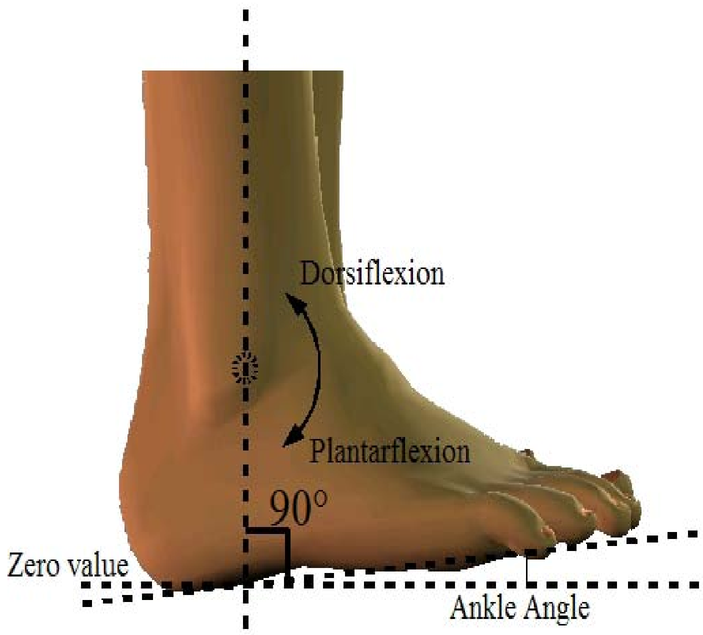
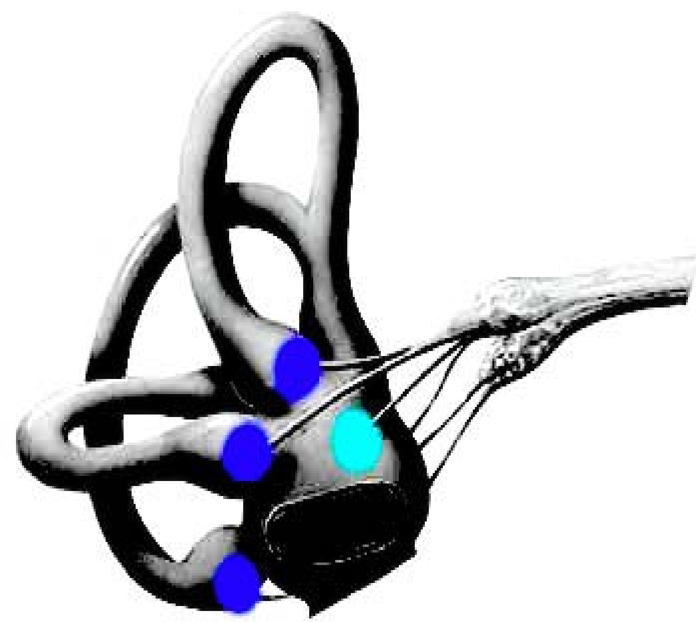
A Computational Biology Approach to Modelling and Identification of Human Physiology

Keynote Talk

Sunil L. Kukreja





Outline

- Introduction: System Identification
- Non-linear Model Form
- Identification Problem
  - Order Estimation, Parameter Estimation, Structure Detection & Model Validation
- System Identification for the Study of Human Biology
  - Modelling & Parameter Estimation Techniques for Human Biomechanics & Vestibulo-Ocular Dynamics
  - Structure Detection of Non-linear Systems for the Study of Human Ankle Dynamics
- Conclusions

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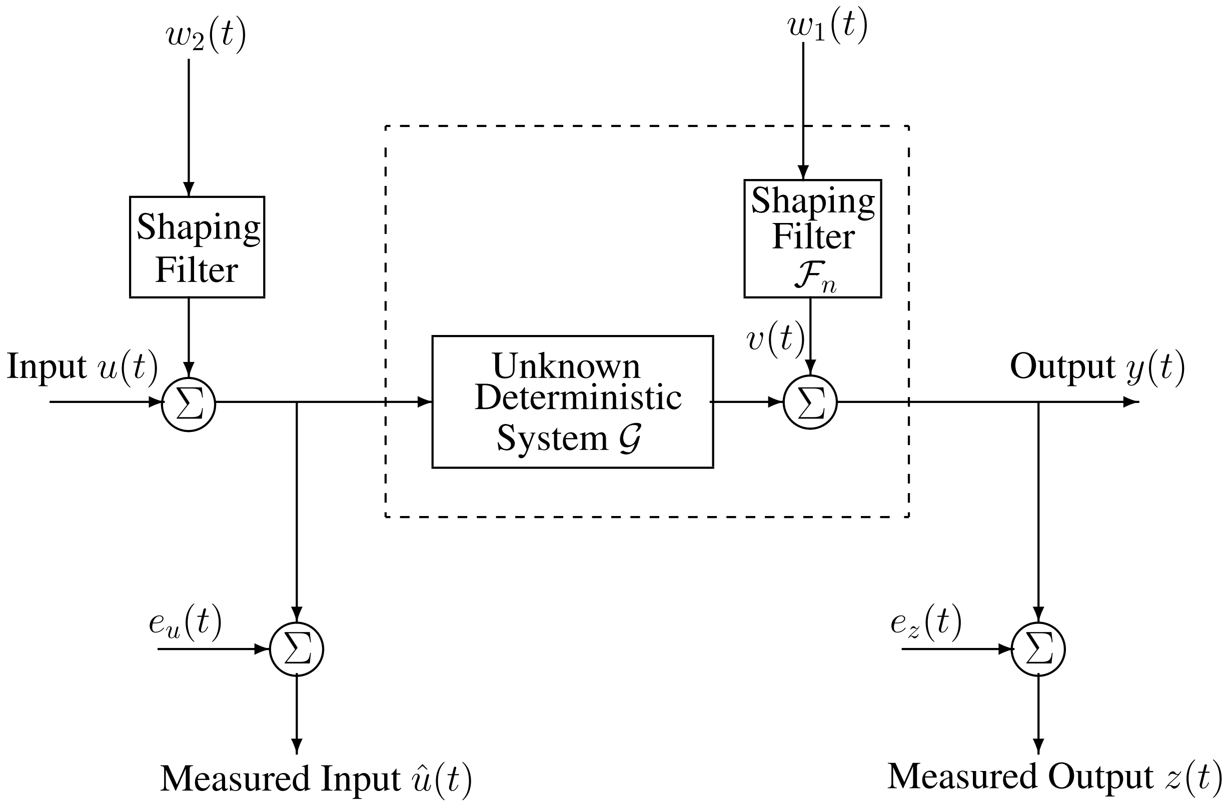
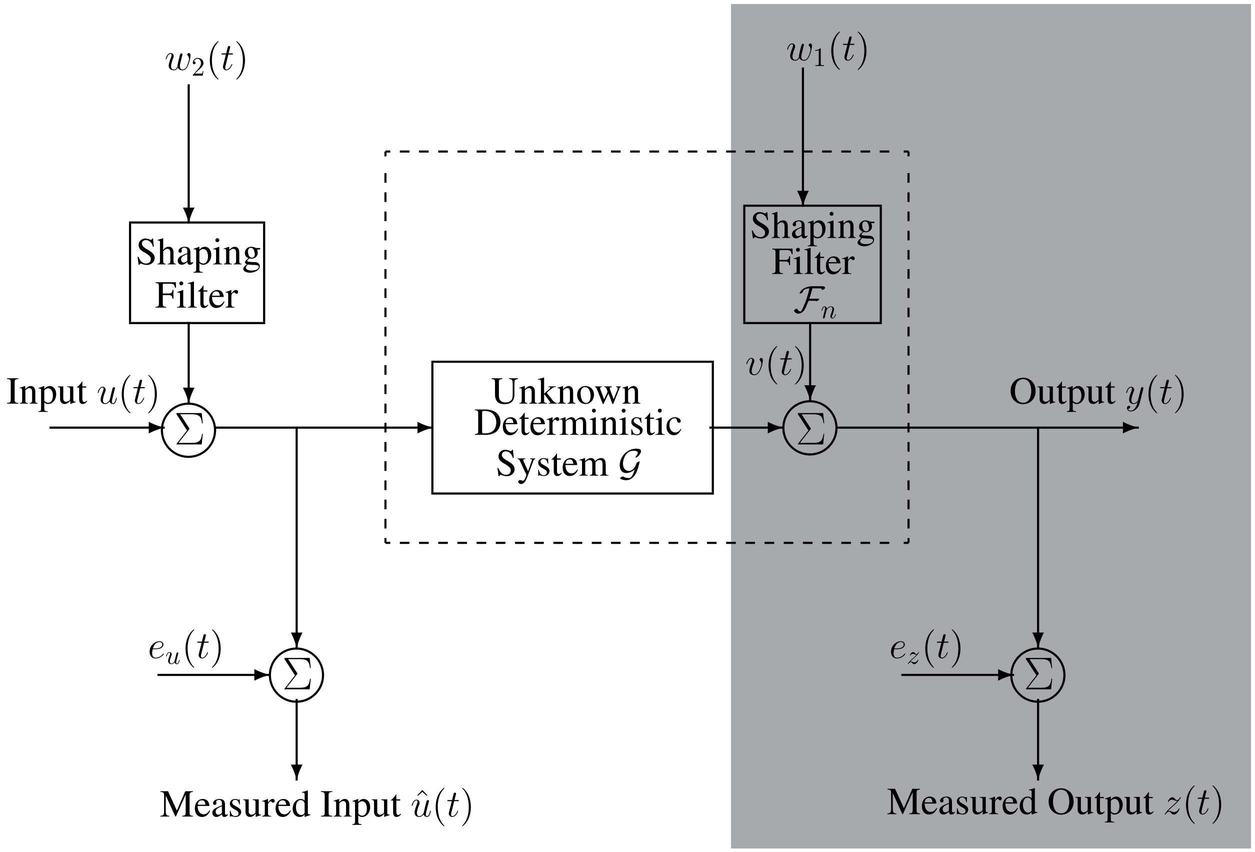
The Identification Problem

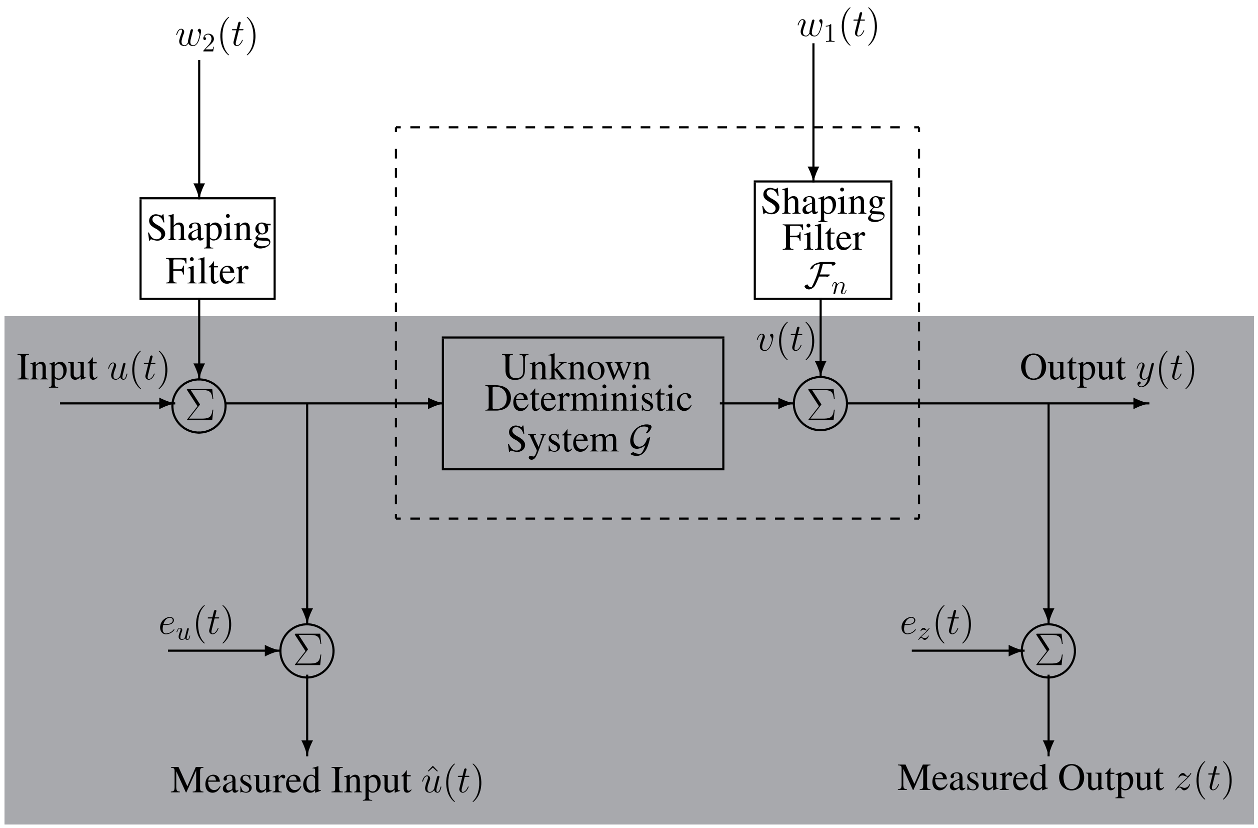


Identification of Stochastic or Noise Model



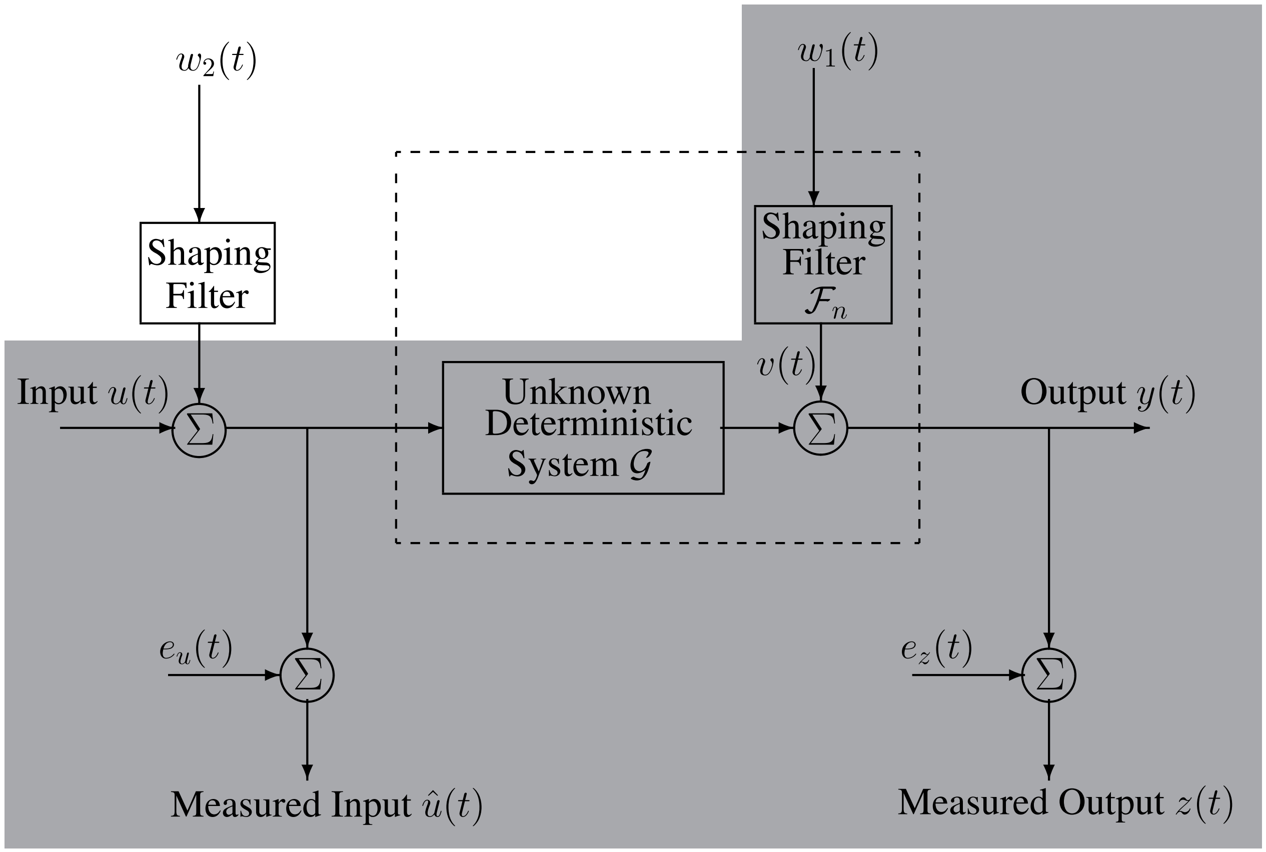
- Relationship between $w_1(t)$ and $z(t)$, given only system output, $z(t)$
- $u(t)$ assumed zero or constant
- Commonly known as time-series analysis
- Economic analysis, geophysical or astronomical phenomena, biological data (e.g. EKG, EEG), etc.

