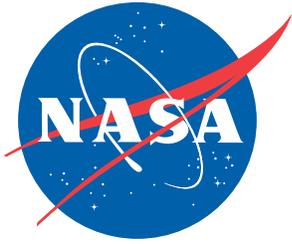


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# Reliability Analysis of Systems Subject to First-Passage Failure

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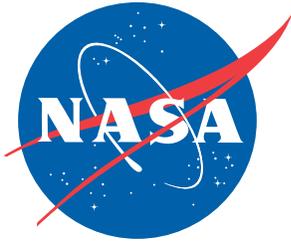
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# RELIABILITY ANALYSIS OF SYSTEMS SUBJECT TO FIRST-PASSAGE FAILURE

## INTRODUCTION

An obvious goal of reliability analysis is the avoidance of system failure. However, it is generally recognized that it is often not feasible to design a practical or useful system for which failure is impossible. Thus it is necessary to use techniques that estimate the likelihood of failure based on modeling the uncertainty about such items as the demands on and capacities of various elements in the system. This usually involves the use of probability theory, and a design is considered acceptable if it has a sufficiently small probability of failure. In some applications, the analysis uses some rule-of-thumb, such as three-sigma, six-sigma, etc., rather than numerical estimates of the probability of failure, but these approaches are based on known correlation between the rule-of-thumb and estimated probability of failure in a random-variable model.

A random variable is a mathematical concept to model the uncertainty about any quantity that has more than one possible numerical value. That is, it models the likelihood that the observed outcome will be any particular one of the possible outcomes. The situation becomes more complicated when the time of likely failure is also an uncertain quantity, such as when the demand on the system varies randomly with the passage of time. Then, one must use a method that models not only the uncertainty about quantities at any particular time, but also the uncertainty about how these quantities vary from one time to the next. The probabilistic model that combines these two types of uncertainty is called a stochastic process (or random process). Thus, the uncertain stress at one instant of time  $t$  will be classified as a random variable  $X(t)$  but the uncertain history of stress over a range of time values will be a stochastic process  $\{X(t)\}$ . The practice of denoting a stochastic process by putting the notation for the associated random variables in braces will be used to indicate that the stochastic process is a family of random variables—one for each  $t$  value.

Study of first-passage failure is appropriate when the system of interest is considered to have performed unsatisfactorily if some measure of response has ever reached some critical value. Thus, for example, first-passage failure might be considered to have occurred if the stress or strain at some critical location has ever exceeded the yield level or if a particular displacement has exceeded a value resulting in collision with some other element of the system, possibly causing damage.

To calculate the probability of first-passage failure during a given time interval  $[0, t]$ , one needs terms such as  $P[\hat{X}(t) > x_{critical}]$  in which  $\hat{X}(t)$  is defined as the maximum of  $X(s)$  over the set of values  $0 \leq s \leq t$ . For a given value of  $t$ , the first-passage reliability can be formulated in terms of the random variable  $\hat{X}(t)$ , but finding (or estimating) the probability distribution of  $\hat{X}(t)$  requires a significant amount of information about the stochastic process  $\{X(s) : 0 \leq s \leq t\}$ .

A fundamental difference between a random variable and a stochastic process relates to what constitutes one observation of an outcome. An outcome for a scalar random variable is a scalar. An outcome for a vector random variable is a vector. But an outcome for a scalar stochastic process is a time history. Furthermore, most stochastic processes have the ergodic property, which assures that any particular time history  $x(t)$  must eventually approach every possible value of the stochastic process  $\{X(t)\}$ . While it is relatively rare that an observation of a random variable will exceed the mean plus four standard deviations of the variable, for example, it is expected that any one observation of the associated process will eventually reach that four-sigma level if the observation time is sufficiently long. Thus, the length of time  $t$  is a crucial variable in first-passage analysis.

It may be noted that there is a sort of “orthogonality” between the idea that a stochastic process is characterized by an “ensemble” containing all possible time histories and the idea that the process is a family of random variables. A time history is a single observation including many values of  $t$ . Many observations at a single value of  $t$  give a statistical sample of the random variable  $X(t)$ . In most practical problems, it is not feasible to describe a stochastic process in terms of the probability of occurrence of particular time histories, but it is always possible to characterize the process by using information about the joint probability distribution of the random variables of which it is composed.

## PROPERTIES OF STOCHASTIC PROCESSES

Almost every property of a stochastic process can be viewed as a generalization of a corresponding property of a random variable. Of course, all the information about the probabilities of a single random variable  $X(t)$  can be encapsulated in a cumulative distribution function

$$F_{X(t)}(u) \equiv P[X(t) \leq u] \quad (1)$$

or by its derivative, which is the probability density function

$$p_{X(t)}(u) \equiv \frac{\partial F_{X(t)}(u)}{\partial u} = \lim_{\Delta u \rightarrow 0} \frac{P[u < X(t) \leq u + \Delta u]}{\Delta u} \quad (2)$$

Note that both the functions  $F_{X(t)}(u)$  and  $p_{X(t)}(u)$  must be defined on the set of all possible values of  $X(t)$ . In order to generalize (1) and (2) to describe a stochastic process  $\{X(t)\}$  one must consider the probabilities of events that are the intersections of occurrences at multiple values of  $t$ . In particular, a complete description requires knowledge of

$$F_{X(t_1) \dots X(t_n)}(u_1, \dots, u_n) \equiv P[X(t_1) \leq u_1, \dots, X(t_n) \leq u_n] \quad (3)$$

or

$$\begin{aligned}
p_{X(t_1)\cdots X(t_n)}(u_1, \dots, u_n) &\equiv \frac{\partial^n F_{X(t_1)\cdots X(t_n)}(u_1, \dots, u_n)}{\partial u_1 \cdots \partial u_n} \\
&= \lim_{\Delta u_1 \rightarrow 0, \dots, \Delta u_n \rightarrow 0} \frac{P[u_1 < X(t_1) \leq u_1 + \Delta u_1, \dots, u_n < X(t_n) \leq u_n + \Delta u_n]}{\Delta u_1 \cdots \Delta u_n}
\end{aligned} \tag{4}$$

and one of these functions must be known for all possible choices of  $n$ ,  $t_1, \dots, t_n$ , and  $u_1, \dots, u_n$ .

In many situations it is possible, and much easier, to use an analysis that requires less information than is given by (1)-(4). In particular, the idea of the mean and variance values of random variable are generalized to a mean-value function

$$\mu_X(t) \equiv E[X(t)] = \int_{-\infty}^{\infty} u p_{X(t)}(u) du \tag{5}$$

and a covariance function

$$\begin{aligned}
K_{XX}(t, s) &\equiv E[(X(t) - \mu_X(t)][X(s) - \mu_X(s)]) \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [u_1 - \mu_X(t)][u_2 - \mu_X(s)] p_{X(t)X(s)}(u_1, u_2) du_1 du_2
\end{aligned} \tag{6}$$

Note, in particular, that the covariance function is defined on a two-dimensional set of  $(t, s)$  values. It includes the much simpler variance of all the random variables, since  $\sigma_{X(t)}^2 = K_{XX}(t, t)$ , but it also includes crucial information about the correlation of the two random variables  $X(t)$  and  $X(s)$ . In some situations it is also necessary to consider covariance functions involving two different stochastic processes. One such example is the cross-covariance of a process and its time-derivative

$$K_{X\dot{X}}(t, s) \equiv E[(X(t) - \mu_X(t)][\dot{X}(s) - \mu_{\dot{X}}(s)]] = \frac{\partial K_{XX}(t, s)}{\partial s} \tag{7}$$

The property of stationarity (or homogeneity) of a stochastic process  $\{X(t)\}$  always refers to some aspect of the description of the process being unchanged by any arbitrary shift along the  $t$  axis. There are many types of stationarity depending on which characteristic of the process has this property of being invariant under a time shift.

The simplest type of stationarity involves only invariance of the mean value function for the process. In particular,  $\{X(t)\}$  is mean-value stationary if  $\mu_X(t+r) = \mu_X(t)$  for any value of the time-shift parameter  $r$ . Clearly this can be true only if  $\mu_X(t)$  is the same for all  $t$  values, so one can say that  $\{X(t)\}$  is mean-value stationary if  $\mu_X(t) = \mu_X$ , in which the absence of a  $t$  argument on the right-hand side conveys the information that the mean value is independent of time. Although the notation  $\mu_X$  is the

same as for the mean value of a random variable, it is here used to refer to the mean value function of a stochastic process. Of course, having the mean value be independent of  $t$  does not imply that the  $\{X(t)\}$  random variables are all the same at different values of  $t$  or that the probability distributions of these random variables are all the same—only that they all have the same mean value.

Similarly, one can define a more rigorous stationarity related to the covariance function. The process  $\{X(t)\}$  is said to be covariant stationary if the covariance function is invariant under a time shift so that  $K_{XX}(t+r, s+r) = K_{XX}(t, s)$  for any value of  $r$ . Choosing  $r = -s$ , then gives  $K_{XX}(t, s) = K_{XX}(t-s, 0)$ , showing that the stationary covariance function depends on only one time argument—the difference between the two times of interest. Thus, one can define a covariance function of only one time argument as  $G_{XX}(\tau) = K_{XX}(t+\tau, t)$  so that

$$K_{XX}(t, s) = G_{XX}(t-s) \quad \text{for any values of } t \text{ and } s \quad (8)$$

Similarly, two stochastic processes  $\{X(t)\}$  and  $\{Y(t)\}$  are jointly covariant stationary if  $K_{XY}(t+r, s+r) = K_{XY}(t, s)$ , and it is convenient to define  $G_{XY}(\tau) = K_{XY}(t+\tau, t)$  so that

$$K_{XY}(t, s) = G_{XY}(t-s) \quad \text{for any values of } t \text{ and } s \quad (9)$$

Although a given stochastic process may simultaneously have various types of moment stationarity, this is not necessary. In particular, a process may be mean-value stationary and covariant nonstationary, and another process may be mean-value nonstationary and covariant stationary. It is common, however, to have situations in which both the mean-value and the covariance are stationary.

There are also forms of stationarity that are not defined in terms of moment functions. Rather, they are defined in terms of probability distributions being invariant under a time shift. The general relationship is that  $\{X(t)\}$  is  $j$ th-order stationary if

$$p_{X(t_1+r)\dots X(t_j+r)}(u_1, \dots, u_j) = p_{X(t_1)\dots X(t_j)}(u_1, \dots, u_j) \quad (10)$$

for all values of  $\{t_1, \dots, t_j, u_1, \dots, u_j\}$  and the shift parameter  $r$ . This includes, as special cases,  $\{X(t)\}$  being first-order stationary if  $p_{X(t+r)}(u) = p_{X(t)}(u)$  and second-order stationary if  $p_{X(t_1+r)X(t_2+r)}(u_1, u_2) = p_{X(t_1)X(t_2)}(u_1, u_2)$  for all values of  $\{t_1, t_2, u_1, u_2\}$  and the shift parameter  $r$ . First-order stationarity always implies mean-value stationarity, and second-order stationarity implies both mean-value and covariance stationarity. In some situations (such as the well-known Gaussian process) mean-value and covariant stationarity also imply second-order stationarity, but this is not true in general.

The most restrictive type of stationarity is called strict stationarity. The process  $\{X(t)\}$  is said to be strictly stationary if it is  $j$ th-order stationary for any value of  $j$ . This implies that any order probability density function has time shift invariance and any order

moment function has time shift invariance. In common usage, one can also say that the word stationary without qualifier simply means that all moments and/or probability distributions being used in the given problem are invariant under a time shift.

An ergodic stochastic process is one for which any infinitely long time history is representative of the entire process. For example, any expected value of some function of the process can be written as the limit of a time-average obtained from one time history of the process. Thus,

$$\mu_X \equiv E[X(t)] = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T x(t) dt \quad (11)$$

and

$$\begin{aligned} G_{XX}(\tau) &\equiv E([X(t+\tau) - \mu_X][X(t) - \mu_X]) \\ &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T [x(t+\tau) - \mu_X][x(t) - \mu_X] dt \end{aligned} \quad (12)$$

with  $x(t)$  representing any one time history of  $\{X(t)\}$  in both (11) and (12). Note that the process is required to be mean-value and covariant stationary in (11) and (12), respectively, since the infinite time average could not possibly converge to a function of  $t$ . Similarly, strict ergodicity implies that the probability distribution can also be found from one time history:

$$F_{X(t)}(u) = E(U[u - X(t)]) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^{T/2} U[u - x(t)] dt \quad (13)$$

in which the unit step function  $U(\cdot)$  is defined to be zero when its argument is negative and unity otherwise. Thus, the integral gives exactly the amount of time for which  $x(t) \leq u$  within the interval  $[0, T]$ . This condition that the probability that  $X(t) \leq u$  at any time  $t$  is the same as the fraction of the time that any time history  $x(t)$  is less than or equal to  $u$  is the basis of the statement in the introduction that any time history of an ergodic process must eventually approach every one of its possible values. Although it is not necessary for all processes to be ergodic, it is common practice to assume ergodicity unless there is some obvious physical reason why that would be inappropriate for the problem of interest.

In the remainder of this presentation, the process  $\{X(t)\}$  will be considered to be mean-zero. The primary motivation for doing this is that it simplifies the presentation without losing any generality in almost all instances. In order to analyze a process  $\{Y(t)\}$  that is not mean-zero, one simply defines  $X(t) = Y(t) - \mu_Y(t)$ . Furthermore, this transformation is generally not difficult, since the evaluation or estimation of the mean-value function is generally much easier than evaluation of other quantities such as covariance. This is illustrated in (11) and (12) for the stationary ergodic situation, and it is also generally true in the analysis of the response of dynamical systems [e.g., Lutes and Sarkani, 2004].

The Gaussian stochastic process occupies a very special place in analysis of practical problems. It has been found that it both gives a reasonable model of many physical processes and has some very desirable mathematical characteristics. The definition of a Gaussian stochastic process  $\{X(t)\}$  is simply that any finite set  $\{X(t_1), X(t_2), \dots, X(t_n)\}$  of random variables from that process has the jointly Gaussian distribution. The general form for the joint probability density function for a vector  $\mathbf{V}$  of jointly Gaussian mean-zero components is

$$p_{\mathbf{V}}(\mathbf{u}) = \frac{1}{(2\pi)^{n/2} |\mathbf{K}_{\mathbf{V}\mathbf{V}}|^{1/2}} \exp\left(-\frac{1}{2} \mathbf{u}^T \mathbf{K}_{\mathbf{V}\mathbf{V}}^{-1} \mathbf{u}\right) \quad (14)$$

which is a function of the square covariance matrix  $\mathbf{K}_{\mathbf{V}\mathbf{V}} = E(\mathbf{V}^T \mathbf{V})$ . Applying this model to the Gaussian process, then, only involves using the vector  $\mathbf{V} = [X(t_1), X(t_2), \dots, X(t_n)]^T$ .

