping width scales. The total damping is a product of the amplitude and width, so that a small damping amplitude needs a corespondingly wide damping layer to provide a given total damping. The very smallest errors are produced by the very smallest damping amplitudes, and they provide very little damping except over very wide damping layers.

Table 3: $\|p - p_R\|_{\infty,\Omega_N}$ at $t = 25$, for PML BC, with σ_2 and $w_G = 3$.

w_C	$w_B = 1$	$w_B = 3$	$w_B = 5$	$w_B = 7$
1	1.2269×10^{-6}	1.2225×10^{-6}	1.2278×10^{-6}	1.2290×10^{-6}
3	4.1117×10^{-7}	4.1009×10^{-7}	4.0892×10^{-7}	4.1163×10^{-7}
5	1.4603×10^{-7}	1.4482×10^{-7}	1.4424×10^{-7}	1.4563×10^{-7}
7	4.4231×10^{-8}	4.2385×10^{-8}	4.2241×10^{-8}	4.3100×10^{-8}
9	2.0671×10^{-8}	1.9703×10^{-8}	1.9697×10^{-8}	1.9843×10^{-8}

Note that $\|p_R\|_{\infty,\Omega_N} = O[10^{-4}]$.

Table 4: $\|p - p_R\|_{\infty,\Omega_N}$ at $t = 25$, for PML BC, with σ_1 and $w_G = 5$.

w_C	$w_B = 1$	$w_B = 3$	$w_B = 5$	$w_B = 7$
1	6.8972×10^{-7}	6.8669×10^{-7}	6.8458×10^{-7}	6.8825×10^{-7}
3	7.2773×10^{-8}	7.1685×10^{-8}	7.1769×10^{-8}	7.2137×10^{-8}
5	1.6277×10^{-8}	1.6506×10^{-8}	1.6558×10^{-8}	1.6707×10^{-8}
7	5.4811×10^{-9}	4.0522×10^{-9}	4.0495×10^{-9}	4.2073×10^{-9}
9	5.0071×10^{-9}	1.5652×10^{-9}	1.2252×10^{-9}	1.3843×10^{-9}

Note that $\|p_R\|_{\infty,\Omega_N} = O[10^{-4}]$.

The second series of computations uses the combined PML BC damping layer, with results presented in Tables 3 and 4. These calculations are for the numerical domain

$$\Omega_N = [-6, 14] \times [-10, 10], \quad \text{with} \quad t_F = 25.$$

The algorithm parameters are $\triangle x = 1/24 = \triangle y$, and $\triangle t = 1/48$, the damping domain widths are $w_L = w_R = w_B = w_T = 10$, the damping terms are fully propagated, and an additive corner treatment is used. The propagating signal does not return from the outer boundary of the damping domain by the final simulation time $t_F = 25$. Recall that σ_2 is a pulse, so that PML B is only active close to the boundary, and that PML C applies only at

a distance greater than w_G from the damping domain interface. The damping amplitudes $\delta_B = 1$ for the PML B terms and $\delta_C = 1$ for the PML C terms have been chosen as moderate but relatively effective damping amplitudes. Note that the data in both tables shows that the PML BC combination is insensitive to the length scale of the PML B terms. Note also that the PML C terms decrease the effectivness of the PML B terms, and that the PML BC errors are from one to three orders of magnitude larger than the comparable errors for PML B by itself. Comparing the results in Table 3 with Table 2 shows that these PML BC results are slightly better than the PML C results by a factor of about three, and that the PML BC results are worse than the PML B results by from one to three orders of magnitude. This suggests that the PML BC error in these computations is dominated by the effect of the PML C terms. Comparing the results in Table 4 with Table 1 shows that these PML BC results are worse than the PML B results by about an order of magnitude, and that the PML BC results are better than the PML C results by about an order of magnitude. The best performance of PML BC has PML C fully engaged at a distance of $w_g + w_C = 12$ or 14 from the interface. If PML C had been used by itself with $w_g = 0$ and $w_C = 12$ or 14, then the relative advantage of PML BC would have been significantly reduced. The PML BC combination can provide some damping layer performance improvement over a pure PML C, but it appears from these initial results that this combination will work well only with a relatively large gap before the PML C terms take effect. The best strategy for employing the PML BC combination seems to be to reduce the solution amplitude by about three orders of magnitude with the PML B terms, and then to use the PML C terms only after most or all of this initial damping has been accomplished. The PML BC combination will require wide damping layers to be effective.

