Error Representation in Time[1] for Compressible Flow Calculations

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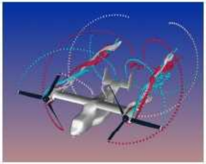
[1] Time is a great teacher, but unfortunately it kills all its pupils – Hector Berlioz

Time Dependent Flow Problems

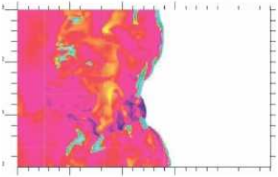
Time plays an essential role in most real world fluid mechanics problems, e.g. turbulence, combustion, acoustic noise, moving geometries, blast waves, etc.

Time dependent calculations now dominate the computational landscape at the various NASA Research Centers but the accuracy of these computations is often not well understood.



Helicopter and Tilt-Rotor Aerodynamics

Launch Vehicle Analysis



Combustion and turbulence

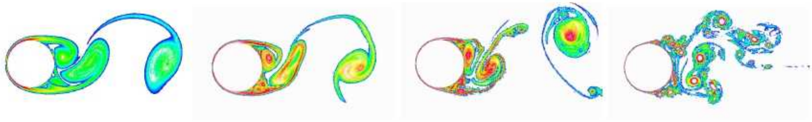
Space-Time Error Representation

In this presentation, we investigate error representation (and error control) for **time-periodic** problems as a prelude to the investigation of feasibility of error control for **stationary statistics** and **space-time averages**.

- These statistics and averages (e.g. time-averaged lift and drag forces) are often the output quantities sought by engineers.
- For systems such as the Navier-Stokes equations, pointwise error estimates deteriorate rapidly which increasing Reynolds number while statistics and averages may remain well behaved.

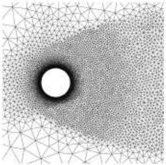
Motivating Example #1: Cylinder Flow

Cylinder flow at Mach = 0.10, logarithm of |vorticity| contours

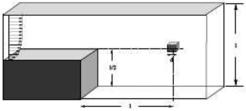


Re=1000 **Re=3900** **Re=10000** **Re=50000**

- Quartic space-time elements
- 25K element mesh
- Viscous walls only imposed on cylinder surface
- Reynolds number based on cylinder diameter



Example: Backward facing step (Re=2000)



Suppose $J(\mathbf{u})$ is the streamwise velocity component averaged in cube in space and over a unit time interval, i.e.

$$J(u) = \int_9^{10} \int_{d\times d\times d} u_1 \, dx^3 \, dt$$

Motivating Example #2: Computability of Outputs

Hoffman and Johnson (2002) have computed solutions of the backward facing step problem using a FEM method with linear elements for incompressible flow.

In velocity and pressure variables, (V, p), the following error estimate for functionals is readily obtained in terms of the dual solution (ψ, ϕ)

$$
\begin{aligned}
|J(V,p) - J(V_h,p_h)| \quad &\leq \quad C\|\dot{\psi}\|\|\Delta t\, r_0(V_h,p_h)\| \\
&+ \quad C\|D^2\psi\|\|h^2\, r_0(V_h,p_h)\| \\
&+ \quad C\|\dot{\phi}\|\|\Delta t\, r_1(V_h,p_h)\| \\
&+ \quad C\|D\phi\|\|h\, r_1(V_h,p_h)\|
\end{aligned}
$$

where r_i are element residuals.

The following stability factors have been computed by Hoffman and Johnson (2002) for the backward facing step problem at Re=2000.

d	$\|\dot{\psi}\|$	$\|\nabla\psi\|$	$\|\nabla\phi\|$	$\|\dot{\phi}\|$
1/8	124.0	836.0	138.4	278.4
1/4	39.0	533.4	48.9	46.0
1/2	10.5	220.3	16.1	25.2

These results clearly show the deterioration in computability as the box width is decreased.

Outline for the Remainder of the Talk

- Review the space-time discontinuous Galerkin (DG) FEM formulation, Reed and Hill (1973), LeSaint and Raviart (1974) and popularized for nonlinear conservation laws by Cockburn and Shu (1990).
- Error representation and estimation for nonlinear hyperbolic systems with and without time
- The space-time discontinuous Galerkin method for the compressible Navier-Stokes equations
- Error representation and estimation for time periodic, and nearly time periodic Navier-Stokes cylinder flow
- (Time Permitting) Recent work moving away from functional error representation/control towards L_p-norm control.