The jointly Gaussian distribution, of course, implies that each of the random components is Gaussian (normal), but it also specifies much more about the interrelationships among the components. One of the very desirable features of the mean-zero Gaussian process is that the distribution depends only on the covariance function  $K_{XX}(t, s)$  or  $G_{XX}(\tau)$ , which provides all components of  $\mathbf{K}_{\mathbf{V}\mathbf{V}}$  for the process. Among the unusual features of the distribution are the facts that a mean-zero covariant stationary Gaussian process is always strictly stationary, and that a mean-zero covariant ergodic Gaussian process is always strictly ergodic.

Particularly important for many applications is the fact that any linear combination of jointly Gaussian random variables is itself Gaussian, and it is also jointly Gaussian with other such linear combinations. This ensures that linear operations on a Gaussian stochastic process always yield other Gaussian processes and that these new processes are jointly Gaussian with the original process. Such linear operations include input-output relationships for any linear dynamical system and calculus operations such as finding the derivative process  $\{\dot{X}(t)\}$  corresponding to a given process  $\{X(t)\}$  or integrating to find the  $\{X(t)\}$  corresponding to a given  $\{\dot{X}(t)\}$ . It should also be noted, though, that there are some processes for which the random variables  $X(t)$  are all individually Gaussian, but not jointly Gaussian. These are not Gaussian processes, and applying a linear-combination operation to them generally does not yield a Gaussian output.

## **FREQUENCY, BANDWIDTH, AND AMPLITUDE**

The Fourier transform provides a classical method for decomposing a time history into its frequency components, and this is often very useful in both computing and interpreting the results of practical models. When applied to a stochastic process  $\{X(t)\}$  the Fourier transform introduces a new stochastic process  $\{\tilde{X}(\omega)\}$  defined by

$$\tilde{X}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(t) e^{-i\omega t} dt \quad (15)$$

The original stochastic process can then be recaptured by using the inverse transform

$$X(t) = \int_{-\infty}^{\infty} \tilde{X}(\omega) e^{i\omega t} d\omega \quad (16)$$

Note that  $\{\tilde{X}(\omega)\}$  gives a complex random variable  $\tilde{X}(\omega)$  for every possible value of  $\omega$ , just as  $\{X(t)\}$  gives a real random variable  $X(t)$  for every possible value of  $t$ .

From (15) one can write  $E[\tilde{X}(\omega_1)\tilde{X}(\omega_2)]$  as a double Fourier transform of  $G_{XX}(\tau)$ . A major difficulty in applying the Fourier transform procedure to many problems of interest, though, is the fact that  $E[\tilde{X}(\omega_1)\tilde{X}(\omega_2)]$  does not exist for all values of frequency when  $\{X(t)\}$  is a stationary stochastic process. That is, the double integral to evaluate  $E[\tilde{X}(\omega_1)\tilde{X}(\omega_2)]$  may be infinite. It is important, though, to have a usable form of  $E[\tilde{X}(\omega_1)\tilde{X}(\omega_2)]$ . To this end, the Fourier transform procedure has been modified as follows. The first step is to truncate the Fourier transform integral in (15) to be over a time interval of finite length  $T$ , which is found to give an approximation of  $E[\tilde{X}(\omega_1)\tilde{X}(\omega_2)]$  that grows in proportion to  $T$ . This is then normalized to give a finite limit, and is defined as the spectral density function of the process:

$$\begin{aligned} S_{XX}(\omega) &\equiv \lim_{T \rightarrow \infty} \frac{2\pi}{T} E \left[ \left[ \frac{1}{2\pi} \int_{-T/2}^{T/2} X(t_1) e^{-i\omega t_1} dt_1 \right] \left[ \frac{1}{2\pi} \int_{-T/2}^{T/2} X(t_2) e^{-i\omega t_2} dt_2 \right] \right] \\ &= \lim_{T \rightarrow \infty} \frac{1}{2\pi T} \int_{-T/2}^{T/2} \int_{-T/2}^{T/2} G_{XX}(t_1 - t_2) e^{-i(\omega_1 t_1 + \omega_2 t_2)} dt_1 dt_2 \end{aligned} \quad (17)$$

For any stationary process, though, it has been shown that (17) gives the spectral density as exactly the Fourier transform of the covariance function for the original process:

$$S_{XX}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G_{XX}(\tau) e^{-i\omega \tau} d\tau \quad (18)$$

Furthermore, this implies that the inverse must also be true:

$$G_{XX}(\tau) = \int_{-\infty}^{\infty} S_{XX}(\omega) e^{i\omega \tau} d\omega \quad (19)$$

The original idea that  $E[\tilde{X}(\omega_1)\tilde{X}(\omega_2)]$  is a double Fourier transform of  $G_{XX}(\tau)$  gives

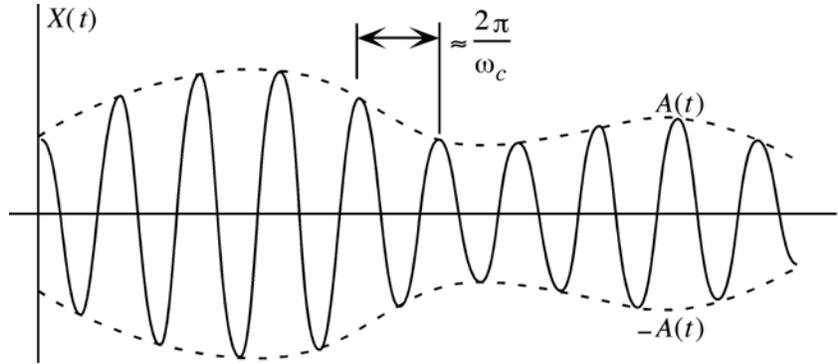
$$G_{XX}(\tau) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E[\tilde{X}(\omega_1)\tilde{X}(\omega_2)] e^{i(\omega_1 + \omega_2)\tau} e^{i\omega_1 \tau} d\omega_1 d\omega_2 \quad (20)$$

and this agrees with the expression in (19) if  $E[\tilde{X}(\omega_1)\tilde{X}(\omega_2)] = S_{XX}(\omega_1)\delta(\omega_1 + \omega_2)$ , in which  $\delta(\cdot)$  denotes the Dirac delta function. This degenerate form has been found to be useful in some calculations.

The spectral density function defined in (17), and commonly computed from (18), has many desirable properties. Most importantly, it does give a complete frequency decomposition of the covariance properties of a stationary  $\{X(t)\}$ . In addition, it is always a real, nonnegative even function of  $\omega$ . Furthermore, setting  $\tau$  equal to zero in (19) shows that the variance  $\sigma_X^2 = G_{XX}(0)$  is simply the area under the  $S_{XX}(\omega)$  curve. It should be noted that there are other forms of spectral density that differ from (18) only by a multiplying factor. One such variation is found by using a cosine Fourier transform in place of the exponential transform used here, and obtaining a spectral density that is twice as large as that of (17)-(19), but exists only for  $\omega \geq 0$ . A less frequent variation is to use frequency measured in Hz (cycles per second) rather than radians per second in saying that  $\sigma_X^2$  is the area under the spectral density curve, which requires a modification of spectral density by a factor of  $2\pi$ .

The cross-spectral density between two processes, such as  $\{X(t)\}$  and  $\{\dot{X}(t)\}$  is defined in a parallel way, and (18) and (19) generalize to give the cross-spectral density and the cross-covariance function as a Fourier transform pair. Taking derivatives of (18) and (19) yields simple results for the cross-spectral density of  $\{X(t)\}$  and  $\{\dot{X}(t)\}$ , the spectral density of  $\{\dot{X}(t)\}$ , etc. In particular,  $S_{X\dot{X}}(\omega) = -i\omega S_{XX}(\omega)$  and  $S_{\dot{X}\dot{X}}(\omega) = \omega^2 S_{XX}(\omega)$ . The results for higher order derivative processes can be written as  $S_{X^{(j)}X^{(k)}}(\omega) = (-1)^k (i)^{j+k} \omega^{j+k} S_{XX}(\omega)$ , in which  $\{X^{(j)}(t)\}$  represents the  $j$ th-order derivative with respect to  $t$ . This includes the special case of  $S_{X^{(j)}X^{(j)}}(\omega) = \omega^{2j} S_{XX}(\omega)$  for the spectral density function of the  $j$ th derivative process.

A process  $\{X(t)\}$  is said to be narrowband if the spectral density  $S_{XX}(\omega)$  is very small except within a narrow band of frequencies. Because  $S_{XX}(\omega)$  is an even function, this really means that the band of significant frequencies appears both for positive and negative  $\omega$  values. One way to state this narrowband condition is to say that  $S_{XX}(\omega) \approx 0$  unless  $|\omega| \approx \omega_c$  for some given characteristic frequency  $\omega_c$ . The limiting narrowband process has the spectral density  $S_{XX}(\omega) = (\sigma_X^2/2)[\delta(\omega + \omega_c) + \delta(\omega - \omega_c)]$ . The corresponding covariance function is  $G_{XX}(\tau) = \sigma_X^2 \cos(\omega_c \tau)$ . Furthermore, the time dependence of the process is of the form  $X(t) = A \cos[\omega_c t + \theta]$  with the amplitude and phase  $A$  and  $\theta$ , respectively, being random variables. That is, any time history is a pure cosine wave with fixed amplitude and frequency. This situation, of course, is not often encountered in practice, but the harmonic nature of  $X(t)$  and  $G_{XX}(\tau)$  is approximated by all narrowband processes. The time histories of  $X(t)$  for a narrowband process generally are of the form of cosine waves with slowly varying amplitude and phase, as shown in Figure 1. Similarly, the corresponding  $G_{XX}(\tau)$  has the form of a cosine wave with amplitude (or envelope) that decays slowly with increasing  $|\tau|$ .



**Figure 1. Typical time history of a narrowband process.**

At the opposite extreme from a narrowband process is the white noise (or delta-correlated) process for which all Fourier components contribute equally, such that the spectral density is the same for all  $\omega$  values. For a process  $\{X(t)\}$  of this type with  $S_{XX}(\omega) = S_0$ , the covariance function is the degenerate case of  $G_{XX}(\tau) = 2\pi S_0 \delta(\tau)$ . The time histories of any delta-correlated process, of course, must be extremely erratic, since they are uncorrelated at times separated by any finite value  $\tau$ . Any practical broadband process has a spectral density that varies slowly with  $\omega$ , a covariance function that tends rapidly to zero with increasing  $|\tau|$ , and very erratic time histories.

Many stochastic processes encountered in practice are not near either the narrowband or the broadband limits. Some of these may be considered to be between narrowband and broadband, while others have distinctly different forms, such as a multimodal spectral density containing a sequence of narrow peaks at distinct values of  $\omega$ . In this case, many but not all frequencies make significant contributions to the time histories. One physical phenomenon that leads to narrowband or multimodal spectral densities relates to the deterministic idea of resonance in dynamical systems. If a system has a resonant frequency, then exciting that system with a broadband process will lead to a response for which the spectral density has a high peak at the resonant frequency. In the same way, multiple resonant frequencies in the system can lead to multimodal spectral densities. In essence, any dynamical system can be regarded as a filter. Some frequencies components of the excitation are attenuated because the system is not sensitive to those frequencies, while other components are amplified due to resonance-like behavior. For a linear system or filter with stochastic input  $\{Y(t)\}$  and output  $\{X(t)\}$  there exists a complex transfer function  $H_{XY}(\omega)$  such that the Fourier transforms satisfy  $\tilde{X}(\omega) = H_{XY}(\omega)\tilde{Y}(\omega)$ . This then gives  $S_X(\omega) = |H_{XY}(\omega)|^2 S_Y(\omega)$  as the description of the filtering effect on the spectral density.

The concepts of characteristic frequency, amplitude and phase that were introduced in discussing narrowband processes can be usefully generalized to provide an alternate characterization of any mean-zero stationary stochastic process. In particular, let

$$X(t) = A(t)\cos[\omega_c t + \theta(t)] \quad (21)$$

The definition of  $\omega_c$  should ideally be chosen such that  $E[\dot{\theta}(t)] = 0$ , because a nonzero value for  $E[\dot{\theta}(t)]$  should, in itself, be a contribution to the characteristic frequency of the  $\{X(t)\}$  process. Next the amplitude and phase are defined as

$$A(t) = [X^2(t) + Z^2(t)]^{1/2} \quad (22)$$

and

$$\theta(t) = -\tan^{-1}[Z(t)/X(t)] - \omega_c t \quad (23)$$

in which  $\{Z(t)\}$  is a process for which every frequency component is  $90^\circ$  out of phase with the corresponding component of  $\{X(t)\}$ . Specifically, this is done by choosing  $\tilde{Z}(\omega) = i g(\omega) \tilde{X}(\omega)$  for some odd real function  $g(\cdot)$  with  $g(\omega_c) = 1$ . This definition of the amplitude and phase is consistent with the intuitive ideas introduced for a narrowband process for which  $\tilde{X}(\omega)$  is nearly zero except in the neighborhood of  $\pm\omega_c$ , and there are many  $g(\cdot)$  functions satisfying the necessary conditions. One of the popular choices is  $g(\omega) = \text{sgn}(\omega)$ , which gives  $Z(t)$  as the Hilbert transform of  $X(t)$ . The amplitude defined in this way was used by Cramer and Leadbetter [1967] and is commonly associated with their names. It will be denoted as  $A_1(t)$ . Another commonly used amplitude follows from using  $g(\omega) = \omega/\omega_c$ , which gives  $Z(t) = \dot{X}(t)/\omega_c$ . This amplitude, which will be denoted as  $A_2(t)$ , is often referred to as the energy-based amplitude, since it can be related to the sum of potential and kinetic energy in some simple oscillators. The  $\omega_c$  values corresponding to the Cramer/Leadbetter and energy-based amplitude are  $\lambda_1/\lambda_0$  and  $(\lambda_2/\lambda_0)^{1/2}$ , respectively, in which the spectral moment  $\lambda_j$  is defined as

$$\lambda_j \equiv \int_{-\infty}^{\infty} |\omega|^j S_{XX}(\omega) d\omega = 2 \int_0^{\infty} \omega^j S_{XX}(\omega) d\omega \quad (24)$$

The term bandwidth parameter is used to refer to any characteristic of a stochastic process that gives an indication of how strongly the process resembles the narrowband or broadband limit. There are many such parameters, but only two will be mentioned here. One natural way to introduce this idea is to consider the rates of change of  $A(t)$  or  $\theta(t)$ , as defined in (22) and (23). In particular, these quantities should change slowly for a narrowband process, but more rapidly for a broadband process. For the Cramer/Leadbetter amplitude it can be shown [e.g., Lutes and Sarkani, 2004] that the rates of change of  $A_1(t)$  and of the corresponding phase are governed by the parameter

$$\alpha_1 = \lambda_1 / (\lambda_0 \lambda_2)^{1/2} \quad (25)$$

Similar results apply for the energy-based amplitude and phase, but the controlling parameter is

$$\alpha_2 = \lambda_2 / (\lambda_0 \lambda_4)^{1/2} \quad (26)$$

It may be noted that the even spectral moments have direct physical meaning since  $\sigma_X^2 = \lambda_0$ ,  $\sigma_{\dot{X}}^2 = \lambda_2$ ,  $\sigma_{\ddot{X}}^2 = \lambda_4$ , etc. This shows that the value of  $\alpha_2$  can be found directly from these variance quantities, which is something of an advantage over  $\alpha_1$ . On the other hand,  $\alpha_2$  depends rather heavily on the high-frequency tail of the spectral density, which is sometimes a disadvantage in comparison to  $\alpha_1$ . Both  $\alpha_1$  and  $\alpha_2$  are defined such that they are in the interval  $[0, 1]$  and tend to unity for narrowband processes.