Table 5: $\|p - p_R\|_{\infty, \Omega_N}$ at $t = 35$, for PML C with PML width 15 and σ_3.

δ_C	$w_C = 1$	$w_C = 5$	$w_C = 9$	$w_C = 13$
10.00	8.6856×10^{-6}	4.4231×10^{-7}	2.6493×10^{-7}	1.6355×10^{-7}
1.000	5.4827×10^{-7}	1.4929×10^{-7}	7.8054×10^{-8}	3.2457×10^{-8}
0.100	6.9008×10^{-8}	3.0104×10^{-8}	1.3658×10^{-8}	6.6209×10^{-9}
0.010	1.0729×10^{-8}	4.0779×10^{-9}	1.7786×10^{-9}	7.3515×10^{-10}
0.001	1.1433×10^{-9}	4.2478×10^{-10}	1.8288×10^{-10}	7.4310×10^{-11}

Note that $\|p_R\|_{\infty, \Omega_N} = O[10^{-4}]$.

The third series of computations is with PML C by itself, and is presented in Table 5. These computations use the σ_3 damping profile, the multidimensional corner treatment, and the Romenski, Titarev, and Toro [21] implementation of the damping terms. The numerical domain is

$$\Omega_N = [-6, 14] \times [-10, 10], \quad \text{with} \quad t_F = 35,$$

and the damping domain widths are $w_L = w_R = w_B = w_T = 15$. The algorithm parameters are $\triangle x = 1/12 = \triangle y$, and $\triangle t = 1/24$. The Damping zone is large enough so that the propagated signal does not return by $t_F = 35$ from the outer boundary of the damping domain, so that the outer boundary does not effect the solution in the numerical domain.

The data in Table 5 is comparable to the results in Tables 1 and 2, in the sense that all three tables show the effect of the damping treatment without any reflection from the outer boundary of the damping layer. Note that the damping profile scales in Table 2 are $w_C = 1, 2, 3, 5,$ and 7, but in Table 5 they are $w_C = 1, 5, 9,$ and 13. It is immediately clear that the results in Table 5 are only slightly different from the comparable results with the same damping profile scale for PML C in Table 2, with errors that are smaller by no more than a factor of two. The slightly better results in Table 5 could possibly be due to the greater decay of the wave front from the initial impulsive start because of the longer simulation time. On the other hand, for any particular damping amplitude δ_C, damping profile σ_3 increases about three times more gradually than σ_1, so that the σ_3 profile scale is effectively three times larger than w_C. This more gradual increase in the damping profile will tend to produce somewhat smaller errors in the numerical domain. Nonetheless, the similarity of the results in Tables 2 and 5 shows that the damping profile, grid resolution, domain size and simulation time, corner treatment, and implementation of the damping terms all have no significant effect upon the accuracy of the damping treatment. Note in this regard, that all of the damping profiles have been chosen so that, at the interface between the numerical and damping domains, all of the spatial derivatives of the profiles are zero up to at least the order of the numerical method. In particular, σ_3 is C^∞ with every spatial derivative equal to zero at the interface, so that the interface between the numerical domain and the damping layer provides a smooth transition between the governing systems in each domain. One of the implications of the data that has been presented so far is that relatively small errors are introduced into the numerical domain solution if the transition from the numerical to the damping zone is gradual as well as smooth. The data in Table 5 also serves to validate the algorithm mix that is used here, and to show that the algorithm mix and simulation parameter selection is representative of the other results that have been discussed.

