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Bounded Linear Stability Analysis - A Time Delay Margin Estimation Approach for Adaptive Control

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This paper presents a method for estimating time delay margin for model-reference adaptive control of systems with almost linear structured uncertainty. The bounded linear stability analysis method seeks to represent the conventional model-reference adaptive law by a locally bounded linear approximation within a small time window using the comparison lemma. The locally bounded linear approximation of the combined adaptive system is cast in a form of an input-time-delay differential equation over a small time window. The time delay margin of this system represents a local stability measure and is computed analytically by a matrix measure method, which provides a simple analytical technique for estimating an upper bound of time delay margin. Based on simulation results for a scalar model-reference adaptive control system, both the bounded linear stability method and the matrix measure method are seen to provide a reasonably accurate and yet not too conservative time delay margin estimation.

I. Introduction

Adaptive control is a potentially promising technology that can improve performance and stability of a conventional fixed-gain controller. The ability to accommodate system uncertainties and to improve fault tolerance of a control system is a major selling point of adaptive control since traditional gain-scheduling or fixed-gain control methods are viewed as being less capable of handling off-nominal operating conditions. Nonetheless, these traditional control methods tend to be robust to disturbances and unmodeled dynamics when operated as intended.

In spite of the advances made in the field of adaptive control, there are several challenges related to the implementation of adaptive control technology in safety-critical systems. The absence of the verification and validation methods of adaptive control systems remain a major hurdle to the implementation of adaptive control in safety-critical systems.[1,2] This hurdle can be traced to the lack of performance and stability metrics for adaptive control which poses a major challenge that prevents adaptive control from being implemented in safety critical systems. The development of verifiable metrics for adaptive control will be important in order to mature adaptive control technology for use in operational safety-critical systems. Of these, stability metrics of adaptive control are an important consideration for assessing system robustness to unmodeled dynamics and exogenous disturbances. In one aspect of verification and validation, a control system is usually certified by demonstrating that it meets an acceptable set of requirements or specifications for stability margins, among other things. Herein lies a major challenge for verification and validation as there is no existing standard tool for stability margin analysis of nonlinear adaptive control. The lack of stability metrics for adaptive control is viewed as a technology barrier to developing certifiable adaptive control for safety-critical systems.[1,2]

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Classical LTI control systems are certified by demonstrating that they meet specifications for stability margins among other things. Typically, certification requirements for flight control systems, such as MIL-F-9490D, are often addressed in terms of phase and gain margins. These margins are used for LTI control laws to provide robustness or safety margins in a control system design as a safeguard against unmodeded effects and unstructured uncertainty. While the gain margin concept has been extended to adaptive control,[3] the phase margin concept does not easily lend itself to adaptive systems due to the inherent nonlinearity in adaptive control. Strictly speaking, phase margin for adaptive control in a global context as in the LTI framework is not possible. However, in a local context, it may be possible to consider phase margin in an approximate local sense, keeping in mind that even a standard gain-scheduling control system design may also contain nonlinear effects due to gain scheduling.

Phase and gain margins can be used as stability metrics in adaptive systems under some circumstances. One possible use would be when an adaptive control process is terminated by switching off the adaptation, essentially freezing the adaptive parameters, or when the adaptive signal converges to a steady state value. Some methods of approximate phase and gain margin analysis for adaptive control have been proposed that could be used without turning off the adaptation. One method for analyzing stability margins is based on a LMI approach by transforming the nonlinear adaptive control into a linear parameter varying form.[4] In another approach, it is proposed to define a LTI system that bounds the closed-loop adaptive system and then evaluate the phase and gain margins for the bounded LTI system in a local time window.[5,6] A potential benefit of this approach is that the adaptation can be "driven" or adjusted on-line to meet an approximate phase margin specification to improve the time delay margin of the closed-loop system.[7] Both approaches in[4] and[5,6] use similar system error dynamics.

Time delay margin has been viewed as a more readily accepted metric for relative stability of nonlinear control. While time delay margin is a suitable stability metric for adaptive control, a current challenge is that there is no well-established analytical tool for computing the time delay margin. Other methods for estimating the time delay margin have been proposed. One such method applies a Pade approximation to approximate a time-delay system.[8] The Pade approximation transforms the original time-delay system into a higher order system without the time delay that can be analyzed by the Lyapunov method to estimate the time delay margin. However, the Lyapunov method with the Pade approximation yields highly conservative estimates of time delay margin even for a simple scalar adaptive control system. The discrepancy between the time delay margin estimated by this method and the numerical evidence from simulations is at least three orders of magnitude.[8]

Despite the fact that new theoretical methods are being developed for computing time delay margin, they are still not ready to be used in a unified framework like the classical phase and gain margins. The most direct way to compute time delay margin is by simulations. The time delay margin is estimated by introducing a time delay at the input of an adaptive control system and then adjusting it until the closed-loop system is on the verge of instability. However, for adaptive control to be accepted in the future, simulation-based time delay margin computation is not considered to be sufficient as long as there is a lack of analytical tools for the same.

This paper presents a new method for estimating time delay margin for model-reference adaptive control of systems with almost linear structured uncertainty. Bounded linear stability analysis method has recently been introduced to represent the conventional model-reference adaptive law by a locally bounded linear approximation within a small time window using the comparison lemma. The locally bounded linear approximation of the combined adaptive system is cast in a form of an input-time-delay system over a small time window. The time delay margin of this system represents a local stability measure and is computed analytically by a matrix measure method, which provides a simple analytical technique for estimating an upper bound of time delay margin. Based on simulation results for a scalar model-reference adaptive control system, both the bounded linear stability method and the matrix measure method are seen to provide a reasonably accurate and yet not too conservative time delay margin estimation.

II. Introduction

III. Time Delay Margin for Linear Time Invariant Control

A. Time Delay Margin for a Simple Scalar System

Consider a scalar time-delay system

$$\dot{x}(t) = ax(t) + bu(t) \tag{1}$$

where $x(t) : [0,\infty) \to \mathbb{R}$, $u(t) : [0,\infty) \to \mathbb{R}$, and $b > 0$.

The system has a feedback control

$$u(t) = -kx(t) \tag{2}$$

with a closed-loop pole $s = a - bk < 0$.

The closed-loop system is designed to be robust to a time delay at the input such that

$$\dot{x}(t) = ax(t) + bu(t - t_d) \tag{3}$$

where t_d is defined as a time delay margin for the original system in Eq. (1).

The Laplace transform of the closed-loop time-delay system is expressed as

$$\left(s - a + bke^{-t_d s}\right) x(s) = x(0) \tag{4}$$

To calculate the time delay margin t_d, a number of approximate methods can be used. Consider the following

1. Taylor's Series Approximation:

   The term $e^{-t_d s}$ can be expanded using the Taylor's series as

   $$e^{-t_d s} = 1 - t_d s + \frac{1}{2} t_d^2 s^2 - \ldots \tag{5}$$

   Expressing in time domain, one gets

   $$u(t - t_d) = u(t) - t_d \dot{u}(t) + \frac{1}{2} t_d^2 \ddot{u}(t) - \ldots \tag{6}$$

   The first-order Taylor's series approximation in effect is a finite-difference approximation of a time derivative since

   $$\dot{u}(t) \approx \frac{u(t) - u(t - t_d)}{t_d} \tag{7}$$

   Then, the time-delay system can be approximated as

   $$\dot{x}(t) = ax(t) + bu(t) - bt_d \dot{u}(t) + \frac{bt_d^2}{2} \ddot{u}(t) - \ldots \tag{8}$$

   So, the effect of time delay shows up as time derivatives of the controller. This linear equivalent system is now conditionally stable. To see this, consider only the first-order approximation of the closed-loop time-delay system as

   $$\dot{x}(t) = (a - bk)x(t) + bkt_d \dot{x}(t) \tag{9}$$

   which can also be written as

   $$\dot{x}(t) = \frac{a - bk}{1 - bkt_d} x(t) \tag{10}$$

   One can see that the system can be not guaranteed to be stable even if $a - bk < 0$ since there is an additional requirement

   $$1 - bkt_d > 0 \tag{11}$$

   that must be fulfilled.

   Thus the time delay margin estimate of the system is given by

   $$t_d^* = \frac{1}{bk} \tag{12}$$

   This indicates that the time delay must be kept low if the feedback gain is large for the closed-loop system to be stable. The result is independent of the parameter a. The Taylor's series approximation of the time delay term does not yield a proper transfer function since the number of zeros is greater than the number of poles.

2. Pade Approximation:

   Pade approximation is frequently used to approximate time delay effects by a rational polynomial function of the form

   $$P(s) = \frac{Q(s)}{R(s)} \tag{13}$$

where the degree of the polynomial $Q(s)$ is less than or equal to that of $R(s)$ for a proper transfer function representation of the time delay.

Consider the following first-order Pade approximation

$$e^{-t_d s} = \frac{2 - t_d s}{2 + t_d s} \tag{14}$$

The system characteristic equation is

$$t_d s^2 + (2 - a t_d - bk t_d) s + 2bk - 2a = 0 \tag{15}$$

which results in a time delay margin estimate of

$$t_d^* = \frac{2}{a + bk} \tag{16}$$

The result now is dependent on all system parameters. The accuracy of the estimation increases with increasing the order of the Pade approximation.

