Global aspects of charged particle motion in axially symmetric multipole magnetic fields

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Abstract

The motion of a single charged particle in the space outside of a compact region of steady currents is investigated. The charged particle is assumed to produce negligible electromagnetic radiation, so that its energy is conserved. The source of the magnetic field is represented as a point multipole. After a general description, attention is focused on magnetic fields with axial symmetry. Lagrangian dynamical theory is utilized to identify constants of the motion as well as the equations of motion themselves. The qualitative method of Størmer is used to examine charged particle motion in axisymmetric multipole fields of all orders. Although the equations of motion generally have no analytical solutions and must be integrated numerically to produce a specific orbit, a topological examination of dynamics is possible, and can be used, á la Størmer, to completely describe the global aspects of the motion of a single charged particle in a space with an axisymmetric multipole magnetic field.

1. Introduction

The problem of determining the motion of a single charged particle in a magnetic field is an old one but remains very challenging. The fundamental difficulty is that the equations describing the motion are generally nonlinear and no analytical solution is possible – computation is required to find specific trajectories. In the years 1907 to 1930, Carl Størmer and his small research group at the University of Oslo investigated the motion of charged particles in the geomagnetic dipole field, since this was important for understanding the behavior of solar particles and cosmic rays near the earth[1]. Although exact solutions to even this basic problem are not generally possible, Størmer found a method for describing global aspects of charged particle motion in a dipole field. His method was to look at *qualitative solutions*, whereby regions of space accessible to a charged particle could be mapped out in terms of certain constants of the motion. One constant is the energy (the charged particle is assumed to produce negligible electromagnetic radiation, so that its energy is conserved). The other invariant is related to rotational symmetry about the dipole axis and is called *Størmer's integral.* Qualitative solutions which describe the partitioning of space into a collection of allowed and forbidden regions, with regard to particle motion, are themselves part of a larger, more modern discipline called *topological dynamics*[2]. (Well-known examples of spatial partitioning are the Van Allen radiation belts.)

Here, the method of Størmer is extended and combined with other methods to investigate motion in axisymmetric magnetic fields other than a dipole field. In particular, point multipoles of all orders are considered. These sources are assumed to lie within a compact region of steady currents, so that any associated magnetic field tends to zero at large distances. At these distances, localized currents are well approximated by point multipoles, so that multipole expansions are the primary approximation far away from such sources. (However, if it is important to characterize charged particle motion near the actual currents that produce the axisymmetric field, then the magnetic fields – more specifically, the magnetic potentials – associated with current loops, rather than point multipoles, must be used; this topic will not be dealt with here.) Størmer's method for analyzing motion in axisymmetric magnetic fields essentially transforms the problem of finding specific three-dimensional trajectories into the problem of finding general two-dimensional topological solutions.

An important aspect of the analysis presented here is the use of *Lagrangian dynamical theory* to identify constants of the motion as well as the equations of motion themselves. In particular, Størmer's integral for any axisymmetric multipole is easily found using Lagrangian methods. In order to do this, the relation between the magnetic scalar and vector potentials for point multipoles is developed. This result is necessary because Lagrangian dynamics explicitly uses the vector potential in its basic formulation, while it is the magnetic scalar potential that is usually expressed as a multipole expansion. Finding the "Størmer integral" for motion in the field of axisymmetric multipoles leads to the definition of an associated topological *quasipotential*. This, in turn, leads to an identification of the *guiding center* of the particle's motion as the absolute minimum of the quasipotential.

The results developed are then applied to problems in astrophysics and in space propulsion system design. In astrophysics, qualitative solutions can be used to predict the shape of trapped particle regions in any multipole field (not just a dipole). In the design of space propulsion systems that use strong, axisymmetric magnetic fields due to current loops, qualitative solutions can predict the 'radiation belts' that are bound to arise around spacecraft using these systems, while numerical solutions can produce the specific paths that charged particles take when leaving the spacecraft (for example, these particles can come from the edge of the exhaust plume of a plasma rocket engine or from interactions with the natural space environment).

2. Lagrangian dynamics

The basic equation of classical mechanics is Newton's 2nd Law:

(2-1) $$\frac{d\mathbf{p}}{dt} = \mathbf{F}.$$

Here, $\mathbf{p}$ is the momentum of a particle of mass m. The most general form of $\mathbf{p}$ for a particle is the relativistic one:

(2-2) $$\mathbf{p} = \frac{m\mathbf{v}}{\sqrt{1-v^2/c^2}}.$$

In the above, the velocity is $\mathbf{v} = d\mathbf{r}/dt$, where $\mathbf{r}$ is the position of the particle; $v^2 = |\mathbf{v}|^2$ and c is the speed of light.

A fundamental example of $\mathbf{F}$ appearing in (2-1) is the force due to an electric field $\mathbf{E} = -\nabla\varphi$ derived from an electrostatic potential φ:

(2-3) $$\mathbf{F} = e\mathbf{E} = -e\nabla\varphi.$$

The electric charge e can be positive or negative. The potential energy is $V = e\varphi$, so that equation (2-1) can be put into the form

(2-4) a) $$\frac{d\mathbf{p}}{dt} = -\frac{\partial V}{\partial \mathbf{r}}$$ b) $$\frac{\partial}{\partial \mathbf{r}} = \nabla = \mathbf{e}^i \frac{\partial}{\partial q^i} \equiv \sum_{i=1}^{3} \mathbf{e}^i \frac{\partial}{\partial q^i}.$$

Here we introduced the *Einstein summation convention*: repeated indices imply summation. Note that, in the *contravariant* generalized components q^i and basis vectors $\mathbf{e}^i$, the superscript i is a coordinate index and not a power. There are also *covariant* quantities q_i and $\mathbf{e}_i$. (The relation between the different bases is $\mathbf{e}^i \cdot \mathbf{e}_j = \delta^i_j$ where δ^i_j is the Kronecker delta. (For those unfamiliar with tensor analysis, please refer to the appropriate texts, for example, Sokolnikoff[3] or Lovelock and Rund[4].)