Conservation law system in $\mathbf{R}^{d\times 1}$

$$\mathbf{u}_{,t} + \mathrm{div}\, \mathbf{f} = 0, \quad \mathbf{u}, \mathbf{f}_i \in \mathbf{R}^m \;\; i = 1, \ldots, d$$

Convex entropy extension

$$U_{,t} + \mathrm{div}\, F \leq 0, \quad U, F_i \in \mathbf{R}$$

Entropy Variables

Existence of a convex entropy-entropy flux pair $\{U, F\}$ implies that the change of variable $\mathbf{u} \mapsto \mathbf{v}$ symmetrizes the original quasilinear system (Mock (1980))

$$\underbrace{\mathbf{u}_{,\mathbf{v}}}_{SPD} \mathbf{v}_{,t} + \underbrace{\mathbf{f}_{i,\mathbf{v}}}_{SYMM} \mathbf{v}_{,x_i} = 0 \quad \text{(implied sum, } i = 1 \ldots d\text{)}$$

so that for smooth solutions

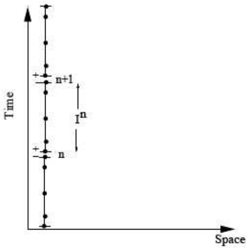
$$\mathbf{v} \cdot (\mathbf{u}_{,t} + \mathrm{div} f) = U_{,t} + \mathrm{div}\, F = 0 \ .$$

with the symmetrization variables (a.k.a. entropy variables) calculated from

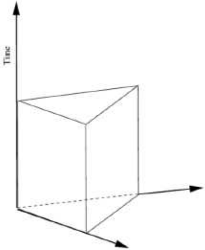
$$\mathbf{v}^T = U_{,\mathbf{u}} \text{ and } \mathbf{v} \cdot \mathbf{f}_{,\mathbf{v}} = F_{,\mathbf{v}} \ .$$

The Discontinuous in Time Approximation Space

- Natural setting for the discontinuous Galerkin (DG) method for hyperbolic problems
- Utilized in the space continuous Galerkin least-squares method (Hughes and Shakib, 1988)
- Often used in the discretization of parabolic problems (Douglas and Dupont, 1976)
- Requires solving the implicit slab equations–no one said it would be easy!



Discontinuous timeslab intervals



Space-time prism element

Piecewise polynomial approximation space:

$$\mathcal{V}^h = \left\{ \mathbf{v}_h \mid \mathbf{v}_h|_{K\times I^n} \in \left(\mathcal{P}_k(K \times I^n)\right)^m \right\}$$

Find $\mathbf{v}_h \in \mathcal{V}^h$ such that for all $\mathbf{w}_h \in \mathcal{V}^h$

$$B(\mathbf{v}_h, \mathbf{w}_h)_{\rm DG} = \sum_{n=0}^{N-1} B^n(\mathbf{v}_h, \mathbf{w}_h)_{\rm DG} = 0 \ ,$$

$$\begin{aligned} B^n(\mathbf{v}, \mathbf{w})_{\rm DG} \quad &= \quad \int_{I^n} \sum_{K\in\mathcal{T}} \int_K -(\mathbf{u}(\mathbf{v}) \cdot \mathbf{w}_{,t} + \mathbf{f}^i(\mathbf{v}) \cdot \mathbf{w}_{,x_i})\, dx\, dt \\ &+ \quad \int_{I^n} \sum_{K\in\mathcal{T}} \int_{\partial K} \mathbf{w}(x_-) \cdot \mathbf{h}(\mathbf{v}(x_-), \mathbf{v}(x_+); \mathbf{n})\, ds\, dt \\ &+ \quad \int_\Omega \left(\mathbf{w}(t_-^{n+1}) \cdot \mathbf{u}(\mathbf{v}(t_-^{n+1})) - \mathbf{w}(t_+^n) \cdot \mathbf{u}(\mathbf{v}(t_-^n)) \right)\, dx \end{aligned}$$

- $\mathbf{u}$ the conservation variables, $\mathbf{v}$ the symmetrization variables
- $\mathbf{h}$ a numerical flux function, $\mathbf{h}(\mathbf{v}_-, \mathbf{v}_+; \mathbf{n}) = -\mathbf{h}(\mathbf{v}_+, \mathbf{v}_-; -\mathbf{n})$, $\mathbf{h}(\mathbf{v}, \mathbf{v}; \mathbf{n}) = \mathbf{f}(\mathbf{v}) \cdot \mathbf{n}$

Theorem E: Global space-time entropy inequality (Cauchy IVP):

$$\int_\Omega U(\mathbf{u}^*(t_-^0))\,dx \le \int_\Omega U(\mathbf{u}(\mathbf{v}_h(x,t_-^N)))\,dx \le \int_\Omega U(\mathbf{u}(\mathbf{v}_h(x,t_-^0)))\,dx$$

$$\mathbf{u}^*(t_-^0) = \frac{1}{\mathrm{meas}(\Omega)} \int_\Omega \mathbf{u}(\mathbf{v}_h(x,t_-^0))\,dx$$

whenever the numerical flux satisfies the system extension of Osher's famous "E-flux" condition

$$\boxed{[\mathbf{v}]_{x_-}^{x_+} \cdot (\mathbf{h}(\mathbf{v}_-,\mathbf{v}_+;\mathbf{n}) - \mathbf{f}(\mathbf{v}(\theta))\cdot\mathbf{n}) \le 0 \ , \ \ \forall\theta\in[0,1] \ , \mathbf{v}(\theta) = \mathbf{v}_- + \theta[\mathbf{v}]_-^+}$$

- Several flux functions satisfy this technical condition when recast in entropy variables, e.g. Lax-Friedrichs, HLLE, Roe with modifications, etc.