The fourth series of computations is with PML C by itself, and is presented in Tables 6, 7, 8 and 9. These computations use the same algorithm mix that produced Table 5, with the same damping profile, multidimensional corner treatment, and implementation of the damping terms. The numerical domain is the same,

$$\Omega_N = [-6, 14] \times [-10, 10],$$

with the same algorithm parameters, $\triangle x = 1/12 = \triangle y$ and $\triangle t = 1/24$. However, the final simulation times and the damping layer widths are all different. Table 6 is for final simulation time $t_F = 55$ with damping domain widths $w_L = w_R = w_B = w_T = 15$, Table 7 is for $t_F = 65$ with $w_L = w_R = w_B = w_T = 20$, Table 8 is for $t_F = 75$ with $w_L = w_R = w_B = w_T = 25$, and Table 9 is for $t_F = 85$ with $w_L = w_R = w_B = w_T = 30$. In each case, the final simulation time is large enough so that the transient effects from the impulsive start can pass out of the numerical domain, and so that the propagated signal can reflect from the outer boundary of the damping domain back through the entire numerical domain. The results in Tables 6-9 are designed to show the effect of reflection from the outer boundary as mediated by the damping profile and the damping zone width. For almost all combination of δ_C and w_C in Tables 6-9, $\|p-p_R\|_{\infty,\Omega_N}$ decreases as the damping layer width increases, with exceptions that appear to be minor variations around a limiting value. In these four tables, insufficient total damping for a fixed δ_C is indicated if $\|p - p_R\|_{\infty,\Omega_N}$ increases with w_C, which stretches the dampng profile and therefore decreases the total damping in the layer. Similarly, insufficient

Table 6: $\|p - p_R\|_{\infty,\Omega_N}$ at $t = 55$, for PML C with PML width 15 and σ_3.

δ_C	$w_C = 1$	$w_C = 5$	$w_C = 9$	$w_C = 13$
10.0	8.4234×10^{-6}	9.9517×10^{-8}	5.9050×10^{-8}	9.2317×10^{-8}
1.0	5.3772×10^{-7}	3.8008×10^{-8}	5.0535×10^{-8}	6.8419×10^{-8}
0.1	1.9056×10^{-7}	6.2012×10^{-7}	1.3412×10^{-6}	2.1143×10^{-6}

Note that $\|p_R\|_{\infty,\Omega_N} = O[10^{-4}]$.

Table 7: $\|p - p_R\|_{\infty,\Omega_N}$ at $t = 65$, for PML C with PML width 20 and σ_3.

δ_C	$w_C = 1$	$w_C = 5$	$w_C = 9$	$w_C = 13$
10.0	8.3917×10^{-6}	1.6884×10^{-8}	2.2645×10^{-8}	3.7814×10^{-8}
1.0	5.4330×10^{-7}	1.5659×10^{-8}	2.6035×10^{-8}	2.3734×10^{-8}
0.1	7.2112×10^{-8}	1.6864×10^{-7}	4.3244×10^{-7}	8.3129×10^{-7}

Note that $\|p_R\|_{\infty,\Omega_N} = O[10^{-4}]$.

Table 8: $\|p - p_R\|_{\infty,\Omega_N}$ at $t = 75$, for PML C with PML width 25 and σ_3.

δ_C	$w_C = 1$	$w_C = 5$	$w_C = 9$	$w_C = 13$
10.0	8.3890×10^{-6}	2.0426×10^{-8}	1.0227×10^{-8}	8.0675×10^{-9}
1.0	5.4134×10^{-7}	9.5706×10^{-9}	6.1355×10^{-9}	1.1910×10^{-8}
0.1	4.7353×10^{-8}	4.7189×10^{-8}	1.3228×10^{-7}	2.9477×10^{-7}

Note that $\|p_R\|_{\infty,\Omega_N} = O[10^{-4}]$.

Table 9: $\|p - p_R\|_{\infty,\Omega_N}$ at $t = 85$, for PML C with PML width 30 and σ_3.

δ_C	$w_C = 1$	$w_C = 5$	$w_C = 9$	$w_C = 13$
10.0	8.3996×10^{-6}	2.9749×10^{-8}	1.2372×10^{-8}	8.4349×10^{-9}
1.0	5.4403×10^{-7}	1.3398×10^{-8}	6.3281×10^{-9}	3.4250×10^{-9}
0.1	4.6392×10^{-8}	1.5651×10^{-8}	4.0343×10^{-8}	9.9061×10^{-8}

Note that $\|p_R\|_{\infty,\Omega_N} = O[10^{-4}]$.