Identification of the Deterministic Model, $\mathcal{G}$



- Relationship between $u(t)$ and $y(t)$ – assumes $w_1(t) = 0$
- Input/output corrupted by noise, $e_u(t)$ & $e_z(t)$ – commonly assumes $e_u(t) = 0$
- Pursued when objective is to gain insight into the functioning of a system
- Automotive industry, chemical plants, pulp & paper, biomedical modelling (e.g. respiratory modelling, drug delivery, disease modelling, etc.)

Identification of Stochastic & Deterministic Models



- Both input/output signals available for identification
- Used when accurate predictions are desired
- Design of model-based control systems for aircraft, spacecraft or robotics.

Representation of the System $\mathcal{G}$

- Linear or Non-linear
  - Systems in nature are inherently non-linear – especially in biology
  - Linear modelling about an operating point is difficult with biological systems due to operating point variability
  - Could lead to misinterpretation of physiology, e.g. disease quantification
  - Biological processes should be modelled in their natural state – non-linearly

- Nonparametric or Parametric

Nonparametric Methods

- Advantage
  - Provide convenient, robust means of characterising the dynamics of linear systems without requiring a priori assumptions regarding the system structure
- Disadvantage
  - Nonparametric estimates of dynamics are difficult to relate to the structure and parameters of the underlying physiological system

Parametric Methods

- Disadvantage
  - Generally require a priori assumptions about the system order

- Advantage
  - Provides a concise description of the system dynamics
  - Yield results that may be related directly to the system structure

Model Form

- Linear statistical model

$$z(n) = \sum_{j=1}^{p} \theta_j f(\varphi_j(n)) + e(n)$$

  - z: observed system output
  - θ_j: unknown system parameter
  - f: non-linear mapping
  - φ_j: regressor
  - e: independent Gaussian variable, zero-mean, σ^2 homoskedastic (constant, finite variance)

- Let $\boldsymbol{\varphi}$ be described as:

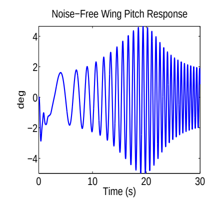
$$\boldsymbol{\varphi}(n) = [1, u(n), \cdots, u(n-n_u), z(n-1), \cdots, z(n-n_z), e(n-1), \cdots, e(n-n_e)]^T$$

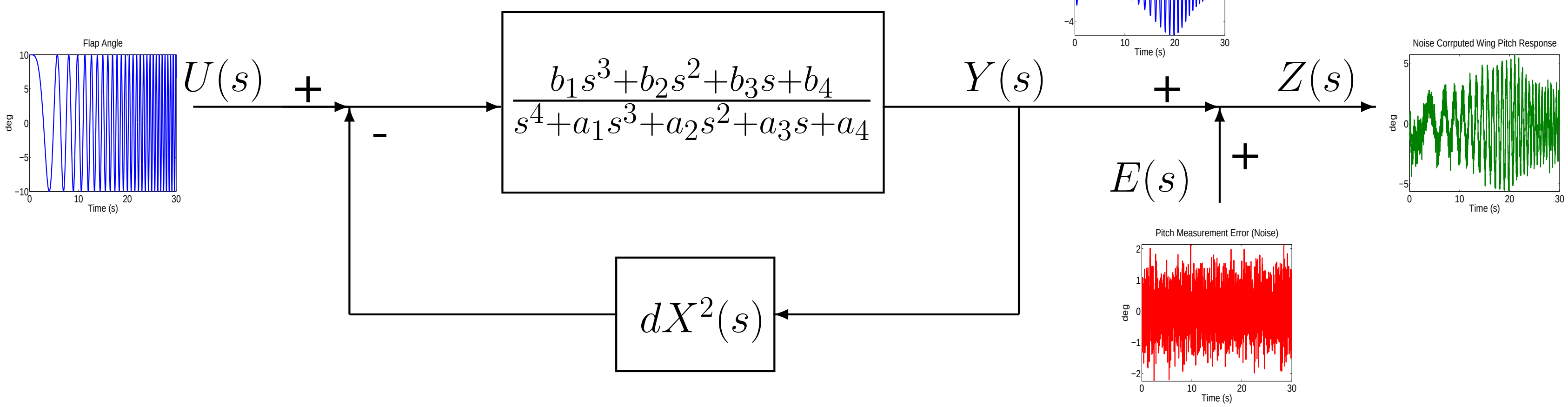
  - Special case f polynomial: $u^2(n-3)$, $u(n)u(n-1)$, $z(n-1)z(n-2)$, $u^2(n-1)z(n-2)$
  - General case f: wide variety of non-linear functions such as a sigmoid
  - NARMAX

- Linear-in-the-parameters
  - Linear or pseudolinear regression techniques

Example of a NARMAX Model

- Non-linear feedback model





- NARMAX representation

$$
\begin{aligned}
z(n) &= \theta_1 z(n-1) + \theta_2 z(n-2) + \theta_3 z(n-3) + \theta_4 z(n-4) + \theta_5 z^2(n-1) + \theta_6 z^2(n-2) \\
&+ \theta_7 z^2(n-3) + \theta_8 z^2(n-4) + \theta_9 z^2(n-5) + \theta_{10} u(n) + \theta_{11} u(n-1) + \theta_{12} u(n-2) \\
&+ \theta_{13} u(n-3) + \theta_{14} u(n-4) - \theta_1 e(n-1) - \theta_2 e(n-2) - \theta_3 e(n-3) - \theta_4 e(n-4) \\
&- 2\theta_5 z(n-1)e(n-1) - 2\theta_6 z(n-2)e(n-2) - 2\theta_7 z(n-3)e(n-3) \\
&- 2\theta_8 z(n-4)e(n-4) - 2\theta_9 z(n-5)e(n-5) + \theta_5 e^2(n-1) + \theta_6 e^2(n-2) \\
&+ \theta_7 e^2(n-3) + \theta_8 e^2(n-4) + \theta_9 e^2(n-5) + e(n)
\end{aligned}
$$

Full Identification Procedure

- Model order selection
  - Determine number of input, output and error lags and non-linearity order/basis function
- Parameter estimation
  - Determine values of unknown parameters
- Structure detection
  - Select parameters to include in model
- Model validation
  - Assess whether identified nominal model can reproduce data from a plant

Model Order

$$z(n) = f^l[1, u(n), \cdots, u(n-n_u), z(n-1), \cdots, z(n-n_z),\\ e(n-1), \cdots, e(n-n_e)] + e(n)$$

- Model order represented as

$$O = [n_u \, n_z \, n_e \, l]$$

Parameter Estimation

- Need an estimate of $\boldsymbol{\theta}$ using standard ℓ_2 minimisation

$$\min_{\boldsymbol{\theta}} \frac{1}{2} \|(\mathbf{Z} - \boldsymbol{\Psi}\boldsymbol{\theta})\|_2^2$$

- NARMAX models provide concise system representation
  - Noise on output enters model as product terms with system input and output making parameter estimation challenging

- Ordinary least-squares yields biased estimate: does not account for noise

$$\hat{\boldsymbol{\theta}} = (\boldsymbol{\Psi}^T\boldsymbol{\Psi})^{-1}\boldsymbol{\Psi}^T\mathbf{Z} \quad \text{where} \quad \boldsymbol{\Psi} = [\boldsymbol{\Psi}_{zu}]$$

- Solution extended least-squares (ELS)

$$\hat{\boldsymbol{\theta}} = (\boldsymbol{\Psi}^T\boldsymbol{\Psi})^{-1}\boldsymbol{\Psi}^T\mathbf{Z} \quad \text{where} \quad \boldsymbol{\Psi} = [\boldsymbol{\Psi}_{zu} \mid \boldsymbol{\Psi}_{zu\hat{\epsilon}} \;\; \boldsymbol{\Psi}_{\hat{\epsilon}}]$$

  - Bias addressed by modelling lagged errors

Structure Detection

- Selection of subset of candidate terms that best describe observed output
- Maximum number of candidate terms

$$p = \sum_{k=1}^{l} p_k + 1$$

$$p_k = \frac{p_{k-1}(n_z + n_u + n_e + k)}{k}, \quad p_0 = 1$$

  - Example: model of order: $O = [4\;4\;4\;2]$
  - $p = 105$ candidate terms
  - The curse of dimensionality!

- Leads to computationally intractable combinatorial optimisation problems

Example of Candidate Terms and Structure Detection

- Model example:

$$z(n) = \theta_i u(n-1) + \theta_i u^2(n-1) + \theta_i z(n-1) + \theta_i e(n-1) + e(n)$$

- Described by: $O \ = \ [1\ 1\ 1\ 2] \Rightarrow p = 15$

- Candidate terms:

$$\begin{aligned}
z(n) \ = \ & \theta_0 + \theta_1 u(n) + \theta_2 u(n-1) + \theta_3 u^2(n) + \theta_4 u(n)u(n-1) + \theta_5 u^2(n-1) \\
+ \ & \theta_6 z(n-1) + \theta_7 u(n)z(n-1) + \theta_8 u(n-1)z(n-1) + \theta_9 z^2(n-1) \\
+ \ & \theta_{10} u(n)e(n-1) + \theta_{11} u(n-1)e(n-1) + \theta_{12} z(n-1)e(n-1) \\
+ \ & \theta_{13} e(n-1) + \theta_{14} e^2(n-1) + e(n)
\end{aligned}$$

NARMAX Representation and Identification of Ankle Dynamics

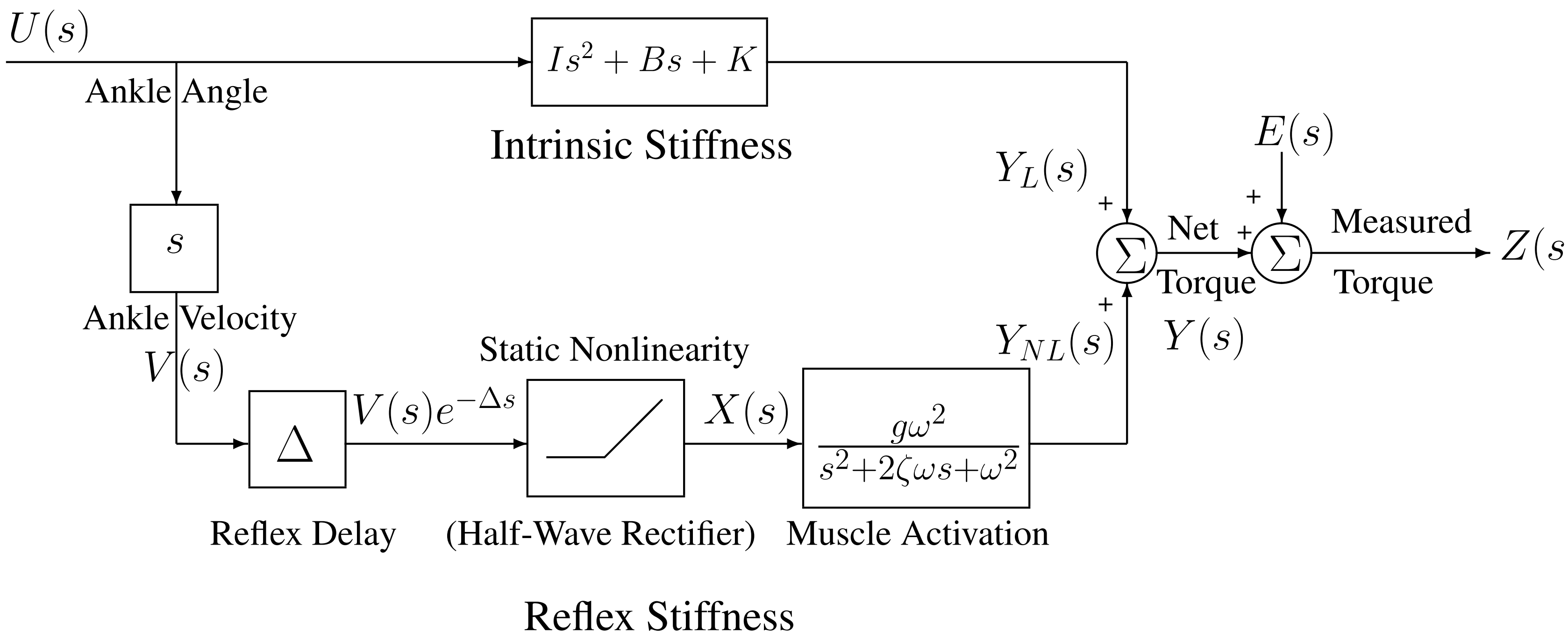
Objectives

- Theoretically derive a non-linear difference equation of a parallel pathway model of ankle dynamics
- Show that the theoretical equation for this ankle model is of the NARMAX form
- Investigate the suitability of NARMAX identification methods applied to ankle dynamics

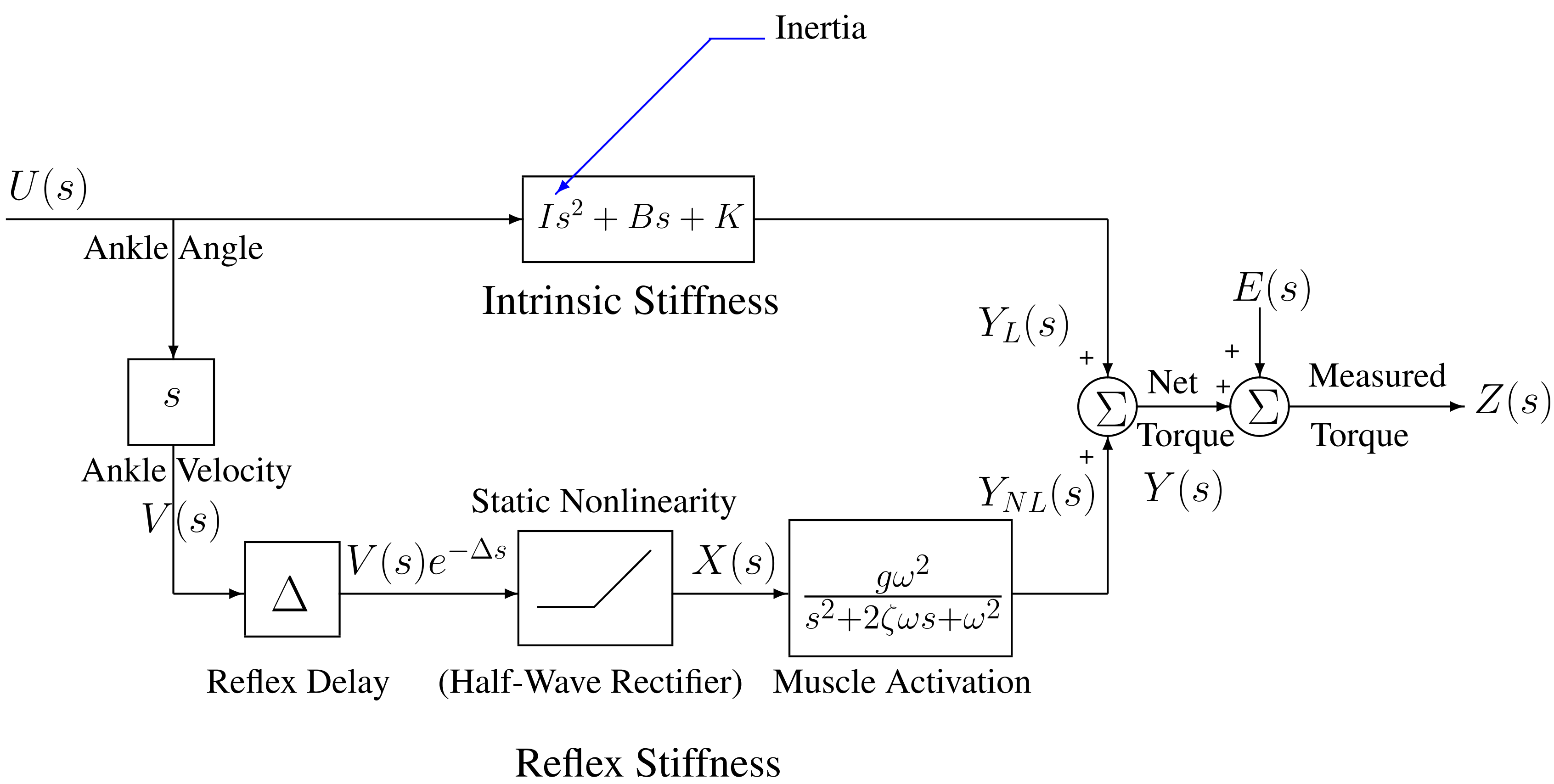
Motivation

- Better design of prosthetics
- Characterisation of normal versus disease subjects
- Early detection of neuromuscular disease
- Design of robotic motor control systems
- Astronaut health to mitigate muscle atrophy – develop optimal exercise regime