Note that if  $\{X(t)\}$  is a Gaussian process, then  $\{\tilde{X}(\omega)\}$ ,  $\{\tilde{Z}(\omega)\}$ , and  $\{Z(t)\}$  are all jointly Gaussian processes, because they are obtained from linear operations on  $\{X(t)\}$ . The Gaussian property then allows evaluation of additional properties of the amplitude and phase. It can be shown that both the Cramer/Leadbetter and energy-based definitions give the random variable  $Z(t)$  to be independent of  $X(t)$  for the same  $t$  value and to have the same variance. This then leads to  $A(t)$  having the common Rayleigh distribution:

$$F_A(u) = 1 - e^{-u^2/(2\sigma_X^2)} \quad p_A(u) = (u/\sigma_X^2) e^{-u^2/(2\sigma_X^2)} \quad (27)$$

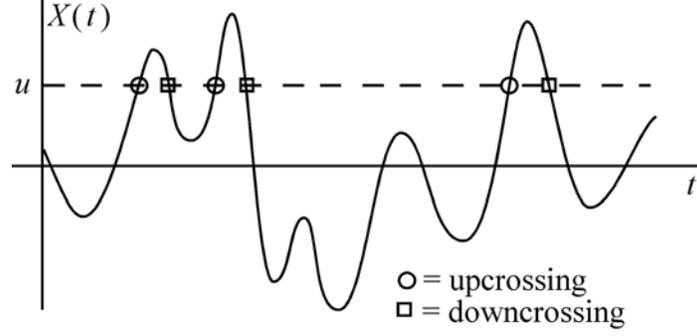
The corresponding phase angle  $\theta(t)$  is independent of  $A(t)$  for the same  $t$  value and is uniformly distributed on  $[0, 2\pi]$ . These simple distributions are sometimes quite useful in evaluating properties of  $\{X(t)\}$  from the amplitude and phase form of (21).

## CROSSING PROBABILITY AND CROSSING RATE

The reliability of a system is closely related to the concept of level-crossings. This is particularly true for first-passage failure, in which the system is considered to fail only when a particular stress process or displacement  $\{X(t)\}$  reaches a critical level  $b$ . In particular, failure cannot then occur before  $X(t)$  crosses  $b$ . Similarly, a crossing of the level zero by the derivative  $\dot{X}(t)$  coincides with a local maximum or minimum of  $X(t)$ , and failure will not occur when all these extrema are below  $b$ . The following paragraphs review important results related to these crossing probabilities, with special emphasis on the stationary Gaussian process so commonly used in practical applications. Rigorous analytical results will be presented for the expected rate of occurrence of such crossings for this process, and for the corresponding probability distribution of local maxima. These results, however, cannot provide an analytical solution for the first-passage probability so further approximations will be introduced for that purpose.

An upcrossing at time  $t$  of the level  $u$  by the process  $\{X(t)\}$  is simply the intersection of the events  $X(t) = u$  and  $\dot{X}(t) > 0$ . Similarly, a downcrossing occurs whenever  $X(t) = u$  with  $\dot{X}(t) < 0$ . These ideas are illustrated in Figure 2. The probabilities of such crossings during a small time increment will be written as

$$v_X^+(u, t) \Delta t \equiv P(\text{upcrossing in } [t, t + \Delta t]) \quad (28)$$



**Figure 2. Crossings of the level  $X(t) = u$**

and the corresponding relationship between a downcrossing and a term  $v_X^-(u, t)$ . Thus,  $v_X^+(u, t)$  and  $v_X^-(u, t)$  are the limits of  $P(\text{crossing in } [t, t + \Delta t]) / \Delta t$  as  $\Delta t$  tends to zero. Note that  $v_X^+(u, t)$  and  $v_X^-(u, t)$  have units of crossings per unit time, and are exactly the expected rate of crossings of the level  $X = u$ . The terms upcrossing rate and downcrossing rate will be used to refer to these quantities, even though the interest here is more focused on the probability of occurrence of the crossing than on the rate.

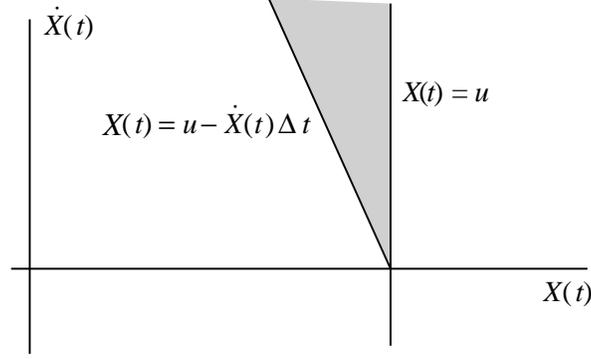
It is easily seen that there can only be an upcrossing of the level  $u$  within the interval  $[t, t + \Delta t]$  if  $X(t)$  at the beginning of the interval is less than  $u$ , but close to  $u$ , and has a positive derivative. Inasmuch as  $\Delta t$  is infinitesimal, one may consider the derivative to be constant at the value  $\dot{X}(t)$  throughout the time interval and conclude that there will be an upcrossing within the interval if  $u - \dot{X}(t)\Delta t < X(t) < u$ . This event is shown shaded on the space of possible values of  $X(t)$  and  $\dot{X}(t)$  in Figure 3. The probability of this event can now be found by integrating the joint probability density of  $X(t)$  and  $\dot{X}(t)$  over the shaded region. However, the fact that  $\Delta t$  is infinitesimal can be used again to argue that  $p_{X(t)\dot{X}(t)}(w, v) \approx p_{X(t)\dot{X}(t)}(u, v)$  for  $u - v\Delta t \leq w \leq u$ , giving:

$$v_X^+(u, t) = \int_0^\infty v p_{X(t)\dot{X}(t)}(u, v) dv = p_{X(t)}(u) \int_0^\infty v p_{\dot{X}(t)}[v | X(t) = u] dv \quad (29)$$

The second form in (29) has been obtained simply by writing, the joint probability density function as the product of a marginal and a conditional density function. Similarly, the downcrossing rate is

$$v_X^-(u, t) = - \int_{-\infty}^0 v p_{X(t)\dot{X}(t)}(u, v) dv = p_{X(t)}(u) \int_{-\infty}^0 |v| p_{\dot{X}(t)}[v | X(t) = u] dv \quad (30)$$

The results in (29) and (30) are rigorous for any stochastic process for which the integrals exist. It should be noted, though, that the values obtained might be infinite if the conditional probability density function of  $\dot{X}(t)$  does not decay sufficiently rapidly. For example, if  $E[\dot{X}(t) | X(t) = u]$  does not exist then  $v_X^+(u, t)$  may be infinite. For a stationary  $\{X(t)\}$  process these rates, of course, are independent of time, and will be denoted simply as  $v_X^+(u)$  and  $v_X^-(u)$ .



**Figure 3. Phase Diagram showing the Event of Upcrossing**

For any covariant stationary process, the random variables  $X(t)$  and  $\dot{X}(t)$  are uncorrelated. In the special case of a Gaussian process, they are then uncorrelated Gaussian random variables, and Gaussian random variables are uncorrelated if and only if they are independent. Thus,  $X(t)$  and  $\dot{X}(t)$  are independent for the stationary Gaussian situation. The integrals can be evaluated explicitly, giving

$$\nu_X^+(u) = \frac{\sigma_{\dot{X}}}{2\pi\sigma_X} e^{-u^2/(2\sigma_X^2)} \quad (31)$$

Note that the ratio  $\sigma_{\dot{X}}/\sigma_X$  is exactly the  $\omega_c$  characteristic frequency defined in connection with the energy-based amplitude  $A_2$  in the preceding section. Also note that the maximum value of the crossing rate occurs when  $u=0$  and it is simply  $\nu_X^+(0) = \omega_c/(2\pi)$ . Thus, the rate of upcrossings of zero by a mean-zero stationary Gaussian process is simply the energy-based average frequency divided by  $2\pi$ . The factor of  $2\pi$ , of course, comes from the fact that  $\omega_c$  represents a frequency in radians per second, while the rate of mean-upcrossings represents a frequency in cycles per second, or Hz. This relationship between mean-crossing rate and characteristic frequency could have been anticipated for any narrowband process, but it holds exactly for a stationary Gaussian process regardless of the bandwidth.

Since a peak of  $X(t)$  occurs whenever  $\dot{X}(t) = 0$  and  $\ddot{X}(t) < 0$ , the rate of occurrence of peaks of  $\{X(t)\}$  is exactly the rate of downcrossings of the level zero by  $\{\dot{X}(t)\}$ :

$$\nu_P(t) \equiv \nu_{\dot{X}}^-(0, t) = \int_{-\infty}^0 |w| p_{\dot{X}(t), \ddot{X}(t)}(0, w) dw = p_{\dot{X}(t)}(0) \int_{-\infty}^0 |w| p_{\ddot{X}(t)}(w | \dot{X}(t) = 0) dw \quad (32)$$

Similarly, the rate of occurrence of valleys of  $\{X(t)\}$  is  $\nu_V(t) \equiv \nu_X^+(0, t)$ . For a stationary Gaussian process  $\nu_P = \nu_V = \sigma_{\dot{X}}/(2\pi\sigma_X)$ , just as  $\nu_X^+(0) = \sigma_{\dot{X}}/(2\pi\sigma_X)$ .

Knowledge of the rates of occurrence of peaks and crossings can also be used to define a measure of bandwidth of a stochastic process. In particular, any sufficiently long

continuous time history of a process must have at least as many peaks as it has upcrossings of any level, and the number of peaks is expected to be only slightly larger than the number of upcrossings of zero for a mean-zero narrowband process. Thus, the irregularity factor, defined as  $IF = \nu_X^+(0,t)/\nu_P(t)$  can be used as a bandwidth measure for any stochastic process. Its range of possible values is from zero to unity, and it tends to unity for a narrowband process, just as for the previously defined bandwidth parameters. For a stationary Gaussian process the irregularity factor becomes  $IF = \sigma_{\dot{X}}^2 / (\sigma_X \sigma_{\ddot{X}})$ . Somewhat surprisingly,  $IF$  for this class of processes is exactly the same as the  $\alpha_2$  bandwidth parameter defined in (26). This allows estimation of its value from occurrence rates for crossings and peaks of a time history, as well as from integrals in the frequency domain, provided that  $\sigma_{\dot{X}}$  is finite and  $\alpha_2 \neq 0$ .

## PROBABILITY DISTRIBUTION OF PEAKS

The probability distribution of the peaks (local maxima) of  $\{X(t)\}$  can be found by a procedure that is basically the same as that used in deriving the rates of occurrence of crossings or peaks. An occurrence rate  $\nu_P[t; X(t) \leq u]$  for peaks below a level  $u$  is defined as  $P(\text{peak} \leq u \text{ during } [t, t + \Delta t]) / \Delta t$  just as  $\nu_P(t) = P(\text{peak during } [t, t + \Delta t]) / \Delta t$ . What is commonly called the distribution of peaks is really the conditional distribution given the existence of a peak at that time:

$$\begin{aligned}
 F_{P(t)}(u) &= \frac{P(\text{peak} \leq u \text{ during } [t, t + \Delta t])}{P(\text{peak during } [t, t + \Delta t])} = \frac{\nu_P[t; X(t) \leq u]}{\nu_P(t)} \\
 &= \frac{\int_{-\infty}^0 \int_{-\infty}^u |z| p_{X(t)\dot{X}(t)\ddot{X}(t)}(w, 0, z) dw dz}{\int_{-\infty}^0 |z| p_{\dot{X}(t)\ddot{X}(t)}(0, z) dz} \tag{33}
 \end{aligned}$$

and taking a derivative with respect to  $u$  gives the probability density function as

$$p_{P(t)}(u) = \frac{\int_{-\infty}^0 |z| p_{X(t)\dot{X}(t)\ddot{X}(t)}(u, 0, z) dz}{\int_{-\infty}^0 |z| p_{\dot{X}(t)\ddot{X}(t)}(0, z) dz} \tag{34}$$

Either (33) or (34) describes the probability distribution of any peak that occurs within the vicinity of time  $t$ . The probability that the peak is within any given interval can be found directly from (33) or from integration of (34), and these formulas are also convenient for evaluating other quantities such as the mean value and variance of  $P(t)$ .