The momentum $\mathbf{p}$ can also be expressed as a gradient – the velocity-gradient of a quantity T:

$$(2\text{-}5) \qquad \text{a) } \mathbf{p} = \frac{\partial T}{\partial \mathbf{v}} \qquad \text{b) } T = -mc^2\sqrt{1-|\mathbf{v}|^2/c^2} \qquad \text{c) } \frac{\partial}{\partial \mathbf{v}} \equiv \mathbf{e}^i \frac{\partial}{\partial \dot{q}^i}.$$

For $v^2/c^2 << 1$, we have $T + mc^2 \cong \frac{1}{2}mv^2$, the non-relativistic kinetic energy.

When the position vector is written as $\mathbf{x}$, its components are specifically the Cartesian coordinates $(x, y, z) = (x^1, x^2, x^3)$, with the associated constant orthonormal basis $\hat{\mathbf{x}}_i$, $i = 1, 2, 3$, while the components of the vector $\mathbf{q}$ are the generalized coordinates, q^i, $i = 1, 2, 3$, and the associated independent basis consists of the vectors $\mathbf{e}_i$, $i = 1, 2, 3$. (The set of available q^k includes the possibility that $q^k = x^k$.) The position vector $\mathbf{r}$ resides in *position* space, while the velocity vector $\mathbf{v} = d\mathbf{r}/dt$ exists in *velocity* space. The components of $\dot{\mathbf{q}}$ are $\dot{q}^i = dq^i/dt$ (in general, $\dot{\mathbf{q}} \neq \mathbf{v}$). The space consisting of vectors $\mathbf{q}$ is the *configuration space* for a single particle; the six-dimensional space $(\mathbf{q}, \dot{\mathbf{q}})$ is often called *phase space*. (In the case of n distinct particles, the full configuration space has 3^n dimensions and the full phase space has 6^n dimensions.)

In phase space, derivatives can be taken with respect to both position and velocity. Consider the following derivatives:

$$(2\text{-}6) \qquad \text{a) } \frac{\partial \dot{x}^i}{\partial \dot{q}_k} = \frac{\partial}{\partial \dot{q}^k}\left(\frac{\partial x^i}{\partial q^j}\dot{q}^j\right) = \frac{\partial x^i}{\partial q^k} \qquad \text{b) } \frac{d}{dt}\frac{\partial x^i}{\partial q^k} = \frac{\partial^2 x^i}{\partial q^j \partial q^k}\dot{q}^j = \frac{\partial \dot{x}^i}{\partial q_k}.$$

These quantities are used in the coordinate transformations that follow.

We take Newton's 2nd Law (2-4a), write it in Cartesian coordinates and transform to generalized coordinates, using the *Jacobian dyadic* (*i.e.*, 2nd rank tensor) $\mathbf{J}$:

$$(2\text{-}7) \qquad \text{a) } \mathbf{J}\cdot\frac{d\mathbf{p}}{dt} = -\mathbf{J}\cdot\frac{\partial V}{\partial \mathbf{x}} \qquad \text{b) } \mathbf{J} = \frac{\partial \mathbf{x}}{\partial \mathbf{q}} = \frac{\partial x^i}{\partial q^k}\mathbf{e}^k\hat{\mathbf{x}}_i\ .$$

Using (2-4b) and (2-7b), the left side of equation (2-7a) becomes

$$(2\text{-}8) \qquad \mathbf{J}\cdot\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\mathbf{x}}}\right) = \mathbf{e}^k\frac{\partial x^i}{\partial q^k}\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}^i}\right) = \mathbf{e}^k\left[\frac{d}{dt}\left(\frac{\partial x^i}{\partial q^k}\frac{\partial T}{\partial \dot{x}^i}\right) - \frac{\partial T}{\partial \dot{x}^i}\frac{d}{dt}\frac{\partial x^i}{\partial q^k}\right].$$

Then, using (2-6b) on the right side of (2-8) gives

$$(2\text{-}9) \qquad \mathbf{J}\cdot\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\mathbf{x}}}\right) = \mathbf{e}^k\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}^k}\right) - \frac{\partial T}{\partial q^k}\right]$$

The right side of (2-7a) is simply

$$-\mathbf{J}\cdot\frac{\partial V}{\partial \mathbf{x}} = -\frac{\partial \mathbf{x}}{\partial \mathbf{q}}\cdot\frac{\partial V}{\partial \mathbf{x}} = -\mathbf{e}^k \frac{\partial V}{\partial q^k}. \tag{2-10}$$

Since the $\mathbf{e}^k$ are independent, equation (2-4) takes the form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}^k}\right) - \frac{\partial T}{\partial q^k} = -\frac{\partial V}{\partial q^k}. \tag{2-11}$$

Finally, in the case where the potential is $V = e\varphi$, where V is only a function of physical space $\mathbf{r}$ (*i.e.*, only of the q^i), we can define a quantity $L = T - V$, called the "Lagrangian function" or just the *Lagrangian*. In generalized coordinates and velocities, equation (2-1), then becomes

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^k}\right) - \frac{\partial L}{\partial q^k} = 0. \tag{2-12}$$

This is the *Euler-Lagrange equation*[3,4,5]. In a Cartesian system $q^i \in \{x, y, z\}$ and $\dot{q}^i \in \{\dot{x}, \dot{y}, \dot{z}\}$, while in a spherical polar coordinate system $q^i \in \{r, \theta, \phi\}$ and $\dot{q}^i \in \{\dot{r}, \dot{\theta}, \dot{\phi}\}$. In fact, the q^i need not be orthogonal coordinates and the $\mathbf{e}_i$ need not be orthonormal, just independent. Furthermore, equation (2-12) remains valid even if force is dependent on velocity, as well as position.