3. Lyapunov-Krasovskii Method:

Stability of time-delay systems can be analyzed using the Lyapunov-Krasovskii method.[9,10] Consider the following Lyapunov-Krasovskii functional

$$V(x(t)) = x^2(t) + \frac{1}{t_d}\int_{t-t_d}^{t} x^2(\tau)\, d\tau > 0 \tag{17}$$

The time derivative of $V(x(t))$ along the solution trajectory is evaluated as

$$\dot{V}(x(t)) = 2ax^2(t) - 2bkx(t)x(t-t_d) + \frac{1}{t_d}x^2(t) - \frac{1}{t_d}x^2(t-t_d) \tag{18}$$

By completing the squares, one obtains

$$\dot{V}(x(t)) = \left(2a + bk + \frac{1}{t_d}\right)x^2(t) + \left(bk - \frac{1}{t_d}\right)x^2(t-t_d) - bk\left[x(t) + x(t-t_d)\right]^2 \tag{19}$$

Since $bk > 0$, the time-delay system is uniformly stable if the following inequalities are satisfied

$$2a + bk + \frac{1}{t_d} < 0 \tag{20}$$

$$bk - \frac{1}{t_d} < 0 \tag{21}$$

The solution of the inequalities is feasible if $a < 0$ and $bk < -a$. This yields

$$-\frac{1}{2a + bk} < t_d < \frac{1}{bk} \tag{22}$$

The result of the time delay margin estimate based on the Lyapunov-Krasovskii method is generally non-unique and is dependent upon the selection of the Lyapunov-Krasovskii functional. For example, suppose the following Lyapunov-Krasovskii functional is selected

$$V(x(t)) = px^2(t) + \frac{1}{t_d}\int_{t-t_d}^{t} px^2(\tau)\, d\tau > 0 \tag{23}$$

where $p > 0$, then

$$\dot{V}(x(t)) = 2pax^2(t) - 2pbkx^2(t) + 2pbkx^2(t) - 2pbkx(t)x(t-t_d) + \frac{p}{t_d}x^2(t) - \frac{p}{t_d}x^2(t-t_d) \tag{24}$$

which becomes

$$\dot{V}(x(t)) = \left(2pa + p^2b^2k^2 + \frac{p}{t_d}\right)x^2(t) + \left(1 - \frac{p}{t_d}\right)x^2(t-t_d) - [pbkx(t) + x(t-t_d)]^2 \tag{25}$$

The solution of the inequalities

$$2pa + p^2b^2k^2 + \frac{p}{t_d} < 0 \tag{26}$$

$$1 - \frac{p}{t_d} < 0 \tag{27}$$

yields

$$t_d < p < \frac{-a}{b^2k^2}\left(1 + \sqrt{1 - \frac{b^2k^2}{a^2}}\right) \tag{28}$$

for $a < 0$ and $bk < -a$.

4. Lyapunov-Razumikhin Method:

The Lyapunov-Razumikhin method can be considered as a subset of the more general Lyapunov-Krasovskii functional approach.[10] However, a nice aspect of the approach is that it utilizes functions as opposed to functionals as the main ingredient. The Lyapunov-Razumikhin theorem states that the system is asymptotically stable if there exists $\eta > 1$ and $P = P^\top > 0$ such that

$$\dot{V}(x(t)) \leq -\varepsilon \|x(t)\|^2 \tag{29}$$

where $\varepsilon > 0$, whenever

$$V(x(t+\theta)) < \eta V(x(t)) \tag{30}$$

for all $\theta \in [-t_d, 0)$.

Consider the following Lyapunov candidate function

$$V(x(t)) = x^2(t) \tag{31}$$

Differentiating $V(x(t))$ along the solution trajectory of $x(t)$ yields

$$\dot{V}(x(t)) = 2ax^2(t) - 2bkx(t)x(t-t_d) \tag{32}$$

Recall from fundamental theorem of calculus that

$$x(t-t_d) = x(t) - \int_{t-t_d}^{t} \dot{x}(\tau)\, d\tau \tag{33}$$

Then

$$\begin{aligned}
\dot{V}(t) &= 2ax^2(t) - 2bkx^2(t) - 2bkx(t)\int_{t-t_d}^{t} [ax(\tau) - bkx(\tau - t_d)]\, d\tau \\
&\leq 2(a-bk)x^2(t) + 2bk|x(t)|\left|\int_{t-t_d}^{t} [ax(\tau) - bkx(\tau - t_d)]\, d\tau\right| \\
&\leq 2(a-bk)x^2(t) + 2|a|bk|x(t)|\int_{t-t_d}^{t} |x(\tau)|\, d\tau + 2b^2k^2|x(t)|\int_{t-t_d}^{t} |x(\tau - t_d)|\, d\tau
\end{aligned} \tag{34}$$

Since $t - t_d \leq \tau \leq t$, then the Lyapunov-Razumikhin theorem gives

$$|x(\tau - t_d)| \leq |x(\tau)| \leq |x(t)| \tag{35}$$

Thus

$$\begin{aligned}
\dot{V}(x(t)) &\leq 2(a-bk)x^2(t) + 2|a|bk|x(t)|\int_{t-t_d}^{t} |x(t)|\, d\tau + 2b^2k^2|x(t)|\int_{t-t_d}^{t} |x(t)|\, d\tau \\
&= 2(a-bk)x^2(t) + 2t_d|a|bkx^2(t) + 2t_db^2k^2x^2(t) \\
&= 2\left[a - bk + t_d\left(|a|bk + b^2k^2\right)\right]x^2(t)
\end{aligned} \tag{36}$$

So the time-delay system is asymptotically stable if

$$a - bk + t_d\left(|a|\,bk + b^2k^2\right) < 0 \tag{37}$$

Hence, the time delay margin can be found as

$$t_d < \frac{bk - a}{bk\,(bk + |a|)} \tag{38}$$

■

It is noted that the time delay margin of the time-delay system of Eq. (3) can actually be found exactly by computing the system pole $s = \sigma + j\omega$ as follows:

$$\sigma + j\omega - a + bke^{-t_d\sigma}e^{-j\omega t_d} = 0 \tag{39}$$

The system is neutrally stable for $\sigma = 0$ so that the following equations result

$$-a + bk\cos\omega t_d^* = 0 \tag{40}$$

$$\omega - bk\sin\omega t_d^* = 0 \tag{41}$$

The solutions of these equations yield the $j\omega$-axis cross-over frequency and the time delay margin as

$$\omega = \sqrt{b^2k^2 - a^2} \tag{42}$$

$$t_d^* = \frac{1}{\sqrt{b^2k^2 - a^2}}\cos^{-1}\frac{a}{bk} \tag{43}$$

The solution actually tends to the Taylor's series approximation for $bk \gg a$. Also, there exists a relationship between a and bk such that the system is stable, independent of time delay. This occurs when $a < 0$ and $bk < -a$.

Example: Given $a = 1$ and $bk = 2$, the Taylor's series approximation yields $t_d^* = 0.5$ sec, whereas the Pade approximation yields $t_d^* = 0.667$ sec. The exact value is $t_d^* = 0.604$ sec. Thus, the Pade approximation gives a better estimate than the Taylor's series approximation, but also over-estimates the time delay margin. The Lyapunov-Krasovskii method has no solution since $a > 0$. The Lyapunov-Razumikhin method gives $t_d^* = 0.167$ sec.

Given $a = -1$ and $bk = 2$, the exact value is now $t_d^* = 1.209$ sec. The Pade approximation over-estimates the time delay margin with $t_d^* = 2$ sec. The Taylor's series approximation yields the same estimate of $t_d^* = 0.5$ sec which is independent of a. The Lyapunov-Krasovskii method also provides no solution since $bk > -a$. The Lyapunov-Razumikhin method gives $t_d^* = 0.5$ sec, which is the same as the Taylor's series approximation. In fact, for $a < 0$, both the Taylor's series approximation and the Lyapunov-Razumikhin method produce the same result.

Given $a = -1$ and $bk = \frac{1}{2}$, the system is stable, independent of time delay. The Taylor's series approximation yields $t_d^* = 2$ sec and the Pade approximation yields $t_d^* = -4$ sec. The time delay margin corresponding to the Lyapunov-Krasovskii functional (17) is $t_d = 2$ sec and that for the Lyapunov-Krasovskii functional (23) is $t_d = 7.464$ sec. The Lyapunov-Razumikhin method gives $t_d^* = 2$ sec, which again is the same as the Taylor's series approximation.

In these examples, both the Lyapunov-Krasovskii and Lyapunov-Razumikhin methods are quite conservative even for a simple linear time invariant scalar system. Relaxation of the conservatism in the Lyapunov-Karsovskii and Lyapunov-Razumikhin is possible and usually requires parameter and functional optimization.