Note that the probability distribution of the peak  $P(t)$  depends on the joint probability distribution of  $X(t)$ ,  $\dot{X}(t)$ , and  $\ddot{X}(t)$ . This joint distribution is relatively simple for the special case of a stationary mean-zero Gaussian process. All that is needed is the covariance matrix of the three random variables, and this involves only the three

standard deviations and the correlation coefficient between  $X(t)$  and  $\ddot{X}(t)$ , since  $\dot{X}(t)$  is uncorrelated with the pair  $[X(t), \ddot{X}(t)]$  for any stationary process. It is easily shown, however, that the correlation coefficient between  $X(t)$  and  $\ddot{X}(t)$  is exactly the negative of the  $\alpha_2$  bandwidth parameter. Thus, this bandwidth parameter plays an important role in the distribution of the peaks of  $\{X(t)\}$ . The resulting formulas are

$$p_P(u) = \frac{(1-\alpha_2^2)^{1/2}}{(2\pi)^{1/2}\sigma_X} e^{-u^2/[2\sigma_X^2(1-\alpha_2^2)]} + \frac{\alpha_2 u}{\sigma_X^2} e^{-u^2/(2\sigma_X^2)} \Phi\left(\frac{\alpha_2 u}{(1-\alpha_2^2)^{1/2}\sigma_X}\right) \quad (35)$$

and

$$F_P(u) = \Phi\left(\frac{u}{(1-\alpha_2^2)^{1/2}\sigma_X}\right) - \alpha_2 e^{-u^2/(2\sigma_X^2)} \Phi\left(\frac{\alpha_2 u}{(1-\alpha_2^2)^{1/2}\sigma_X}\right) \quad (36)$$

in which  $\Phi(y) \equiv [1 + \text{erf}(y/2^{1/2}\sigma_Y)]/2$  is the cumulative distribution function  $F_Y(y)$  for a mean-zero random variable  $Y$ . The formulas in (35) and (36) are commonly referred to as the Rice distribution, in recognition of Rice's pioneering work on this problem [1945].

The limiting forms of this distribution for  $\alpha_2 = 1$  and  $\alpha_2 = 0$  yield important results regarding the peak distribution. For the narrowband situation with  $\alpha_2$  approaching unity, some of the arguments in  $p_P(u)$  and  $F_P(u)$  tend to infinity and the limit for  $\alpha_2 = 1$  is exactly the Rayleigh distribution of (27) that describes both the  $A_1(t)$  and  $A_2(t)$  amplitudes of the Gaussian process. The agreement of the peak distribution and the amplitude distribution of the limiting narrowband process is consistent with the fact that a narrowband process can be considered to be a harmonic function with slowly varying amplitude and phase. Because the narrowband amplitude varies slowly, each peak of the narrowband process is equal to the amplitude of the process at that instant of time, so it is not surprising that the two quantities have the same probability distribution. For the opposite extreme situation with  $\alpha_2 = 0$ , the probability distribution of the peaks becomes exactly the Gaussian distribution of  $X(t)$ , which is consistent with the fact that the rate of occurrence of peaks is infinite for  $\alpha_2 = 0$ . That is, there may be peaks everywhere along the process so the distribution of peaks is same as the distribution of  $X(t)$ .

A convenient feature of the Rice distribution in (35) and (36) is that it also describes a random variable  $R = \alpha_2 R_1 + (1-\alpha_2^2)^{1/2} R_2$ , in which  $R_1$  and  $R_2$  are independent random variables with  $R_1$  having the Rayleigh distribution of the  $A(t)$  amplitude and  $R_2$  having the mean-zero Gaussian distribution of  $X(t)$  [Krenk, 1978].

The distribution of peaks of a mean-zero narrowband process is sometimes approximated by a function that can be obtained from knowledge only of the crossing rates of  $X(t)$ . The rationale is that in the narrowband case one can ignore the possibility of peaks below zero or valleys above zero. With this simplification, a peak occurs within the interval  $[u, u + \Delta u]$  if and only if there is an upcrossing of the level  $u$  that is not followed by an upcrossing of the level  $u + \Delta u$ . This approximation then implies that the

expected number of peaks within an interval  $[u, u + \Delta u]$  is the difference between the number of upcrossings of the level  $u$  and the number of upcrossings of the level  $u + \Delta u$ , and the expected rate of occurrence of peaks in the interval is  $\nu_X^+(u) - \nu_X^+(u + \Delta u)$ . After normalization this gives

$$p_{P(t)}(u) \approx \frac{-1}{\nu_X^+(\mu_X)} \frac{d\nu_X^+(u)}{du} \quad (37)$$

for  $u \geq 0$ . For the special case of a stationary Gaussian process this result is exactly the Rayleigh distribution, which is the true answer only in the limiting case with  $\alpha_2 = 1$ . For a non-Gaussian process, as well, one can anticipate that (37) will be asymptotically correct for  $\alpha_2$  approaching unity. The approximation may be significantly inaccurate for some more broadband situations.

## FIRST-PASSAGE AND GLOBAL MAXIMUM

Finding the probability distribution of a global maximum within an interval  $[0, t]$  is considerably more complicated than finding the distribution for a peak (i.e., a local maximum). The extreme value distribution for the process  $\{X(t)\}$  and any fixed time  $t$  is defined to be the distribution of the random variable

$$\hat{X}(t) = \max_{0 \leq s \leq t} X(s) \quad (38)$$

which makes  $\{\hat{X}(t)\}$  a stochastic process. Note that even for a stationary  $\{X(t)\}$ , one must expect that  $\{\hat{X}(t)\}$  will be nonstationary, since larger and larger values of  $X(t)$  will generally occur if the period of observation is extended. Letting  $L_X(u, t)$  denote the cumulative distribution function of  $\hat{X}(t)$  gives

$$L_X(u, t) \equiv F_{\hat{X}(t)}(u) \equiv P[\hat{X}(t) \leq u] \equiv P[X(s) \leq u; 0 \leq s \leq t] \quad (39)$$

in which the notation on the final term means that the  $X(s) \leq u$  inequality holds for all the given  $s$  values. This  $L_X(u, t)$  function is sometimes called the probability of survival, which is certainly appropriate if  $u$  denotes a critical value for  $\{X(t)\}$  corresponding to some failure mode of the system. The probability density function for the extreme value is simply the derivative  $\partial L_X(u, t) / \partial u$ , and from this information one can also calculate the mean and variance of the extreme value.

An alternative problem that is almost equivalent to extreme value analysis involves the random quantity called first-passage time. Let  $T_X(u)$  denote the first time (after time zero) at which  $X(t)$  has an upcrossing of the level  $u$ . That is,

$X[T_X(u)] = u$ ,  $\dot{X}[T_X(u)] > 0$ , and there has been no upcrossing in the interval  $0 \leq t < T_X(u)$ . This  $T_X(u)$  quantity is a random variable for any given  $u$  value.

It should be noted that some caution is necessary in analyzing either the global maximum response or the first-passage time for a process  $\{Y(t)\}$  that has a nonstationary mean-value function  $\mu_Y(t)$ . In this situation the event  $\{Y(s) \leq v: 0 \leq s \leq t\}$  is  $\{X(s) \leq v + \mu_Y(s): 0 \leq s \leq t\}$  for the mean-zero process  $X(t) = Y(t) - \mu_Y(t)$ . Similarly a crossing of the level  $v$  by  $Y(t)$  coincides with a crossing of  $v - \mu_Y(t)$  by  $X(t)$ . This amounts to a problem with a variable barrier level, and such problems are not explicitly studied here since they are less frequently encountered in practice.

To see the relationship between the first-passage time and the extreme value distribution consider the event  $\{X(s) \leq u: 0 \leq s \leq t\}$  that appears in (39). This event can also be written as  $\{X(0) \leq u, T_X(u) \geq t\}$ , since  $X(s)$  can be less than  $u$  throughout the time interval only if it starts below  $u$  and does not have an upcrossing during the time interval. Thus

$$\begin{aligned} L_X(u, t) &= P[X(0) \leq u]P[T_X(u) \geq t | X(0) \leq u] \\ &= L_X(u, 0)P[T_X(u) \geq t | X(0) \leq u] \end{aligned} \quad (40)$$

and

$$p_{T_X(u)}[t | X(0) \leq u] = \frac{-1}{L_X(u, 0)} \frac{\partial}{\partial t} L_X(u, t) \quad (41)$$

Thus, the partial derivative of  $L_X(u, t)$  with respect to  $t$  gives the conditional probability density of the first-passage time, just as the partial derivative with respect to  $u$  gives the probability density of the extreme response  $\hat{X}(t)$ . This close relationship between the extreme value problem and the first-passage problem is not always mentioned in the literature, with some authors using only one terminology and some using only the other.

In some problems, one can neglect the conditioning in (40) and (41), treating  $T_X(u)$  as independent of  $X(0)$ . In particular, this is true in problems with an initial condition of  $P[X(0) \leq u] = 1$ , such as when the system is known to start at  $X(0) = 0$ . Even in situations in which there is no specific information of independence, it is usually true that the effect of the initial-value conditioning is significant only for small values of  $t$ .

It is often convenient to write the probability of survival in an exponential form of

$$L_X(u, t) = L_X(u, 0) \exp\left(-\int_0^t \eta_X(u, s) ds\right) \quad (42)$$

which follows directly from a definition of  $\eta_X(u, t) \equiv [-1/L_X(u, t)] \partial L_X(u, t) / \partial t$ . This gives

$$\eta_X(u, t) \Delta t = P(\text{upcrossing in } [t, t + \Delta t] | X(0) \leq u, \text{ no upcrossing prior to } t) \quad (43)$$

Comparing (43) with the upcrossing rate in (28) shows that  $\eta_X(u,t)$  is a conditional form of  $v_X^+(u,t)$ . Thus,  $\eta_X(u,t)$  can be considered to be the conditional rate of upcrossings of the level  $u$ , given the initial condition and the fact that there has been no prior upcrossing. The idea of a first-crossing rate seems contradictory, though, since there can only be one first-crossing for a given time history. If exceeding the level  $u$  is considered to correspond to a failure of the system, then  $\eta_X(u,t)$  is what is called the hazard function in reliability theory.

Unfortunately, it is not easy to calculate  $\eta_X(u,t)$  in order to find the probability distribution of either the first-passage time  $T_X(u)$  or the extreme value  $\hat{X}(t)$ . The only rigorous, general relationship for  $\eta_X(u,t)$  is the so-called inclusion-exclusion series, which involves an infinite sequence of rather complicated integrals involving the joint distribution of  $X$  and  $\dot{X}$  at multiple time values [e.g., see Madsen et al., 1986].

In many practical situations  $\eta_X(u,t)$  tends asymptotically to a function  $\eta_X(u)$  that is independent of  $t$  as  $t$  becomes large. In particular, this asymptotic behavior is likely if  $\{X(t)\}$  is stationary, or becomes stationary for large values of  $t$ . The reason is that most physical processes have only a finite memory, in the sense that conditioning by an event far in the past has little effect on the current behavior of the process. When this is true one can use an approximation

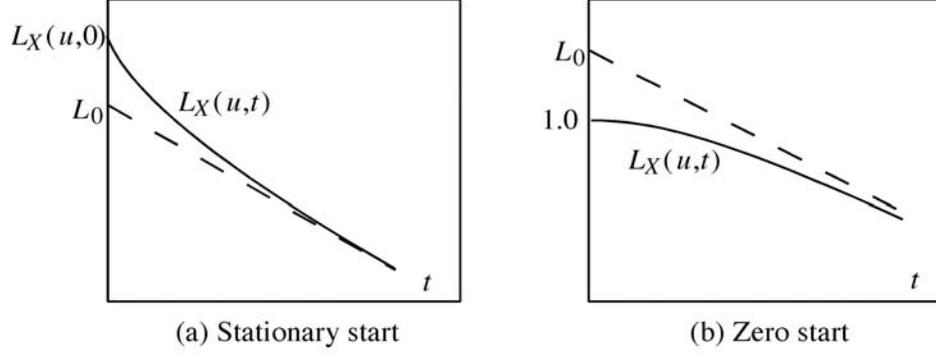
$$L_X(u,t) \approx L_0 e^{-\eta_X(u)t} \quad \text{for large } t \quad (44)$$

If there exists a time value  $T_0$  such that  $\eta_X(u,t) \approx \eta_X(u)$  for  $t \geq T_0$ , then (44) agrees with (42) for  $t \geq T_0$  if

$$L_0 = L_X(u,0) \exp\left(-\int_0^{T_0} [\eta_X(u,s) - \eta_X(u)] ds\right) \quad (45)$$

The value of  $L_0$  then depends on the way in which  $\eta_X(u,t)$  for small  $t$  differs from  $\eta_X(u)$ . For example, if  $\{X(t)\}$  is stationary for all time  $t$ , then  $\eta_X(u,t)$  for small time is usually greater than the stationary value  $\eta_X(u)$  and  $L_0$  is smaller than  $L_X(u,0) < 1$ . On the other hand, a common nonstationary process has the “zero-start” condition of  $X(0) = 0$  and  $\dot{X}(0) = 0$ , as applies to the response of a dynamical system that starts from a condition of rest. In this case,  $\eta_X(u,t)$  starts from zero and increases as the response grows, giving  $L_0 > L_X(u,0) = 1$ . These behaviors are illustrated in Figure 4.

Note that (42) and (45) divide the probability of failure into two parts. The exponential terms relate to the probability of crossing the level  $u$ , while the multiplier  $L_X(u,0)$  accounts for the fact that a process may start above the level  $u$ , which is given by  $P[X(0) \leq u]$ . For the mean-zero narrowband process, note that there is a high probability of an early upcrossing if  $A(0) > u$ . Since the narrowband amplitude varies slowly with time, it is very likely that any  $X(t)$  sample with  $A(0) > u$  will exceed  $u$  during the first cycle. One way to approximate this effect is to treat the event  $A(0) > u$  as



**Figure 4. Effect of initial conditions on  $L_0$  in (44).**

though it were failure at time zero, so that the multiplier  $L_X(u,0)$  in (42) and (45) is taken as  $P[A(0) \leq u]$ . Clearly, this approximation does not accurately describe the reliability during the first cycle of the mean-zero narrowband process, but it can give useful results for later times, which are generally of greater interest.

Any error in estimating  $L_0$  in (44) results in a proportional error in the predicted reliability  $L_X(u,t)$ . An error in  $\eta_X(u)$ , though, is much more important since it leads to an error in  $L_X(u,t)$  that grows exponentially. Thus, there has been considerable effort invested in estimating this limiting  $\eta_X(u)$  function. The following section summarizes some of the results obtained by direct consideration of the first-passage time.