total damping for a fixed w_C is indicated if $\|p - p_R\|_{\infty,\Omega_N}$ increases with a decrease in δ_C, which diminishes the damping profile, decreasing the total damping. Note from the data for $w_C = 1$ in Tables 6-9 that the errors for $\delta_C = 1$ and 10 are the same for any PML width, from 15 to 30, and that these errors are close to the comparable errors in Table 5 where the outer boundary has no effect. We may conclude that a PML width of 15 is sufficient to produce the best result possible with $w_C = 1$ and $\delta_C = 1$ or 10, or that the profile itself creates an error level that is not diminished by further damping farther from the interface. By the time the solution has propagated through a layer of width at least fifteen, it has been damped sufficiently so that whatever error is reflected from the outer boundary is less than the error produced by the damping layer with these profiles. Similarly, for $w_C = 1$ and $\delta_C = 0.1$, a damping layer width of about 20 or slightly more is sufficient to produce the best that is possible with this profile. Table 10 presents crude estimates of damping layer widths that suffice to produce the best possible results with the damping profile and parameter

combinations that were used to produce the error data in Tables 8-9. A total damping can

Table 10: Layer width for smallest errors with PML C and σ_3.

δ_C	$w_C = 1$	$w_C = 5$	$w_C = 9$	$w_C = 13$
10.00	15	20	25	25
1.000	15	25	25	> 30
0.100	20	30	> 30	> 30

be defined as

$$TD(w_D, \delta_C, w_C) = \delta_C \int_0^{w_D} \sigma_3(\xi, w_C) d\xi.$$

If the results in Tables 6-9 are sorted into smaller and greater errors than the comparable cases in Table 5, then

$$TD(w_C, w_D) \geq 1.7 \quad \text{for smaller errors},$$

and

$$TD(w_C, w_D) \leq 1.4 \quad \text{for greater errors}.$$

This in principle permits at least the crude estimation of damping layer parameters that would maintain a specified error bound. It must be remembered that relatively small errors are produced if the transition from the numerical domain to the damping zone is gradual as well as smooth, or with large damping profile scaling w_C and small damping amplitudes δ_C.

6   DL for the LEE with a Nonuniform Base Flow

The LEE in two space dimensions with a nonuniform base flow requires two thermodynamic variables, and we choose to use the disturbance pressure p and specific volume $v_s = 1/\rho$, where ρ is the density. In terms of the perturbation quantities

$$\vec{U} = (p, u, v, v_s)^T,$$

the LEE can be written as: the pressure equation

$$\frac{\partial p}{\partial t} + \bar{u}\frac{\partial p}{\partial x} + \bar{v}\frac{\partial p}{\partial y} + \gamma\bar{p}(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) + u\frac{\partial \bar{p}}{\partial x} + v\frac{\partial \bar{p}}{\partial y} + \gamma p(\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y}) = 0;$$

the velocity equations

$$\frac{\partial u}{\partial t} + \bar{u}\frac{\partial u}{\partial x} + \bar{v}\frac{\partial u}{\partial y} + \frac{1}{\gamma M_R^2}\bar{v}_s\frac{\partial p}{\partial x} + u\frac{\partial \bar{u}}{\partial x} + v\frac{\partial \bar{u}}{\partial y} + \frac{1}{\gamma M_R^2}v_s\frac{\partial \bar{p}}{\partial x} = 0,$$

and

$$\frac{\partial v}{\partial t} + \bar{u}\frac{\partial v}{\partial x} + \bar{v}\frac{\partial v}{\partial y} + \frac{1}{\gamma M_R^2}\bar{v}_s\frac{\partial p}{\partial y} + u\frac{\partial \bar{v}}{\partial x} + v\frac{\partial \bar{v}}{\partial y} + \frac{1}{\gamma M_R^2}v_s\frac{\partial \bar{p}}{\partial y} = 0;$$

and the equation for the specific volume

$$\frac{\partial v_s}{\partial t} + \bar{u}\frac{\partial v_s}{\partial x} + \bar{v}\frac{\partial v_s}{\partial y} - \bar{v}_s\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + u\frac{\partial \bar{v}_s}{\partial x} + v\frac{\partial \bar{v}_s}{\partial y} - v_s\left(\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y}\right) = 0;$$

where the base flow is $(\bar{p}, \bar{u}, \bar{v}, \bar{v}_s)^T$, and where M_R is a reference Mach number. The dimensionless variables for the base flow should satisfy

$$\bar{a}^2 M_R^2 = \bar{p}/\bar{\rho} = \bar{p}\bar{v}_s,$$

so that if we want the nondimensional speed of sound to be $\bar{a} = 1$, then we must have

$$M_R = \sqrt{(\bar{p}/\bar{\rho})} = \sqrt{(\bar{p}\bar{v}_s)}.$$

For this paper we assume a constant base temperature, and choose

$$\bar{p} = 1, \quad \text{and} \quad \bar{v}_s = 1, \quad \text{so that} \quad M_R = 1.$$