Suppose $\mathbf{u}_{,\mathbf{v}}$ remains bounded in the sense

$$0 < c_0 \leq \frac{\mathbf{z} \cdot \mathbf{u}_{,\mathbf{v}}(\mathbf{v}_h(x,t))\,\mathbf{z}}{\|\mathbf{z}\|^2} \leq C_0 \ , \quad \forall \mathbf{z} \neq 0$$

and Theorem E is satisfied for the Cauchy IVP, then following L_2 stability result is readily obtained

L_2 Stability:

$$\|\mathbf{u}(\mathbf{v}_h(\cdot, t_-^N) - \mathbf{u}^*(t_-^0)\|_{L_2(\Omega)} \leq (c_0^{-1} C_0)^{1/2} \|\mathbf{u}(\mathbf{v}_h(\cdot, t_-^0)) - \mathbf{u}^*(t_-^0)\|_{L_2(\Omega)} \ .$$

Space-Time Error Control

Given a system of PDEs with exact solution $u \in \mathbf{R}^m$ in a domain Ω, the overall objective is to construct a locally adapted discretization with numerical solution u_h that achieves

- Norm control [Babuska and Miller, 1984]

$$\|\mathbf{u} - \mathbf{u}_h\| < \text{tolerance}$$

- Functional output control [Becker and Rannacher, 1997]

$$|J(\mathbf{u}) - J(\mathbf{u}_h)| < \text{tolerance} \quad , \quad J(\mathbf{u}) : \mathbf{R}^m \mapsto \mathbf{R}$$

Example functional outputs:

- Time-averaged lift force, drag force, pitching moments
- Average flux rates through specified surfaces
- Weighted-average functionals of the form

$$J_\Psi(\mathbf{u}) = \int_{T_0}^{T_1} \int_\Omega \Psi(x,t) \cdot N(\mathbf{u}) dx\, dt$$

for some user-specified weighting $\Psi(x,t)$ and nonlinear function $N(u)$

Error Representation: Linear Case

Assume $\mathcal{B}(\cdot,\cdot)$ bilinear and $J(\cdot)$ linear.

Primal Numerical Problem: Find $\mathbf{u}_h \in \mathcal{V}_h^{\mathrm{B}}$ such that

$$B(\mathbf{u}_h, \mathbf{w}) = F(\mathbf{w}) \quad \forall\, \mathbf{w} \in \mathcal{V}_h^{\mathrm{B}}.$$

Auxiliary Dual Problem: Find $\Phi \in \mathcal{V}^{\mathrm{B}}$ such that

$$B(\mathbf{w}, \Phi) = J(\mathbf{w}) \quad \forall\, \mathbf{w} \in \mathcal{V}^{\mathrm{B}}.$$

$$\begin{aligned}
J(\mathbf{u}) - J(\mathbf{u}_h) &= J(\mathbf{u} - \mathbf{u}_h) && \text{(linearity of } J\text{)}\\
&= B(\mathbf{u} - \mathbf{u}_h, \Phi) && \text{(dual problem)}\\
&= B(\mathbf{u} - \mathbf{u}_h, \Phi - \pi_h\Phi) && \text{(Galerkin orthogonality)}\\
&= B(\mathbf{u}, \Phi - \pi_h\Phi) - B(\mathbf{u}_h, \Phi - \pi_h\Phi) && \text{(linearity of } B\text{)}\\
&= F(\Phi - \pi_h\Phi) - B(\mathbf{u}_h, \Phi - \pi_h\Phi) && \text{(primal problem)}
\end{aligned}$$

Final error representation formula:

$$J(\mathbf{u}) - J(\mathbf{u}_h) = F(\Phi - \pi_h\Phi) - B(\mathbf{u}_h, \Phi - \pi_h\Phi)$$

Estimating $\Phi - \pi_h \Phi$:

Various techniques in use for estimating $\Phi - \pi_h \Phi$:

- Higher order solves [Becker and Rannacher, 1998],[B. and Larson, 1999], [Süli and Houston, 2002], [Houston and Hartman, 2002]
- Patch postprocessing techniques [Cockburn, Luskin, Shu, and S uli, 2003]
- Extrapolation from coarse grids

Mean-value linearized forms:

$$\begin{aligned} \mathcal{B}(\mathbf{u},\mathbf{v}) &= \mathcal{B}(\mathbf{u}_h,\mathbf{v}) + \overline{\mathcal{B}}(\mathbf{u}-\mathbf{u}_h,\mathbf{v}) \quad \forall\, \mathbf{v} \in \mathcal{V}^{\mathrm{B}} \\ J(\mathbf{u}) &= J(\mathbf{u}_h) + \overline{J}(\mathbf{u}-\mathbf{u}_h), \end{aligned}$$