## FIRST-PASSAGE FORMULATIONS

The simplest approximation of  $\eta_X(u,t)$  is related to the similarity between (43) and (28). Since  $\eta_X(u,t)$  is a conditional form of the upcrossing rate, simply neglecting the conditioning gives  $\eta_X(u,t) \approx v_X^+(u,t)$ . For any process this gives

$$L_X(u,t) \approx L_X(u,0) \exp\left(-\int_0^t v_X^+(u,s) ds\right) \quad (46)$$

and in the particular case of a stationary mean-zero process it reduces to the form in (44) with  $L_0 = L_X(u,0)$  which may be taken as either  $P[X(0) \leq u]$  or  $P[A(0) \leq u]$ . The formula in (46) is commonly called the Poisson approximation of the first-passage problem. This name comes from the fact that if the integer-valued process that counts the number of upcrossings by  $X(t)$  were a Poisson process, then the crossing rate would be independent of the past history of the process and the conditioning neglected in (46) would be irrelevant.

For a stationary  $\{X(t)\}$  process, the Poisson approximation gives the first passage time  $T_X(u)$  as having a simple exponential distribution with  $P[T_X(u) \leq t] = 1 - e^{-v_X^+(u)t}$ . This then gives the mean and standard deviation of  $T_X(u)$  as both being equal to

$[\nu_X^+(u)]^{-1}$ . In the particular case of a stationary mean-zero Gaussian process, the crossing rate in (31) then gives

$$L_X(u,t) \approx L_X(u,0) \exp\left(\frac{-\omega_c t}{2\pi} e^{-u^2/(2\sigma_X^2)}\right) \quad (47)$$

in which  $\omega_c$  is again the characteristic frequency defined in connection with the energy-based amplitude  $A_2$ . Also, the value of  $P[X(0) \leq u]$  or  $P[A(0) \leq u]$  for use as  $L_X(u,0)$  is easily calculated for the Gaussian process, for which  $A$  has the Rayleigh distribution. It should be noted that (47) can only be considered to be an approximation of  $L_X(u,t)$  for  $u \geq 0$ . In particular,  $P[\hat{X}(t) \leq u] \equiv L_X(u,t)$  must be an increasing function of  $u$ , and this condition is not met by (47) for  $u < 0$ . Thus, one must consider (47) to give a finite probability that  $\hat{X}(t)$  is equal to zero:  $P[\hat{X}(t) = 0] = L_X(0,0) e^{-\omega_c t/(2\pi)}$ . This discontinuity in  $F_{\hat{X}(t)}(u)$  decays toward zero as  $t$  increases.

The error of the Poisson approximation is most serious when the  $\{X(t)\}$  process is very narrowband. In that situation, an upcrossing of level  $u$  at time  $t$  is very likely to be associated with another upcrossing approximately one period later, due to the slowly varying amplitude of  $\{X(t)\}$ . Such a ‘‘clumping’’ of the upcrossing times is inconsistent with the Poisson approximation that the times between upcrossings are independent. On the other hand, when  $u$  is very large it is found that the independence assumption seems to be better. There are few general results that apply to all  $\{X(t)\}$  processes, but for stationary Gaussian processes it has been demonstrated that  $\eta_X(u,t)$  does tend asymptotically to  $\nu_X^+(u)$  as  $u$  tends to infinity [Cramer 1966]. Thus, the Poisson approximation is best when the  $\{X(t)\}$  process is very broadband and/or the level  $u$  is very large. In some narrowband situations, it may be significantly in error.

From the form of (42) it is clear that an overestimation of  $\eta_X(u,t)$  results in an underestimation of  $L_X(u,t)$ . An error of this type is usually considered to be conservative, because it overestimates the probability of failure due to large excursions. Furthermore, in most practical situations it is true that  $\eta_X(u,t) \leq \nu_X^+(u,t)$ , so that the Poisson approximation does underestimate  $L_X(u,t)$ . Some caution is appropriate, though, since there are situations in which  $\eta_X(u,t) > \nu_X^+(u,t)$ . In particular, if the level  $u$  is so small that  $P[X(t) < u]$  is also small and if there is an initial condition that  $X(0) < u$ , then it is very likely that  $X(t)$  will quickly have an upcrossing of  $u$ . Mathematically this requires that  $L_X(u,t)$  approach zero as  $u \rightarrow -\infty$ , for any finite  $t$  value. This is ensured for any choice of  $L_X(u,0)$  only if  $\eta_X(u,t)$  tends to infinity for  $u \rightarrow -\infty$ , at least for  $t \approx 0$ , clearly requiring that  $\eta_X(u,t) > \nu_X^+(u,t)$ . This small- $u$  situation, though, is not often of practical importance in reliability analysis.

There are several ways in which the Poisson estimate has been modified to give somewhat better results. One of the simplest is an attempt to address, in an approximate

way, the noted underestimation of  $\eta_X(u,t)$  for situations in which  $u$  is so low that the average time between upcrossings may be much larger than the time until the first upcrossing. Instead of the usual Poisson assumption that the probability distribution of the first-passage time  $T_X(u)$  for a stationary process is the same as that of the total time interval between successive upcrossings, the modification is based on the idea that  $T_X(u)$  has more in common with the time spent below the level  $u$  between successive upcrossings. This reduces the estimate of the time until first passage and gives more conservative estimates of failure probabilities for small values of  $u$  but has little effect for large  $u$  values. A simple approximation of this time increment below  $u$  gives  $\eta_X(u,t) \approx v_X^+(u,t)/F_{X(t)}(u)$  [Lutes and Sarkani, 2004]. This approximation does give the desired behavior that  $L_X(u,t)$  tends to zero as  $u$  tends to negative infinity. This same approximation of  $\eta_X(u,t)$  was earlier obtained by Ditlevsen [1986] by a somewhat different method of reasoning.

Attention will now be focused on more significant modifications to the Poisson approximation. In particular, the aim of these approaches is to give better results for  $\eta_X(u)$  for a narrowband process, in which the upcrossing times cannot be independent because of the slowly varying amplitude  $\{A(t)\}$ . The simplest extreme value approximation based on the slowly varying amplitude process amounts to assuming that the extreme value of the mean-zero  $\{X(t)\}$  process is the same as that of  $\{A(t)\}$ , and that the Poisson approximation applies to  $\{A(t)\}$ . This gives  $L_X(u,t) \approx L_A(u,t)$ , in which  $L_A(u,t)$  is the same as (46) except that  $v_X^+(u)$  is replaced by  $v_A^+(u)$ . By exactly the same reasoning as was used in suggesting that  $v_X^+(u)/F_{X(t)}(u)$  was preferable to  $v_X^+(u)$  for the original Poisson approximation, one can also say that  $v_A^+(u)/F_{A(t)}(u)$  is preferable to  $v_A^+(u)$  for this Poisson-amplitude approximation.

For the special case of a stationary mean-zero Gaussian process, the  $v_A^+(u)$  crossing rate has been evaluated for both the Cramer/Leadbetter and energy-based amplitude [e.g., Lutes and Sarkani, 2004]. The results are

$$v_{A_1}^+(u) = \frac{u}{\sigma_X} e^{-u^2/(2\sigma_X^2)} \left( \frac{\sigma_{\dot{X}}}{\sigma_X} \right) \left( \frac{1 - \alpha_1^2}{2\pi} \right)^{1/2} \quad (48)$$

and

$$v_{A_2}^+(u,t) = \frac{u}{\sigma_X} e^{-u^2/(2\sigma_X^2)} \left( \frac{\sigma_{\ddot{X}}}{\sigma_{\dot{X}}} \right) \left( \frac{4(1 - \alpha_2^2)^{1/2}}{(2\pi)^{3/2}} \right) \quad (49)$$

and the probability estimate for first-cycle crossing of a narrowband process is  $F_{A(t)}(u) = 1 - e^{-u^2/(2\sigma_X^2)}$ . Thus, implementation of the Poisson-amplitude approximation is very straightforward.

It is generally presumed that the Poisson assumption is much better for  $\{A(t)\}$  than for  $\{X(t)\}$ , and that the amplitude-crossing approach is conservative, since the maximum of  $X(t)$  cannot exceed the maximum of  $A(t)$ . Numerical studies show that the Poisson-amplitude approximation does give significantly improved estimates of  $\eta_X(u, t)$  for small to moderate values of  $u$  values, but it is seriously in error when  $u$  is large. Whereas the original Poisson approximation of  $\eta_X(u) \approx \nu_X^+(u)$  becomes asymptotically correct as  $u$  goes to infinity,  $\nu_A^+(u)/\nu_X^+(u)$  grows without limit as  $u$  increases, giving grossly overly conservative estimates of  $\eta_X(u)$ .

An improved approximation of the extreme value distribution can be obtained by estimating the fraction of the upcrossings by  $A(t)$  that are accompanied by upcrossings by  $X(t)$ . The  $\eta_X(u, t)$  conditional crossing rate can then be taken to represent the rate of occurrence of this subset of the amplitude-upcrossings. Vanmarcke introduced such a scheme [Vanmarcke, 1972; Corotis et al., 1972; Vanmarcke, 1975] and Madsen et al. [1986] derived a very similar result. Some of the assumptions in these approximations will be summarized here, along with the formulas.

Let the random variable  $T_1$  denote the time between an upcrossing of  $u$  by  $A(t)$  and the subsequent downcrossing by  $A(t)$ . Then  $T_1$  represents the duration of an interval with  $A(t) > u$ . If  $T_1$  is large, then it seems almost certain that  $X(t)$  will have an upcrossing of  $u$  within the interval, but if  $T_1$  is small, then it seems quite likely that no upcrossing by  $X(t)$  will occur. Vanmarcke approximated this relationship by

$$P[\text{no upcrossing by } X(t) | T_1 = \tau] \approx [1 - \nu_X^+(0, t) \tau] U[1 - \nu_X^+(0, t) \tau] \quad (50)$$

Considering  $[\nu_X^+(0, t)]^{-1}$  to represent the period of an average cycle of the  $\{X(t)\}$  process, this approximation amounts to saying that an upcrossing by  $X(t)$  is sure if  $T_1$  exceeds the period, and the probability of its occurrence grows linearly with  $T_1$  for  $T_1$  less than the period. Even though this approximation is crude, it is substantially better than simply assuming that an upcrossing by  $X(t)$  occurs in connection with every upcrossing by  $A(t)$ . To calculate the unconditional probability of an upcrossing in the  $T_1$  interval, it is necessary to have a probability distribution for  $T_1$ , and this is assumed to be the exponential distribution  $p_{T_1}(\tau) = e^{-\tau/E(T_1)}/E(T_1)$ . Based on the usual formulation that the mean time between upcrossings is  $[\nu_A^+(u, t)]^{-1}$ , the mean time below  $u$  during the interval is estimated as  $E(T_1) \approx P[A(t) > u]/\nu_A^+(u, t)$ . Using this along with  $\eta_X(u, t) \approx \nu_A^+(u, t) P[\text{upcrossing by } X(t) \text{ during } T_1]$  gives

$$\eta_X(u, t) \approx P[A(t) > u] \nu_X^+(0, t) \left( 1 - \exp \left[ \frac{-\nu_A^+(u, t)}{P[A(t) > u] \nu_X^+(0, t)} \right] \right) \quad (51)$$

In the limit as  $u$  tends to zero,  $P[A(t) > u]$  tends to unity and  $\eta_X(u, t)$  tends to  $v_A^+(u, t)$ , as when considering each upcrossing by  $A(t)$  to correspond to a crossing by  $X(t)$ . For large values of  $u$ ,  $P[A(t) > u]$  is very small, so  $v_A^+(u, t) \gg P[A(t) > u]v_X^+(0, t)$  and (51) gives  $\eta_X(u, t) \approx P[A(t) > u]v_X^+(0, t)$ . For the special case of a Gaussian process it can be shown that this latter limit is identical to  $v_X^+(u, t)$ , so the approximation agrees with the results from the assumption of Poisson crossings by  $X(t)$ . For a non-Gaussian process, these two results for large  $u$  values may not be identical, although they are expected to be quite similar.

As in our other approximations, one can expect to obtain better results for small  $u$  values by including the effect of the initial condition. Because this estimate of  $\eta_X(u, t)$  is a modified version of the amplitude-crossing rate, it is reasonable to do this by dividing by  $P[A(t) < u]$ , giving

$$\eta_X(u, t) \approx \frac{P[A(t) > u]v_X^+(0, t)}{P[A(t) < u]} \left( 1 - \exp \left[ \frac{-v_A^+(u, t)}{P[A(t) > u]v_X^+(0, t)} \right] \right) \quad (52)$$

which agrees with  $v_A^+(u, t)/F_{A(t)}(u)$  in the limit for  $u$  near zero.

It should be noted that for a general non-Gaussian  $\{X(T)\}$  process, (52) is not identical to Vanmarcke's result and that his derivation uses somewhat more sophisticated assumptions about the behavior of  $\{X(T)\}$ . For the special case of the Gaussian process, though, it can be shown that  $P[A_1(t) > u]v_X^+(0, t) = v_X^+(u, t)$ , so (52) with the Cramer and Leadbetter definition of amplitude does become identical to Vanmarcke's form of

$$\eta_X(u, t) \approx v_X^+(u, t) \left( 1 - \exp \left[ \frac{-v_{A_1}^+(u, t)}{v_X^+(u, t)} \right] \right) \left( 1 - \frac{v_X^+(u, t)}{v_X^+(0, t)} \right)^{-1} \quad (53)$$

It can be expected that the two approximations will also give similar results for other processes that do not differ greatly from the Gaussian distribution.