The principle of least action

Equation (2-12) can be shown to result from requiring that a certain integral, called the *action*, be a minimum with respect to variation of the generalized coordinates q^i and velocities $\dot{q}^i$ about the trajectory of a particle (allowing no variation at the beginning and ending times t_1 and t_2.) The action S is (where q and $\dot{q}$ are shorthand for q^i and $\dot{q}^i$, $i = 1,2,3$, respectively):

$$S = \int_{t_1}^{t_2} L(q, \dot{q})\, dt. \tag{2-13}$$

The development of the Euler-Lagrange equation from the *principle of least action* proceeds as follows[3,4,5].

The principle of least action requires that the variation of the action S with respect to any independent variables be zero to first order. Let us vary the generalized coordinates and velocities within the action as given in (2-13), but not at the end points t_1 and t_2:

$$\delta S = \int_{t_1}^{t_2} \delta L(q, \dot{q})\, dt = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q^i}\delta q^i + \frac{\partial L}{\partial \dot{q}^i}\delta \dot{q}^i\right) dt = 0. \tag{2-14}$$

Variation of the generalized velocity satisfies

$$\delta\dot{q}^i = \delta\frac{dq^i}{dt} = \frac{d\delta q^i}{dt}. \tag{2-15}$$

Upon integrating (2-14) by parts and using (2-15), we have

$$\delta S = \int_{t_1}^{t_2}\left(\frac{\partial L}{\partial q^i}\delta q^i + \frac{\partial L}{\partial \dot{q}^i}\frac{d\delta q^i}{dt}\right)dt = \frac{\partial L}{\partial \dot{q}^i}\delta q^i\bigg|_{t_1}^{t_2} - \int_{t_1}^{t_2}\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i}\right)\delta q^i\,dt. \tag{2-16}$$

Since there is no variation at the endpoints, we have $\delta q^i(t_1) = \delta q^i(t_2) = 0$, so that the first term on the right side of equation (2-16) is zero. Finally, since δq^i is small but otherwise arbitrary for $t_1 < t < t_2$, we see that (2-12) is required for δS to be zero, to first order.

Lagrangian for a charged particle

In electrodynamics[6,7], we proceed as follows. First, in the case when $\mathbf{F}$ is due to an electrostatic potential φ, as in (2-3), the Lagrangian is

$$L = -mc^2\sqrt{1-|\mathbf{v}|^2/c^2} - e\varphi. \tag{2-17}$$

The potential φ does not include a contribution from the particle itself, but is due to other charges; if there are no others, then the potential is set to zero (or some other constant).

In general, a magnetic field $\mathbf{B}$, as well as an electric field $\mathbf{E}$, will exist. Then, in addition to the scalar potential φ, a vector potential $\mathbf{A}$ will be needed:

$$\text{a) } \mathbf{B} = \nabla\times\mathbf{A} \qquad \text{b) } \mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t} - \nabla\varphi. \tag{2-18}$$

Note that the divergence of the first equation and the curl of the second equation above produce two of Maxwell's equations:

$$\text{a) } \nabla\cdot\mathbf{B} = 0 \qquad \text{b) } \nabla\times\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}. \tag{2-19}$$

For completeness, and for later use, the other two Maxwell's equations are

$$\text{a) } \nabla\cdot\mathbf{E} = 4\pi\rho \qquad \text{b) } \nabla\times\mathbf{B} = \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t} + \frac{4\pi}{c}\mathbf{j}. \tag{2-20}$$

Equations (2-19) are known as the *homogeneous* Maxwell equations, while those in (2-20) are called the *inhomogeneous* Maxwell equations because they have as sources the charge density ρ and the current density $\mathbf{j}$. For a single particle, $\rho = q\delta(\mathbf{x}-\mathbf{x}_{(1)})$ and $\mathbf{j} = q\mathbf{v}_{(1)}\delta(\mathbf{x}-\mathbf{x}_{(1)})$, where $\delta(\mathbf{x})$ is the 3-dimensional Dirac delta function, $\mathbf{x}_{(1)}$ is the particle's actual position and $\mathbf{v}_{(1)}$ its velocity at time t. (In the present work, we will be interested, eventually, only in one restricted case: the magnetic field in a source-free region, where, except for the single charge itself, the current density $\mathbf{j}$ and charge density ρ are both zero.)

In the many-particle case, the Lagrangian for a collection of n charged particles takes the form (the summation here is over particles rather than coordinate indices):

(2-21)

$$L = -\sum_{i=1}^{n}\left(m_{(i)}c^2\sqrt{1-v_{(i)}^2/c^2} + e_{(i)}\varphi(\mathbf{x}_{(i)}) - \frac{e_i}{c}\mathbf{A}(\mathbf{x}_{(i)})\,\mathbf{v}_{(i)} \right) + \frac{1}{8\pi}\int [|\mathbf{E}(\mathbf{x}')|^2 - |\mathbf{B}(\mathbf{x}')|^2]\,d^3x' .$$

The potentials $\varphi(\mathbf{x}_{(i)})$ and $\mathbf{A}(\mathbf{x}_{(i)})$ do not include contributions from the i-th particle. The last term is the Lagrangian for the electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$; it is a constant under variation with respect to the spatial variable $\mathbf{x}_i$ because the integration is over all space. However, variation of S in (2-13), with L given by (2-21), with respect to φ, the components of $\mathbf{A}$, and all their derivatives (which serve the same roles for a continuous field as the q^i and $\dot{q}^i$ do for a point particle), yields the electromagnetic field equations (2-19) and (2-20). This is a result of classical field theory[6], which, although highly interesting and important, is outside the scope of this work.