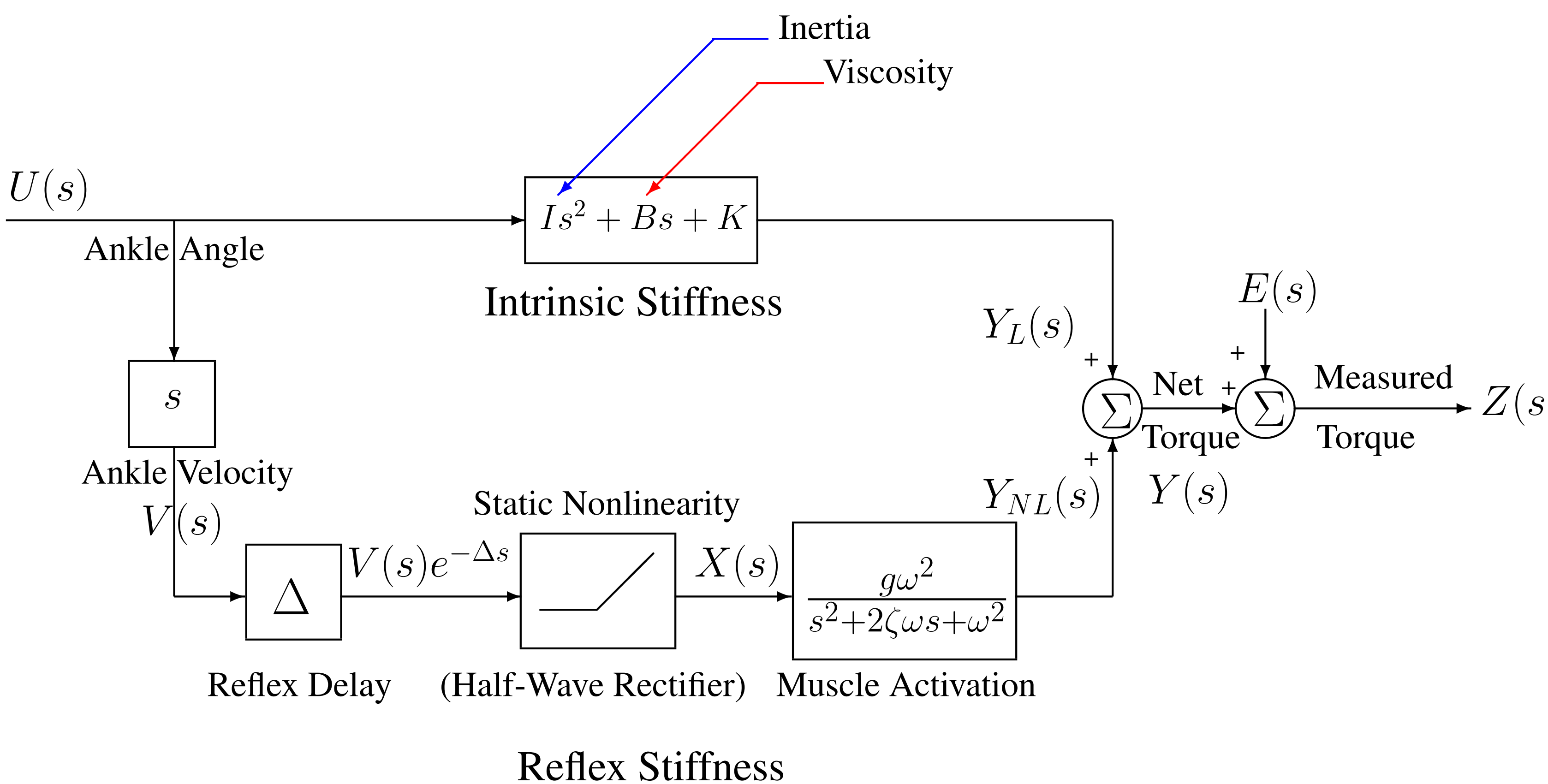
Parallel Pathway Model of Ankle Dynamics



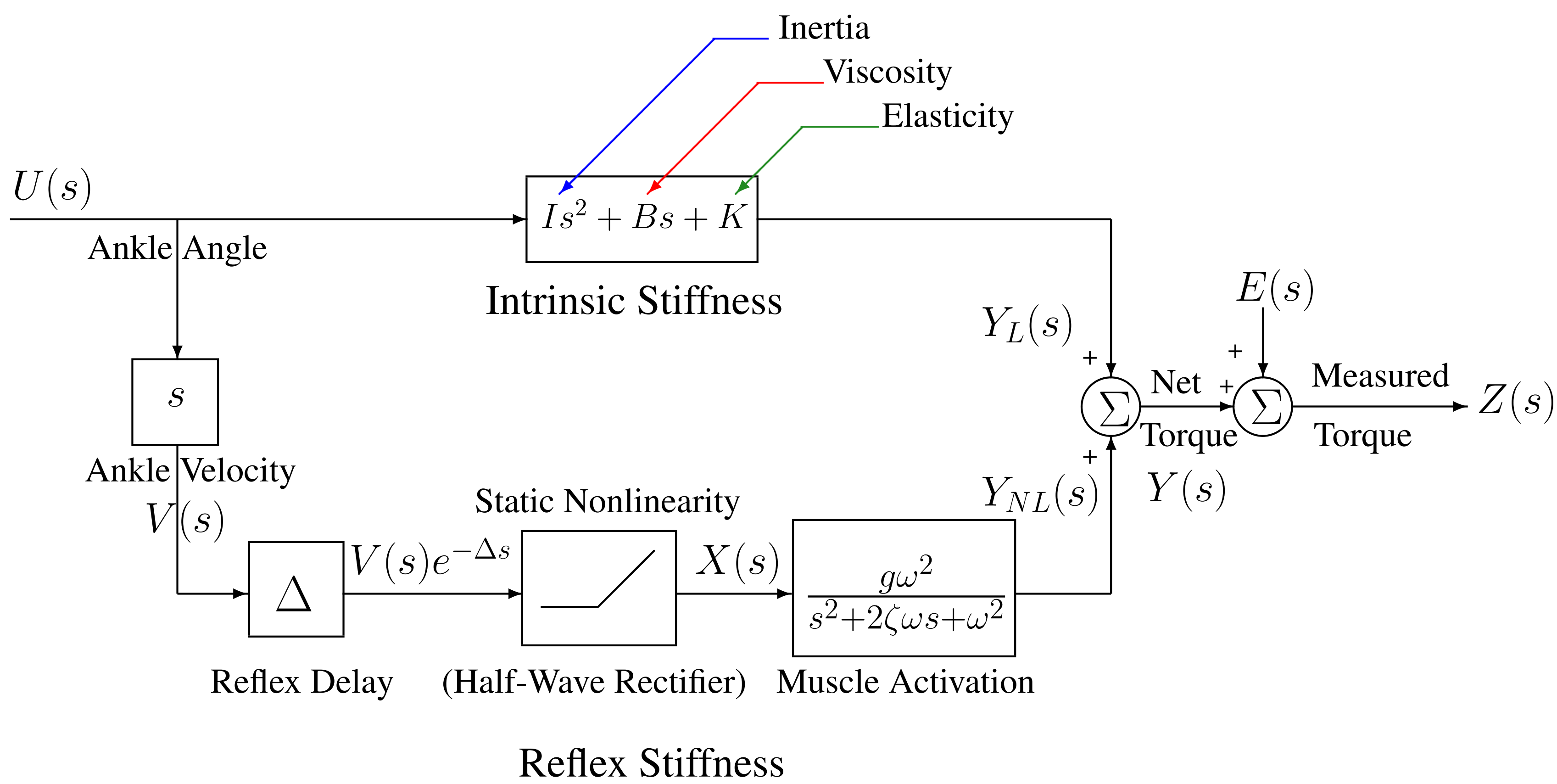
Parallel Pathway Model of Ankle Dynamics



Parallel Pathway Model of Ankle Dynamics



Parallel Pathway Model of Ankle Dynamics



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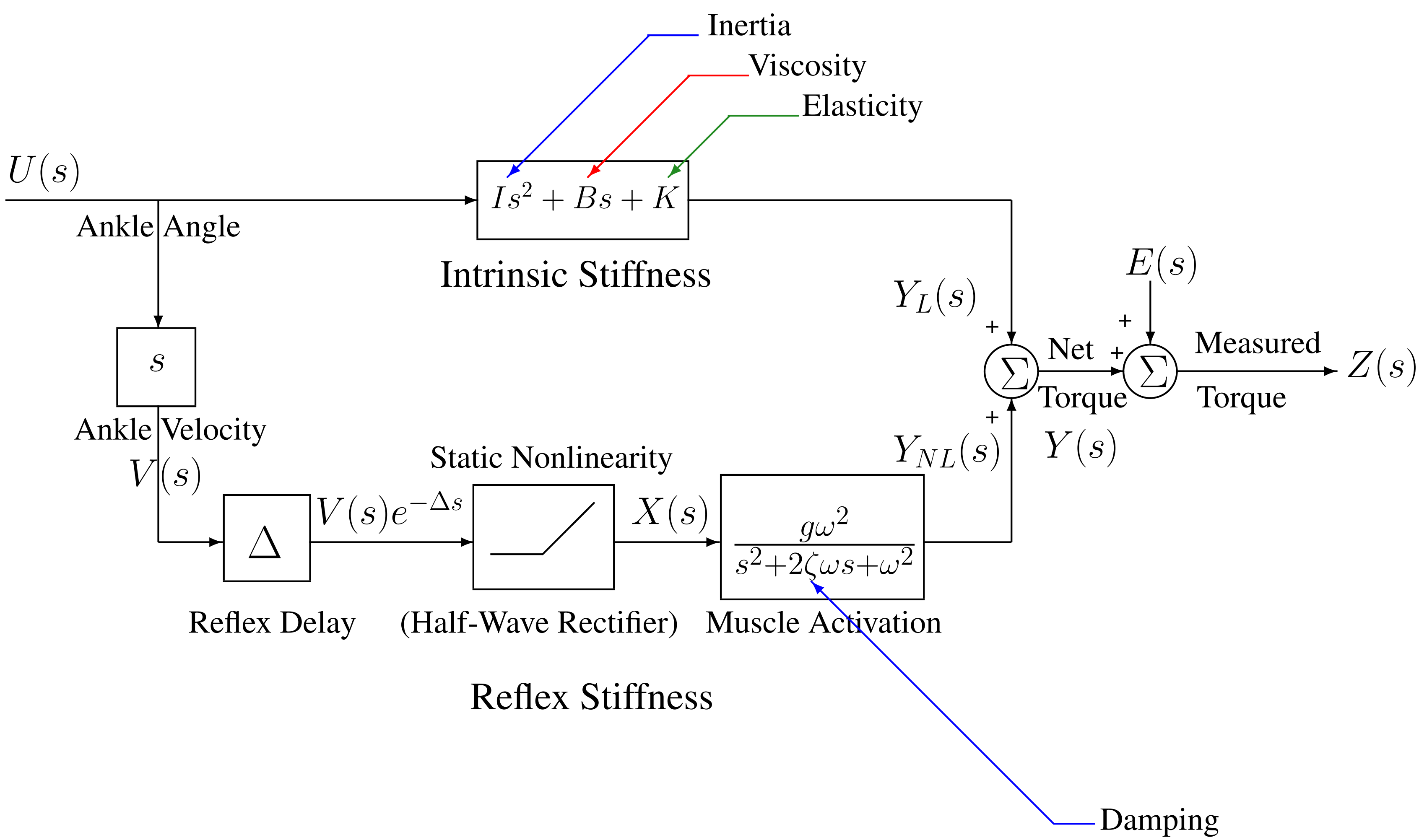