Example: $\mathcal{B}(u,v) = (L(u),v)$ with $L(u)$ differentiable

$$\begin{aligned} L(u_B) - L(u_A) &= \int_{u_A}^{u_B} dL = \int_{u_A}^{u_B} \frac{dL}{du}\, du \\ &= \int_0^1 \frac{dL}{du}(\tilde{u}(\theta))\, d\theta \cdot (u_B - u_A) = \overline{L}_{,u} \cdot (u_B - u_A) \end{aligned}$$

with $\tilde{u}(\theta) \equiv u_A + (u_B - u_A)\,\theta$.

$$\begin{aligned} \mathcal{B}(\mathbf{u},\mathbf{w}) &= \mathcal{B}(\mathbf{u}_h,\mathbf{w}) + (\overline{L}_{,\mathbf{u}} \cdot (\mathbf{u}-\mathbf{u}_h), \mathbf{w}) \\ &= \mathcal{B}(\mathbf{u}_h,\mathbf{w}) + \overline{\mathcal{B}}(\mathbf{u}-\mathbf{u}_h,\mathbf{w}) \quad \forall v \in \mathcal{V}^{\mathrm{B}} \end{aligned}$$

Error Representation: Nonlinear Case

Semilinear form $\mathcal{B}(\cdot,\cdot)$ and nonlinear $J(\cdot)$.

Primal numerical problem: Find $\mathbf{u}_h \in \mathcal{V}_h^{\mathrm{B}}$ such that

$$\mathcal{B}(\mathbf{u}_h, \mathbf{w}) = F(\mathbf{w}) \quad \forall\, \mathbf{w} \in \mathcal{V}^{\mathrm{B}}.$$

Linearized auxiliary dual problem: Find $\Phi \in \mathcal{V}^{\mathrm{B}}$ such that

$$\overline{\mathcal{B}}(\mathbf{w}, \Phi) = \overline{J}(\mathbf{w}) \quad \forall\, \mathbf{w} \in \mathcal{V}^{\mathrm{B}}.$$

$$\begin{aligned}
J(\mathbf{u}) - J(\mathbf{u}_h) &= \overline{J}(\mathbf{u} - \mathbf{u}_h) && \text{(mean value } J\text{)}\\
&= \overline{\mathcal{B}}(\mathbf{u} - \mathbf{u}_h, \Phi) && \text{(dual problem)}\\
&= \overline{\mathcal{B}}(\mathbf{u} - \mathbf{u}_h, \Phi - \pi_h\Phi) && \text{(Galerkin orthogonality)}\\
&= \mathcal{B}(\mathbf{u}, \Phi - \pi_h\Phi) - \mathcal{B}(\mathbf{u}_h, \Phi - \pi_h\Phi) && \text{(mean value } \mathcal{B}\text{)}\\
&= F(\Phi - \pi_h\Phi) - \mathcal{B}(\mathbf{u}_h, \Phi - \pi_h\Phi), && \text{(primal problem)}
\end{aligned}$$

Final error representation formula:

$$J(\mathbf{u}) - J(\mathbf{u}_h) = F(\Phi - \pi_h\Phi) - \mathcal{B}(\mathbf{u}_h, \Phi - \pi_h\Phi)$$

Space-time error representation formula

$$B_{\mathrm{DG}}(\mathbf{v}_h, w) - F_{\mathrm{DG}}(\Phi - \pi_h\Phi) = \sum_{n=0}^{N-1} \sum_{Q^n} B_{\mathrm{DG},Q^n}(\mathbf{v}_h, \Phi - \pi_h\Phi) - F_{\mathrm{DG},Q^n}(\Phi - \pi_h\Phi)$$

Stopping Criteria:

$$|J(\mathbf{u}) - J(\mathbf{u}_h)| = \left| \sum_{n=0}^{N-1} \sum_{Q^n} B_{\mathrm{DG},Q^n}(\mathbf{v}_h, \Phi - \pi_h\Phi) - F_{\mathrm{DG},Q^n}(\Phi - \pi_h\Phi) \right|$$

Refinement/Coarsening Indicator:

$$|J(\mathbf{u}) - J(\mathbf{u}_h)| \leq \sum_{n=0}^{N-1} \sum_{Q^n} \underbrace{\left| B_{\mathrm{DG},Q^n}(\mathbf{v}_h, \Phi - \pi_h\Phi) - F_{\mathrm{DG},Q^n}(\Phi - \pi_h\Phi) \right|}_{\text{refinement indicator}}$$

- This provides a unified framework for both stationary and time dependent problems