3. Lagrangian of a particle in a steady magnetic field with axial symmetry

The Lagrangian for a single particle in a steady magnetic field

In the present work, the primary concern is with only one particle in an externally imposed, time-independent magnetic field (*i.e.*, no electric field $\mathbf{E}$ is acting on the particle). The Lagrangian is given by (2-21) for one particle ($n = 1$). We also set $\varphi = 0$ and omit the last term (which is merely a constant with respect to the particle's generalized coordinates and velocities, q_i and $\dot{q}_i$).

In this case, (2-21) becomes the effective Lagrangian for a single charged particle in a steady external magnetic field (with $\mathbf{x} = \mathbf{x}_{(1)}$ and $\mathbf{v} = \mathbf{v}_{(1)}$):

(3-1) $$L = -mc^2\sqrt{1-v^2/c^2} + e\,\mathbf{A}(\mathbf{x})\cdot\mathbf{v}/c .$$

The constant magnetic field $\mathbf{B} = \nabla\times\mathbf{A}$ may be found by using (2-18) along with $\varphi = 0$ and the fact that $\mathbf{A}$ is not a function of the time t.

We also assume that any radiation fields that the particle may produce are negligible. Otherwise we would need to include the last term in (2-21), as $\mathbf{A}$ would no longer represent a steady (*i.e.*, time-independent) magnetic field. A single particle in a steady magnetic field has an important constant of the motion: its energy. This is shown by developing a general result.

The first constant of the motion

To begin, we sum the product of $\dot{q}_i$ with the expression found in equation (2-12):

$$\dot{q}^i \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \dot{q}^i \frac{\partial L}{\partial q^i} = \frac{d}{dt}\left(\dot{q}^i \frac{\partial L}{\partial \dot{q}^i}\right) - \left(\ddot{q}^i \frac{\partial L}{\partial \dot{q}^i} + \dot{q}^i \frac{\partial L}{\partial q^i}\right) = 0 . \tag{3-2}$$

Above, we have differentiated by parts (and again used the summation convention). The quantity in the last parenthesis on the right of (3-2) is the total time derivative of L:

$$\frac{dL}{dt} = \ddot{q}^i \frac{\partial L}{\partial \dot{q}^i} + \dot{q}^i \frac{\partial L}{\partial q^i} . \tag{3-3}$$

If L were an explicit function of the time, then it would be necessary to add the term $\partial L / \partial t$ to the right-hand-side of (3-3).

Next, using (3-3), equation (3-2) becomes

$$\frac{d}{dt}\left(\dot{q}^i \frac{\partial L}{\partial \dot{q}^i} - L\right) = 0 . \tag{3-4}$$

The conserved quantity in parenthesis is the energy H:

$$H = \dot{q}^i \frac{\partial L}{\partial \dot{q}^i} - L . \tag{3-5}$$

Thus, once the form of the Lagrangian (3-1) is specified, the energy H immediately follows.

Equation (3-5) is a general result that depends only on the fact that L is not an explicit function of time. [Although we will only consider cases with only a magnetic field present, an electric field could also be included. As long as both electric and magnetic field are steady, then the energy H, as determined from (3-5), still remains constant during a particle's motion.]

Axially symmetric magnetic fields

Now, consider the case where the magnetic field is axially symmetric or *axisymmetric*, that is, invariant upon rotation about some axis. In this case, two appropriate coordinate systems are the spherical polar coordinate system ($q_i \in \{r, \theta, \phi\}$) and the circular cylindrical coordinate system

($q_i \in \{\rho, \phi, z\}$). Specifically, it will be assumed that the magnetic field does not change when ϕ varies, *i.e.*, **B** is not a function of ϕ (or t).

In what follows, we will see the usefulness of the Lagrangian method, as embodied in (2-14), when L is not a function of the coordinate ϕ. (Here, the absent coordinate is ϕ, but in many other physical situations, it may be some other generalized coordinate q_i.) The importance of a missing coordinate in the Lagrangian is that this leads to a second constant of the motion (as will be seen in the next section).

4. Lagrangian with axial symmetry in spherical polar coordinates

In a spherical polar coordinate system, the coordinates are $q_i \in \{r, \theta, \phi\}$ and an axially symmetric magnetic field has the form:

(4-1) $$\mathbf{B} = B_r(r,\theta)\hat{\mathbf{r}} + B_\theta(r,\theta)\hat{\boldsymbol{\theta}} \quad \Leftrightarrow \quad \mathbf{A} = \acute{A}(r,\theta)\hat{\boldsymbol{\phi}}.$$

In the above equation, $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$, $\hat{\boldsymbol{\phi}}$ are the unit vectors in the r, θ, ϕ directions, respectively.