B. Time-Delay Margin by Matrix Measure Method

For a vector time-delay system

$$\dot{x}(t) = Ax(t) - BKx(t - t_d) \tag{44}$$

where $x(t) : [0,\infty) \to \mathbb{R}^n$ and $\lambda(A - BK) \in \mathbb{C}^-$, i.e., $A - BK$ is Hurwitz, the time delay margin can be found from the following characteristic equation

$$\det\left(j\omega I - A + BKe^{-j\omega t_d}\right) = 0 \tag{45}$$

The bounds on ω and t_d can be estimated by a matrix measure method.[9] Defining μ as an eigenvalue of a symmetric part of a complex matrix such that

$$\mu_i(C) = \lambda_i\left(\frac{C + C^*}{2}\right) \tag{46}$$

where C $\in \mathbb{C}$ is a complex matrix and C^* is its complex conjugate transpose, then μ has the following properties

$$\mu_i(C) \in \mathbb{R} \tag{47}$$

$$\overline{\mu}(C) = \max_{1\le i\le n} \lambda_i\left(\frac{C+C^*}{2}\right) = \lim_{\varepsilon\to 0}\frac{\|I+\varepsilon C\|-1}{\varepsilon} \tag{48}$$

$$\underline{\mu}(C) = \min_{1\le i\le n} \lambda_i\left(\frac{C+C^*}{2}\right) = \lim_{\varepsilon\to 0}\frac{1-\|I-\varepsilon C\|}{\varepsilon} \tag{49}$$

$$\overline{\mu}(C) = -\underline{\mu}(-C) \tag{50}$$

$$\overline{\mu}(jC) = -\underline{\mu}(jC) \tag{51}$$

$$\underline{\mu}(C) \le \mathrm{Re}\lambda_i(C) \le \overline{\mu}(C) \tag{52}$$

$$\mathrm{Im}\lambda(C) \le \overline{\mu}(-jC) \tag{53}$$

$$\overline{\mu}(C) \le \|C\| \tag{54}$$

$$\overline{\mu}(C+D) \le \overline{\mu}(C) + \overline{\mu}(D) \tag{55}$$

$$\underline{\mu}(C+D) \ge \underline{\mu}(C) + \underline{\mu}(D) \tag{56}$$

$$\overline{\mu}(C-D) \ge \overline{\mu}(C) - \overline{\mu}(D) \tag{57}$$

$$\underline{\mu}(C-D) \le \underline{\mu}(C) - \underline{\mu}(D) \tag{58}$$

The matrix measure μ affords a simple way to estimate the bounds on the system poles for a MIMO system.

Lemma 1: The time-delay system (44) is asymptotically stable if the following inequalities hold

$$t_d < \frac{1}{\omega}\cos^{-1}\frac{\overline{\mu}(A)+\overline{\mu}(jBK)}{\|BK\|} \tag{59}$$

$$\omega < \overline{\mu}(-jA) + \|BK\| \tag{60}$$

where $\|.\| = \|.\|_2$ is the $\mathscr{L}_2$-norm.

Proof: The real parts of the system poles are bounded from above by

$$\sigma = \mathrm{Re}\lambda_i\left(A - BKe^{-j\omega t_d}\right) \le \overline{\mu}(A) + \overline{\mu}\left(-BKe^{-j\omega t_d}\right) \le \overline{\mu}(A) + \overline{\mu}(-BK)\left|\cos\omega t_d\right| + \overline{\mu}(jBK)\left|\sin\omega t_d\right| \tag{61}$$

Let $0 \le \omega t_d \le \frac{\pi}{2}$, then the time-delay system is stable if $\sigma < 0$ which implies

$$\begin{aligned}\overline{\mu}(A) < -\overline{\mu}(-BK)\cos\omega t_d - \overline{\mu}(jBK)\sin\omega t_d = \underline{\mu}(BK)\cos\omega t_d - \overline{\mu}(jBK)\sin\omega t_d \\ < \overline{\mu}(BK)\cos\omega t_d - \overline{\mu}(jBK)\sin\omega t_d\end{aligned} \tag{62}$$

Upon some algebra, this can be expressed as

$$\left[\overline{\mu}^2(BK) + \overline{\mu}^2(jBK)\right]\cos^2\omega t_d - 2\overline{\mu}(A)\overline{\mu}(BK)\cos\omega t_d + \overline{\mu}^2(A) - \overline{\mu}^2(jBK) > 0 \tag{63}$$

The solution yields a bound on time delay margin t_d as

$$t_d < \frac{1}{\omega}\cos^{-1}\frac{\overline{\mu}(A)\overline{\mu}(BK) + \overline{\mu}(jBK)\sqrt{\overline{\mu}^2(BK)+\overline{\mu}^2(jBK)-\overline{\mu}^2(A)}}{\overline{\mu}^2(BK)+\overline{\mu}^2(jBK)} \tag{64}$$

But

$$\overline{\mu}^2(BK) \le \overline{\mu}^2(BK) + \overline{\mu}^2(jBK) \le \|BK\|^2 \tag{65}$$

So

$$t_d < \frac{1}{\omega}\cos^{-1}\frac{\overline{\mu}(A)\|BK\| + \overline{\mu}(jBK)\sqrt{\|BK\|^2 - \overline{\mu}^2(A)}}{\|BK\|^2} < \frac{1}{\omega}\cos^{-1}\frac{\overline{\mu}(A)+\overline{\mu}(jBK)}{\|BK\|} \tag{66}$$

The imaginary parts of the system poles are bounded from above by

$$\omega = \mathrm{Im}\lambda_i\left(-jA + jBKe^{-j\omega t_d}\right) \le \overline{\mu}(-jA) + \overline{\mu}\left(jBKe^{-j\omega t_d}\right)$$
$$\le \overline{\mu}(-jA) + \overline{\mu}(jBK)\left|\cos\omega t_d\right| + \overline{\mu}(BK)\left|\sin\omega t_d\right| \quad (67)$$

which can be expressed as

$$\omega \le \overline{\mu}(-jA) + \sqrt{\overline{\mu}^2(BK) + \overline{\mu}^2(jBK)} < \overline{\mu}(-jA) + \|BK\| \quad (68)$$

where for added conservatism the less than or equal sign is replaced by the less than sign.

Corollary: The time-delay system (44) is asymptotically stable independent of time delay if

$$\overline{\mu}(A) < \|BK\| < -\underline{\mu}(A) \quad (69)$$

Proof: The time-delay system is stable, independent of time delay, if

$$\underline{\mu}(A) + \underline{\mu}(-BK)\cos\omega t_d + \underline{\mu}(jBK)\sin\omega t_d < \underline{\mu}(A) - \overline{\mu}(BK)\cos\omega t_d + \underline{\mu}(jBK)\sin\omega t_d$$
$$< \underline{\mu}(A) + \sqrt{\overline{\mu}^2(BK) + \underline{\mu}^2(jBK)} = \underline{\mu}(A) + \sqrt{\overline{\mu}^2(BK) + \overline{\mu}^2(jBK)} < \underline{\mu}(A) + \|BK\| < 0 \quad (70)$$

This implies

$$\underline{\mu}(A) < -\|BK\| \quad (71)$$

Note that this condition is in addition to the requirement that $A - BK$ is Hurwitz, which can easily be shown by a similar argument that

$$\overline{\mu}(A) < \|BK\| \quad (72)$$

Therefore

$$\overline{\mu}(A) < \|BK\| < -\underline{\mu}(A) \quad (73)$$

■

Example: Given

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix},\ BK = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$$

The bounds on ω and t_d^* are computed as follows:

$$\overline{\mu}(A) = 1$$

$$\overline{\mu}(jBK) = 0$$

$$\|BK\| = 2$$

$$\omega < \overline{\mu}(-jA) + \|BK\| = 3 \text{ rad/sec}$$

$$t_d < \frac{1}{\omega}\cos^{-1}\frac{\overline{\mu}(A)}{\|BK\|} = \frac{\pi}{9} = 0.349 \text{ sec}$$

The exact results can be determined from

$$\begin{vmatrix} j\omega & -1 \\ 1 & j\omega - 1 + 2\left(\cos\omega t_d^* - j\sin\omega t_d^*\right) \end{vmatrix} = -\omega^2 + 2\omega\sin\omega t_d^* + 1 - j\omega\left(1 - 2\cos\omega t_d^*\right) = 0$$

$$\omega = \frac{\sqrt{3}+\sqrt{7}}{2} = 2.189 \text{ rad/sec}$$

$$t_d^* = \frac{2\pi}{3\left(\sqrt{3}+\sqrt{7}\right)} = 0.478 \text{ sec}$$

Thus, the time delay margin estimated by the matrix measure method is reasonably conservative but not overly conservative that renders it impractical for design and analysis purposes.