From the error representation formula, weighted estimates are obtained in space-time

$$J(\mathbf{u}) - J(\mathbf{u}_h) = \sum_{n=0}^{N} \sum_{Q^n} \left((\mathbf{r}_h, \Phi - \pi_h \Phi)_{Q^n} + \langle \mathbf{j}_h, \Phi - \pi_h \Phi \rangle_{\partial Q^n} \right)$$

where $\mathbf{r}_h$ denotes the residual on element interiors

$$\mathbf{r}_h \equiv \mathbf{u}_{h,t} + \mathrm{div}(\mathbf{f}(\mathbf{u}_h)) \ .$$

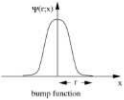
and $\mathbf{j}_h$ denotes one of four possible jump terms

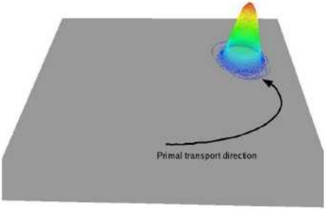
$$\mathbf{j}_h \equiv \begin{cases} \mathbf{f}(n; \mathbf{u}_h(x_-)) - \mathbf{h}(n; \mathbf{u}_h(x_-), \mathbf{u}_h(x_+)), & \partial Q^n \backslash \Gamma \ , \ t \neq 0 \\ \mathbf{f}(n; \mathbf{u}_h(x_-)) - \mathbf{h}(n; \mathbf{u}_h(x_-), \mathbf{g}(x_+)), & \partial Q^n \cap \Gamma \\ (\mathbf{u}_h(x, t_+) - \mathbf{u}_h(x, t_-)), & \partial Q^n \cap [t]_-^+ \\ (\mathbf{u}_h(x, t) - \mathbf{u}_0(x)), & \partial Q^0 \ , \ t = 0 \end{cases}$$

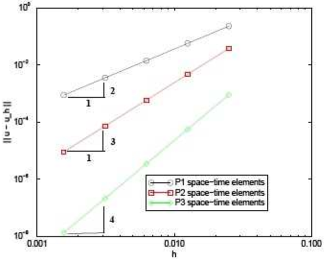
Example: A Scalar Time-Dependent PDE

Circular transport, $\lambda = (y, -x)$, of bump data

$$u_t + \lambda \cdot \nabla u = 0 \ , \qquad x \in [-1, 1]^2$$
$$u(x, 0) = \Psi(1/10; x - x_0) \ , \qquad x_0 = (7/10, 0, 0)$$







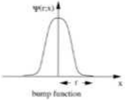
Primal numerical problem

Convergence, $\|u - u_h\|_{L_2(\Omega \times [0,T])}$

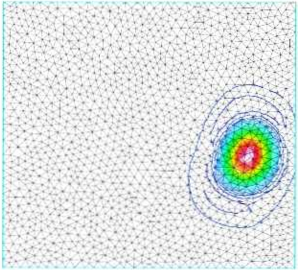
Space-Time DG Method

Example: Circular transport of bump data, $\lambda = (y, -x)$

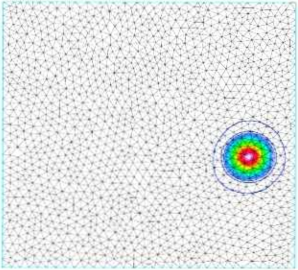
$$u_t + \lambda \cdot \nabla u = 0 \ , \quad x \in [-1,1]^2$$



3K element mesh



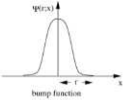
$\mathcal{P}_1$ in space-time



$\mathcal{P}_2$ in space-time

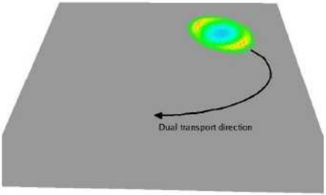
A functional is chosen that averages the solution data in the space-time ball of radius 1/10 located at $x_c = (1/2, 1/2, 1.05)$ in space-time

$$J(\mathbf{u}) = \int_0^{1.15} \int_\Omega \Psi(1/10; x - x_c)\, \mathbf{u}\, dxdt$$

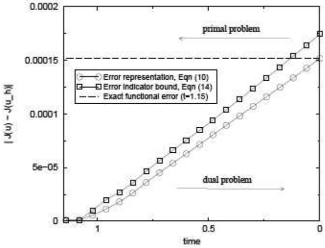


$$J(\mathbf{u}) - J(\mathbf{u}_h) = \sum_{n=N-1}^{0} \sum_{K} F_{\mathrm{DG},Q^n}(\Phi - \pi_h \Phi) - B_{\mathrm{DG},Q^n}(\mathbf{v}_h, \Phi - \pi_h \Phi)$$

$$|J(\mathbf{u}) - J(\mathbf{u}_h)| \le \sum_{n=N-1}^{0} \sum_{K} \left| F_{\mathrm{DG},Q^n}(\Phi - \pi_h \Phi) - B_{\mathrm{DG},Q^n}(\mathbf{v}_h, \Phi - \pi_h \Phi) \right|$$