Using **A** as defined in (4-1), the Lagrangian (3-1) becomes

(4-2) $$L = -mc^2\sqrt{1 - v^2/c^2} + eA(r,\theta)v_\phi/c.$$

Here, $v^2 = v_r^2 + v_\theta^2 + v_\phi^2$, with $v_r \equiv \dot{r}$, $v_\theta \equiv r\dot{\theta}$ and $v_\phi \equiv r\sin\theta\dot{\phi}$. Since L does not contain ϕ (it is said to be "cyclic in ϕ"), then the ϕ component of (2-14) can be immediately integrated to produce a constant quantity:

(4-3) $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = 0 \quad \Rightarrow \quad \frac{\partial L}{\partial \dot{\phi}} = P_\phi = \text{constant} .$$

We can now use (3-5), (4-2) and (4-3) to produce the explicit form of the constants of the motion for the case of an axially symmetric magnetic field. First, using equations (3-5) and (4-2), as well as $v_\phi = r\sin\theta\dot{\phi}$, gives an expression for the constant energy H:

(4-4) $$H = \frac{mc^2}{\sqrt{1 - v^2/c^2}} .$$

Second, placing (4-2) into (4-3) yields the constant conjugate momentum P_ϕ:

$$(4\text{-}5) \qquad \frac{\partial L}{\partial \dot{\phi}} = r\sin\theta\left[\frac{m\, r\sin\theta\,\dot{\phi}}{\sqrt{1-v^2/c^2}} + \frac{e}{c}A(r,\theta)\right] = P_\phi\,.$$

Equation (4-4) can then be used with (4-5) to write P_ϕ in the form

$$(4\text{-}6) \qquad P_\phi = \frac{r\sin\theta}{c}\left[\frac{H}{c} r\sin\theta\,\dot{\phi} + eA(r,\theta)\right].$$

Thus, in the case of an axisymmetric, steady magnetic field, there are two constants of the motion: the energy H, as defined by (3-5), and P_ϕ, known as *Størmer's integral.*

5. Equations of motion in spherical polar coordinates

In this section, exact and qualitative equations of motion will be discussed.

Equations for a single trajectory

The equations of motion of a charged particle in an axially symmetric magnetic field, specified by $A(r,\theta)$, are found by putting the Lagrangian (4-2) into the Euler-Lagrange equations (2-12), with $q_i \in \{r,\theta,\phi\}$; the equation for $\dot{\phi}$ comes from (4-6). The results (remember that H and P_ϕ are constants) are:

$$(5\text{-}1) \qquad \text{a)}\ \frac{d\dot{r}}{dt} = \frac{ec}{H}\frac{\partial(rA)}{\partial r}\sin\theta\,\dot{\phi} + \frac{v^2-\dot{r}^2}{r} \qquad \text{b)}\ \frac{dr}{dt} = \dot{r}$$

$$(5\text{-}2) \qquad \text{a)}\ \frac{d\left(r^2\dot{\theta}\right)}{dt} = r\dot{\phi}\left[\frac{ec}{H}\frac{\partial(\sin\theta A)}{\partial\theta} + r\sin\theta\cos\theta\,\dot{\phi}\right] \qquad \text{b)}\ \frac{d\theta}{dt} = \dot{\theta}$$

$$(5\text{-}3) \qquad \frac{d\phi}{dt} = \dot{\phi} = \frac{v\,h(r,\theta)}{r\sin\theta} \qquad \text{b)}\ h(r,\theta) = \frac{c}{Hv}\left[\frac{cP_\phi}{r\sin\theta} - eA(r,\theta)\right].$$

The velocity magnitude v is introduced into (5-3) for convenience later. Some mathematical manipulation[6] can show that (5-1), (5-2), and (5-3) are equivalent to $\dot{\mathbf{p}} = e\,\mathbf{v}\times\mathbf{B}/c$.

Equations (5-1) – (5-3) comprise five highly nonlinear equations that serve to determine the exact trajectory of a charged particle in an arbitrary axisymmetric magnetic field $\mathbf{B}(r,\theta)$ with a vector potential $\mathbf{A} = A(r,\theta)\hat{\boldsymbol{\phi}}$, once a set of initial conditions are given. One constant of the motion, Størmer's integral, is used to produce equation (5-3), while the other constant of the motion, the energy H, can be used to reduce the number of independent equations from five to four. It is well known that such a set of equations describes chaotic motion[11,12], *i.e.*, if the initial conditions are changed only slightly, then the resulting trajectory diverges quickly from the one flowing out of the original initial conditions. In chaotic motion, no exact analytical solution is generally possible if **A** is nontrivial – a numerical solution is required. However, a *qualitative solution* is possible, *i.e.*, one that allows a mapping of the regions accessible and inaccessible to particle motion. We now turn to this topic.

Qualitative solutions

In the following, we use well-established (though not necessarily widely-known) results[1,11] that were developed by Stormer in his studies of the polar aurorae. These results allow a qualitative solution of the equations of the highly non-linear motion of a charged particle.

Since $H = mc^2(1-v^2/c^2)^{-1/2}$ is constant, so is v^2. In a spherical polar coordinate system, v^2 has the form

$$(5\text{-}4) \qquad v^2 = v_r^2 + v_\theta^2 + v_\phi^2 = \dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2 .$$

Now, if we divide the expression in (5-4) by v^2, and use $ds = vdt$ (ds is an element of the path length along the particle's trajectory), along with (5-3), we get

$$(5\text{-}5) \qquad \left(\frac{dr}{ds}\right)^2 + \left(r\frac{d\theta}{ds}\right)^2 + h^2 = 1 .$$

This can be thought of as the total energy equation for a surrogate particle[9,12] – one of mass 2, moving in a two-dimensional potential energy field h^2.