Example: Given

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{3}{2} & -\frac{3}{2} \end{bmatrix}, \; BK = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$$

This time-delay system is stable, independent of time delay, since

$$\overline{\mu}(A) = 0.041$$

$$\underline{\mu}(A) = -1.541$$

$$\|BK\| = 1.414$$

$$\overline{\mu}(A) < \|BK\| < -\underline{\mu}(A)$$

C. Time-Delay Margin by Lyapunov-Karsovskii Method

Stability of time-delay differential equations based on the Lyapunov-Karsovskii method have been studied exhaustively by many authors.[9,10] As shown above, different Lyapunov-Karsovskii functionals lead to different results. Invariably, the negative-definiteness of the time derivative of a Lyapunov-Karsovskii functional results in a linear matrix inequality that can be solved for a time delay margin. While this study does not focus on the Lyapunov-Karsovskii method, it is instructive to illustrate this technique for estimating time delay margin.

For the same time-delay system (44) with an assumption $\lambda(A) \in \mathbb{C}^-$, then consider the following Lyapunov-Karsovskii functional

$$V(t) = x^\top(t) P x(t) + \frac{1}{t_d} \int_{t-t_d}^{t} x^\top(\tau) P x(\tau) \, d\tau > 0 \tag{74}$$

where $P = P^\top > 0$.

Evaluating $\dot{V}(t)$ yields

$$\begin{aligned} \dot{V}(t) &= \dot{x}^\top(t) P x(t) + x^\top(t) P \dot{x}(t) + \frac{1}{t_d} x^\top(t) P x(t) - \frac{1}{t_d} x^\top(t-t_d) P x(t-t_d) \\ &= x^\top(t) A^\top P x(t) - x^\top(t-t_d) K^\top B^\top P x(t) + x^\top(t) P A x(t) - x^\top(t) P B K x(t-t_d) \\ &\qquad + \frac{1}{t_d} x^\top(t) P x(t) - \frac{1}{t_d} x^\top(t-t_d) P x(t-t_d) \end{aligned} \tag{75}$$

For stability, $\dot{V}(t) < 0$ is satisfied by the following LMI

$$\begin{bmatrix} A^\top P + PA + \frac{1}{t_d} P & -PBK \\ -K^\top B^\top P & -\frac{1}{t_d} P \end{bmatrix} < 0 \tag{76}$$

IV. Time Delay Margin for Adaptive Control

A. Model Reference Adaptive Control

Given a nonlinear plant

$$\dot{x}(t) = Ax(t) + B[u(t) + f(x(t)) + \Delta(x(t), u(t), z(t), t)] \tag{77}$$

$$\dot{z}(t) = g(z(t), x(t), u(t), t) \tag{78}$$

where $x(t) : [0,\infty) \to \mathbb{R}^n$ is a measurable or observable state vector, $z(t) : [0,\infty) \to \mathbb{R}^q$ is an unobservable state vector, $u(t) : [0,\infty) \to \mathbb{R}^p$ is a control vector, $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times p}$ are known such that the pair (A,B) is controllable, $f(x(t)) : \mathbb{R}^n \to \mathbb{R}^p$ is a matched structured uncertainty, $\Delta(x(t), u(t), z(t), t)$ is a matched unstructured uncertainty or unmodeled dynamics, and $g(z(t), x(t), u(t), t) : [0,\infty) \times \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^q$ represents the unmodeled dynamics.

The unstructured uncertainty $\Delta(x(t), u(t), z(t), t)$ can be due to numerous unmodeled effects and are generally cannot be captured in the control model due to modeling difficulty. In flight vehicles, these effects can represent various complex modes of interactions including aeroservoelastic modes, pilot interactions, nonlinear unsteady aerodynamics near stall and post-stall, atmospheric disturbances such as sharp-edged wind gusts and wake vortices; just to name a few. The tendency for the unstructured uncertainty $\Delta(x(t), u(t), z(t), t)$ is to destabilize a control system since it cannot be accounted for in a control design. In a typical control design framework, stability margins are built into a

control system to accommodate for the unstructured uncertainty $\Delta(x(t),u(t),z(t),t)$ while the structured uncertainty $f(x(t))$ is kept as small as possible by increasing the accuracy of the control model. The certification for stability of a control system is based on meeting well-accepted specifications for stability margins, such as MIL-F-9040D standards used for certifying flight control systems.

The matched structured uncertainty $f(x)$ has a form of

$$f(x) = \Theta^{*\top}\Phi(x) \tag{79}$$

where $\Theta^* \in \mathbb{R}^{m\times p}$ is an unknown constant weight matrix that represents a parametric uncertainty, and $\Phi(x) : \mathbb{R}^n \to \mathbb{R}^m$ is a vector of known functions

The objective is to design a controller that enables the plant to follow a reference model

$$\dot{x}_m(t) = A_m x_m(t) + B_m r(t) \tag{80}$$

where $A_m \in \mathbb{R}^{n\times n}$ is Hurwitz and known, $B_m \in \mathbb{R}^{n\times p}$ is also known, and $r(t) : [0,\infty) \to \mathbb{R}^p \in \mathscr{L}_\infty$ is a command vector with $\dot{r} \in \mathscr{L}_\infty$.

Defining the tracking error as $e(t) = x_m(t) - x(t)$, then the controller $u(t)$ is specified by

$$u(t) = K_x x(t) + K_r r(t) - u_{ad}(x(t)) \tag{81}$$

where $K_x \in \mathbb{R}^{p\times n}$ and $K_r \in \mathbb{R}^{p\times p}$ are known nominal gain matrices, and $u_{ad}(x(t)) :\to \mathbb{R}^p$ is a direct adaptive signal.

Then, the tracking error equation becomes

$$\dot{e}(t) = \dot{x}_m(t) - \dot{x}(t) = A_m e(t) + (A_m - A - BK_x)x(t) + (B_m - BK_r)r(t) + B\left[u_{ad}(x(t)) - \Theta^{*\top}\Phi(x(t))\right] \tag{82}$$

We choose the gain matrices K_x and K_r to satisfy the model matching conditions $A + BK_x = A_m$ and $BK_r = B_m$. The adaptive signal u_{ad} is an estimator of the parametric uncertainty in the plant such that

$$u_{ad}(x) = \Theta^\top \Phi(x) \tag{83}$$

where $\Theta \in \mathbb{R}^{m\times p}$ is an estimate of the parametric uncertainty Θ^*.

Let $\tilde{\Theta} = \Theta - \Theta^*$ be an estimation error of the parametric uncertainty. Then the tracking error equation can be expressed as

$$\dot{e}(t) = A_m e(t) + B\tilde{\Theta}^\top(t)\Phi(x(t)) \tag{84}$$

The system can be designed to follow the reference model with a direct model reference adaptive control update law

$$\dot{\Theta}(t) = -\Gamma\Phi(x(t))e^\top(t)PB \tag{85}$$

where $P = P^\top > 0$ solves the Lyapunov equation

$$PA_m + A_m^\top P = -Q \tag{86}$$

where $Q = Q^\top > 0$.

If $\Delta(x(t),u(t),z(t),t) = 0$, then the adaptive law (85) can be shown to be stable and results in $e(t) \to 0$ as $t \to \infty$. As the adaptive gain Γ increases, the tracking error further decreases. The upper limit of the adaptive gain is set by the sampling frequency of the discrete-time implementation of the adaptive law. Concomitant with the increase in the adaptive law is an increase in the high frequency content in the adaptive signal. The estimate $\Theta(t)$ is essentially a nonlinear integral gain since $\Theta(t)$ can be expressed as

$$\Theta(t) = -\int_0^t \Gamma\Phi(x(t))e^\top(t)PBd\tau \tag{87}$$

For a LTI system, as an integral gain increases, the closed-loop poles move away from the real axis along the imaginary axis. Hence, the frequency of the closed-loop system increases. The phase margin for an LTI system generally decreases with increasing the integral gain. Even though $\Theta(t)$ is a nonlinear integral gain, the behavior is similar to that of an LTI system.

In real systems, because the unstructured uncertainty $\Delta(x,u,z,t)$ is not always zero, the adaptive law (85) is not robust and cannot guarantee boundedness of tracking error. Many robust modification schemes can be incorporated into the adaptive law (85) to improve its robustness to unmodeled dynamics such as σ-modification[11]

$$\dot{\Theta}(t) = -\Gamma\left[\Phi(x(t))e^{\top}(t)PB + \sigma\Theta(t)\right] \tag{88}$$

e-modification[12]

$$\dot{\Theta}(t) = -\Gamma\left[\Phi(x(t))e^{\top}(t)PB + \mu\left\|e^{\top}(t)PB\right\|\Theta(t)\right] \tag{89}$$

or optimal control modification which has recently been introduced[13]

$$\dot{\Theta}(t) = -\Gamma\left[\Phi(x(t))e^{\top}(t)PB - \nu\Phi(x(t))\Phi^{\top}(x(t))\Theta(t)B^{\top}PA_m^{-1}B\right] \tag{90}$$

where $\sigma,\mu,\nu > 0$ are tuning parameters in the modification schemes.

All these modifications invariably introduce new parameters to adjust the adaptive law, for example, σ is such a parameter in the above adaptive law. The modification parameters effectively add a damping mechanism to the adaptive law to ensure that the adaptive signal is bounded. However, in general there exists a trade-off between performance and robustness. In the adaptive law above, increasing the adaptive gain Γ results in a better tracking performance but poorer robustness, while increasing the parameter σ improves robustness but results in a poorer tracking ability. Thus, a current challenge in adaptive control design is to be able to select appropriately the tuning parameters that can achieve stability and performance specifications. Currently, there is no well-accepted stability and performance metrics for adaptive control design and analysis.