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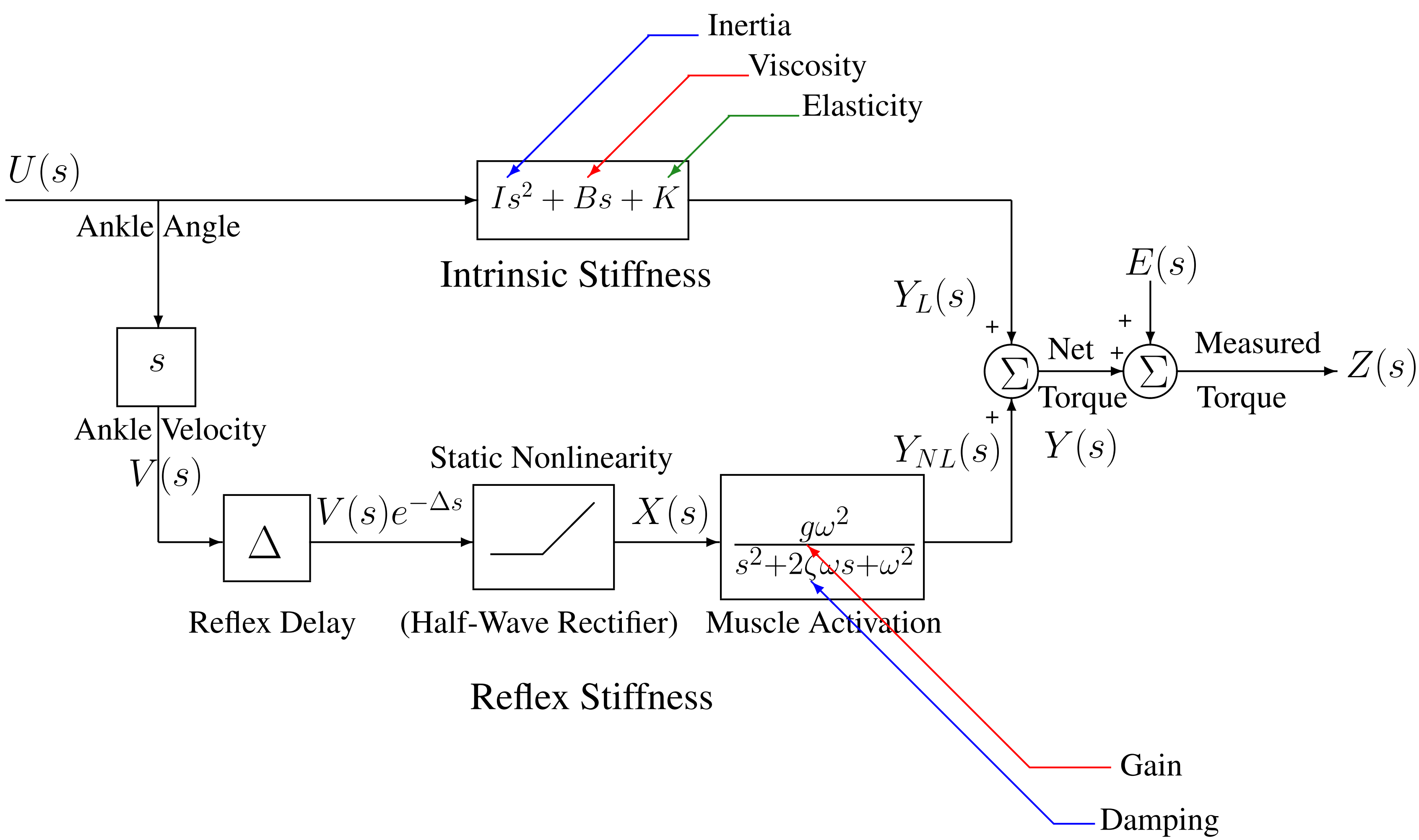
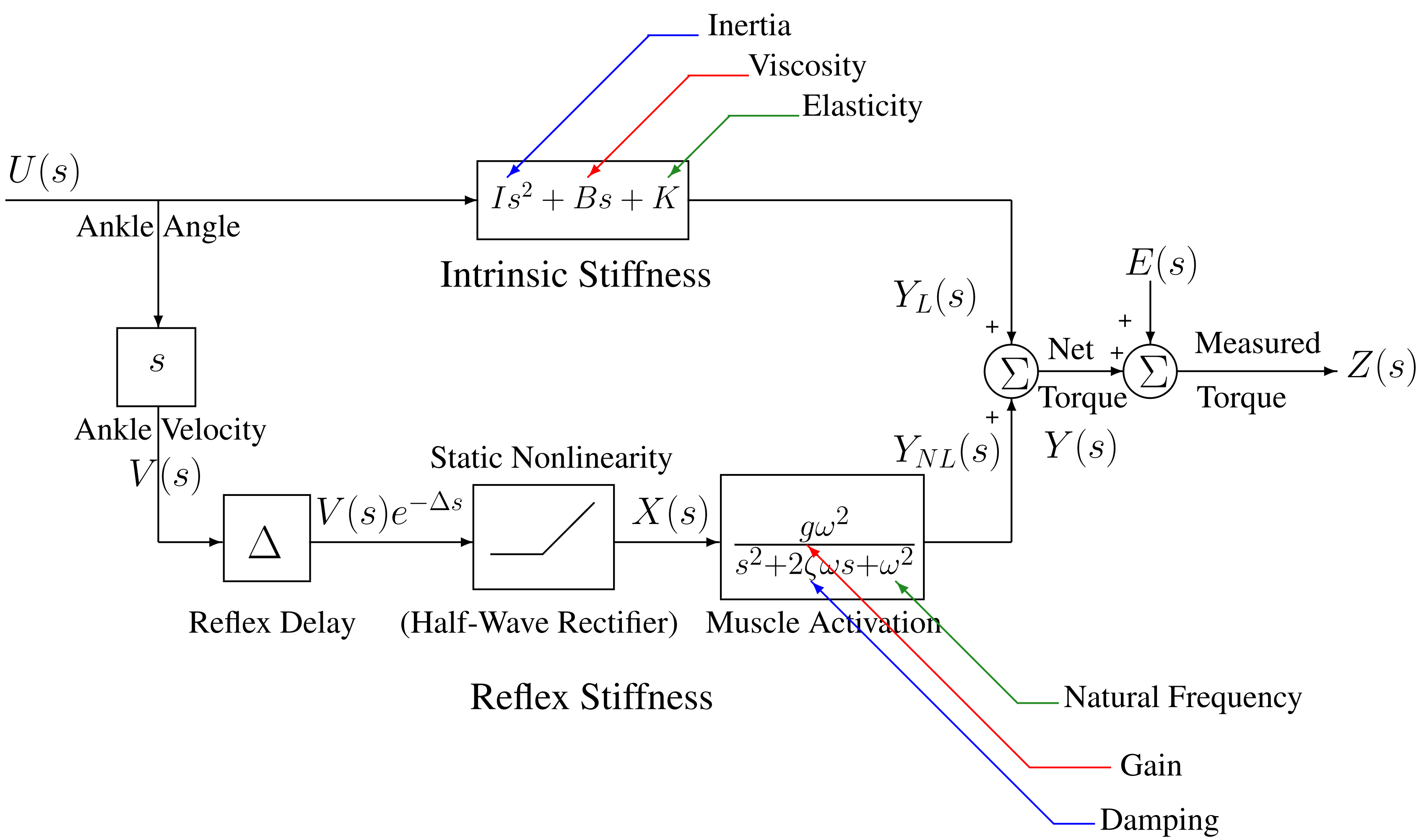
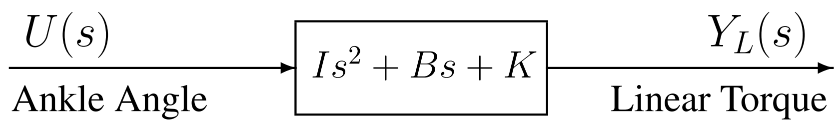


Parallel Pathway Model of Ankle Dynamics



Theoretical Analysis: Linear Pathway

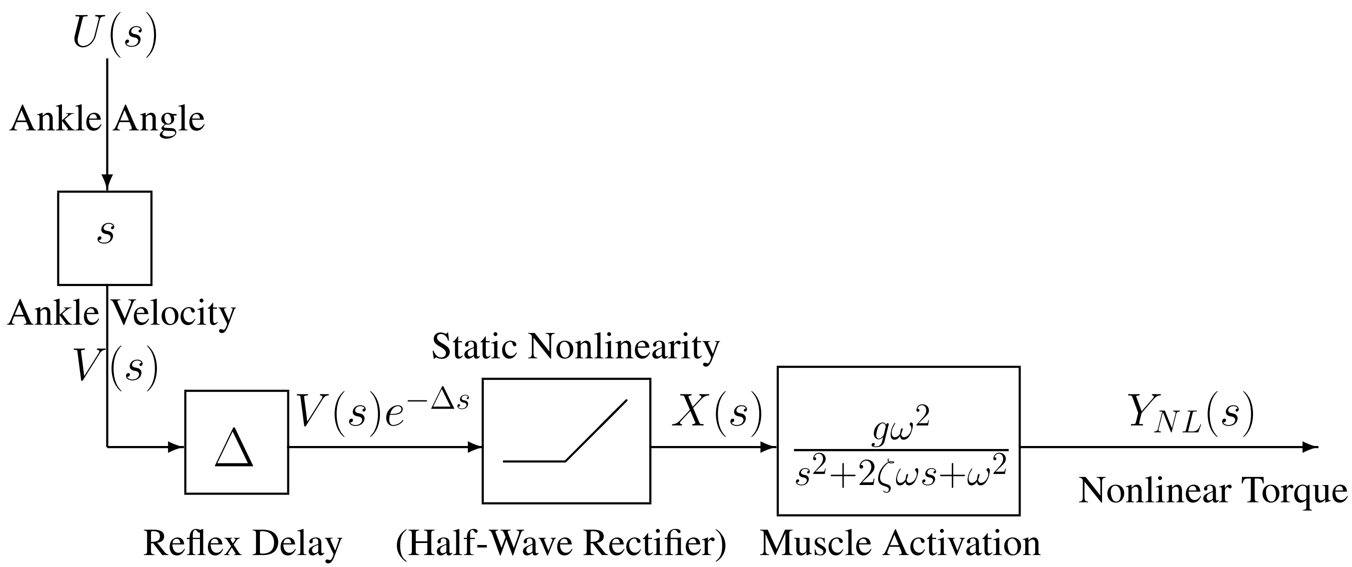


- Discrete-domain approximation to a derivative (Newton's backwards formula) was used to approximate the intrinsic pathway dynamics

$$s = \frac{d\ u(t)}{dt} \approx \frac{u(n) - u(n-1)}{T}$$

where $T \equiv$ sampling rate and $n \equiv$ sampled data point index

Theoretical Analysis: Non-linear Pathway



- Continuous-time delay converted to discrete-time as $\tau = \lceil \frac{\Delta}{T} \rceil$ where Δ is the continuous-time delay
- Half-wave rectifier approximated as a second-order static polynomial: $c_0 + c_1 x(n) + c_2 x^2(n)$
  - Second-order fit accounted for over 98% of the output variance of static non-linearity
- Activation dynamics converted to discrete-time via the bilinear transform

$$s = \frac{2}{T}\left(\frac{z-1}{z+1}\right)$$

Theoretical Analysis: Overall Non-linear Model

- Collecting terms and combining yielded the overall non-linear model as

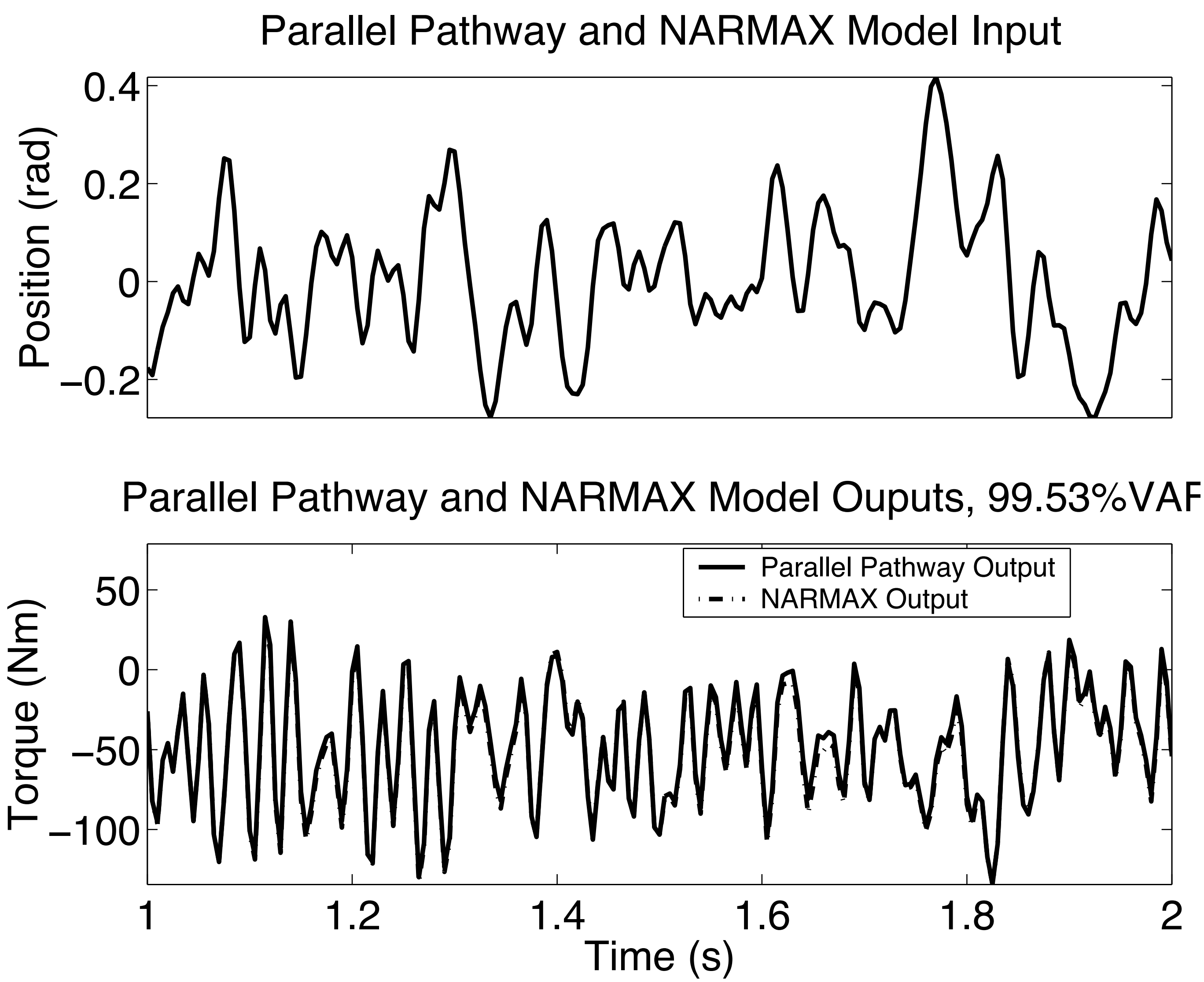
$$
\begin{aligned}
z(n) &= b_0 + b_1 z(n-1) + b_2 z(n-2) + b_3 u(n) + b_4 u(n-1) + b_5 u(n-2) \\
&+ b_6 u(n-3) + b_7 u(n-4) + b_8 u(n-\tau) + b_9 u(n-\tau-1) \\
&+ b_{10} u(n-\tau-2) + b_{11} u(n-\tau-3) + b_{12} u^2(n-\tau) + b_{13} u^2(n-\tau-1) \\
&+ b_{14} u^2(n-\tau-2) + b_{15} u^2(n-\tau-3) + b_{16} u(n-\tau) u(n-\tau-1) \\
&+ b_{17} u(n-\tau-1) u(n-\tau-2) + b_{18} u(n-\tau-2) u(n-\tau-3) \\
&+ b_{19} e(n-1) + b_{20} e(n-2) + e(n)
\end{aligned}
$$

- This is a NARMAX model since
  - (i) it includes input-output terms that are combinations of linear, non-linear integer powers and cross-products and
  - (ii) is linear-in-the-parameters

Validation of NARMAX Ankle Model

- Simulate the NARMAX description model's response and compare it to the continuous-time model's response for typical parameter values found in experiments
- Input sequence uniformly distributed, white, zero-mean, random input, bandlimited to 30 Hz
- Operating range between ± 0.40 rad
- Sampling rate 200 Hz ($T = 0.005$ s)

Results


Parallel Pathway and NARMAX Model Input
Position (rad)
0.4
0.2
0
−0.2
Parallel Pathway and NARMAX Model Ouputs, 99.53%VAF
Parallel Pathway Output
NARMAX Output
Torque (Nm)
50
0
−50
−100
1
1.2
1.4
1.6
1.8
2
Time (s)

Identification of NARMAX Representation of Ankle Dynamics

- Assess the utility of methods developed for identifying NARMAX models

- Noise on the output enters the model as product terms with the system input and output

- Ordinary least-squares yields biased estimate $\Rightarrow$ Does not account for noise

- One solution extended least-squares (ELS) $\Rightarrow$ Bias addressed by modelling lagged errors

Identification of Full NARMAX Representation Yields Biased Estimates

- Reduce number of terms
- Not a general reduction of terms to describe the data but rather a minimisation of the number of regressors or degrees of freedom used to form the regressor matrix
- Reduction provides a regressor matrix that is more stable in terms of invertibility since the coefficients will no longer be interrelated
- Similar to reconditioning a matrix via normalisation