Note that since v is a constant, so is $p = |\mathbf{p}|$, where $\mathbf{p}$ is given by equation (2-2). Using $pc^2 = vH$ and (5-3b), we see that h can be expressed as

$$(5\text{-}6) \qquad h = \frac{P_\phi}{pr\sin\theta} - \frac{eA(r,\theta)}{pc}.$$

We will call the quantity h^2 the *pseudo-potential*. We must have $0 \le h^2 \le 1$ during a particle's motion, as (5-5) shows, so the motion of a charged particle in an axisymmetric magnetic field is restricted to those areas of the r-θ plane for which $-1 \le h \le 1$; these are *allowed* or accessible regions for particle motion. Regions for which $h^2 > 1$ are *forbidden* or inaccessible regions This fact will allow a qualitative global analysis of charged particle motion once a functional form for $A(r,\theta)$ is chosen, a topic which will be addressed next.

6. Multipole Fields

In the case of a steady magnetic field in a space $r > r_0$ where there are no sources and no electric field, equations (2-19a) and (2-20b) yield the following field equations for $\mathbf{B}$:

(6-1) a) $\nabla \cdot \mathbf{B} = 0$ b) $\nabla \times \mathbf{B} = 0$.

Thus, $\mathbf{B}$ can be expressed as either

(6-2) a) $\mathbf{B} = \nabla \times \mathbf{A}$ or b) $\mathbf{B} = -\nabla \Phi$.

Equation (6-2a) was already seen in (2-18a), where $\mathbf{A}$ was identified as the magnetic *vector* potential. In (6-2b), Φ is termed the magnetic *scalar* potential. If we place $\mathbf{B} = -\nabla \Phi$ into (2-19a), *i.e.*, $\nabla \cdot \mathbf{B} = 0$, we obtain Laplace's equation:

(6-3) $\nabla^2 \Phi = 0$.

Requiring that $\Phi \to 0$ as $r \to \infty$ in a spherical polar coordinate system leads to the following well-known general solution[7,8]:

(6-4) $$\Phi = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \frac{c_{nm}}{r^{n+1}} Y_{nm}(\theta, \phi).$$

Here, the c_{nm} are complex constants satisfying $c_{nm}^* = (-1)^m c_{n,-m}$ (the * indicates complex conjugation), and the Y_{nm} are the well-known *spherical harmonics*:

(6-5) $$Y_{nm}(\theta, \phi) = \sqrt{\frac{2n+1}{4\pi} \frac{(n-m)!}{(n+m)!}} P_n^m(\cos\theta) e^{im\phi}.$$

The *associated Legendre functions* P_n^m satisfy

(6-6) $$P_n^m(x) = \frac{(-1)^m}{2^n n!} \left(1 - x^2\right)^{\frac{m}{2}} \frac{d^{n+m}}{dx^{n+m}} \left(x^2 - 1\right)^n = (-1)^m \left(1 - x^2\right)^{\frac{m}{2}} \frac{d^m}{dx^m} P_n(x).$$

The $P_n(x) \equiv P_n^0(x)$ are the Legendre polynomials. Furthermore, we have

(6-7) a) $P_n^{-m}(x) = (-1)^m \dfrac{(n-m)!}{(n+m)!} P_n^m(x)$ b) $Y_{n,-m}(\theta, \phi) = (-1)^m Y_{nm}^*(\theta, \phi)$.

In P_n^m, n is the *degree* and m is the *order* the of the associated Legendre function.

The vector potential equivalent to a scalar potential

In (6-2), two possibilities are given: $\mathbf{B} = \nabla \times \mathbf{A}$ or $\mathbf{B} = -\nabla\Phi$. Here, we will find the vector potential $\mathbf{A}_{nm}$ equivalent to the scalar potential Φ_{nm}, where Φ_{nm} is one of the terms in the spherical harmonic expansion (6-4) of Φ:

(6-8) a) $\Phi = \sum_{n,m} \Phi_{nm}$ b) $\Phi_{nm} = \dfrac{c_{nm}}{r^{n+1}} Y_{nm}(\theta, \phi) = \dfrac{f_{nm}}{r^{n+1}} P_n^m(\cos\theta)\, e^{im\phi}$.

First, we will assume that $\mathbf{A}$ has the form

(6-9) a) $\mathbf{A} = \sum_{n,m} \mathbf{A}_{nm}$ b) $\mathbf{A}_{nm} = \dfrac{a_{nm}}{r^{n+1}} \left[F_n^m(\theta)\hat{\boldsymbol{\theta}} + G_n^m(\theta)\hat{\boldsymbol{\phi}}\right] e^{im\phi}$.

Using (6-8) and (6-9), along with the requirement $\nabla \times \mathbf{A}_{nm} = -\nabla\Phi_{nm}$, yields

(6-10) $\dfrac{d}{d\theta}\left(\sin\theta\, G_n^m\right) - imF_n^m = (n+1)\sin\theta\, P_n^m$

(6-11) a) $nG_n^m = -\dfrac{dP_n^m}{d\theta}$ b) $nF_n^m = \dfrac{im}{\sin\theta} P_n^m$.

Here, $P_n^m = P_n^m(\cos\theta)$. Placing F_n^m and G_n^m as given in (6-11) into equation (6-10) yields

(6-12) $\dfrac{1}{\sin\theta}\dfrac{d}{d\theta}\left(\sin\theta \dfrac{dP_n^m}{d\theta}\right) + \left[n(n+1) - \dfrac{m^2}{\sin^2\theta}\right] P_n^m = 0$.

This is the differential equation satisfied by the associated Legendre function $P_n^m(\cos\theta)$. Thus, the three equations in (6-10) and (6-11) are consistent.

Therefore, if the magnetic scalar potential Φ contains a term Φ_{nm} as given in (6-8), the magnetic vector $\mathbf{A}$ contains a term $\mathbf{A}_{nm}$ given by

(6-13) $\mathbf{A}_{nm} = \dfrac{f_{nm}}{nr^{n+1}} \left[\dfrac{imP_n^m(\cos\theta)}{\sin\theta}\hat{\boldsymbol{\theta}} - \dfrac{dP_n^m(\cos\theta)}{d\theta}\hat{\boldsymbol{\phi}}\right] e^{im\phi}$.