B. Bounded Linear Stability Method

Global stability is the ultimate requirement for any control system including linear time invariant and nonlinear adaptive control. Global stability is difficult to prove even by the Lyapunov method since it requires a complete detail information of a system. Because of uncertainty, it is more tractable to design a control system to satisfy specified stability margins rather than attempting to prove that the system is globally stable. Linear time-invariant control systems have been designed with the classical phase and gain margins which are well accepted and understood.

In dealing with nonlinear adaptive control, time delay margin has been proposed by numerous authors as a substitution for the classical phase margin to indicate relative stability of a nonlinear adaptive control system. Unfortunately, there is no existing nonlinear theory for global stability of time-delay nonlinear adaptive control. Attempts to overcome this theoretical gap including methods that approximate a time-delay system with a Pade approximation and then invokes the Lyapunov method to obtain an estimate of the time delay margin. The global stability requirement based on the Lyapunov method inherently results in highly conservative estimates of a time delay margin. To relax the global stability, linearization of the nonlinear adaptive control system can provide some local stability estimates using the classical linear stability margins. To obtain an equivalent LTI system, the adaptive law can be linearized at a certain point in time when the weights have reached their steady state values, usually long after initial transients have settled down. However, transient responses during adaptation can be important and can compromise system robustness.

As an alternative to linearization, the bounded linear stability analysis method has recently been introduced to approximate a nonlinear adaptive law with its bounded linear version over a short analysis time window using a comparison lemma. The bounded linear approximation of the adaptive law then allows the LTI stability concept to be analyzed for the adaptive law locally within the analysis window. The resulting time delay margin estimated by the bounded linear stability method using the phase margin and the gain cross-over frequency has been shown to have a reasonable agreement with simulation results. In this study, the bounded linear stability analysis provides an approximate locally bounded linear version of a nonlinear adaptive law. Using this locally bounded linear system, a time delay margin can be estimated using the matrix measure method as derived previously.

The bounded linear stability analysis method is based on a version of the comparison lemma[14] stated as follows:

Lemma 2: The equilibrium state $y(t) = 0$ of the differential equation

$$\dot{y}(t) = -\Phi^{\top}(t)\Gamma\Phi(t)y(t) \tag{91}$$

where $y(t) : [0,\infty) \to \mathbb{R}$, $\Phi(t) \in \mathscr{L}_2 : [0,\infty) \to \mathbb{R}^n$ is a piecewise continuous and bounded function, and $\Gamma > 0 \in \mathbb{R}^{n\times n}$, is uniformly asymptotically stable, if there exists a constant $\gamma > 0$ such that

$$\frac{1}{T_0}\int_t^{t+T_0} \Phi^{\top}(\tau)\Gamma\Phi(\tau)\,d\tau \geq \gamma \tag{92}$$

which implies that $y(t)$ is locally bounded by the solution of a linear differential equation

$$\dot{z}(t) = -\gamma z(t) \tag{93}$$

for $t \in [t_i, t_i + T_0)$, where $t_i = t_{i-1} + T_0$ and $i = 1, 2, \ldots, n \to \infty$.

Proof: Choose a Lyapunov candidate function and evaluate its time derivative

$$V(t) = \frac{1}{2} y^2(t) \tag{94}$$

$$\dot{V}(t) = -\Phi^\top(t)\Gamma\Phi(t) y^2(t) = -2\Phi^\top(t)\Gamma\Phi(t) V(t) \tag{95}$$

Then, there exists $\gamma > 0$ for which V is uniformly asymptotically stable since

$$V(t + T_o) = V(t) \exp\left(-2\int_t^{t+T_0} \Phi^\top(\tau)\Gamma\Phi(\tau)\, d\tau\right) \leq V(t) e^{-2\gamma T_0} \tag{96}$$

This implies that

$$\exp\left(-2\int_t^{t+T_0} \Phi^\top(\tau)\Gamma\Phi(\tau)\, d\tau\right) \leq e^{-2\gamma T_0} \tag{97}$$

Thus, the equilibrium $y(t) = 0$ is uniformly asymptotically stable if

$$\frac{1}{T_0}\int_t^{t+T_0} \Phi^\top(\tau)\Gamma\Phi(\tau)\, d\tau \geq \gamma \tag{98}$$

provided $\Phi(t) \in \mathscr{L}_2$ is bounded.

Then $y(t) \in \mathscr{L}_2 \cap \mathscr{L}_\infty$ since

$$V(t \to \infty) - V(0) \leq -2\gamma \int_0^\infty y^2(t)\, dt \Rightarrow 2\gamma \int_0^\infty y^2(t)\, dt \leq V(0) - V(t \to \infty) < \infty \tag{99}$$

It follows that

$$\dot{V}(t) \leq -2\gamma V(t) \Rightarrow y(t)\dot{y}(t) \leq -\gamma y^2(t) \tag{100}$$

which implies that the solution of Eq. (91) is bounded from above if $y(t) \geq 0$ and from below if $y(t) \leq 0$ by the local solution of

$$\dot{z}(t) = -\gamma z(t) \tag{101}$$

for $t \in [t_i, t_i + T_0)$, where $t_0 = 0$, $t_i = t_{i-1} + T_0$, and $i = 1, 2, \ldots, n \to \infty$.

Equation (101) also applies for $\Phi = \Phi(y(t))$ since the condition $\Phi(y(t)) \in \mathscr{L}_2$ is identically satisfied given that $y(t) \in \mathscr{L}_2 \cap \mathscr{L}_\infty$. This is shown by evaluating $\dot{V}(t)$ as

$$\dot{V}(t) = \dot{y}(t)\frac{dV}{dy} = -\Phi^\top(y(t))\Gamma\Phi(y(t)) y(t)\frac{dV}{dy} = -2\Phi^\top(y(t))\Gamma\Phi(y(t)) V(t) \tag{102}$$

Thus

$$\frac{dV}{V(t)} = 2\frac{dy}{y(t)} \tag{103}$$

Suppose there exists γ such that

$$\frac{dy}{y(t)} \leq -\gamma dt \tag{104}$$

Then multiplying both sides of Eq. (104) by $y^2(t)$ and dividing by dt result in the same equation as Eq. (100). Thus, $V(t)$ is uniformly asymptotically stable and $y(t)$ is bounded by the same equation as Eq. (101). Therefore, γ given by Eq. (92) satisfies Eq. (104).

In this study, the comparison lemma allows the stability of nonlinear adaptive control to be analyzed in a local sense using its bounded linear approximation.

■

The adaptive law (85) thus now can be expressed by its bounded linear version as

$$\dot{\Theta}^\top(t)\Phi(x(t)) = -B^\top P e(t)\Phi^\top(x(t))\Gamma\Phi(x(t)) \leq -\gamma B^\top P e(t) \tag{105}$$

where $\gamma = \inf_{t\in[t_i,t_i+T_0)}\left(\frac{1}{T_0}\int_{t_i}^{t_i+T_0}\Phi^\top(x(\tau))\Gamma\Phi(x(\tau))\, d\tau\right) > 0 \in \mathbb{R}$ for $t \in [t_i, t_i + T_0)$, where $t_0 = 0$, $t_i = t_{i-1} + T_0$, and $i = 1, 2, \ldots, n \to \infty$.

C. Time Delay Margin by Matrix Measure Method

In the presence of unstructured uncertainty, sufficient robustness must be built into the design of an adaptive controller. Time delay margin can be thought of as a measure of stability robustness for adaptive control. If the bound on the unstructured uncertainty $\Delta(x,u,z,t)$ could be determined, then a time delay margin can be estimated for an input time-delay system with an equivalent stability behavior. Thus, instead of studying the stability of the original system (77), one seeks to investigate stability robustness of the following equivalent input time-delay system

$$\dot{x}(t) = Ax(t) + B\left[u(t-t_d) + \Theta^{*\top}\Phi(x(t))\right] \tag{106}$$

The following two problem statements are equivalent: 1) for a specified time delay margin, what is the largest adaptive gain Γ that can be used in the adaptive law (85) for a stable adaptation, and 2) for a given adaptive gain Γ in the adaptive law (85), what is the estimate of the time delay margin?

Consider a special case when the uncertainty is almost linear in structure; i.e.,

$$\Phi(x) = x + g(x) \tag{107}$$

where $g(x)$ is a vector function whose magnitudes of higher order derivatives are much smaller than that of the first-order derivative.