Reduced Model Representation

- Recombining all terms according to coefficients of the static non-linearity yields an overall non-linear model represented by 12 terms as

$$
\begin{aligned}
z(n) &= b_0 + b_1 z(n-1) + b_2 z(n-2) + b_3 u(n) + b_4 u(n-1) + b_5 u(n-2) \\
&+ b_6 u(n-3) + b_7 u(n-4) + m_1 \upsilon(n) + m_2 \chi(n) + b_{19} e(n-1) \\
&+ b_{20} e(n-2) + e(n)
\end{aligned}
$$

where

$$
\upsilon(n) = u(n-\tau) + u(n-\tau-1) - u(n-\tau-2) - u(n-\tau-3)
$$

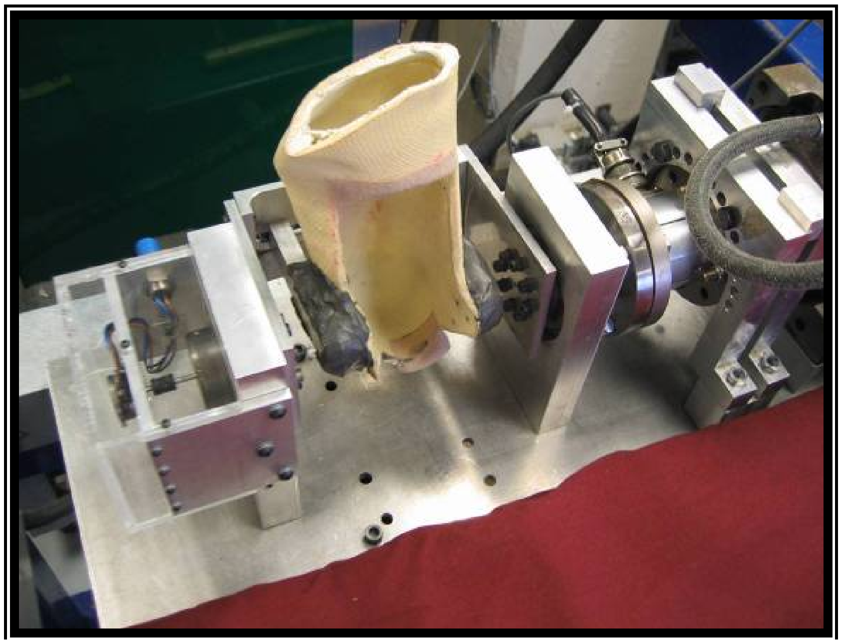
and

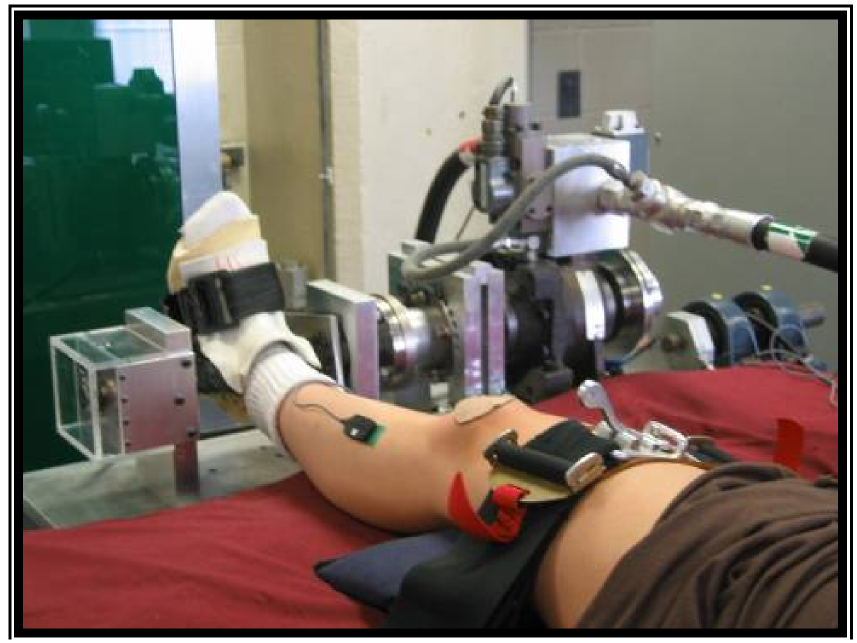
$$
\begin{aligned}
\chi(n) &= u^2(n-\tau) + 3u^2(n-\tau-1) + 3u^2(n-\tau-2) + u^2(n-\tau-3) \\
&- 2u(n-\tau)u(n-\tau-1) - 4u(n-\tau-1)u(n-\tau-2) \\
&- 2u(n-\tau-2)u(n-\tau-3)
\end{aligned}
$$

Experimental Data

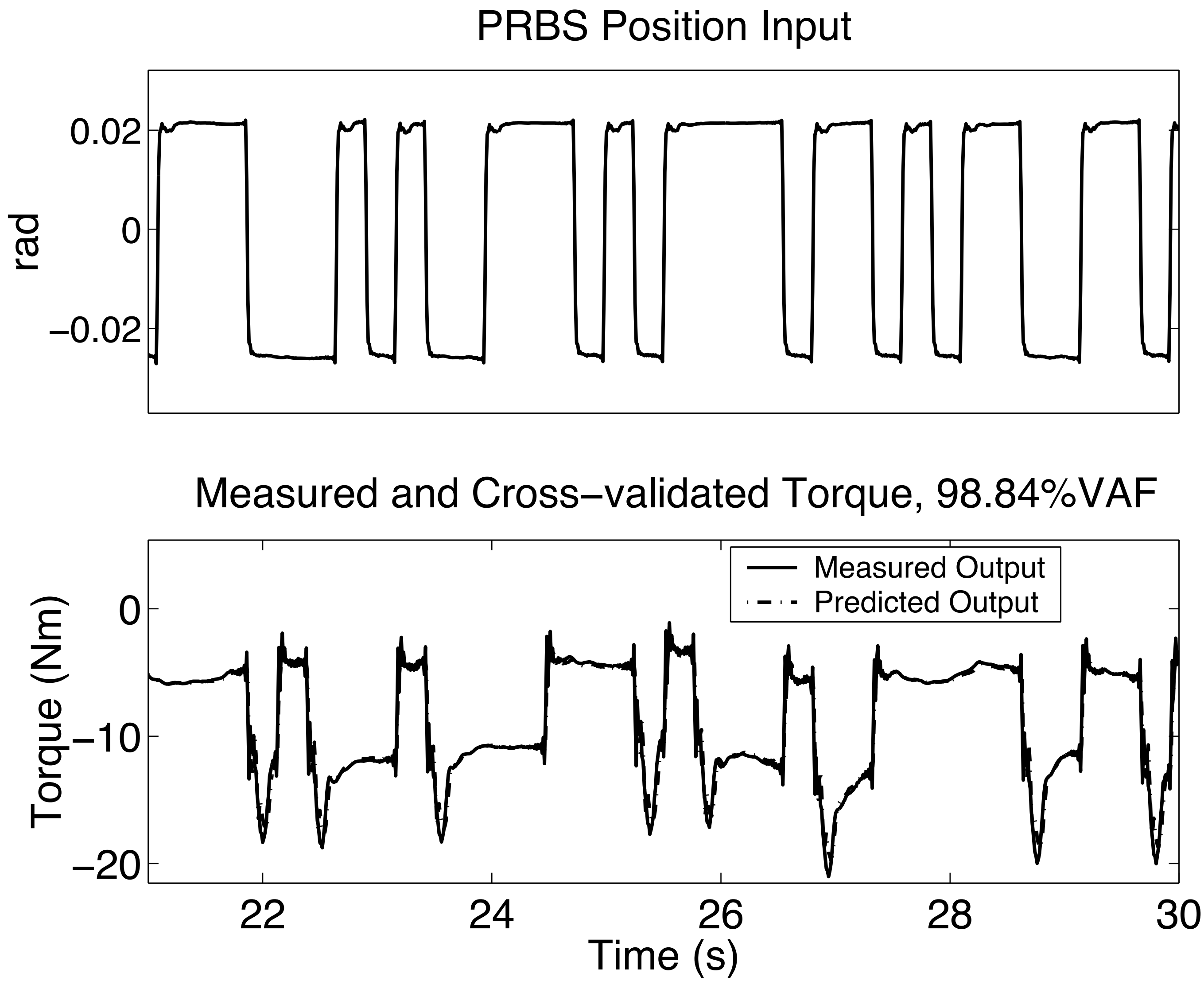
- Eight trials of experimental human ankle data analyzed from single subject with no history of neuromuscular disease
- Pseudo-random binary sequences (PRBS) of 0.05 rad and 260 ms switching rate
- Subject maintained a mean contraction of -5 Nm

Experimental Setup

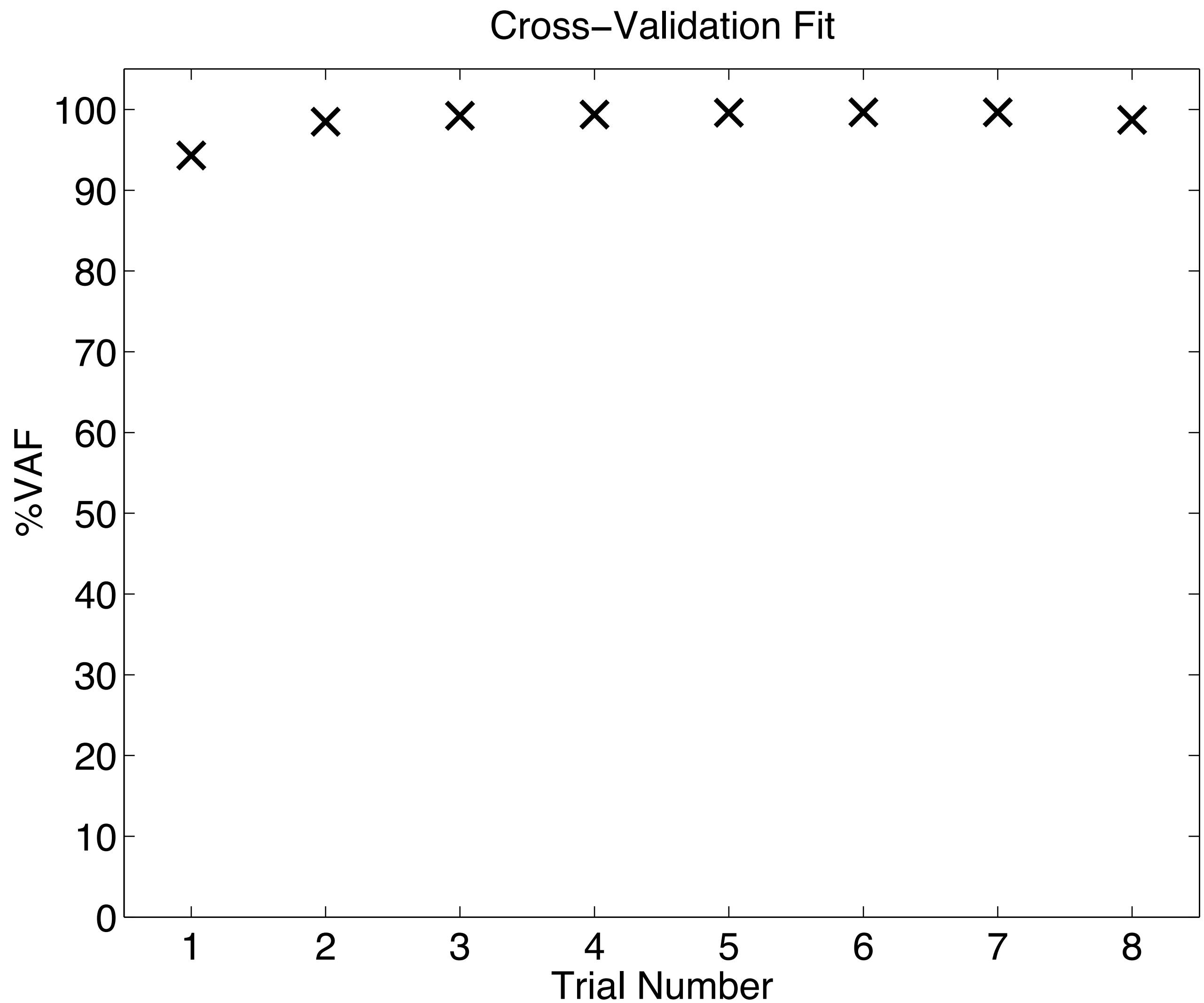




Typical Position Input and Cross-Validated Torque Output



Cross-Validation %VAF for Each Trial



Conclusions

- Non-linear difference equation describing parallel pathway model is a NARMAX model
- Simulation results show that the NARMAX model matches the continuous-time response well
- Identification methods developed for NARMAX models can be used to identify human ankle dynamics parametrically
- Experimental data shows that estimated parameters explain the input-output data well
- The NARMAX form is clearly amenable to the study of a wide range of biological systems

A Least-Squares Parameter Estimation Algorithm for Switched Hammerstein Systems With Applications to the VOR

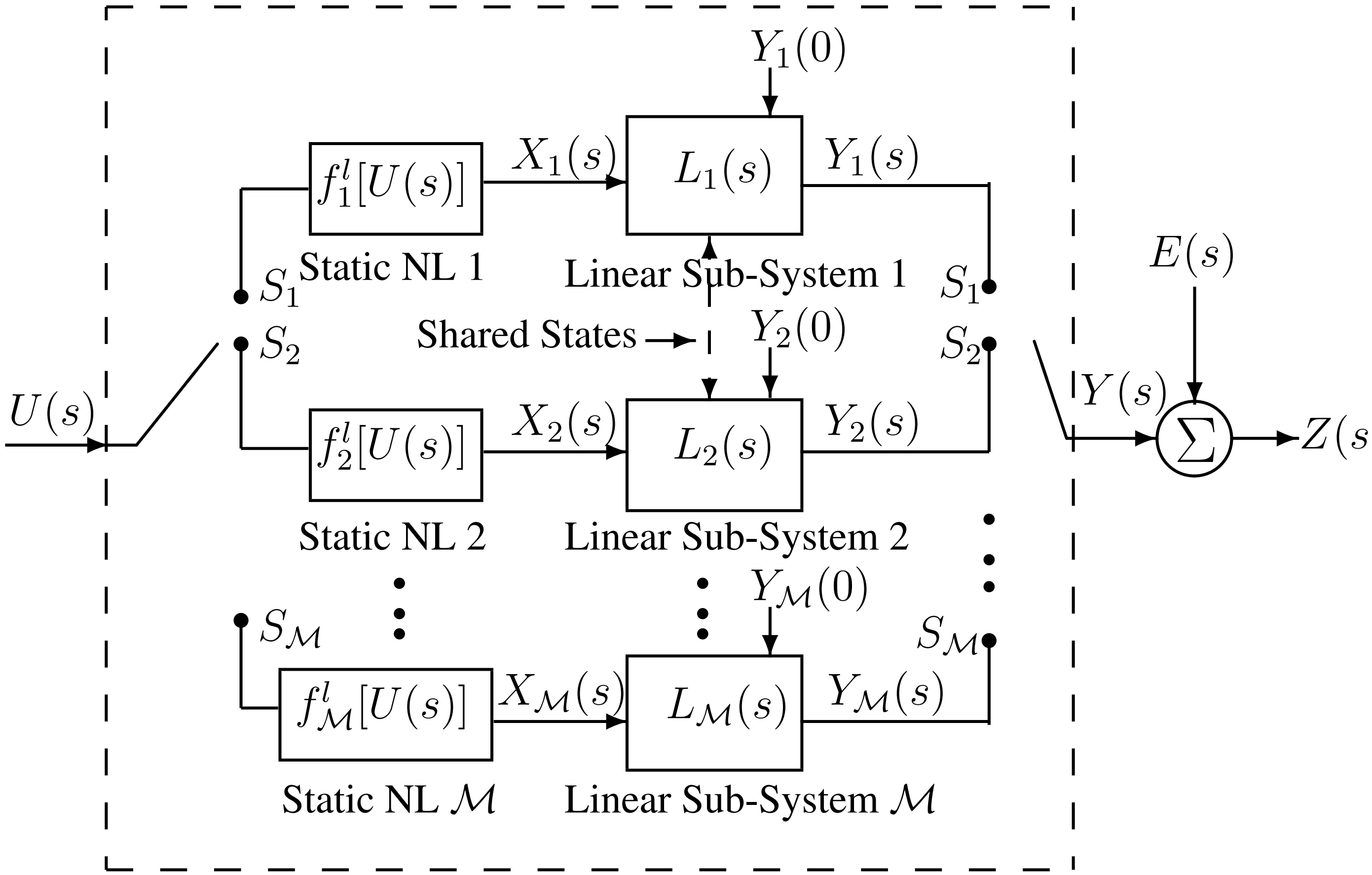
Objectives

- Can Vestibular Ocular Reflex (VOR) be described by a NARMAX model?
- Develop identification technique to estimate parameters of non-linear hybrid (switched) systems
- Apply identification technique to experimental human VOR data