Another way of writing this is

(6-14) $\mathbf{A}_{nm} = \dfrac{1}{n}\left[\dfrac{\hat{\boldsymbol{\theta}}}{\sin\theta}\dfrac{\partial}{\partial\phi} - \hat{\boldsymbol{\phi}}\dfrac{\partial}{\partial\theta}\right]\Phi_{nm} = -\dfrac{1}{n}\mathbf{r} \times \nabla\Phi_{nm}$.

The vector potential, in terms of a given scalar potential, can also be expressed as

$$\text{(6-15)} \qquad \text{a) } \mathbf{A}_{nm} = \nabla \times \mathbf{Ø}_{nm} \qquad \text{b) } \mathbf{Ø}_{nm} = \tfrac{1}{n}\mathbf{r}\Phi_{nm}.$$

The vector potential $\mathbf{A}_{nm}$ clearly satisfies $\nabla \cdot \mathbf{A}_{nm} = 0$.

Axisymmetric multipoles

Now that we have the 3-dimensional form of **A**, we will reduce this to the form of **A** in (4-1), *i.e.*, $A_\theta = 0$, $A_\phi = A(r,\theta)$, which is appropriate for a magnetic field with axial symmetry. In this case, we use (6-13) with $m = 0$, $\mathbf{A}_n = \mathbf{A}_{n0}$, and $a_n = f_{n0}$:

$$\text{(6-16)} \qquad \text{a) } \mathbf{A}_n = A(r,\theta)\hat{\boldsymbol{\phi}} \qquad \text{b) } A(r,\theta) = -\frac{a_n}{nr^{n+1}}\frac{dP_n(\cos\theta)}{d\theta}.$$

The associated Legendre functions satisfy (among others) the following recursion relation:

$$\text{(6-17)} \qquad \frac{d}{d\theta}P_n^{m-1}(\cos\theta) = P_n^m(\cos\theta) + (m-1)\cot\theta P_n^{m-1}(\cos\theta).$$

$$\text{(6-18)} \qquad P_n^{m+2}(\cos\theta) + 2(m+1)\cot\theta\, P_n^{m+1}(\cos\theta) + (n-m)(n+m-1)P_n^m(\cos\theta) = 0.$$

If we set $m = 1$, we can use (6-17) to express $A(r,\theta)$ in (6-16b) as

$$\text{(6-19)} \qquad A(r,\theta) = -\frac{a_n}{nr^{n+1}}P_n^1(\cos\theta).$$

This is the n^{th} order multipole component of an axisymmetric magnetic vector potential: $n = 1$ is a dipole, $n = 2$ is a quadrupole, and $n = 3$ is an octupole, etc.

7. Lagrangian with axial symmetry in circular cylindrical coordinates

In a circular cylindrical coordinate system the coordinates are $q^i \in \{\rho, \theta, z\}$ and an axially symmetric magnetic field has the form:

$$\text{(7-1)} \qquad \mathbf{B} = B_r(\rho,z)\hat{\mathbf{n}} + B_z(\rho,z)\hat{\mathbf{z}} \quad \Leftrightarrow \quad \mathbf{A} = \acute{A}(\rho,z)\hat{\boldsymbol{\phi}}.$$

In the above equation, $\hat{\mathbf{n}}$, $\hat{\boldsymbol{\phi}}$, $\hat{\mathbf{z}}$ are the unit vectors in the ρ, ϕ, z directions, respectively.

Using (7-1), the Lagrangian (3-1) becomes (with $v^2 = v_r^2 + v_\phi^2 + v_z^2$; $v_\rho = \dot{\rho}$, $v_\phi = \rho\dot{\phi}$, $v_z = \dot{z}$)

(7-2) $$L = -mc^2\sqrt{1 - v^2/c^2} + eA(\rho, z)v_\phi/c.$$

Again, since L does not contain ϕ, then the ϕ component of the Euler-Lagrange equations (2-12) leads to

(7-3) $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = 0 \quad \Rightarrow \quad \frac{\partial L}{\partial \dot{\phi}} = P_\phi = \text{constant} \; .$$

Putting (7-2) into (7-3) again yields Størmer's integral, this time in cylindrical coordinates:

(7-4) $$P_\phi = \frac{\rho}{c}\left[\frac{H}{c}\rho\dot{\phi} + eA(\rho, z)\right].$$

The energy H of the charged particle is defined in terms of L by (3-5). After, putting (7-2) into the general expresson (3-5), we see that it takes the same form as (4-5):

(7-5) $$H = \frac{mc^2}{\sqrt{1 - v^2/c^2}} \; .$$

Next, we will discuss the equations of motion in circular cylindrical coordinates, as it often useful to phrase the solution in terms of a cylindrical rather than a spherical polar coordinate system.