Dual defect, $\Phi - \pi\Phi$



Error estimate buildup

Software Implementation and extension to the Navier-Stokes Eqns

Space-Time FEM:

- Implements the discontinuous Galerkin discretization in entropy variables.
- Unconditionally stable for all time step sizes
- Parallel implementation using overlapping domain decomposition and ILU preconditioned GMRES subdomain solves.
- Solves both the primal numerical problem and the jacobian linearized dual problem arising in space-time error estimation.
- High-order accuracy demonstrated in both space and space-time
- DG extension to the compressible Navier-Stokes equations using the symmetric interior penalty method of Douglas and Dupont, 1976) as described in Hartmann and Houston (2006)

Find $\mathbf{v}_h \in \mathcal{V}^{\mathrm{B}}_{h,p}$ such that

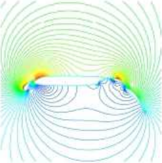
$$B_{\mathrm{DG}}(\mathbf{v}_h, \mathbf{w}) = \underbrace{\sum_{n=0}^{N-1}}_{\text{time}} \underbrace{\sum_{K}}_{\text{space}} B_{\mathrm{DG},Q^n}(\mathbf{v}_h, \mathbf{w}) = 0 \ , \quad \forall \mathbf{w} \in \mathcal{V}^{\mathrm{B}}_{h,p}$$

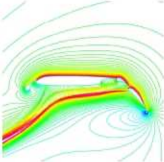
with

$$\begin{aligned}
B_{\mathrm{DG},Q^n}(\mathbf{v},\mathbf{w}) = & \int_K \int_{I^n} \mathbf{w} \cdot \left(\mathbf{u}_{,t} + \mathbf{F}^{\mathrm{inv}}_{i,x_i} - \mathbf{F}^{\mathrm{vis}}_{i,x_i}\right) dt\, dx \\
& + \int_{\partial K \setminus \Gamma} \int_{I^n} \mathbf{w}(x_-) \cdot (h(n; \mathbf{v}_+, \mathbf{v}_-) - n_i\, \mathbf{F}^{\mathrm{inv}}_i(\mathbf{v}_-))\, dt\, dx \\
& + \int_{\partial K \cap \Gamma_{\mathrm{wall}}} \int_{I^n} \mathbf{w}(x_-) \cdot n_i\, (\mathbf{F}^{\mathrm{inv\ wall}}_i - \mathbf{F}^{\mathrm{inv}}_i)(\mathbf{v}_-))\, dt\, dx \\
& + \int_{\partial K \cap \Gamma_{\mathrm{farfield}}} \int_{I^n} \mathbf{w}(x_-) \cdot (\mathbf{h}(\mathbf{n}; \mathbf{g}_\infty, \mathbf{v}_-) - n_i\, \mathbf{F}^{\mathrm{inv}}_i(\mathbf{v}_-))\, dt\, dx \\
& - \int_{\partial K \cap \Gamma_{\mathrm{N}}} \int_{I^n} \mathbf{w}(x_-) \cdot n_i\, (\mathbf{F}^{\mathrm{vis}}_i(\mathbf{g}_N) - \mathbf{F}^{\mathrm{vis}}_i(\mathbf{v}_-))\, dt\, dS \\
& - \int_{\partial K \setminus \Gamma} \int_{I^n} \frac{1}{2} \mathbf{w}(x_-) \cdot [n_i\, \mathbf{F}^{\mathrm{vis}}_i]^{x_+}_{x_-}\, dt\, dx \\
& + \int_{\partial K \setminus \Gamma} \int_{I^n} \frac{1}{2} [\mathbf{v}]^{x_+}_{x_-} \cdot n_j M_{ij}(x_-) \mathbf{w}_{,x_j}(x_-)\, dt\, dx \\
& + \int_{\partial K \cap \Gamma_{\mathrm{D}}} \int_{I^n} (\mathbf{g}_D - \mathbf{v}(x_-)) \cdot n_j M_{ij}(x_-) \mathbf{w}_{,x_j}(x_-)\, dt\, dx \\
& - \int_{\partial K \setminus \Gamma} \int_{I^n} \langle \kappa p^2 / h \rangle^{x_+}_{x_-}\, \mathbf{w}(x_-)) \cdot n_i\, n_j\, M_{ij}\, [\mathbf{v}]^{x_+}_{x_-}\, dt\, dx \\
& - \int_{\partial K \cap \Gamma_{\mathrm{D}}} \int_{I^n} (\kappa p^2 / h) x_-\ \mathbf{w}(x_-)) \cdot n_i\, n_j\, M_{ij}\, (\mathbf{g}_D - \mathbf{v}(x_-)))\, dt\, dx \\
& + \int_{K, n \neq 0} \mathbf{w}(t^n_+) \cdot [\mathbf{u}(\mathbf{v})]^{t^n_+}_{t^n_-}\, dx + \int_{K, n=0} \mathbf{w}(t^0_+) \cdot (\mathbf{u}(\mathbf{v}(t^0_-)) - \mathbf{u}_0)\, dx
\end{aligned}$$