For the special case of a stationary, mean-zero, Gaussian  $\{X(t)\}$  process, Vanmarcke [1975] also offered an empirical correction that improves the approximation of the conditional crossing rate, at least for small to moderate values of  $u$ . For this situation, the ratio  $v_A^+(u)/v_X^+(u)$  of (53) is  $[2\pi(1-\alpha_1^2)]^{1/2}(u/\sigma_X)$  in which the term  $(1-\alpha_1^2)$  introduces the effect of the bandwidth of the process. As an empirical correction for effects not included in the derivation of these equations, Vanmarcke suggested replacing the  $(1-\alpha_1^2)$  term with  $(1-\alpha_1^2)^{1.2}$ . This gives the modified Vanmarcke approximation as

$$\frac{\eta_X(u)}{v_X^+(u)} \approx \left( 1 - \exp \left[ -(1 - \alpha_1^2)^{0.6} (2\pi)^{1/2} \frac{u}{\sigma_X} \right] \right) \left( 1 - \exp \left[ \frac{-u^2}{2\sigma_X^2} \right] \right)^{-1} \quad (54)$$

In many problems, failure may occur due to large excursions of  $X(t)$  in either the positive or negative direction, whereas all the development up until now has been concerned only with the probability that  $X(t)$  remains below  $+u$ . However, the event of  $X(t)$  remaining within  $[-u, +u]$  is exactly the same as the event of  $|X(t)|$  remaining below the level  $u$ , and this allows the new problem to be written in the form of (42), but with  $L_{|X|}(u, 0)$  and  $\eta_{|X|}(u, s)$  used in place of  $L_X(u, 0)$  and  $\eta_X(u, s)$ , respectively. The terms double-barrier problem and single-barrier problem are often used to distinguish between upcrossings by  $|X(t)|$  and  $X(t)$ , respectively. Of course, one can also consider double-barrier problems in which the levels of interest are not symmetric. Modifications to the various first-passage approximations will be summarized for the symmetric double-barrier problem.

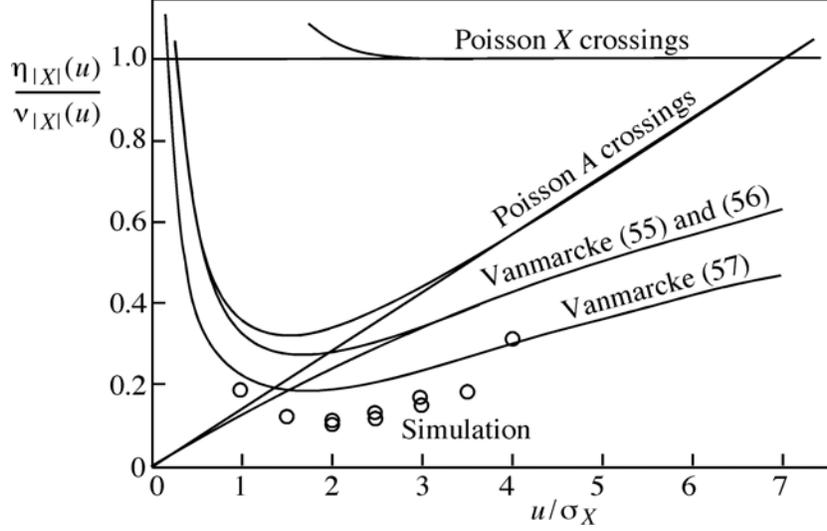
The double-barrier Poisson approximation is that  $\eta_{|X|}(u, s) = v_X^+(u, s) + v_X^-(-u, s)$ . If the distribution of  $X(t)$  and  $\dot{X}(t)$  is symmetric, this then gives  $\eta_{|X|}(u, s) \approx 2v_X^+(u, s)$ , indicating that the decay of  $L$  with increasing  $t$  is twice as fast as for the single-barrier problem. The small-time modification of dividing  $v_{|X|}^+(u, s)$  by  $F_{|X|}(u)$  has more effect than the corresponding adjustment for the single barrier problem, since  $F_{|X|}(u)$  tends to zero as  $u$  tends to zero. The approximation of  $\eta$  by  $v_A^+(u, s)$  or  $v_A^+(u, s)/F_{|X|}(u)$  is equally valid for the double-barrier problem as for the single-barrier situation. That is, for the mean-zero process, the extreme distribution of  $A(t)$  can be considered an approximation of the extreme distribution of  $|X(t)|$ , just as well as for the extreme distribution of  $X(t)$ . In fact, it seems likely that the approximation will be better for  $|X(t)|$  than for  $X(t)$ .

Adapting the Vanmarcke approximation to the double-barrier problem requires estimation of the probability that an upcrossing by  $A(t)$  is accompanied by an upcrossing by  $|X(t)|$  during the interval of length  $T_1$  between an upcrossing of  $u$  by  $A(t)$  and the subsequent downcrossing. This probability is estimated as zero for  $T_1 = 0$  and growing linearly to unity for  $T_1$  equal to the half-period  $[v_X^+(0, t)]^{-1}/2$ . Other assumptions are the same as for the single-barrier situation and the result corresponding to (52) is

$$\eta_{|X|}(u, t) \approx \frac{2P[A(t) > u] v_X^+(0, t)}{P[A(t) < u]} \left( 1 - \exp \left[ \frac{-v_A^+(u, t)}{2P[A(t) > u] v_X^+(0, t)} \right] \right) \quad (55)$$

The corresponding modification of Vanmarcke's formula in (53) is

$$\eta_{|X|}(u, t) \approx v_{|X|}^+(u, t) \left( 1 - \exp \left[ \frac{-v_{A_1}^+(u, t)}{v_{|X|}^+(u, t)} \right] \right) \left( 1 - \frac{v_X^+(u, t)}{v_X^+(0, t)} \right)^{-1} \quad (56)$$



**Figure 5. Simulation and various approximations.**

which is identical to using the Cramer and Leadbetter amplitude in (55) for a mean-zero Gaussian  $\{X(t)\}$  process. Using the empirical adjustment in (54) for a stationary process changes this to

$$\frac{\eta_{|X|}(u)}{\nu_{|X|}^+(u)} \approx \left( 1 - \exp \left[ - (1 - \alpha_1^2)^{0.6} (\pi/2)^{1/2} \frac{u}{\sigma_X} \right] \right) \left( 1 - \exp \left[ \frac{-u^2}{2\sigma_X^2} \right] \right)^{-1} \quad (57)$$

An illustration is given in Figure 5 of the effects of the various assumptions on the double-barrier estimates of  $\eta_{|X|}(u)$  for a stationary, mean-zero, Gaussian, narrowband process with  $\alpha_1 = 0.994$ . The results are presented as the ratio  $\eta_{|X|}(u)/\nu_{|X|}^+(u)$ , a normalization introduced by Crandall et al. [1966] and commonly used since then. Thus, the results for the assumption of Poisson crossings by  $X(t)$  appear as unity on this plot, even though they are inaccurate for such a narrowband process. The results for the  $\nu_A^+(u)$  approximation are based on the Cramer/Leadbetter amplitude. There are two curves each for Poisson  $X$  crossings and Poisson  $A$  crossings, showing the increase in the value of  $\eta$  when the crossing rates are divided by  $F_{|X|}(u)$  and  $F_A(u)$ , respectively. Only the results for Vanmarcke's approximation give all the desired tendencies of  $\eta_X(u)$ : tending to infinity for  $u$  going to zero, being less than  $\nu_X^+(u)$  for intermediate values of  $u$ , and tending to  $\nu_X^+(u)$  for large values of  $u$ .

The plot in Figure 5 also includes some simulation results for one particular stationary, mean-zero, Gaussian process with the specified value of  $\alpha_1$  [Lutes and Sarkani, 2004]. It is seen that the simulated values of  $\eta_X^+(u)$  are smaller than the predictions from any of the approximations, except when  $u$  is less than about  $1.2\sigma_X$ . Also, the simulation data clearly show that the  $\eta_X(u)/\nu_X^+(u)$  ratio has a minimum value when  $u$  is approximately  $2\sigma_X$ . The analytical form due to Vanmarcke comes closest to

fitting the simulation data. In particular, the modified Vanmarcke form of (57) gives a minimum value of  $\eta_{|X|}(u)/\nu_X^+(u)$  at approximately the right  $u$  value, and the values it gives for the ratio in this vicinity are better than those of any of the other approximations that have reasonable behavior for smaller  $u$  values. Nonetheless, it must be noted that there is sometimes a significant discrepancy between Vanmarcke's formula and the simulation data. For  $u=2\sigma_X$ , for example, the approximation is about 70% above the value of 0.10 or 0.11 obtained from simulation, even though the modified approximation agrees almost perfectly with the data point from simulation for  $u=4\sigma_X$ . Any overestimation of  $\eta_{|X|}(u)$  gives  $L_X(u,t)$  values that decay more rapidly with increasing  $t$  than do the values from simulation. For  $u=2\sigma_X$  and large values of time, Vanmarcke's formula will significantly overpredict the probability that  $|X(t)|$  has ever reached the level  $u$ , but this discrepancy is much smaller than that of the commonly used assumption of Poisson crossings by  $X(t)$ . Also, the discrepancy is smaller when  $u$  is large, and this is often the region of primary interest in predicting failure. It should also be noted that the simulation data suggest the possibility that the modified Vanmarcke approximation in (57) may be nonconservative for  $u$  values greater than  $4\sigma_X$ . That is, the empirical adjustment in (54) consistently improves the results for moderate values of  $u$ , but it may be excessive for very large values of  $u$ .

## GLOBAL EXTREME FORMULATIONS

Recall that  $\hat{X}(t)$  was defined in (38) as the maximum value of the process  $\{X(t)\}$  within the interval  $[0,t]$ . There is an obvious relationship between the problem of describing this global maximum and the classical problem of describing the maximum of a set of random variables:  $\hat{R} = \max\{R_1, R_2, \dots, R_n\}$ . The simplest version of this problem is when the  $R_j$  random variables are independent and identically distributed, which gives the distribution of  $\hat{R}$  as

$$F_{\hat{R}}(u) = F_R^n(u) \quad p_{\hat{R}}(u) = n F_R^{n-1}(u) p_R(u) \quad (58)$$

The fact that  $F_R(u)$  must tend from zero to unity as  $u$  increases, means that  $F_R^{n-1}(u)$  has a similar form, but is shifted further and further to the right along the  $u$  axis as  $n$  increases. Thus, the probability distribution of  $\hat{R}$  for very large  $n$  values depends only on the behavior of the right-hand tail of  $p_R(u)$ .

One way to apply this random variable idea to the stochastic process problem is to note that  $\hat{X}(t)$  is the maximum peak of  $\{X(t)\}$  during the interval  $[0,t]$ . For the mean-zero Gaussian process,  $\omega_c/(2\pi\alpha_2)$  is the expected peak-occurrence rate, with  $\omega_c = \sigma_{\dot{X}}/\sigma_X$ . Thus, the expected number of peaks is  $n_p = (\omega_c t)/(2\pi\alpha_2)$ . If one now assumes that these  $n_p$  peaks are all independent random variables then

$$L_X(u,t) \equiv F_{\hat{X}(t)}(u) \approx F_P^{n_p}(u) = \exp(\omega_c t \ln[F_P(u)]/[2\pi\alpha_2]) \quad (59)$$

which is exactly the same as the asymptotic crossing-rate form in (44) with  $L_0 = 1$  and  $\eta_X(u) = [-\omega_c/(2\pi\alpha_2)] \ln[F_P(u)]$ .

It has been shown that the large- $u$  results of the Poisson-crossings approximation can also be obtained by assuming that all the peaks within the  $[0, t]$  interval are independent random variables [Crandall, 1970]. To see this, recall that the Rice distribution of the peak  $P$  of a Gaussian process, as given in (36) is exactly the same as the weighted sum of two independent random variables, one having the Rayleigh distribution of the amplitude of  $\{X(t)\}$  and the other having the Gaussian distribution of  $\{X(t)\}$ . The weighting factors for the Rayleigh and Gaussian components, respectively, are the bandwidth parameter  $\alpha_2$  and  $(1-\alpha_2^2)^{1/2}$ . The probability density of the Rayleigh distribution, as given in (27), is proportional to  $ue^{-u^2/(2\sigma_X^2)}$  whereas that of the Gaussian distribution is proportional to  $e^{-u^2/(2\sigma_X^2)}$ . From this comparison it is clear that the Rice distribution will be dominated by the Rayleigh component when  $u$  is very large. Since large- $u$  is the situation of interest when  $n$  is large, this gives  $F_P(u)$  from (36) tending to  $1-\alpha_2 e^{-u^2/(2\sigma_X^2)}$  and  $\ln[F_P(u)]$  tending to  $(-\alpha_2 e^{-u^2/(2\sigma_X^2)})$ . The asymptotic first-crossing rate is then  $\eta_X(u) = (\omega_c/2\pi) e^{-u^2/(2\sigma_X^2)} \equiv \nu_X^+(u)$ , which is exactly the same as the result of the Poisson-crossings assumption for  $X(t)$ .

For a narrowband Gaussian process there is an even simpler approach that gives asymptotic equivalence between (58) and the Poisson  $X$ -crossing approximation. Consider the interval  $[0, t]$  to be divided into  $n_c$  cycles of period  $(2\pi/\omega_c)$ , giving  $n_c = (\omega_c t)/(2\pi)$ . Assume that the amplitude  $A(t)$  can be considered to be constant over each of these cycles and independent in different cycles so that (59) applies, giving

$$L_X(u, t) \approx F_A^{n_c}(u) = \exp(\omega_c t \ln[F_A(u)]/[2\pi]) \quad (60)$$

For the Rayleigh random variables,  $\ln[F_{A(t)}(u)] = \ln[1 - e^{-u^2/(2\sigma_X^2)}]$  and for large  $u$  this tends to  $(-e^{-u^2/(2\sigma_X^2)})$ . Again the asymptotic behavior is the same as the Poisson-crossings result with  $\eta_X(u) = \nu_X^+(u)$ . The formulas in (59) and (60) both give  $\eta_X(u) \geq \nu_X^+(u)$  for small values of  $u$ . In fact, the curves are generally similar to those for  $\eta_X(u) = \nu_X^+(u)/F_X(u)$ .