Motivation

- Characterisation of normal versus disease subjects
- Early detection and quantification of ocular disease and balance disorder
- Design of robotic motor control systems
- Pilot/astronaut health and safety

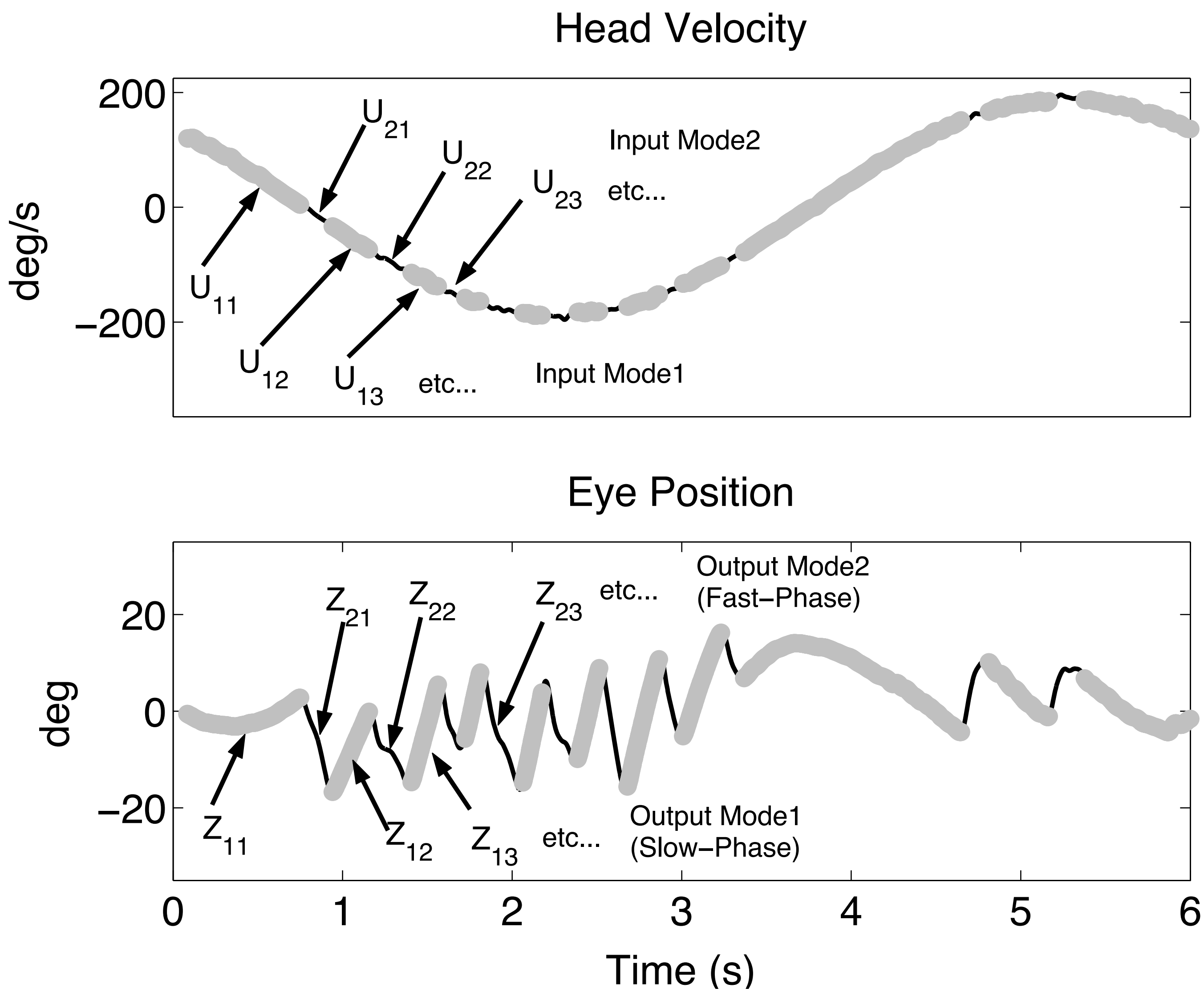
General Hammerstein Structure for Switched System



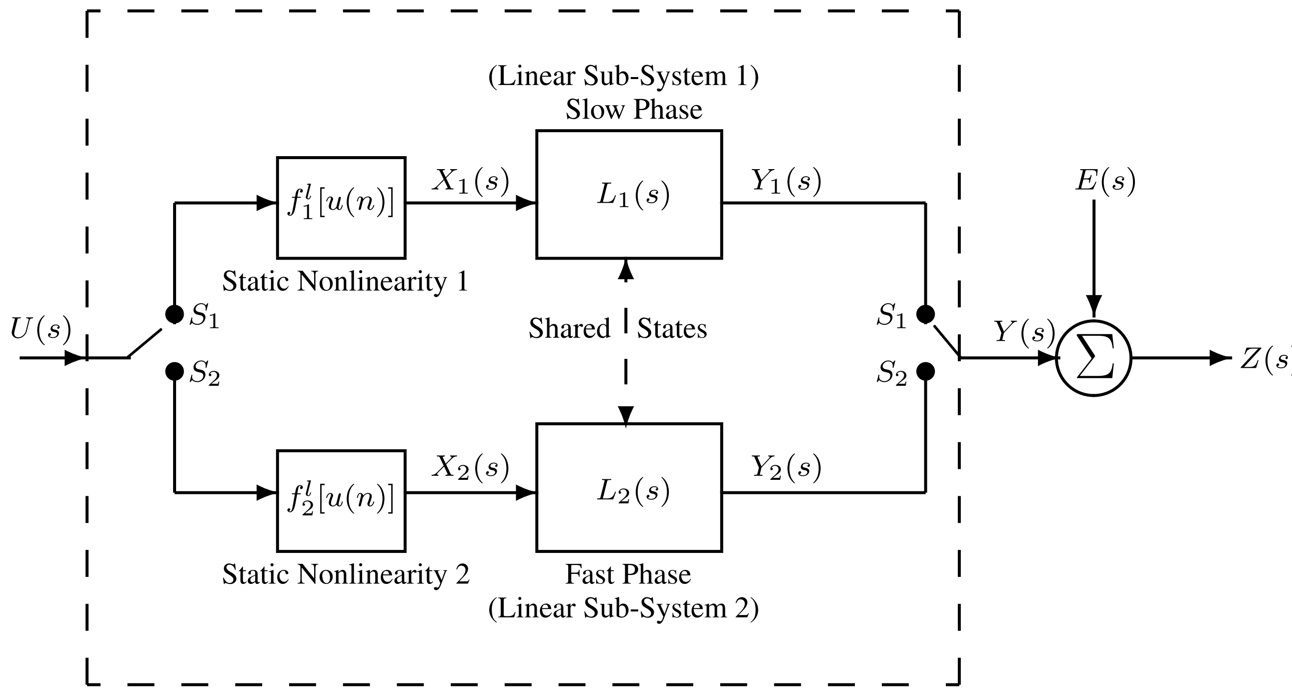
Extension of NARMAX Model for Non-Zero-Initial-State

$$
\begin{aligned}
z_m(n) &= f_m^l[u(n), \ldots, u(n-n_u)] + \mathcal{L}[z(n-1), \ldots, z(n-n_z), e(n-1), \ldots, e(n-n_e)] + e(n) \\
zr_m(n) &= f_m^l[u(1), \ldots, u(n-1), u(n)] + \mathcal{L}[\delta(n)] \quad \text{for} \quad m = 1, 2, \ldots, \mathcal{M} \quad \forall \text{ finite } \mathcal{M}
\end{aligned}
$$

Simulated VOR Data



VOR Model



$$
\begin{aligned}
f_1^l\left[U(s)\right] &= a_1 + b_1 U(s) + c_1 U^2(s) + d_1 U^3(s) = X_1(s) \\
f_2^l\left[U(s)\right] &= a_2 + b_2 U(s) + c_2 U^2(s) + d_2 U^3(s) = X_2(s)
\end{aligned}
$$

$$
\begin{aligned}
Y_1(s) &= \frac{K_1}{\tau_1 s + 1} X_1(s) + \frac{Y_{1i}(0)}{\tau_1 s + 1} = \frac{G_1}{s + p_1} X_1(s) + \frac{Y_{1i}(0)/\tau_1}{s + p_1}; \quad i = 1, \cdots, q \\
Y_2(s) &= \frac{K_2}{\tau_2 s + 1} X_2(s) + \frac{Y_{2\ell}(0)}{\tau_2 s + 1} = \frac{G_2}{s + p_2} X_2(s) + \frac{Y_{2\ell}(0)/\tau_2}{s + p_2}; \quad \ell = 1, \cdots, r
\end{aligned}
$$

NARMAX Model of VOR

- Dynamics converted to discrete-time via the bilinear transform
- Collecting terms and combining yielded an overall non-linear model with two NARMAX sub-models:

$$y(n) = \begin{cases} y_1(n) & \text{Switch Position } S_1 \\ y_2(n) & \text{Switch Position } S_2 \end{cases}$$

$$\begin{aligned} y_1(n) &= \beta_1 + \beta_2 y(n-1) + \beta_3[u(n) + u(n-1)] \\ &+ \beta_4[u^2(n) + u^2(n-1)] + \beta_5[u^3(n) + u^3(n-1)] \\ &+ \kappa_i \delta_i(n_i); \quad i = 1, \cdots, q \end{aligned}$$

$$\begin{aligned} y_2(n) &= \vartheta_1 + \vartheta_2 y(n-1) + \vartheta_3[u(n) + u(n-1)] \\ &+ \vartheta_4[u^2(n) + u^2(n-1)] + \vartheta_5[u^3(n) + u^3(n-1)] \\ &+ \lambda_\ell \delta_\ell(n_\ell); \quad \ell = 1, \cdots, r \end{aligned}$$

Biased Parameter Estimate

- Define: $\mathbf{Z} = \mathbf{\Psi}_{zu\epsilon}\hat{\boldsymbol{\theta}}_{ELS} + \boldsymbol{\xi}$

  $\boldsymbol{\xi} = \lambda_1\delta_1(n - t_1) + \lambda_2\delta_2(n - t_2) + \cdots + \lambda_r\delta_r(n - t_r) + e(n)$

- $E\left[\hat{\boldsymbol{\theta}}_{ELS}\right] = (\mathbf{\Psi}^T\mathbf{\Psi})^{-1}\mathbf{\Psi}^T E\left[\mathbf{Z}\right] = \hat{\boldsymbol{\theta}}_{ELS} + (\mathbf{\Psi}^T\mathbf{\Psi})^{-1}\mathbf{\Psi}^T\boldsymbol{\xi}$

- Bias: $E\left[(\mathbf{\Psi}^T\mathbf{\Psi})^{-1}\mathbf{\Psi}^T\boldsymbol{\xi}\right] \neq 0$

Modified Extended Least Squares

- Modify ELS algorithm to correct bias due to initial conditions

- Include columns in the regressor matrix to account for the initial conditions
  $\mathbf{\Phi} = [\mathbf{\Psi}_{\mathbf{zu}\hat{\boldsymbol{\epsilon}}} \mid \mathbf{\Psi}_{\boldsymbol{\delta}}]$

- Extended parameter set based on this model formulation defined as:
  $\hat{\boldsymbol{\theta}}_{MELS} = (\mathbf{\Phi}^T\mathbf{\Phi})^{-1}\mathbf{\Phi}^T\mathbf{Z}$ where $\hat{\boldsymbol{\theta}}_{MELS} = [\hat{\boldsymbol{\theta}}_p \; \hat{\boldsymbol{\theta}}_\delta]$.

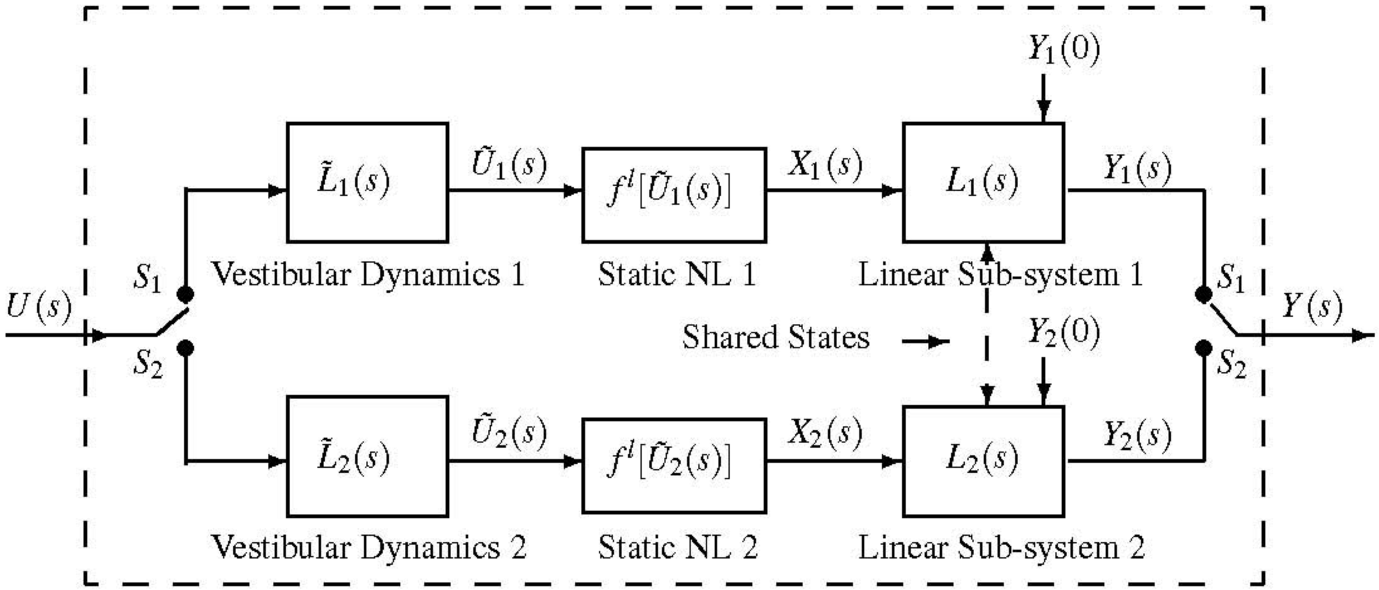
Algorithm

- Segment all data according to mode
- Concatenate data from each mode
- Form regressor matrix for each mode
- Add a column modelled as an impulse when new segment is used to form the regressor matrix
- Estimate $\boldsymbol{\theta}_{MELS}$ using ℓ_2 minimisation

Experimental Data

- Data analyzed from single patient with history of peripheral vestibular disease – no function in the right inner ear following surgery
  - Often associated with large non-linearity in the VOR response and abnormally small slow-phase time constant
- Experimental protocol used a sinusoidal rotation at 1/6 Hz, with a peak head velocity ~ 200 deg/s
  - Test 52 s, last 32 s recorded to measure VOR properties during sensory steady state
- Signals sampled at 500 Hz then digitally low-pass filtered to 15 Hz to reduce high frequency content