Primal-Dual Problems in Fluid Mechanics

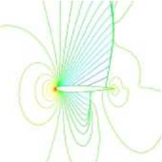
Subsonic Euler flow
$M = .10,\ 5^\circ$ AOA
Primal Mach contours

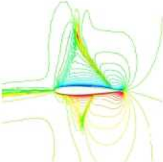




Lift force functional
Dual x-momentum contours

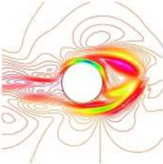
Transonic Euler flow
$M = .85,\ 2^\circ$ AOA
Primal density contours





Lift force functional
Dual density contours

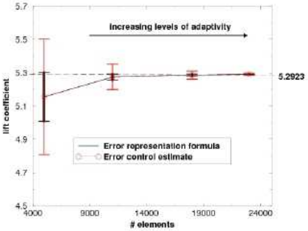
Viscous cylinder flow
$M = .15,\ Re = 300$
Primal vorticity contours



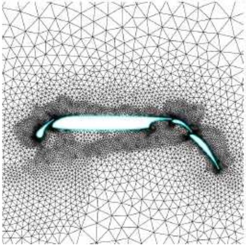
Drag force functional
Dual x-momentum contours

Example: Euler flow past multi-element airfoil geometry. $M = .1$, $5°$ AOA.

lift coefficient (error representation)	lift coefficient (error control)	refinement level	# elements	equivalent uniform refinement # elements
$5.156 \pm .147$	$5.156 \pm .346$	0	5000	5000
$5.275 \pm .018$	$5.275 \pm .076$	1	11000	20000
$5.287 \pm .006$	$5.287 \pm .024$	2	18000	80000
$5.291 \pm .002$	$5.291 \pm .007$	3	27000	320000



Error reduction during mesh adaptivity



Adapted mesh (18000 elements)

Computing dual (backwards in time) problems looks expensive in terms of both storage and computation

- Storage of the primal time slices for use in the locally linearized dual problem.
- Approximation of the infinite-dimensional dual problem for the backwards in time dual problem.

Tremendous simplification arising for periodic flow problems with period P when phase-independent functionals are utilized, e.g. mean drag

- Functional independent of the startup transient
- Only a small number of periods of the primal problem need be stored or recreated.

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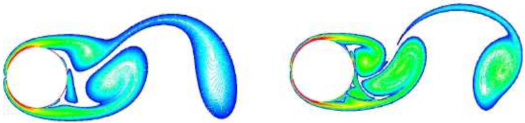
Cylinder flow at Mach = 0.10, logarithm of |vorticity| contours



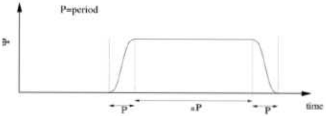
Re=300 **Re=1000**

Task: Represent and estimate the error in the mean drag force coefficient

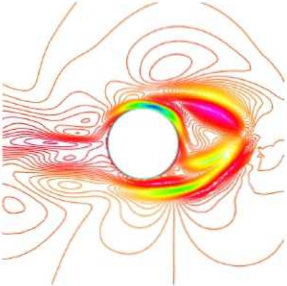
- Solve the primal problem using linear space-time elements
- Construct a smooth phase invariant functional measuring the mean drag force coefficient
- Solve the dual (backwards in time) problem using quadratic space-time elements
- Calculated the estimated functional error and compare with a reference calculation using cubic elements

Mean Drag for Cylinder Flow

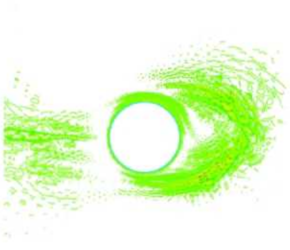
$$J_{\text{drag}}(u) = \int_0^T \int_{\Gamma_{\text{wall}}} (\text{Force} \cdot \hat{t}_{\text{drag}})\, \Psi(t)\, dx\, dt$$



Example: Cylinder flow at Re=300

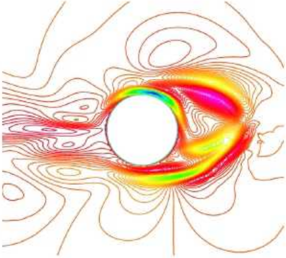


Dual problem, $\phi^{(x-mom)}$

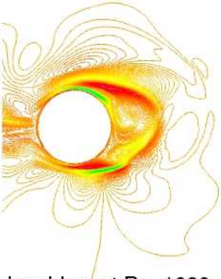


Dual defect, $\phi^{(x-mom)} - \pi_h \phi^{(x-mom)}$.