Naess and Gaidai [2008] recently introduced another approximation that may be viewed as a modification of the formula in (58). The approximation is based on the peaks of the time history, as in (59), but including conditioning based on the values of prior peaks. In particular, the approximation is that

$$L_X(u, t) \approx F_{P_n}^{(n_P-k)}(u | B_k) \quad (61)$$

in which the conditioning event for peak  $P_n$  is  $B_k = \{P_{n-1} \leq u, P_{n-2} \leq u, \dots, P_{n-k} \leq u\}$  (i.e., the preceding  $k$  peaks have all been below  $u$ ). The method used by Naess and Gaidai to estimate the critical quantity  $F_{P_n}(u|B_k)$  involves simulation of one or more relatively long time histories of  $X(t)$ . It is easy to create a computer program to count peaks in a simulated time history and determine the fraction of peaks above  $u$  for which there are  $k$  preceding peaks that are all below  $u$ . This fraction is used as the estimate of  $[1 - F_{P_n}(u|B_k)]$ . One difficulty of such a simulation approach is that it may require a very long time history to find any significant number of peaks above a high  $u$  level. Naess and Gaidai, though, found that  $[1 - F_{P_n}(u|B_k)]$  for large  $u$  is approximated by the form  $q_0 \exp[-a(u-b)^c]$ . After evaluating the parameters to fit observed data, this form could be used to extrapolate  $[1 - F_{P_n}(u|B_k)]$  to larger  $u$  values. The method has been demonstrated to give very good approximations of empirical data for  $k$  values of 3 or 4. It also has been demonstrated for a process with bimodal spectral density comprised of two narrow-band components. In this situation, though, the value of  $k$  must be large enough to include peaks over a selected number of lower-frequency periods. This can be a significant number when the frequencies of the components are well separated.

A modified version of the largest cycle-amplitude approach is based on including some effect of correlation of the cycle amplitudes [Lutes, 2008]. The major simplification is to include only the effect of knowing that  $A(t - \tau_c) < u$  when estimating the probability of  $A(t) < u$ , in which  $\tau_c$  is a representative period of the process. Obviously, this is very similar to the peak conditioning in (61) with  $k=1$ , but with the advantage of working with the distribution of amplitudes rather than peaks. The only new information needed is

$$q(u) \equiv P[A(t) < u | A(t - \tau_c) < u] = \frac{1}{F_A(u)} \int_0^u \int_0^u p_{A(t-\tau_c)A(t)}(v_1, v_2) dv_1 dv_2 \quad (62)$$

or, in a form that is more accurate for numerical evaluations,

$$q(u) = \frac{1}{F_A(u)} \left( 1 - 2 \int_u^\infty \int_0^{v_2} p_{A(t-\tau_c)A(t)}(v_1, v_2) dv_1 dv_2 \right) \quad (63)$$

The result of this approximation is

$$L_X(u, t) \equiv F_{\hat{X}(n_c \tau_c)}(u) \approx F_A(u) q^{(n_c-1)}(u) = [F_A(u)/q(u)] \exp(t \ln[q(u)]/\tau_c) \quad (64)$$

which gives the asymptotic first-crossing rate as  $\eta_X(u) = -\ln[q(u)]/\tau_c$ . Note that  $q(u)$  is bounded as  $F_A(u) \leq q(u) \leq 1$ , and it approaches the upper limit for a narrowband process. This gives an estimate of  $\eta_X(u)$  that can be significantly smaller than that of the Poisson assumption for a narrowband process.

Under certain circumstances the joint probability density for the amplitudes in (62) and (63) reduces to a relatively simple form. In particular, if the  $\{X(t)\}$  process is

Gaussian so that the amplitude has the Rayleigh distribution, and if  $G_{ZZ}(\tau_c) = G_{XX}(\tau_c)$ , for the  $\{Z(t)\}$  used in defining the amplitude, then the jointly Rayleigh distribution is

$$P_{A(t-\tau)A(t)}(u_1, u_2) = \frac{u_1 u_2}{(1-\rho)\sigma_X^4} \exp\left(\frac{-(u_1^2 + u_2^2)}{2(1-\rho)\sigma_X^2}\right) I_0\left(\frac{\rho^{1/2} u_1 u_2}{(1-\rho)\sigma_X^2}\right) \quad (65)$$

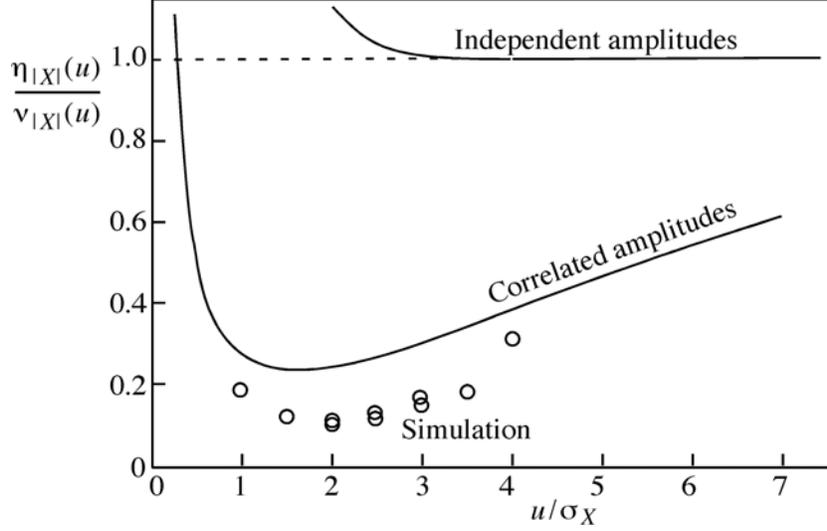
in which  $I_0(\cdot)$  denotes the modified Bessel function of order zero. The new parameter  $\rho$  in (65) is the correlation coefficient of  $A^2(t)$  and  $A^2(t + \pi/\omega_c)$ , so is always in the range  $[0,1]$ . It can also be written as  $\rho = [G_{XX}^2(\tau_c) + G_{XZ}^2(\tau_c)]/\sigma_X^4$ . Although no closed-form solution has been found for the double integration in (62) or (63) for this probability distribution, numerical evaluation is relatively simple. For the Cramer/Leadbetter amplitude,  $G_{ZZ}(\tau) = G_{XX}(\tau)$  for all values of  $\tau$  so that the  $G_{ZZ}(\tau_c) = G_{XX}(\tau_c)$  condition on (64) is automatically satisfied and one can use the natural choice of  $\tau_c = 2\pi/\omega_c$ . For the energy-based amplitude, the condition can sometimes be met by making a small adjustment in  $\tau_c$  [Roberts, 1976].

For the Cramer/Leadbetter amplitude, the quantity  $G_{XZ}(\tau)$  is the negative of the Hilbert transform of  $G_{XX}(\tau)$  and for the energy-based amplitude it is  $-G'_{XX}(\tau)/\omega_c$ . These quantities can be further simplified for the situation of a narrowband process. In particular, it is known that  $G_{XX}(\tau)$  is nearly harmonic with period  $\tau_c$  for a narrowband process, and this nearly harmonic nature of the functions gives  $G_{ZZ}(\tau_c) \approx G_{XX}(\tau_c)$  and  $G_{XZ}(\tau_c) \ll G_{XX}(\tau_c)$ . Thus, the distribution in (64) with  $\rho \approx G_{XX}^2(\tau_c)/\sigma_X^4$  is a good approximation for any narrowband process.

This method of considering conditioning of cycle-amplitudes as opposed to peaks, as in (64), has the advantage of avoiding numerical simulation of time histories. On the other hand, the method does require numerical integration of (63). In principle, the method can also be extended to include some amplitude-correlation over multiples of  $\tau_c$ , giving something even closer to (61), but this would be significantly more difficult.

The modifications needed in order to apply the methods in this section to the double-barrier problem are quite straightforward. The independent peak approximation of (59), now counts both peaks and valleys and half-cycles replace cycles in the cycle-amplitude approaches. This requires replacing  $n_p$  with  $2n_p$  in (59) and replacing  $n_c$  with  $2n_c$  for cycle-amplitudes. The only change in the Naess and Gaidai approach is in the simulation to find the conditional distribution function in (61).

Numerical results for the independent-amplitude method of (60) and the correlated-amplitude method of (64) are shown in Figure 6, along with the same simulation data as in Figure 5. The independent-peaks method of (59) and the correlated-peaks method of (61) are not useful for the process producing this simulation data, since the data are for a process with  $\alpha_2 = 0$  due to  $\sigma_{\ddot{X}} = \infty$ . This limiting condition gives the peak distribution as Gaussian, so that it does not have the necessary tendency to the Rayleigh form for



**Figure 6. Simulation and amplitude approximations.**

large  $u$ , as in all other situations. Figures 5 and 6 show that for this narrowband process the correlated-amplitude approach gives  $\eta$  values that are quite similar to those of the Vanmarcke method, but are somewhat more conservative for large  $u$  values.

## NUMERICAL RESULTS

The significance of the preceding analytical results will now be illustrated by presenting figures and tables demonstrating the probability of  $|X(s)|$  exceeding a level  $u$  at least once during the interval  $0 \leq s \leq t$ :

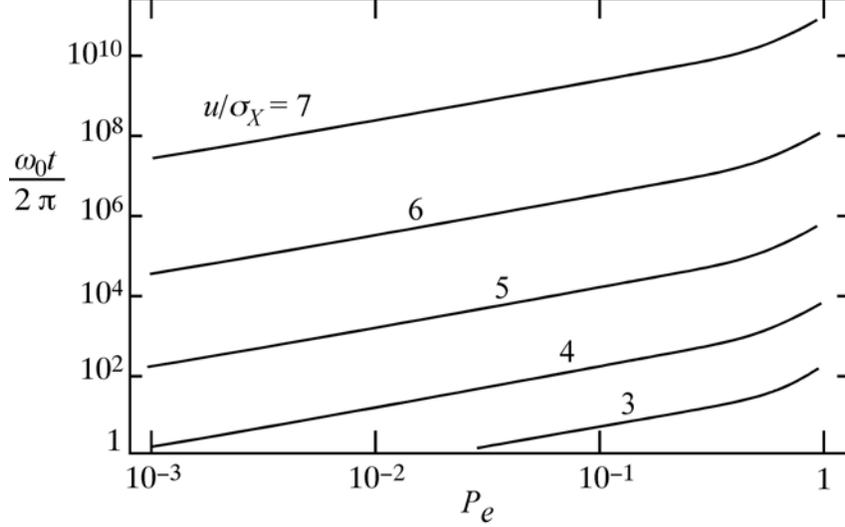
$$P_e(u, t) = 1 - L_{|X|}(u, t) = P[\hat{X}(t) > u] = P[T_X(u) < t] \quad (66)$$

in which,  $\hat{X}(t)$  is the largest value of  $|X(s)|$  within the set  $0 \leq s \leq t$  and  $T_X(u)$  is the time at which  $|X(t)|$  first exceeds the level  $u$ . These results will be limited to the situation in which it is given that the process begins with  $|X(0)| \leq u$  and initial conditions will be ignored, so that (44) gives  $L_{|X|}(u, t) = \exp[-\eta_{|X|}(u)t]$ .

The most comprehensive results will be presented for the Poisson approximation of the double-barrier first-passage probability, since it is relatively simple and is asymptotically correct for large  $u$  values. In particular, letting  $\eta_{|X|}(u) = 2\nu_X(u)$  and solving for time gives

$$\frac{\omega_c t}{2\pi} = \frac{-1}{2} \ln[1 - P_e(u, t)] e^{u^2/(2\sigma_X^2)} \quad (67)$$

Note that the left-hand side of (67) is a normalized time that can be taken as the number of “cycles” (or upcrossings of zero) of the mean-zero Gaussian process during the time interval  $[0, t]$ . Figure 7 then shows this cycle-count versus  $P_e$  for several integer values of



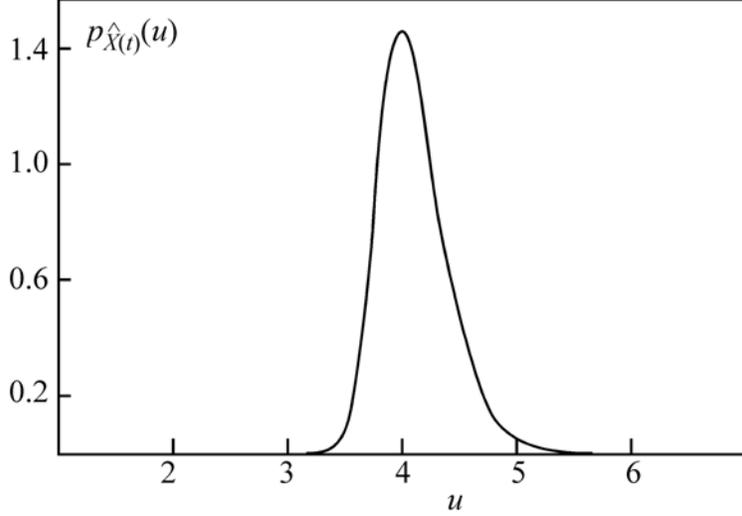
**Figure 7. Time for  $P[\hat{X}(t) > u]$  to reach level  $P_e$ .**

$u/\sigma_X$ . This can be viewed as the time required for  $P[\hat{X}(t) > u]$  to reach level  $P_e$  for a stationary process or, equivalently, as the duration of a finite time history for which  $P[\hat{X}(t) > u] = P_e$ . As would be expected, the time value grows with increasing  $u$  and also with increasing  $P_e$ . What is more interesting is the magnitude of the numerical values. For the 5-sigma level, for example, there is a 0.1% probability of exceedance within a time history of about 134 cycles, increasing to 1% probability for 1,350 cycles, and to 50% probability for 93,000 cycles. Thus, there may be many situations in which exceeding 5-sigma should not be considered to be a very rare event. Similarly, the exceedance of 3-sigma may be considered to be quite common, having a 50% probability of occurrence within a time history of only 31 cycles.

The likelihood of seemingly large extreme values can be illustrated in another way by using (67) to find the probability distribution of the largest extreme for a given value of  $t$ . In particular, the probability density of  $\hat{X}(t)$  is simply the derivative of  $L_{|X|}(u, t)$  with respect to  $u$ , but with the modification that the Poisson approximation requires a finite value for  $L_{|X|}(0, t)$ , as discussed following (47). For the Poisson approximation of the double-barrier problem, this gives the probability density as

$$p_{\hat{X}(t)}(u) = e^{-\omega_c t / \pi} \delta(u) + \left( \frac{\omega_c t}{\pi} \right) \frac{u}{\sigma_X} \exp \left( \frac{-u^2}{2\sigma_X^2} - \frac{\omega_c t}{\pi} e^{-u^2 / (2\sigma_X^2)} \right) \quad (68)$$

This probability distribution can then be used to evaluate numerically the mean ( $\mu_{\hat{X}}$ ) and the standard deviation ( $\sigma_{\hat{X}}$ ) of the largest extreme within a time history of a given length. For illustration consider a time history of 1,350 cycles, for which it was previously found that there is a 1% probability of finding an extreme exceeding  $5\sigma_X$ . The probability density of  $\hat{X}(t)$  for this situation is shown in Figure 8, and it is found numerically that  $\mu_{\hat{X}} \approx 4.03\sigma_X$  and  $\sigma_{\hat{X}} \approx 0.31\sigma_X$ . Thus,  $\mu_{\hat{X}} + 3\sigma_{\hat{X}} \approx 5\sigma_X$ . For this particular example



**Figure 8. Probability density of  $\hat{X}(t)$ .**

there is a 1% probability of the largest extreme exceeding  $\mu_{\hat{X}} + 3\sigma_{\hat{X}}$ —the mean plus three standard deviations of the random extreme. Again, it seems that  $5\sigma_X$  may be viewed as a value that is relatively likely to be exceeded in many studies of extreme values.