Then by Taylor's series expansion

$$\begin{aligned}\Phi(x) = x + g(x_0) + \left.\frac{dg(x)}{dx}\right|_{x=x_0}(x-x_0) + \frac{1}{2!}\left.\frac{d^2g(x)}{dx^2}\right|_{x=x_0}(x-x_0)^2 + \ldots \\ = x + g(x_0) + \left.\frac{dg(x)}{dx}\right|_{x=x_0}(x-x_0) + \mathscr{O}\left(x^2\right)\end{aligned} \tag{108}$$

The error equation corresponding to the time-delay system (106) can be derived by substituting the time-delay version of the controller from Eq. (81), thus resulting in

$$\dot{e}(t) = A_m x_m(t) + B_m r(t) - Ax(t) - B\left[K_x x(t-t_d) + K_r r(t-t_d) - \Theta^{\top}(t-t_d)\Phi(x(t-t_d)) + \Theta^{*\top}\Phi(x(t))\right] \tag{109}$$

which upon simplification can be expressed as

$$\dot{e}(t) = Ae(t) + BK_x e(t-t_d) + Bu_{ad}(t-t_d) - B\Theta^{*\top}\Phi(x(t)) + BK_x[x_m(t) - x_m(t-t_d)] + BK_r[r(t) - r(t-t_d)] \tag{110}$$

Using the bounded linear approximation of the adaptive law (85), one gets a piecewise locally bounded linear approximation of the adaptive law (85)

$$\dot{u}_{ad}(t) = \dot{\Theta}^{\top}(t)\Phi(x(t)) + \Theta^{\top}(t)\dot{\Phi}(x(t)) \approx -\gamma B^{\top}Pe(t) + \Theta^{\top}(t)\dot{\Phi}(x(t)) \tag{111}$$

for $t \in [t_i, t_i + T_0)$, where $t_0 = 0$, $t_i = t_{i-1} + T_0$, and $i = 1,2,\ldots,n \to \infty$.

The second term in the right hand side can be locally approximated by a first -order Taylor's series as

$$\begin{aligned}\Theta^{\top}(t)\dot{\Phi}(x(t)) = \Theta^{\top}(t_i)\dot{\Phi}(x(t_i)) + \dot{\Theta}^{\top}(t_i)\dot{\Phi}(x(t_i))\Delta t + \Theta^{\top}(t_i)\left[\dot{\Phi}(x(t)) - \dot{\Phi}(x(t_i))\right] + \ldots \\ = \Theta_i^{\top}\dot{\Phi}(x(t)) + \dot{\Theta}_i^{\top}\dot{\Phi}_i\Delta t + \ldots\end{aligned} \tag{112}$$

where $\Theta_i = \Theta(t_i)$ and $\dot{\Theta}_i = \dot{\Theta}(t_i)$.

The piecewise bounded linear local approximation of the adaptive law (85) then becomes

$$\dot{u}_{ad}(t) \approx -\gamma B^{\top}Pe(t) + \Theta_i^{\top}\dot{\Phi}(x(t)) + \dot{\Theta}_i^{\top}\dot{\Phi}_i\Delta t \tag{113}$$

Furthermore, the term $\dot{\Phi}(x(t))$ can be approximated by a first -order Taylor's series as

$$\begin{aligned}\dot{\Phi}(x(t)) = \frac{\partial\Phi(x(t))}{\partial x}\dot{x}(t) = \frac{\partial\Phi(x(t_i))}{\partial x}\dot{x}(t_i) + \frac{\partial^2\Phi(x(t_i))}{\partial x^2}\dot{x}(t_i)[x(t) - x(t_i)] + \frac{\partial\Phi(x(t_i))}{\partial x}[\dot{x}(t) - \dot{x}(t_i)] + \ldots \\ = \Phi_i^{'}[\dot{x}_m(t) - \dot{e}(t)] + \Phi_i^{''}\dot{x}_i[x_m(t) - e(t) - x_i] + \ldots\end{aligned} \tag{114}$$

where $\Phi_i^{'} = \partial\Phi(x(t_i))/\partial x$ and $\Phi_i^{''} = \partial^2\Phi(x(t_i))/\partial x^2$.

Differentiating the error equation (110) yields

$$\begin{aligned}\ddot{e}(t) = A\dot{e}(t) + BK_x\dot{e}(t-t_d) - \gamma BB^\top Pe(t-t_d) + B\Theta_i^\top\Phi_i^{'}[\dot{x}_m(t-t_d) - \dot{e}(t-t_d)] \\ + B\Theta_i^\top\Phi_i^{''}\dot{x}_i[x_m(t-t_d) - e(t-t_d) - x_i] + B\dot{\Theta}_i^\top\dot{\Phi}_i\Delta t - B\Theta^{*\top}\Phi_i^{'}[\dot{x}_m(t) - \dot{e}(t)] \\ - B\Theta^{*\top}\Phi_i^{''}\dot{x}_i[x_m(t) - e(t) - x_i] + BK_x[\dot{x}_m(t) - \dot{x}_m(t-t_d)] + BK_r[\dot{r}(t) - \dot{r}(t-t_d)]\end{aligned} \tag{115}$$

for $t \in [t_i, t_i + T_0)$, where $t_0 = 0$, $t_i = t_{i-1} + T_0$, and $i = 1, 2, \ldots, n \to \infty$.

Thus, the locally bounded linear approximation of the error equation can be expressed as

$$\begin{aligned}\begin{bmatrix} \ddot{e}(t) \\ \dot{e}(t) \end{bmatrix} = \begin{bmatrix} A + B\Theta^{*\top}\Phi_i^{'} & B\Theta^{*\top}\Phi_i^{''}\dot{x}_i \\ I & 0 \end{bmatrix}\begin{bmatrix} \dot{e}(t) \\ e(t) \end{bmatrix} \\ + \begin{bmatrix} BK_x - B\Theta_i^\top\Phi_i^{'} & -\gamma BB^\top P - B\Theta_i^\top\Phi_i^{''}\dot{x}_i \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \dot{e}(t-t_d) \\ e(t-t_d) \end{bmatrix} + \begin{bmatrix} d_1(t) + d_2(t-t_d) + d_3 \\ 0 \end{bmatrix}\end{aligned} \tag{116}$$

where

$$d_1(t) = -B\Theta^{*\top}\Phi_i^{'}\dot{x}_m(t) - B\Theta^{*\top}\Phi_i^{''}\dot{x}_i x_m(t) + BK_x\dot{x}_m(t) + BK_r\dot{r}(t) \tag{117}$$

$$d_2(t-t_d) = B\Theta_i^\top\Phi_i^{'}\dot{x}_m(t-t_d) + B\Theta_i^\top\Phi_i^{''}\dot{x}_i x_m(t-t_d) - BK_x\dot{x}_m(t-t_d) - BK_r\dot{r}(t-t_d) \tag{118}$$

$$d_3 = -B\Theta_i^\top\Phi_i^{''}\dot{x}_i x_i + B\dot{\Theta}_i^\top\dot{\Phi}_i\Delta t + B\Theta^{*\top}\Phi_i^{''}\dot{x}_i x_i \tag{119}$$

are treated as time varying disturbances and therefore do not affect the closed-loop adaptive system stability.

Equation (116) shows that the stability of the locally bounded linear approximation depends not only on the adaptive gain Γ but also on the trajectory of the state vector $x(t)$ together captured in the parameter γ. Furthermore, it also depends on the initial values of the state vector x_i and its time derivative $\dot{x}_i$, the weight Θ_i, and the derivatives $\Phi_i^{'}$ and $\Phi_i^{''}$, as well as the unknown parameter Φ^*. Using the matrix measure method as defined previously, the time delay margin of the adaptive system can be estimated as

$$\omega_i < \overline{\mu}(-jC_i) + \|D_i\| \tag{120}$$

$$t_{d_i} < \frac{1}{\omega_i}\cos^{-1}\frac{\overline{\mu}(C_i) + \overline{\mu}(jD_i)}{\|D_i\|} \tag{121}$$

where

$$C_i = \begin{bmatrix} A + B\Theta^{*\top}\Phi_i^{'} & B\Theta^{*\top}\Phi_i^{''}\dot{x}_i \\ I & 0 \end{bmatrix} \tag{122}$$

$$D_i = \begin{bmatrix} -BK_x + B\Theta_i^\top\Phi_i^{'} & \gamma BB^\top P + B\Theta_i^\top\Phi_i^{''}\dot{x}_i \\ 0 & 0 \end{bmatrix} \tag{123}$$

for $t \in [t_i, t_i + T_0)$, where $t_0 = 0$, $t_i = t_{i-1} + T_0$, and $i = 1, 2, \ldots, n \to \infty$.

It is noted that the computation of the local time delay margin is retrospective in that the estimate is computed for a time window for which most recent data have been collected for analysis.