Mean Drag Dual Problems at Re=300 and Re=1000



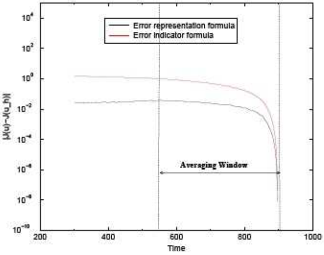
Dual problem at Re=300

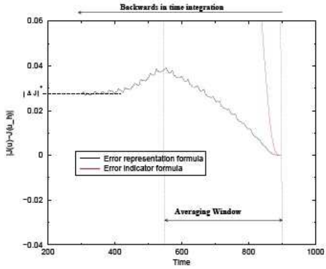


Dual problem at Re=1000

Mean Drag for Cylinder Flow at Re=1000

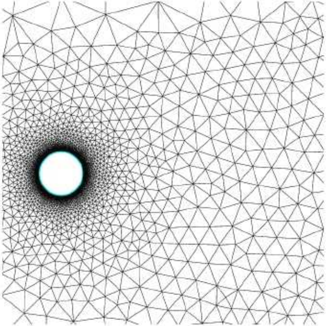
Error representation buildup during the backward in time dual integration



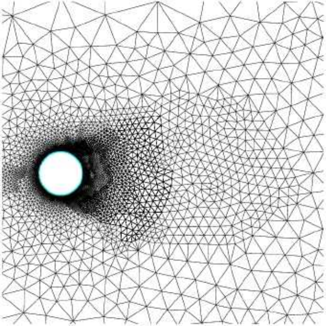


Mean Drag for Cylinder Flow at Re=1000

Adapted mesh from element indicators averaged over a period P



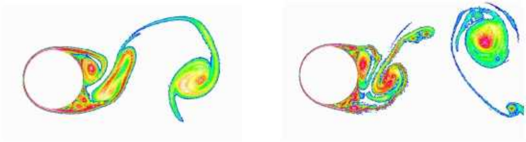
Coarse mesh (12K elements)



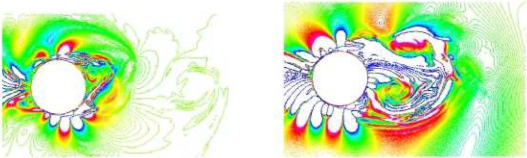
2 level refined mesh (20K elements)

Cylinder flow at Re=3900 and Re=10000.

- Choosing measurement problems that are not genuinely stationary produces rapidly growing dual problems and dependency on the initial data.



Dual solution corresponds to the average drag force over 3 approximate "periods".

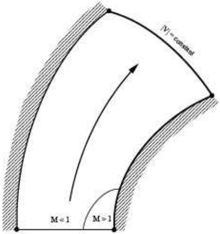


Re=3900 **Re=10000**

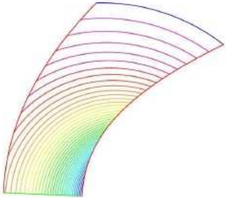
Concluding Technical Remarks

- Including time as "just another dimension" has many merits
  - Arbitrary order approximation
  - Provable non-linear stability
  - Simplified space-time error estimation
- But it also comes at a price
  - Increased arithmetic operations
  - Increased memory storage
  - More complex code implementation
- Error representation/estimation results presented today barely scratch the surface
  - Error control for general transient problems.
  - Dual problems in the presence of flow bifurcations
  - Computability and deterioration of functionals with increasing Reynolds number
  - Computer memory and storage constraints.

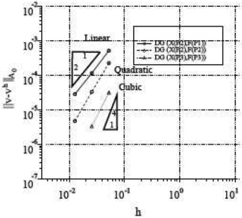
Example: Ringleb Flow



Schematic of Ringleb flow



Iso-Density contours

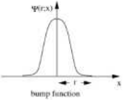


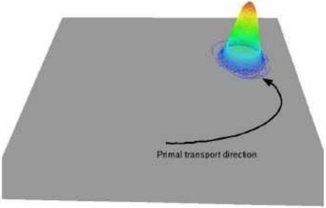
Discontinuous Galerkin

back

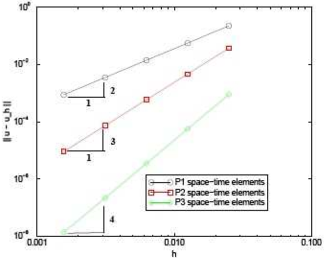
Circular transport, $\lambda = (y, -x)$, of bump data

$$u_t + \lambda \cdot \nabla u = 0 \ , \qquad x \in [-1, 1]^2$$
$$u(x, 0) = \Psi(1/10; x - x_0) \ , \qquad x_0 = (7/10, 0, 0)$$





Primal numerical problem



Convergence, $\|u - u_h\|_{L_2(\Omega \times [0,T])}$

back