Data Analysis

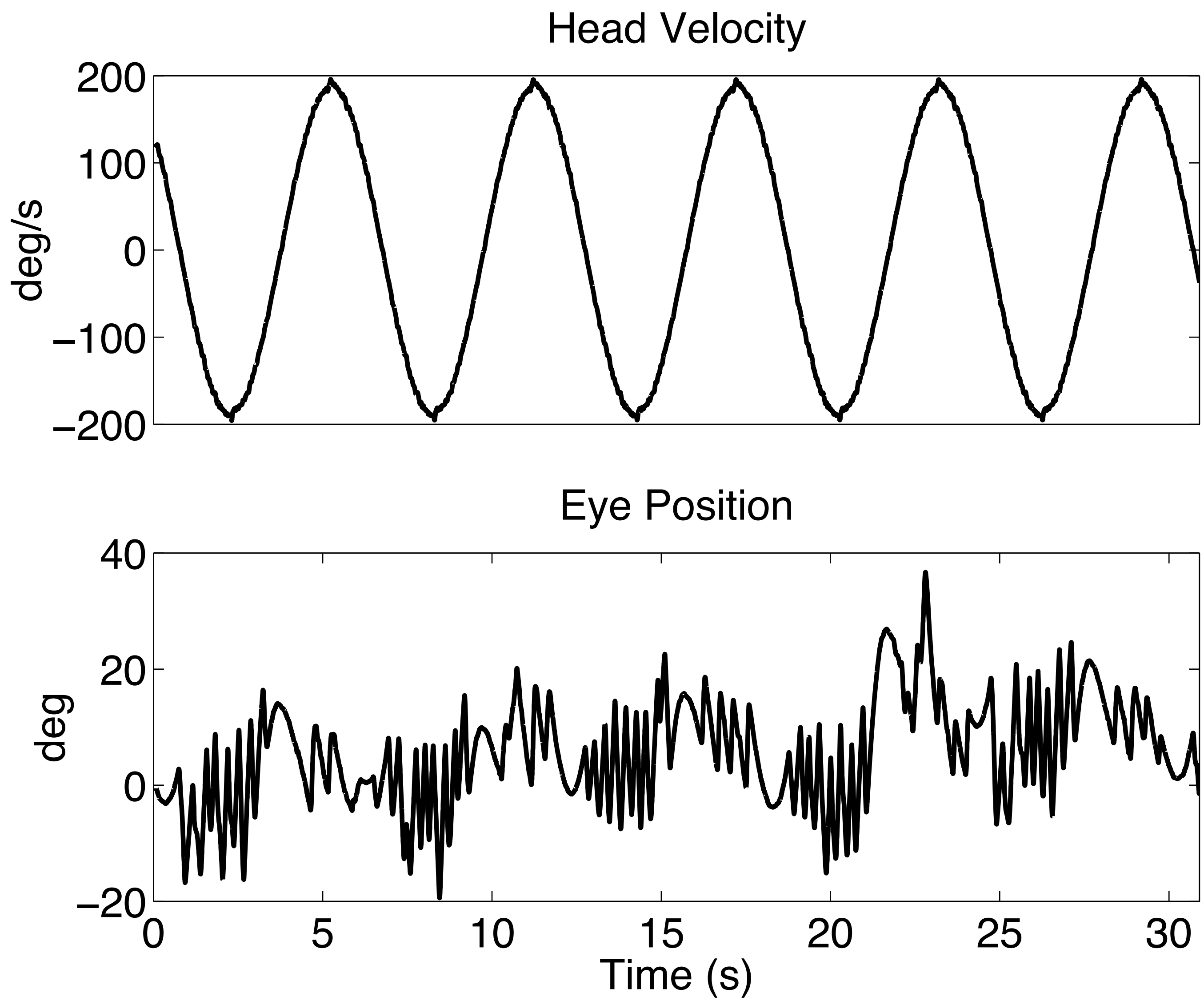


$$\tilde{U}_1(s) = \frac{\tau_1 s}{\tau_1 s + 1} U(s) = \frac{s}{s + p_1} U(s) \quad \tilde{U}_2(s) = \frac{\tau_2 s}{\tau_2 s + 1} U(s) = \frac{s}{s + p_2} U(s)$$

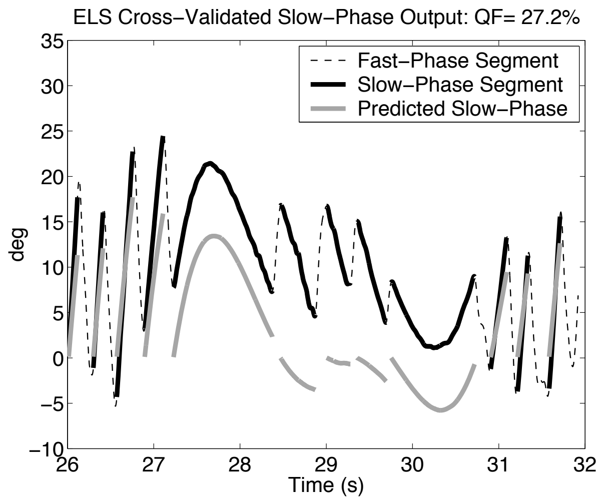
- Canal dynamics pre-process head velocity which serve as sensor for head movement
- Sensory vestibular process well described by first-order, high-pass system which transmits signals to the central circuits acting as a switched system
- Estimate $\tau_{1,2}$ associated with vestibular dynamics, assuming $\tau_{1,2}$ =1 -15 s in 1 s increments
  - Combination of vestibular time constant and switched system time constant which yielded highest cross-validation %QF was deemed the best-fit model
- Compared the results of MELS to ELS (not modelling initial conditions)

Experimental Setup

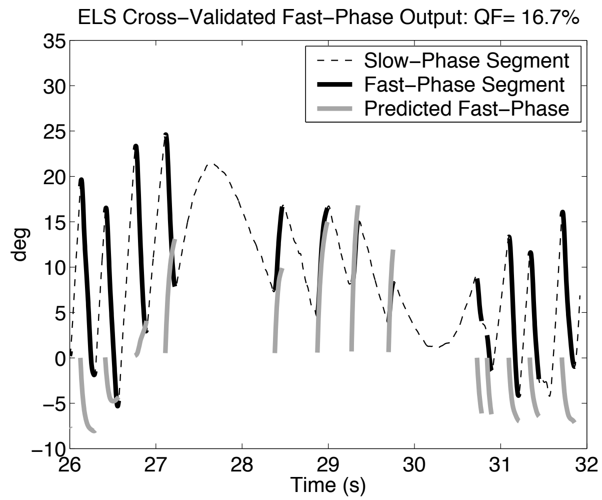
Experimental VOR Data



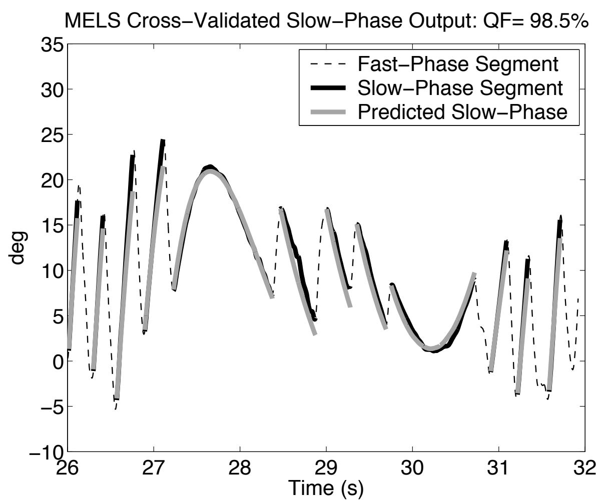
Results: Cross-Validated Eye Position



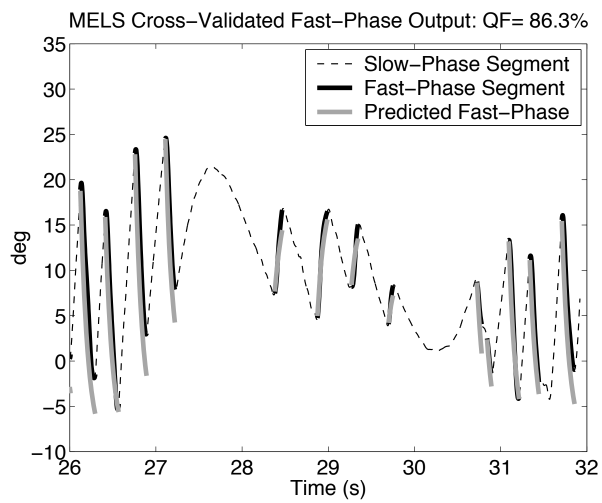
(a)



(b)

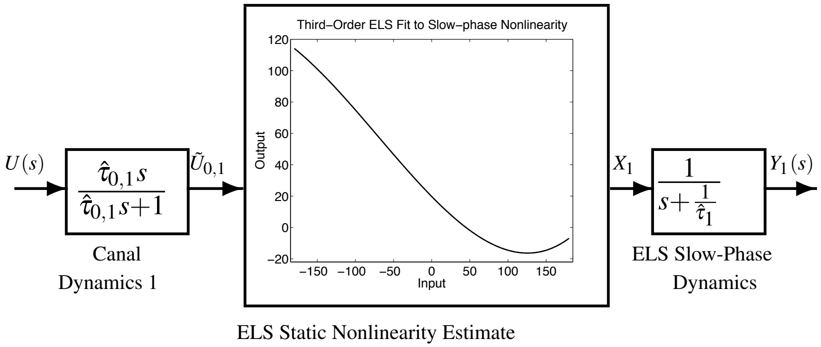


(c)

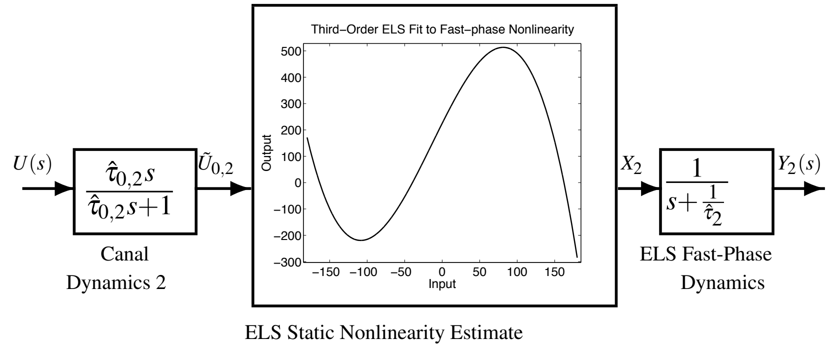


(d)

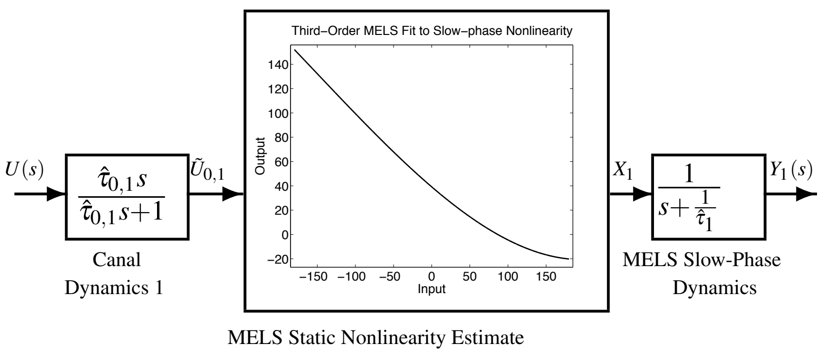
Estimated Vestibular & Switched System Parameters



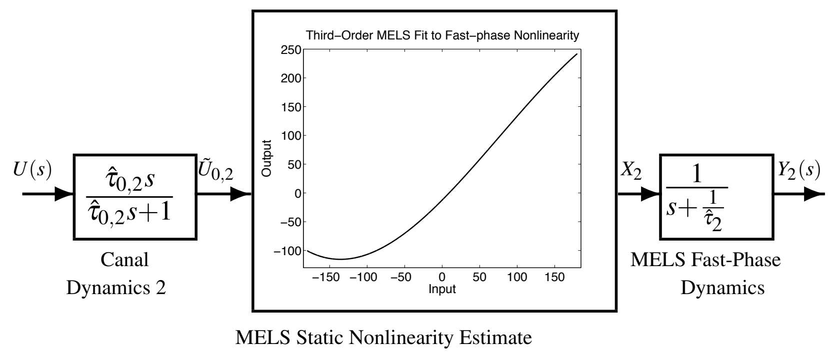
(a) $\hat{\tau}_{0,1} = 14.0$ s, $\hat{\tau}_1 = 0.849$ s



(b) $\hat{\tau}_{0,2} = 1.00$ s, $\hat{\tau}_2 = 0.0366$ s



(c) $\hat{\tau}_{0,1} = 10.0$ s, $\hat{\tau}_1 = 0.915$ s



(d) $\hat{\tau}_{0,2} = 2.00$ s, $\hat{\tau}_2 = 0.0831$ s

Conclusions

- MELS method provides accurate estimates of parameters since it takes advantage of an entire data record even though the individual segments are short

- Results may have a clinical significance in the analysis of ocular nystagmus of all types (pursuit, optokinetic, etc.)

- Technique here allows greater insight into the functionality of various ocular reflexes, by providing quantitative measures of both saccadic and slow ocular dynamics from a single experimental record

- MELS method may be useful to estimate the coefficients of complex Hammerstein structure switched systems in biology

A Suboptimal Bootstrap Method for Structure Detection of Non-linear Output-Error Models with Application to Human Ankle Dynamics

Objectives

- Develop a structure detection method to determine a parsimonious model description which best fits the observed output

- Gain insight into the underlying process describing ankle model

- Assess whether morphological ankle model is accurate

Two Fundamental Approaches to Structure Detection

- Exhaustive search
  - Every possible subset of the full model is considered
  - Requires a large number of computations and known to not converge to true underlying system
- Parameter variance
  - Parameter variance estimates computed from model residuals
  - Often inaccurate when number of candidate terms large

Bootstrap

- Numerical procedure for estimating parameter statistics
- Mild conditions on sample errors
  - Errors independent and identically distributed (i.i.d.)
  - Zero-mean

Hypothesis

The bootstrap might be useful for structure detection of over-parameterised non-linear models

Theoretical Analysis of Bootstrap

- Bickel and Freedman (1981) analysed linear regression model where N and p both large

- Showed for full p-dimensional distribution of least-squares estimates, bootstrap distribution will converge to true unknown distribution when

$$\gamma = \frac{p^2}{N} \to 0 \approx 0.1$$

- Initially, p cannot change; accuracy of bootstrap estimate determined by data length, N, available for estimation

Data Length

- Model order: $O = [4\ 4\ 4\ 2]$ has $p = 105$ candidate terms

- Data points: $N = \frac{105^2}{0.1} = 110,250$

- Desirable to reduce number of candidate terms

Noise Model

$$\begin{aligned}
y(n) &= \theta_1 y(n-4) + \theta_2 y^2(n-4) + \theta_3 u^2(n-4) y(n-4) + \theta_4 u(n-4) \\
z(n) &= y(n) + e(n) \\
z(n) &= \theta_1 [z(n-4) - e(n-4)] + \theta_2 [z(n-4) - e(n-4)]^2 \\
&+ \theta_3 u^2(n-4) [z(n-4) - e(n-4)] + \theta_4 u(n-4) + e(n)
\end{aligned}$$

- Simple output additive noise can result in complex noise model
- Number of noise terms often can be larger than process terms
- Useful to avoid estimating noise model whilst yielding a unbiased estimate
- Estimator?

Instrumental Variables

- Based on selecting instrument matrix $\mathbf{V}$ which satisfies conditions

$$\lim_{N\to\infty} \frac{1}{N}\mathbf{V}^T\boldsymbol{\Psi}_{zu} \;=\; \mathbf{R}; \quad \text{where } \mathbf{R} \text{ is nonsingular}$$
$$\lim_{N\to\infty} \frac{1}{N}\mathbf{V}^T(\mathbf{Z} - \boldsymbol{\Psi}_{zu}\boldsymbol{\theta}_0) \;=\; 0.$$

  - (i) the matrix product, $\mathbf{R} = \mathbf{V}^T\boldsymbol{\Psi}_{zu}$, has full rank, and
  - (ii) the errors have zero-mean and be uncorrelated with $\mathbf{V}$

- This ensures the estimate

$$\hat{\boldsymbol{\theta}}_{\boldsymbol{IV}} = (\mathbf{V}^T\boldsymbol{\Psi}_{zu})^{-1}\mathbf{V}^T\mathbf{Z}$$

is unbiased since the instrument matrix is not correlated with the errors

NARMAX with Linear Map

- IV gives unbiased estimates when noise represented as linear map

$$\begin{aligned} z(n) &= f^l[u(n), \cdots, u(n-n_u), z(n-1), \cdots, z(n-n_z)] \\ &+ \mathcal{L}[e(n-1), \cdots, e(n-n_e)] + e(n) \end{aligned}$$

- Assuming output additive noise

$$\begin{aligned} z(n) = f^l[u(n), \cdots, u(n-n_u)] \;+\; \mathcal{L}[z(n-1), \cdots, z(n-n_z), \\ e(n-1), \cdots, e(n-n_e)] \;+\; e(n) \end{aligned}$$

- Restricted class of NARMAX models
  - Blocked structured N-L models (static non-linearity followed by a causal, linear, time-invariant, dynamic system) such as Hammerstein models, bilinear models, etc.