Recall that the Vanmarcke approximations and the correlated-amplitude approximation all give results with  $L_{|X|}(u, t)$  decaying exponentially as  $\eta_{|X|}(u)t$ . Thus, any of these methods gives results similar to Figure 7 except that the time for  $P[\hat{X}(t) > u]$  to reach level  $P_e$  is increased by the ratio  $v_{|X|}(u)/\eta_{|X|}(u)$ . For the Vanmarcke approximations this ratio depends on the bandwidth parameter  $\alpha_1$  of  $X(t)$ , and for the correlated-amplitude approximation it depends on  $\rho \equiv [G_{XX}^2(\pi/\omega_c) + G_{XZ}^2(\pi/\omega_c)]/\sigma_X^4$ .

Tables 1 and 2 show the extent to which the time until exceedance is increased by use of the Vanmarcke approximations instead of the Poisson approximation. In particular, the numbers in the tables are multipliers to apply to the Poisson time values, as in (67) and Figure 7, to obtain a Vanmarcke estimate of the time for  $P[\hat{X}(t) > u]$  to reach level  $P_e$  for several values of  $u$ , and for several different narrowband processes. Note that Table 1 is for the original version of the Vanmarcke approximation, without the empirical correction factor introduced in (54) and (57), with this latter form being shown in Table 2. Table 3 presents the same information for the approximation of correlated amplitudes, as in (64) and (65).

The values of the parameters  $\alpha_1$  and  $\rho$  in the tables, of course, are related to the bandwidth of the process. To illustrate their significance, it may be helpful to consider a relatively well-known problem that does yield such narrowband processes as those presented here. In particular, consider the response of a single degree-of-freedom oscillator subjected to a white noise excitation, for which the bandwidth is governed by

the fraction of critical damping  $\zeta$  in the oscillator. For this example, the parameter  $\alpha_1$  is found to be

$$\alpha_1 = \frac{1}{(1-\zeta^2)^{1/2}} \left( 1 - \frac{2}{\pi} \tan^{-1} \left( \frac{\zeta}{(1-\zeta^2)^{1/2}} \right) \right) \quad (69)$$

Similarly, the correlated-amplitude approximation is easily applied to this situation by using the  $A_2$  definition of amplitude and choosing the time increment  $\tau_c$  for one half-cycle to be given by  $\omega_0 \tau_c = \pi / (1 - \zeta^2)^{1/2}$ , in which  $\omega_0$  is the undamped frequency of the oscillator. This gives

$$\rho = \exp[-2\zeta \pi / (1 - \zeta^2)^{1/2}] \quad (70)$$

The results in (69) and (70) are also approximately correct for any broadband excitation of the simple oscillator, and they tend to  $\alpha_1 \approx 1 - 2\zeta/\pi$  and  $\rho \approx 1 - 2\zeta\pi$  for very small values of  $\zeta$ .

**Table 1.** Increase in time for Vanmarcke correction of (55) and (56)

	$u = 3$	$u = 4$	$u = 5$	$u = 6$	$u = 7$	$u = 8$
$\alpha_1 = 0.999$	6.39	4.98	4.09	3.50	3.08	2.77
$\alpha_1 = 0.99$	2.40	1.97	1.70	1.53	1.41	1.32
$\alpha_1 = 0.95$	1.43	1.26	1.16	1.11	1.07	1.05
$\alpha_1 = 0.9$	1.23	1.13	1.07	1.04	1.02	1.01

**Table 2.** Increase in time for Vanmarcke correction of (57)

	$u = 3$	$u = 4$	$u = 5$	$u = 6$	$u = 7$	$u = 8$
$\alpha_1 = 0.999$	11.45	8.81	7.16	6.05	5.26	4.67
$\alpha_1 = 0.99$	3.28	2.63	2.22	1.95	1.76	1.62
$\alpha_1 = 0.95$	1.63	1.41	1.27	1.18	1.13	1.09
$\alpha_1 = 0.9$	1.32	1.19	1.11	1.07	1.04	1.03

**Table 3.** Increase in time for correlated-amplitude correction of (64)

	$u = 3$	$u = 4$	$u = 5$	$u = 6$	$u = 7$	$u = 8$
$\rho = 0.99$	8.27	6.29	5.05	4.23	3.65	3.17
$\rho = 0.9$	2.64	2.07	1.72	1.49	1.34	1.24
$\rho = 0.6$	1.38	1.18	1.08	1.03	1.01	1.00
$\rho = 0.3$	1.11	1.03	1.01	1.00	1.00	1.00

The values of  $\alpha_1$  in Tables 1 and 2 and the values of  $\rho$  in Table 3 have been chosen to correspond to similar values of  $\zeta$  in the single degree-of-freedom oscillator. In particular, the  $\alpha_1$  values of 0.999, 0.99, 0.95, and 0.9 correspond to  $\zeta$  values of 0.0016, 0.016, 0.084, and 0.18, respectively, while the  $\rho$  values of 0.99, 0.9, 0.6, and 0.3 correspond to  $\zeta$  values of 0.0016, 0.017, 0.081, and 0.19. The numbers in the first two rows of each table can be considered to be for very narrowband processes. Those in the first row ( $\zeta = 0.0016$ ), in particular, are for a situation that is so narrowband as to be extremely rare in practice.

The modified Vanmarcke results in Table 2, in particular, show that it is possible for this correction to give exceedance times that are an order of magnitude larger than those predicted by the Poisson approximation. The increases in exceedance time for the correlated-amplitude approximation in Table 3 are also significant for the same situations. However, these large increases only occur for very narrowband processes and a relatively low crossing level. Thus, the simple Poisson approximation is quite adequate for many practical situations.

## TRANSLATION PROCESSES

There is one class of non-Gaussian processes for which it is particularly simple to apply all the results developed for the Gaussian situation. In particular, consider a non-Gaussian process  $\{X(t)\}$  that can be written as

$$X(t) = g[Y(t)] \quad (71)$$

in which  $\{Y(t)\}$  is a normalized stationary Gaussian process and  $g(\cdot)$  is a monotonically increasing function. The cumulative distribution of such a so-called translation process  $\{X(t)\}$  [Grigoriu, 1995] is then given by

$$F_{X(t)}[g(y)] \equiv P[X(t) \leq g(y)] = P[Y(t) \leq y] = \Phi(y) \quad (72)$$

Furthermore, the peaks of  $X(t)$  occur at the same time as those of  $Y(t)$  and the relationship between the peak magnitudes is  $P_X = g(P_Y)$ . In addition, the global maxima within a time interval  $[0, t]$  satisfy  $\hat{X}(t) = g[\hat{Y}(t)]$ , with a similar relationship for the global minima. Note that the global maximum and the global minimum of  $X(t)$  may have quite different probability distributions if  $g(\cdot)$  is not an odd function.

In practice it is often useful to apply the translation procedure of (71) in order to approximate a non-Gaussian process  $\{X(t)\}$  by a translation process  $\{g[Y(t)]\}$ . One can always find an appropriate translation process by choosing  $g(\cdot)$  to match the cumulative distribution function of  $g[Y(t)]$  to that of  $X(t)$ . In particular, the inverses of the two distribution functions give  $u = F_X^{-1}[P(X \leq u)]$  and  $y = \Phi^{-1}[P(Y \leq y)]$ . Plotting  $u$  versus  $y$  for a range of probability values then gives the desired  $g(\cdot)$  function. It should be noted,

though, that this procedure does not necessarily give  $\{g^{-1}[X(t)]\}$  as a Gaussian process. Rather, it assures that  $\{g^{-1}[X(t)]\}$  can be viewed as a collection of Gaussian random variables, but these random variables may or may not be jointly Gaussian, as is required for a Gaussian process.

It is known that the peaks of the normalized Gaussian process  $\{Y(t)\}$  are given by the Rice distribution

$$P(Y \leq y) = \Phi\left(\frac{y}{(1-\alpha_2^2)^{1/2}}\right) - \alpha_2 e^{-y^2/2} \Phi\left(\frac{\alpha_2 y}{(1-\alpha_2^2)^{1/2}}\right) \quad (73)$$

The fact that  $g(y)$  is a monotonically increasing function assures that every peak of  $Y(t)$  maps into a peak of  $g[Y(t)]$ . If  $\{g^{-1}[X(t)]\}$  truly is a normalized Gaussian process, then the probability distribution of the peaks of  $g[Y(t)]$  agrees with that of the peaks of  $X(t)$ . Otherwise,  $\{X(t)\}$  is not truly a translation process. Of course, the Rice distribution of peaks is a necessary but not sufficient condition for Gaussianity, but it is one of the easier tests to apply in practice.

Even in a situation in which the translation concept gives only an approximate description of the data being analyzed, it is fairly likely that the approximation will be of some value in predicting the probability of first-passage failure for a non-Gaussian process.

## SUMMARY

The reliability problem can be formulated either as finding an estimate of the first-passage time  $T_X(u)$  for a given value  $u$  of the stochastic process  $\{X(t)\}$ , or as estimating the largest value  $\hat{X}(t)$  of  $\{X(s): 0 \leq s \leq t\}$  for a given value of  $t$ . The probability distributions of these two random quantities are governed by the one function  $L_X(u,t) \equiv P[\hat{X}(t) \leq u]$ , since the probability density of  $\hat{X}(t)$  is the  $u$ -derivative of  $L_X(u,t)$  and the probability density of  $T(u)$  is proportional to its  $t$ -derivative. The role of time is critical in the evaluations. For an ergodic unbounded process, such as a Gaussian process, the mean value of  $\hat{X}(t)$  grows without limit as  $t$  becomes very large.

One can always write  $L_X(u,t)$  as an exponential function containing a time-integral of a quantity  $\eta_X(t,u)$ . For a stationary process, this new quantity depends on  $t$  only when  $t$  is small. Thus, it is efficient to concentrate on estimating the stationary value  $\eta_X(u)$ . For other than very small values of  $t$  this gives  $L_X(u,t) \approx L_0 e^{-\eta_X(u)t}$ . Since  $L_X(u,t)$  can be viewed as a probability of survival, a conservative procedure is one that underestimates  $L_X(u,t)$ , and for large  $u$  this requires an over-estimation of  $\eta_X(u)$ . On the same basis, it is desirable to under-estimate the value of  $L_0$ . For a zero-start process that tends to a stationary condition this conservative estimate can be taken as  $L_0 \approx 1$ . For a truly

stationary process a reasonable estimate is  $L_0 \approx P[A(t) < u]$  based on the amplitude of the process. This is more conservative than using  $P[X(t) < u]$  to estimate  $L_0$ .

The most commonly used method for approximating  $\eta_X(u)$  involves Poisson crossing times for  $X(t)$ , and gives  $\eta_X(u)$  as the same as the expected up-crossing rate  $\nu_X^+(u)$ . This approximation is conservative for all but small values of  $u$ , and is asymptotically correct for a Gaussian process as  $u$  becomes very large. A more conservative and better estimate for small  $u$  is obtained by using  $\eta_X(u) \approx \nu_X^+(u)/F_X(u)$ . The major shortcoming of the Poisson X-crossings approach is that its conservatism may be regarded as excessive for some narrowband processes for some values of  $u$ , with the greatest discrepancy occurring for  $u \approx 2\sigma_X$ . Using an alternate assumption of Poisson crossing times by  $A(t)$  leads to  $\eta_X(u) \approx \nu_A^+(u)/F_A(u)$ . This approximation provides a significant improvement for some values of  $u$ , but it can give the nonconservative result of  $\eta_X(u) > \nu_X^+(u)$  for large  $u$  values. A conservative estimate of  $\eta_X(u)$  can be written as  $\min[\nu_A^+(u)/F_A(u), \nu_X^+(u)]$ , but this slightly complicates the computation of such quantities as the mean or standard deviation of  $\hat{X}(t)$ , since they involve integrations of the probability density of  $\hat{X}(t)$  over all positive values of  $u$ .

The Vanmarcke approximation of (52)-(57) provides the best available estimates of  $\eta_X(u)$  using only analytical expressions. The formulas are significantly more complicated than for the Poisson crossing approaches, and they involve a factor  $\alpha_1$  that relates to the bandwidth of the spectral density of  $\{X(t)\}$ . Nonetheless, the method is easily implemented in a computer program. Similar, but slightly more conservative results are obtained from the correlated-amplitude approach of (62)-(65), which involves some straightforward numerical integration.

The recently introduced correlated-peaks method of Naess and Gaidai seems to be capable of providing very good estimates of  $\eta_X(u)$  for a variety of processes, including ones that are narrowband. Significant simulation of time histories of  $\{X(t)\}$  is required, though, to determine the values of key parameters in the formulation.

Other analytical techniques summarized herein show the similarities and differences between the results of various assumptions, but none show improvements over the Poisson, Vanmarcke, and correlated-amplitude approximations. A plot and tables have been included to provide a simple overview of the length of time history for which  $P[\hat{X}(t) > u] = P_e$  for chosen values of  $u$  and  $P_e$  for each of these approximations.

The simple concept of a translation process has been introduced to provide an approximate method for applying the Gaussian techniques of first-passage analysis in a situation in which the process of interest is found to be non-Gaussian.

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<b>14. ABSTRACT</b> An obvious goal of reliability analysis is the avoidance of system failure. However, it is generally recognized that it is often not feasible to design a practical or useful system for which failure is impossible. Thus it is necessary to use techniques that estimate the likelihood of failure based on modeling the uncertainty about such items as the demands on and capacities of various elements in the system. This usually involves the use of probability theory, and a design is considered acceptable if it has a sufficiently small probability of failure. This report contains findings of analyses of systems subject to first-passage failure.						
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