D. Estimation of Time Delay Margin for Scalar Adaptive Systems

Consider a scalar system with linear structured uncertainty

$$\dot{x}(t) = ax(t) + b[u(t) + \theta^* x(t)] \tag{124}$$

The reference model is given by

$$\dot{x}_m(t) = a_m x_m(t) + b_m r(t) \tag{125}$$

The controller is given by

$$u(t) = k_x x(t) + k_r r(t) - \theta(t)x(t) \tag{126}$$

$$\dot{\theta}(t) = -\Gamma x(t)\, pbe(t) \tag{127}$$

The input time delay version of the system is

$$\dot{x}(t) = ax(t) + b\left[u(t-t_d) + \theta^* x(t)\right] \tag{128}$$

From the response of the delay-free scalar system, the parameter γ is evaluated as

$$\gamma = \frac{\Gamma}{T_0}\int_{t_i}^{t_i+T_0} x^2(\tau)\,d\tau \tag{129}$$

The matrices C_i and D_i are

$$C_i = \begin{bmatrix} a+b\theta^* & 0 \\ 1 & 0 \end{bmatrix} \tag{130}$$

$$D_i = \begin{bmatrix} -bk_x + b\theta_i & \gamma b^2 p \\ 0 & 0 \end{bmatrix} \tag{131}$$

The following parameters are computed analytically as

$$\overline{\mu}(C_i) = \frac{a+b\theta^* + \sqrt{(a+b\theta^*)^2+1}}{2} \tag{132}$$

$$\overline{\mu}(-jC_i) = \frac{1}{2} \tag{133}$$

$$\overline{\mu}(jD_i) = \frac{\gamma b^2 p}{2} \tag{134}$$

$$\|D_i\| = \sqrt{(bk_x - b\theta_i)^2 + \gamma^2 b^4 p^2} \tag{135}$$

The cross-over frequency and time delay margin are then estimated as

$$\omega_i < \overline{\mu}(-jC_i) + \|D_i\| = \frac{1}{2} + \sqrt{(bk_x - b\theta_i)^2 + \gamma^2 b^4 p^2} \tag{136}$$

$$t_{d_i} < \frac{1}{\omega_i}\cos^{-1}\frac{\overline{\mu}(C_i)+\overline{\mu}(jD_i)}{\|D_i\|} = \frac{2}{1+2\sqrt{(bk_x-b\theta_i)^2+\gamma^2b^4p^2}}\cos^{-1}\frac{a+b\theta^*+\sqrt{(a+b\theta^*)^2+1}+\gamma b^2 p}{2\sqrt{(bk_x-b\theta_i)^2+\gamma^2b^4p^2}} \tag{137}$$

The "exact" values of the cross-over frequency and time delay margin for the locally bounded linear approximation of the error equation can be determined as follows:

$$\det\left(j\omega_i - C_i + D_i e^{-j\omega_i t_{d_i}}\right) = 0 \tag{138}$$

This results in two equations

$$-\omega_i^2 - (bk_x - b\theta_i)\,\omega_i \sin\omega_i t_{d_i} + \gamma b^2 p\cos\omega_i t_{d_i} = 0 \tag{139}$$

$$-(a+b\theta^*)\,\omega_i - (bk_x - b\theta_i)\,\omega_i\cos\omega_i t_{d_i} - \gamma b^2 p\sin\omega_i t_{d_i} = 0 \tag{140}$$

The cross-over frequency equation is obtained as

$$\omega^4 + \left[(a+b\theta^*)^2 - (bk_x - b\theta_i)^2\right]\omega^2 - \gamma^2 b^4 p^2 = 0 \tag{141}$$

The solution gives

$$\omega_i = \sqrt{\frac{(bk_x-b\theta_i)^2 - (a+b\theta^*)^2 + \sqrt{4\gamma^2b^4p^2 + \left[(a+b\theta^*)^2-(bk_x-b\theta_i)^2\right]^2}}{2}} \tag{142}$$

$$t_{d_i}^* = \frac{1}{\omega_i}\cos^{-1}\frac{\gamma b^2 p\omega_i^2 - (a+b\theta^*)(bk_x - b\theta_i)\omega_i^2}{\gamma^2 b^4 p^2 + (bk_x - b\theta_i)^2 \omega_i^2} \tag{143}$$

The adaptive gain for which the adaptive system is stable, independent of time delay, can be estimated as

$$\|D_i\| = \sqrt{(-bk_x + b\theta_i)^2 + \gamma^2 b^4 p^2} < -\underline{\mu}(C_i) = \frac{-a - b\theta^* + \sqrt{(-a-b\theta^*)^2 + 1}}{2} \tag{144}$$

$$\Gamma \le \frac{T_0\sqrt{\frac{1}{2} + (a+b\theta^*)^2\left[1 + \sqrt{1 + (a+b\theta^*)^{-2}}\right]} - (bk_x - b\theta_i)^2}{b^2 p\int_{t_i}^{t_i+T} x^2(\tau)\,d\tau} \tag{145}$$

provided $x(t) \neq 0$ for all $t \in [t_i, t_i + T_0)$.

Metrics-driven adaptive control is a concept whereby the adaptive gain Γ can be adjusted for each time window in order to achieve a desired or specified time delay margin. This type of adaptation can allow a trade-off between transient performance and stability robustness. Let τ_d be a desired time delay margin, then the metrics-driven adaptive gain can be computed from

$$\tau_d = \frac{1}{\omega_{i-1}(\gamma_i)}\cos^{-1}\frac{\overline{\mu}(C_{i-1}) + \overline{\mu}(jD_{i-1}(\gamma_i))}{\|D_{i-1}(\gamma_i)\|} = f\left(\frac{\Gamma_i}{T_0}\int_{t_{i-1}}^{t_{i-1}+T} x^2(\tau)\,d\tau\right) \tag{146}$$

This is a nonlinear equation which can be solved for the metrics-driven adaptive gain Γ_i for a current time window $t \in [t_i, t_i + T_0)$ using the information from a previous time window $t \in [t_{i-1}, t_{i-1} + T_0)$. It is noted that the adaptive gain Γ is inversely proportional to the mean-square value of the system state $\frac{1}{T_0}\int_{t_{i-1}}^{t_{i-1}+T} x^2(\tau)\,d\tau$, which has a notion of the system state "energy". Thus, stability of the adaptive system is dependent on the product of the adaptive gain and the system state mean-square value. For metrics-driven adaptive control to achieve a desired time delay margin, this product needs to be kept at a desired value. Therefore, if the system state mean-square value is high, then the adaptive gain needs to be reduced, and vice versa.

Example: Given $a = 1$, $b = 1$, $\theta^* = 0.1$, $a_m = -1$, $b_m = 1$, $p = 1$, $\theta(0) = 0$, $r(t) = \sin(t)$. The control gains are computed to be $k_x = -2$ and $k_r = 1$.

For $\Gamma = 1$, the time histories of the state $x(t)$, control $u(t)$, and the weight $\theta(t)$ for the delay-free system are plotted in Fig. 1. It is noted that the weight $\theta(t)$ converges to the correct value after about $t = 40$ sec.

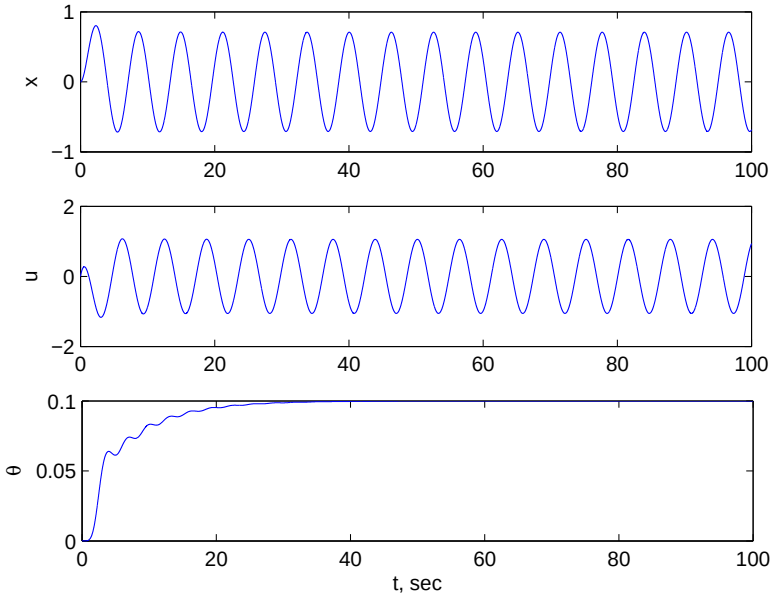

Fig. 1 - Time Histories of Delay-Free Adaptive Control System

Select a time window $T_0 = 1$ sec. For the first window $0 \leq t < T_0$, from the response of the delay-free scalar system, the parameter γ is evaluated numerically at $t = T_0$ as

$$\gamma = \frac{1}{T_0}\int_0^{T_0} \Gamma x^2(\tau)\, d\tau = 0.0266$$

and

$$\overline{\mu}(C_1) = 1.293$$

$$\overline{\mu}(-jC_1) = \frac{1}{2}$$

$$\overline{\mu}(jD_1) = 0.0133$$

$$\|D_1\| = 2.000$$

The bounds on the cross-over frequency and time delay margin are calculated to be

$$\omega_1 < 2.500 \text{ rad/sec}$$

$$t_{d_1} < 0.344 \text{ sec}$$

For comparison, the "exact" values of the cross-over frequency and time delay margin for the locally bounded linear approximation of the error equation are computed to be

$$\omega_1 = 1.670 \text{ rad/sec}$$

$$t_{d_1} = 0.591 \text{ sec}$$

and for the non-adaptive LTI system for which $\theta(t) = 0$ for all t are

$$\omega = 1.670 \text{ rad/sec}$$

$$t_d = 0.592 \text{ sec}$$

The "exact" results are almost the same since the estimates using the locally bounded linear approximation of the error equation are for the first time window for whixh $\theta_i = 0$.