In this case, we shall write the LEE with nonuniform base flows as

$$\frac{\partial \vec{U}}{\partial t} + A_v\frac{\partial \vec{U}}{\partial x} + B_v\frac{\partial \vec{U}}{\partial y} + C_v\vec{U} = 0,$$

where A_v, B_v, and C_v are

$$A_v = \begin{pmatrix} \bar{u} & \gamma\bar{p} & 0 & 0 \\ \bar{v}_s/\gamma & \bar{u} & 0 & 0 \\ 0 & 0 & \bar{u} & 0 \\ 0 & -\bar{v}_s & 0 & \bar{u} \end{pmatrix}, \quad B_v = \begin{pmatrix} \bar{v} & 0 & \gamma\bar{p} & 0 \\ 0 & \bar{v} & 0 & 0 \\ \bar{v}_s/\gamma & 0 & \bar{v} & 0 \\ 0 & 0 & -\bar{v}_s & \bar{v} \end{pmatrix},$$

and

$$C_v = \begin{pmatrix} \gamma\left(\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y}\right) & \frac{\partial \bar{p}}{\partial x} & \frac{\partial \bar{p}}{\partial y} & 0 \\[2mm] 0 & \frac{\partial \bar{u}}{\partial x} & \frac{\partial \bar{u}}{\partial y} & \frac{1}{\gamma}\frac{\partial \bar{p}}{\partial x} \\[2mm] 0 & \frac{\partial \bar{v}}{\partial x} & \frac{\partial \bar{v}}{\partial y} & \frac{1}{\gamma}\frac{\partial \bar{p}}{\partial y} \\[2mm] 0 & \frac{\partial \bar{v}_s}{\partial x} & \frac{\partial \bar{v}_s}{\partial y} & -\left(\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y}\right) \end{pmatrix},$$

and where $\vec{U} = (p, u, v, v_s)^T$ are the unsteady perturbation quantities. Note that these coefficient matrices could be nonconstant in time as well as in space, and that a minor modification would permit a nonuniform base temperature. We considered the LEE with the pressure source

$$g(x, y, t) = 0.01\sin[2\pi t]\exp[-36(x^2 + y^2)],$$

in the pressure equation. With this pressure source, the LEE with nonuniform flow can be written as

$$\frac{\partial \vec{U}}{\partial t} + A_v\frac{\partial \vec{U}}{\partial x} + B_v\frac{\partial \vec{U}}{\partial y} + C_v\vec{U} = \vec{G},$$

where $\vec{G} = (g, 0, 0, 0)^T$.

In the case of a nonuniform base flow, a PML B treatment perpendicular to the x axis will be adapted in the form

$$\frac{\partial \vec{U}}{\partial t} + A_v \frac{\partial \vec{U}}{\partial x} + B_v \frac{\partial \vec{U}}{\partial y} + C_v \vec{U} + \delta\sigma(x)A_v\vec{U} = 0,$$

where $\sigma(x)$ is a variable damping profile, δ is a damping scale, and the damping coefficient matrix is just the x propagation matrix A_v. A similar formulation can be written down for a PML B treatment perpendicular to the y axis,

$$\frac{\partial \vec{U}}{\partial t} + A_v \frac{\partial \vec{U}}{\partial x} + B_v \frac{\partial \vec{U}}{\partial y} + C_v \vec{U} + \delta\sigma(y)B_v\vec{U} = 0.$$

Note that since PML B is derived from a directional eigenvector analysis, δ should be positive for a layer on the right, and negative for a layer on the left, and similarly for up and down. A PML C treatment can be written simply as

$$\frac{\partial \vec{U}}{\partial t} + A_v \frac{\partial \vec{U}}{\partial x} + B_v \frac{\partial \vec{U}}{\partial y} + C_v \vec{U} + \delta\sigma\vec{U} = 0,$$

where σ would depend on the variable across the layer in either coordinate direction, and $\delta \geq 0$.