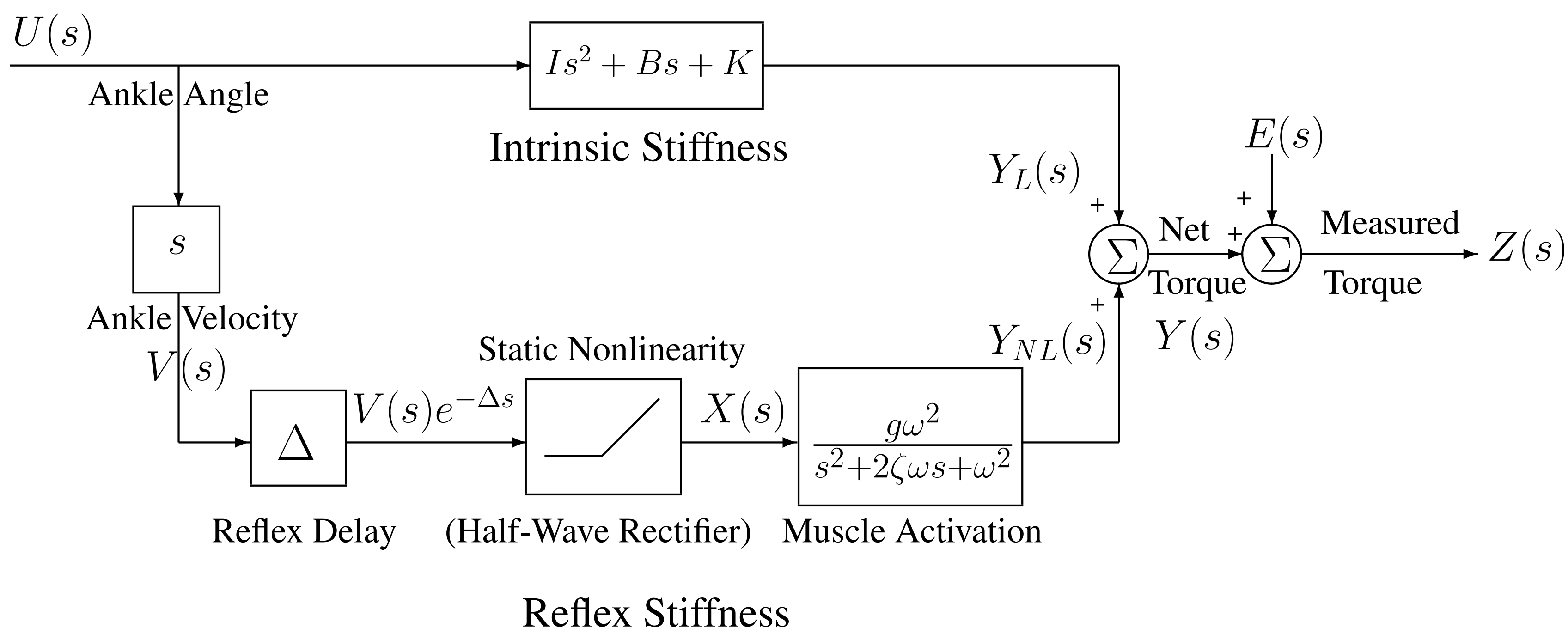
Model Order & Candidate Terms

- Redefine model order as $O = [n_u\, n_z\, l]$
- Maximum number of candidate terms:

$$\begin{aligned} p &= n_z + \sum_{k=1}^{l} p_k + 1 \\ p_k &= \frac{p_{k-1}(n_u + k)}{k}, \quad p_0 = 1 \end{aligned}$$

- Consider again earlier example: model of order: $O = [4\ 4\ 4\ 2] \Rightarrow p = 105$ candidate terms
- Order redefined as $O = [4\ 4\ 2] \Rightarrow p = 25$ terms need be considered; a significant reduction

Parallel Pathway Model of Ankle Dynamics Revisited



Overall Non-linear Model of Ankle Dynamics

$$
\begin{aligned}
z(n) &= b_0 + b_1 z(n-1) + b_2 z(n-2) + b_3 u(n) + b_4 u(n-1) + b_5 u(n-2) \\
&+ b_6 u(n-3) + b_7 u(n-4) + b_8 u(n-\tau) + b_9 u(n-\tau-1) \\
&+ b_{10} u(n-\tau-2) + b_{11} u(n-\tau-3) + b_{12} u^2(n-\tau) + b_{13} u^2(n-\tau-1) \\
&+ b_{14} u^2(n-\tau-2) + b_{15} u^2(n-\tau-3) + b_{16} u(n-\tau) u(n-\tau-1) \\
&+ b_{17} u(n-\tau-1) u(n-\tau-2) + b_{18} u(n-\tau-2) u(n-\tau-3) \\
&+ b_{19} e(n-1) + b_{20} e(n-2) + e(n)
\end{aligned}
$$

- Reflex delay 50-100 ms; corresponding to a discrete-time delay $\tau = \lceil \frac{\Delta}{T} \rceil$
  - Maximum model order $O = [8\ 2\ 2\ 2]$
- Gives full model description with 105 candidate terms
  - True system described by 21 parameters
- Computationally intractable combinatorial optimisation problem

Overall Non-linear Model of Ankle Dynamics

$$
\begin{aligned}
z(n) \;=\;& b_0 + b_1 z(n-1) + b_2 z(n-2) + b_3 u(n) + b_4 u(n-1) + b_5 u(n-2) \\
&+ b_6 u(n-3) + b_7 u(n-4) + b_8 u(n-\tau) + b_9 u(n-\tau-1) \\
&+ b_{10} u(n-\tau-2) + b_{11} u(n-\tau-3) + b_{12} u^2(n-\tau) + b_{13} u^2(n-\tau-1) \\
&+ b_{14} u^2(n-\tau-2) + b_{15} u^2(n-\tau-3) + b_{16} u(n-\tau) u(n-\tau-1) \\
&+ b_{17} u(n-\tau-1) u(n-\tau-2) + b_{18} u(n-\tau-2) u(n-\tau-3) \\
&\cancel{+ \; b_{19} e(n-1) + b_{20} e(n-2)} + e(n)
\end{aligned}
$$

- Reflex delay 50-100 ms; corresponding to a discrete-time delay $\tau = \left\lceil \frac{\Delta}{T} \right\rceil$
  - Maximum model order $O = [8\ 2\ 2\ 2]$
- Use *a priori* knowledge to eliminate unrealistic candidate terms, increasing computational efficiency
  - Eliminate all nonlinear and cross-terms associated with discrete-time delays between $\tau = [0\ 2]$
- Gives full model description with 33 candidate terms
  - True system described by 19 parameters
- Computationally *tractable* combinatorial optimisation problem

Assumptions

- System order assumed known
  - Small signal perturbations to estimate a LTI model order

- SOBSD algorithm limits possible class of models but it is reasonable to consider this limited set since the physical model suggests that the true system lies within this class

- Structure detection provide useful process insights that can be used in subsequent development or refinement of physical models

Experimental Data

- Data analysed two data sets from a subject with no history of neuromuscular disease

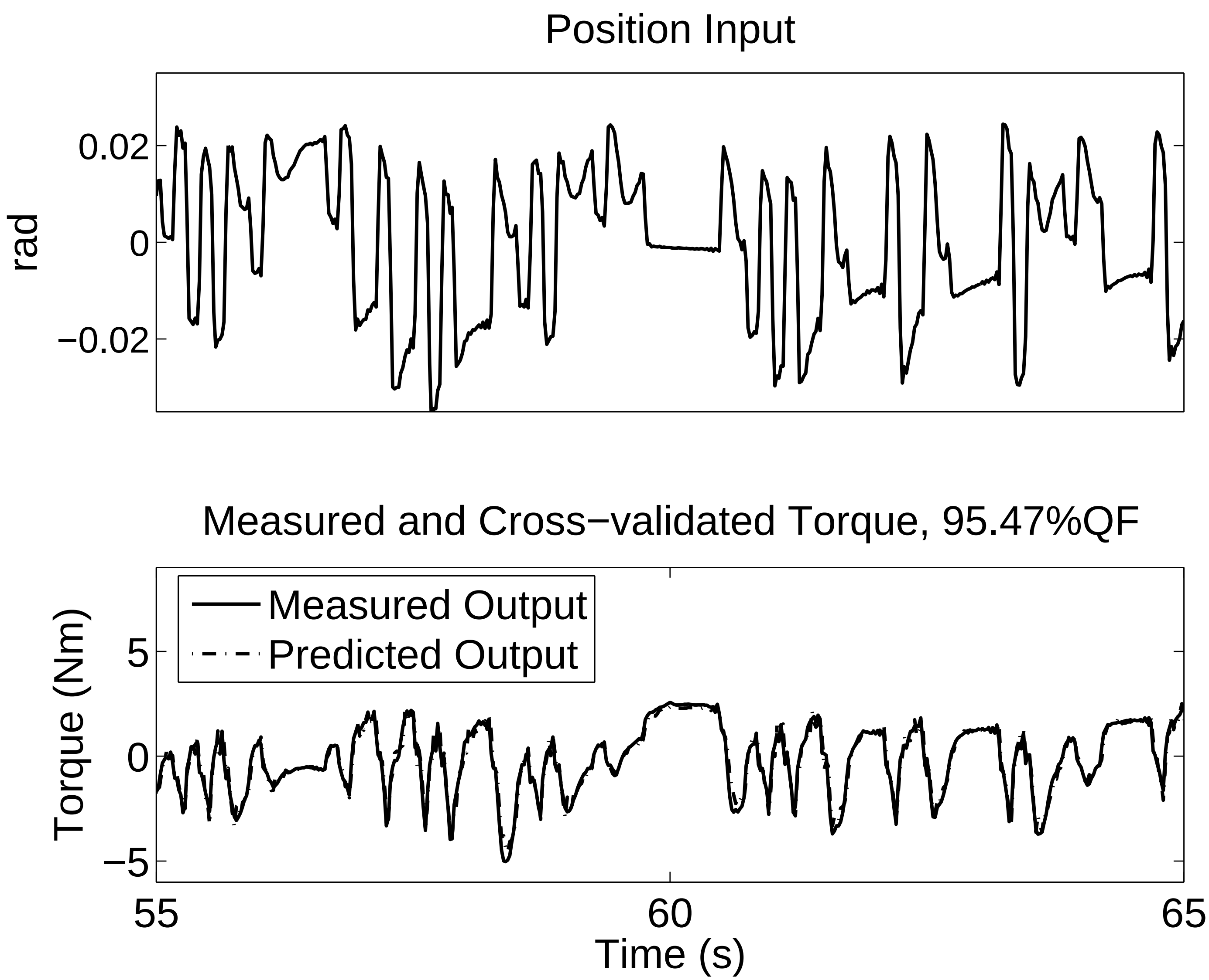
- PRBS input with 0.06 rad peak-to-peak amplitude and switching rate of 125 ms and 260 ms whilst subject maintained a mean contraction of -5 Nm

- Input-output recorded for 50 seconds

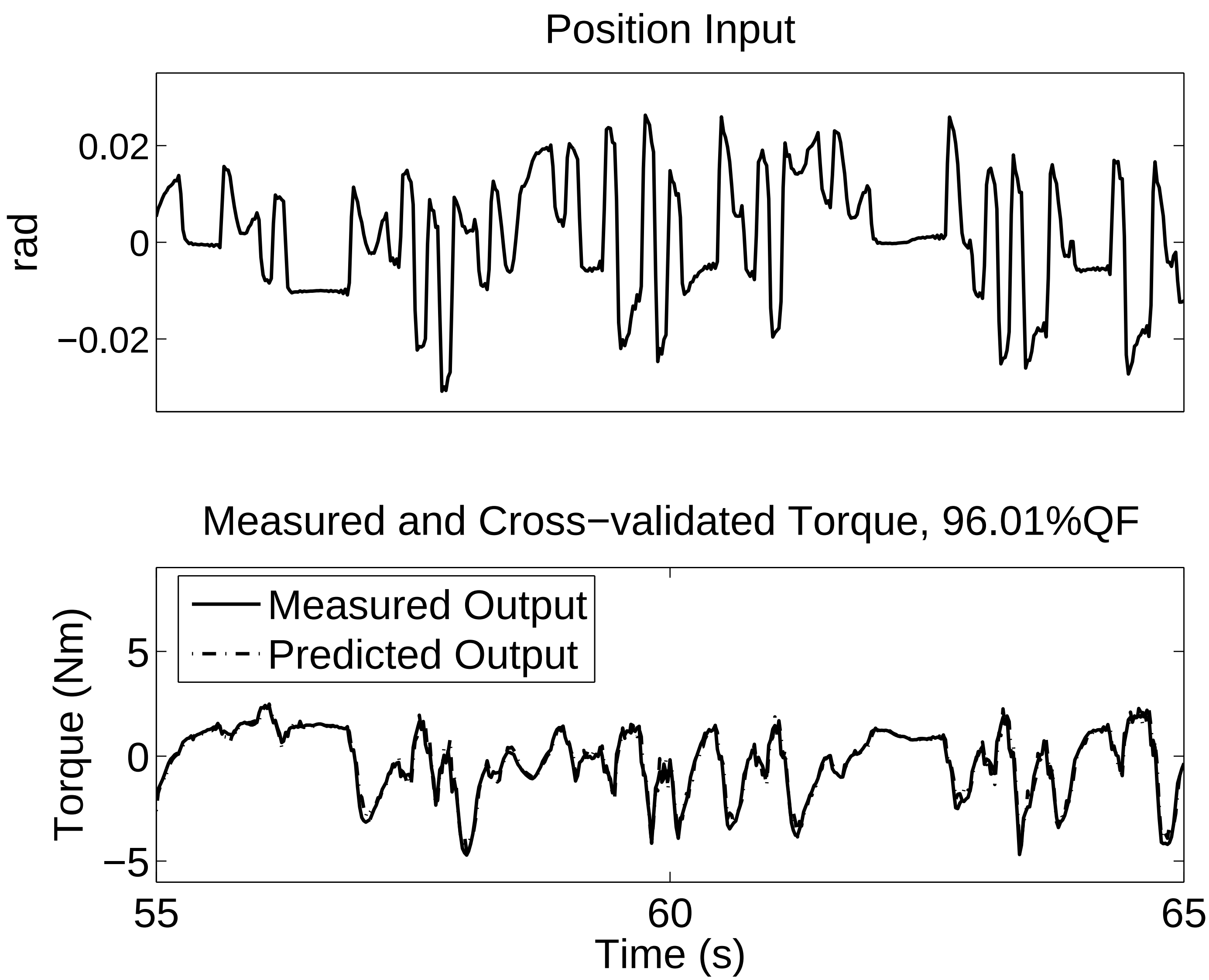
- Measured data sampled at 500 Hz, for estimation data decimated to 50 Hz

- Estimation data $N_e = 2,000$, validation data $N_v = 1,000$ and $B = 100$ bootstrap replications to estimate parameter statistics

Results of Identifying Experimental Human Ankle Data Set 1



Results of Identifying Experimental Human Ankle Data Set 2



Conclusions

- SOBSD algorithm provides a novel approach to the model selection problem without computing a noise model and, hence, reducing the dimensionality of this ill-posed combinatorial optimisation problem

- Reduction in dimensionality comes at cost of limited model structures that can be considered

- Study illustrates usefulness of structure detection as an approach to validate morphological models via analysis of input-output data

- Results show identified model structure matches the theoretically expected structure well

- Indicates morphological modeling studies may be accurate for this model describing ankle dynamics

Acknowledgments

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