The numerical evidence of the time delay margin is estimated to be $t_d^* \approx 0.407$ sec. Thus, the estimated local time delay margin for the first time window is in a reasonable agreement with the numerical evidence. On the other hand, the "exact" value of the time delay margin for the locally bounded linear approximation of the error equation over-estimates the time delay margin of the adaptive system. The process is then repeated for the next time window and so on.

Figure 2 is a plot of the variation of the local time delay margin estimates within the time interval for three different sizes of time window; $T_0 = 1$ sec, $T_0 = 5$ sec, and $T_0 = 10$ sec. It is noted that as the window size increases, the variation in the local time delay margin estimates decreases. The longer time window allows a more uniform average value of the parameter γ to be computed, thus reducing the local variation of time delay margin estimate from one time window to another. It appears that the mean value of the computed time delay margins is relatively insensitive to the window size. In fact, the mean estimate of the time delay margin for the three time window sizes $T_0 = 1$ sec, $T_0 = 5$ sec, and $T_0 = 10$ sec are 0.319 sec, 0.319 sec, and 0.318 sec, respectively.

Figure 3 is a plot of the mean value of the time delay margin estimates as a function of the unknown parameter $-1 \leq \theta^* \leq 1$ for $T_0 = 1$ sec. Generally, θ^* is not known, so in a verification setting, time delay margin should be computed over all possible parameter variations within their physical bounds. As can be seen in Fig. 3, for $\theta^* < 0$, the time delay margin estimate of the adaptive system is greater than that when $\theta^* > 0$. This is consistent with the observation that for $\theta^* < 0$, the open-loop system is more stable than that when $\theta^* > 0$. Also plotted is the time delay numerical evidence from simulations. The numerical evidence is larger than the mean value of the time delay margin estimates as computed from the bounded linear stability analysis method by about 20 to 30 percent. Nonetheless, the estimation of the time delay margin by the bounded linear stability analysis method is quite reasonable and, more importantly, is not too overly conservative. This is important from a practical perspective since any analytical tool for estimating time delay margin for an adaptive system must be realistic with reliability, good accuracy, and sufficient conservatism.

Figure 4 is a plot of the mean value of the time delay margin estimates as a function of the adaptive gain $1 \leq \Gamma \leq 100$ for $T_0 = 1$ sec. It can be seen that as the adaptive gain Γ increases, the time delay margin of the adaptive

system decreases. This is a well-known phenomenon in the conventional model reference adaptive control. Thus, there exists a trade-off between transient performance and stability robustness. Increasing the adaptive gain Γ gives better transient performance but at the expense of robustness. Metrics-driven adaptive control could provide a way to maintain consistent time delay margin throughout adaptation in exchange for lower transient performance.

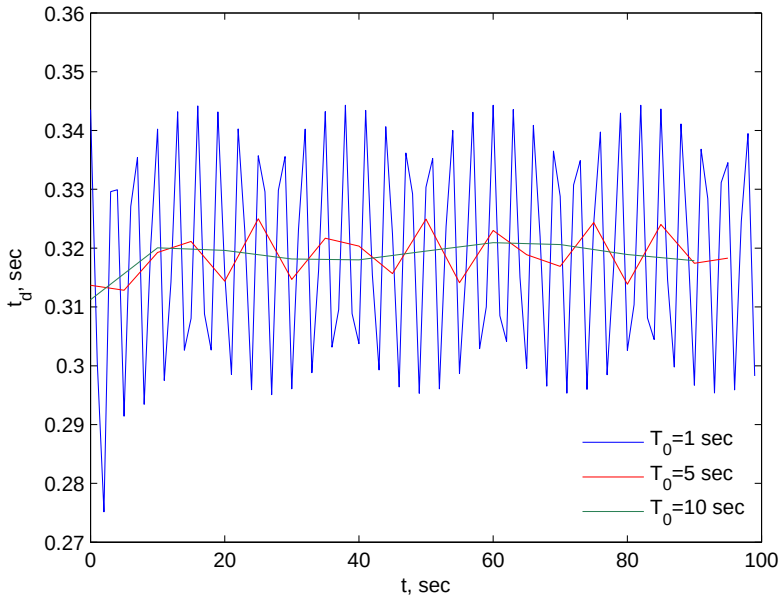

Fig. 2 - Time Delay Margin Estimates

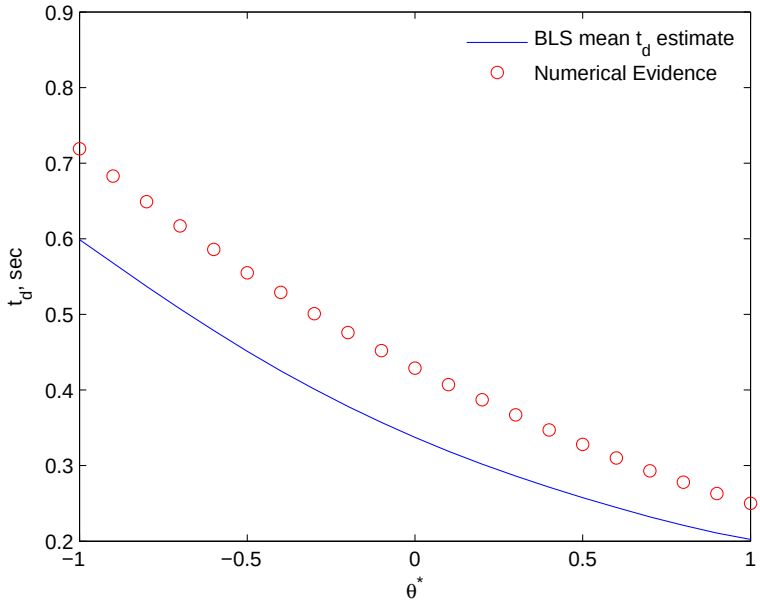

Fig. 3 - Time Delay Margin Variation with Unknown Parameter θ^*

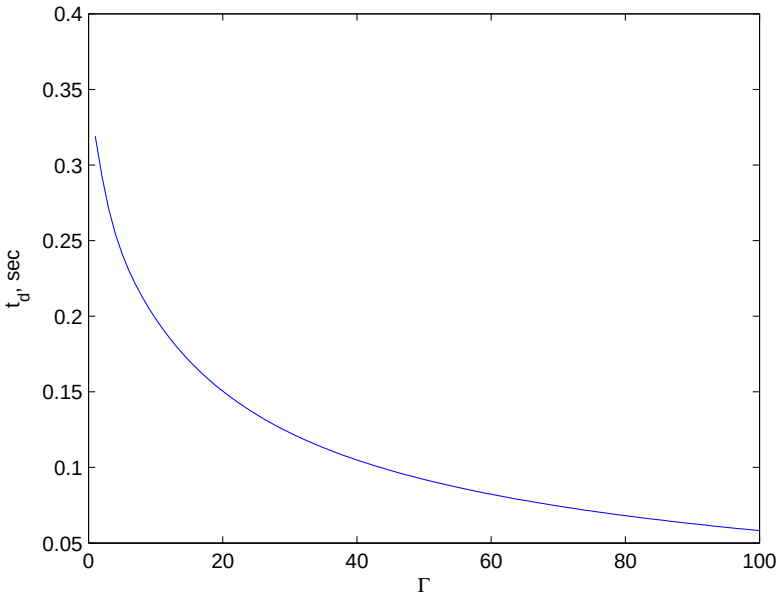

Fig. 4 - Time Delay Margin Variation with Adaptive Gain

V. Conclusions

This paper presents a new method for estimating time delay margin for model reference adaptive control. The bounded linear stability analysis method provides a locally bounded linear approximation of the conventional model reference adaptive law. In effect, the adaptive law is transformed into a locally bounded linear approximation within small time windows for which local time delay margins are to be estimated. A matrix measure approach provides a simple analytical method for computing an upper bound of time delay margin for a linear system is introduced. This method is shown to provide a good estimate of time delay margin for a linear system without incurring too much conservatism. To analyze the time delay margin, the adaptive system is formulated as an input-time-delay error equation. Using this method for the locally bounded linear approximation of the input-time-delay error equation, time delay margin for a model reference adaptive control can be estimated. A special case for a scalar model reference adaptive control system is studied. The method was able to provide a reasonable, yet not too conservative estimate of the time delay margin for the scalar adaptive system. The effect of the time window size was examined. As the time window size increases, the variation in the estimates of the local time delay margins for each time window decreases. It is found that the mean value of the local time delay margin estimates over the time interval is quite insensitive to the time window size. Thus, the mean value of the local time delay margin estimates could be considered as a representative time delay margin estimate for the entire time interval. The method also correctly predicts a typical behavior in model reference adaptive control whereby the time delay margin decreases with an increase in the adaptive gain. Future work will relax the restriction of almost linear uncertainty and also extend this method to robust modification schemes in adaptive control.

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