7 Results for the LEE with a Parallel Jet

The next set of experiments are for the LEE with a nonuniform base flow and a pressure source. The pressure source is the same time harmonic oscillation of a narrow Gaussian spatial profile that was used above,

$$g(x, y, t) = 0.01 \sin[2\pi t] \exp[-36(x^2 + y^2)].$$

The nonuniform base flow that we use here is a simple parallel jet flow, with

$$\vec{U} = (\bar{p}, \bar{u}, \bar{v}, \bar{v}_s)^T = (1, 0.4 + 0.4\exp[-36y^2], 0, 1).$$

The numerical domain is

$$\Omega_N = \{(x, y) \in [-3, 7] \times [-5, 5]\},$$

and the third order c2o1 HCKT algorithm is used, with $\triangle x = 1/24 = \triangle y$, and $\triangle t = 1/60$. Solutions are computed for $t \leq 10$. A PML C is used with damping profile σ_1 and zone width $w_R = w_L = w_B = w_T = 5$. The damping zone uses the PML C formulation for nonuniform flows in the inflow and outflow damping zones

$$\Omega_{DI} = \{(x, y) \in [-8, -3] \times [-5, 5]\} \quad \text{and} \quad \Omega_{DO} = \{(x, y) \in [7, 12] \times [-5, 5]\},$$

Table 10: $\|p - p_R\|_{\infty, \Omega_N}$ at $t = 10$, for PML C with σ_1 and PML width 5.

δ_C	$w_C = 1$	$w_C = 3$	$w_C = 5$	$w_C = 7$
10.000	1.7287×10^{-5}	2.4439×10^{-6}	4.1319×10^{-7}	1.2125×10^{-7}
1.000	2.4723×10^{-6}	4.1704×10^{-7}	6.4439×10^{-8}	1.5573×10^{-8}
0.100	3.2806×10^{-7}	4.5465×10^{-8}	7.2817×10^{-9}	1.5973×10^{-9}
0.010	3.5859×10^{-8}	4.5992×10^{-9}	7.3718×10^{-10}	7.3718×10^{-10}
0.001	3.6219×10^{-9}	4.6049×10^{-10}	7.3809×10^{-11}	1.6018×10^{-11}

Note that $\|p_R\|_{\infty, \Omega_N} = O[10^{-4}]$.

and the PML C formulation for uniform flows in the top and bottom damping zones

$$\Omega_{DT} = \{(x, y) \in [-8, 12] \times [5, 10]\} \quad \text{and} \quad \Omega_{DB} = \{(x, y) \in [-8, 12] \times [-10, -5]\}.$$

The corners are treated additively and as if both side zones used the uniform flow PML C formulation. The data that will be reported is the maximum absolute error in the numerical domain $\|p - p_R\|_{\infty, \Omega_N}$, where p_R is a reference numerical solution at $t = 10$. The reference pressure solution p_R is $O[10^{-4}]$ in Ω_N at $t_F = 10$.

The data in Table 10 for PML C with a nonuniform flow is comparable to the data in Table 2 for PML C with a uniform flow. Both sets of data are for the same numerical domain, and the same spatial grid resolution. Because of the higher maximum velocity of the parallel jet the time resolution has been increased from $\triangle t = 1/48$ to $\triangle t = 1/60$. Note that this jet is superimposed upon the same uniform flow as was used for the experiments that produced Table 2. Recall that the data in Table 2 is with a damping zone width $w_D = 10$ and final time $t_F = 15$, while the data in Table 10 is with a damping zone width $w_D = 5$ and final time $t_F = 10$. The same damping profile is used in both sets of calculations, with the same space scales for the profiles, and the same set of damping amplitudes (except that $w_C = 9$ has not been used for the parallel jet case). The damping terms have been implemented with full time evolution, and not with the simpler postprocessing algorithm, just as for the simulations that produced the data in Table 2. We note that almost all of the data for $\|p - p_R\|_{\infty, \Omega_N}$ in Table 10 is smaller than the corresponding data in Table 2 by up to almost an order of magnitude. As in Table 2, the errors in Table 10 range in absolute terms from $O[10^{-5}]$ to $O[10^{-11}]$, or in relative terms from $O[10^{-1}]$ to $O[10^{-7}]$. Note here as well that the PML error decreases linearly with the damping amplitude, and linearly with an increase in the damping profile width scale. As in the uniform flow case, the error induced by a PML C damping zone with a nonuniform flow is controllable. Overall, as demonstrated by the data in Table 10, the performance of PML C with a nonuniform flow is remarkably close to the performance of PML C with a uniform flow.

8   Conclusions

Computations for the Linearized Euler Equations (LEE) have been reported for acoustic propagation with a uniform flow and a parallel jet. Two damping layer treatments for the

Linearized Euler Equations have been considered, Perfectly Matched Layer (PML) PML B based upon directional damping [17, 11], and Damping Layer (DL) PML C [21, 11]. The multilayer PML BC has been introduced, with a damping matrix that combines PML B for accuracy near a PML interface with PML C for damping in an outer layer. A series of numerical experiments have been conducted with all three boundary layers, primarily for the LEE with a uniform base flow, but also with a jet flow. The intent of these experiments has been to try and understand what are the critical parameters that effectively control the errors produced by a PML/DL, and what is the practically achievable level of error that can be obtained. Various combinations of numerical domain size, simulation time, grid resolution, damping layer size, treatments for layer corners, implementation of damping terms, damping profiles types, and damping profile parameter mix have been used in the numerical experiments. The data from the numerical experiments shows that:

1. The errors introduced into a numerical solution by either a PML B or a PML C can be controlled. Results from the numerical experiments show relative error reductions by as little as $O[10^{-1}]$ to as much as $O[10^{-10}]$. It seems clear that smaller errors can be obtained with sufficient effort.

2. Control of the error from a PML/DL appears to be effected most strongly by the layer width, and the amplitude and width scale of the damping profile. If the transition from the numerical domain to the damping layer is sufficiently smooth, then effect of both the damping amplitude and width is linear. Differences in the damping profile, grid resolution, domain size and simulation time, corner treatment, and implementation of the damping terms all have much less effect upon the accuracy of the damping treatment. If the damping treatment offers insufficient total damping, then reflection from the outer boundary of the damping layer can become the dominant source of the error produced by the layer.

3. PML B produces approximately two orders of magnitude less disturbance in the numerical domain than PML C, but PML B amplifies distortions back towards the numerical domain, while PML C dampens omnidirectionally.

4. The best strategy for employing the PML BC combination seems to be to reduce the solution amplitude by about three orders of magnitude with the PML B terms, and then to use the PML C terms only after most or all of this initial damping has been accomplished. Because of this, the PML BC combination appears to require wide damping layers to be effective.

5. Estimation of the total damping as a function of layer width, damping amplitude, and damping spatial scale permit estimation of layer parameters needed to maintain a specified error bound. Total damping is a product of the amplitude and width, so that a small damping amplitude requires a corespondingly wide damping layer.

Relatively small errors are introduced into the numerical domain solution if the transition from the numerical to the damping zone is gradual as well as smooth. Accurate damping layers with $O[10^5]$ relative errors are not difficult to achieve. Very accurate damping layers are difficult to achieve and appear to require wide layers and substantial computational effort.

The treatment of the outer boundary of the damping layer and its effect on controlling the error produced by the layer as a whole has not been considered here. Every order of magnitude decrease in the error produced by the outer boundary treatment will have a significant effect on the error produced by the damping layer, or on the cost of producing any specified error level.

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14. ABSTRACT
In this paper we show by means of numerical experiments that the error intruduced in a numerical domain because of a Perfectly Matched Layer or Damping Layer boundary treatment can be controlled. These experimental demonstrations are for acoustic propagation with the Linearized Euler Equations with both uniform and steady jet flows. The propagating signal is driven by a time harmonic pressure source. Combinations of Perfectly Matched and Damping Layers are used with different damping profiles. These layer and profile combinations allow the relative error introduced by a layer to be kept as small as desired, in principle. Tradeoffs between error and cost are explored.

15. SUBJECT TERMS
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