8. Equations of motion in circular cylindrical coordinates

Equations for a single trajectory

The equations of motion of a charged particle in an axially symmetric magnetic field, specified by $A(\rho, z)$, are found by putting the Lagrangian (7-2) into the Euler-Lagrange equations (2-14), with $q_i \in \{\rho, \phi, z\}$; the equation for $\dot{\phi}$ is determined from (7-4). The results (again, remember that H and P_ϕ are constants) are:

(8-1) a) $$\frac{d\dot{\rho}}{dt} = \frac{ec}{H}\frac{\partial(\rho A)}{\partial \rho}\dot{\phi} + \rho\dot{\phi}^2$$ b) $$\frac{d\rho}{dt} = \dot{\rho}$$

(8-2) a) $$\frac{d\dot{z}}{dt} = \frac{ec}{H}\rho\frac{\partial A}{\partial z}\dot{\phi}$$ b) $$\frac{dz}{dt} = \dot{z}$$

$$(8\text{-}3) \qquad \frac{d\phi}{dt} = \dot{\phi} = \frac{v\,h}{\rho}. \qquad\qquad \text{b)}\ h = \frac{c}{vH}\left[\frac{cP_\phi}{\rho} - eA(\rho, z)\right].$$

Again, some mathematical manipulation[6] will show that these are equivalent to $\dot{\mathbf{p}} = e\,\mathbf{v}\times\mathbf{B}/c$.

Equations (8-1) – (8-3) are equivalent to (5-1) – (5-3) and also comprise five highly nonlinear equations, that serve to determine the exact trajectory of a charged particle, in an arbitrary axisymmetric magnetic field $\mathbf{B}(\rho, z)$ with a vector potential $\mathbf{A} = A(\rho, z)\hat{\boldsymbol{\phi}}$, once a set of initial conditions are given. One constant of the motion, Størmer's integral (7-4), is used to produce equation (8-3), while the remaining constant of the motion, the energy H as given in (7-5), can be again used to reduce the number of independent equations from five to four. These equations again define chaotic motion, in which no analytical solution is generally possible if $\mathbf{A}$ is non-trivial, leading to the necessity of numerical solution.

<u>Qualitative solutions</u>

Once again, a *qualitative solution* is possible, *i.e.*, one that allows a mapping of the regions accessible and inaccessible to particle motion. At large distances from the assembly of current loops, the magnetic field and vector potential can be represented in terms of multipoles and the discussion presented in Section 5 is again pertinent. At closer distances, the vector potential appropriate to a set of current loops is required (and will be provided in the next section).

As before, v^2 is constant. In circular cylindrical coordinates, v^2 has the form

$$(8\text{-}4) \qquad v^2 = v_r^2 + v_\phi^2 + v_z^2 = \dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2.$$

Dividing the expression in (8-4) by v^2 and using (8-3b), along with $ds = vdt$ (ds is again an element of the path length along the particle's trajectory), gives

$$(8\text{-}5) \qquad \left(\frac{d\rho}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2 + h^2 = 1.$$

As with (5-2), this can be thought of as the total energy equation for a surrogate particle of mass 2. Here, the two-dimensional pseudo-potential h^2 is a function of ρ and z.

We have already seen that if v^2 is constant, so is $p = |\mathbf{p}|$. Using $pc^2 = vH$, along with equations (8-3b), we see that the function h can be expressed in cylindrical coordinates as

$$(8\text{-}6) \qquad h = \rho\frac{d\phi}{ds} = \frac{P_\phi}{p\rho} - \frac{eA(\rho, z)}{pc}.$$

As before, $0 \le h^2 \le 1$ during a particle's motion, so that the motion of a charged particle in an axisymmetric magnetic field is restricted to those areas of the ρ-z plane for which $-1 \le h \le 1$

(these are regions where particle motion is allowed). A particle is forbidden to travel in regions in which $h^2 > 1$, while $h^2 = 1$ defines the boundary between the two regions. As in the case of spherical polar coordinates, this will enable us to perform a qualitative global analysis of charged particle motion once a functional form for $A(\rho, z)$ is chosen.

9. Global structure of the pseudo-potential

The pseudo-potential h^2 will be expressed either in spherical polar or cylindrical coordinates, where h is given by (5-6) for spherical polar and by (8.6) for cylindrical coordinates. Since the pseudo-potential is the square of h, some general comments can be made on its global structure. Let ($q_1 = r$, $q_2 = \theta$ or $q_1 = \rho$, $q_2 = z$); then the extrema of h^2 are found by requiring

$$(9\text{-}1) \qquad \frac{\partial h^2}{\partial q_i} = 2h\frac{\partial h}{\partial q_i} = 0, \qquad i = 1,2.$$

Obviously, $h = 0$ is an absolute minimum of h^2. There is also the case where $\partial h/\partial q_i = 0$.

The nature of the extremum is found by considering the second derivatives of h^2: (again, we use $q_1 = r$, $q_2 = \theta$ or $q_1 = \rho$, $q_2 = z$)

$$(9\text{-}2) \qquad \frac{\partial^2 h^2}{\partial q_i \partial q_j} = 2\left(h\frac{\partial^2 h}{\partial q_i \partial q_j} + \frac{\partial h}{\partial q_i}\frac{\partial h}{\partial q_j}\right), \qquad i = 1,2;\ j = 1,2.$$

If h^2 is expanded in a Taylor series about the point $q_i = q_{i,o} + dq_i$ and (9-1) holds, we have, to second order,

$$(9\text{-}3) \qquad h^2 - h_o^2 = \left.\frac{\partial^2 h^2}{\partial q_1^2}\right|_o dq_1^2 + 2\left.\frac{\partial^2 h^2}{\partial q_1 \partial q_2}\right|_o dq_1 dq_2 + \left.\frac{\partial^2 h^2}{\partial q_2^2}\right|_o dq_2^2.$$

If $h_o = 0$, then (9-3) becomes

$$(9\text{-}4) \qquad h^2 = 2\left(\left.\frac{\partial h}{\partial q_1}\right|_o dq_1 + \left.\frac{\partial h}{\partial q_2}\right|_o dq_2\right)^2$$

Using (9-4), we see that the equation of the line along which $h = 0$ is

$$(9\text{-}5) \qquad \frac{dq_1}{dq_2} = -\left.\frac{\partial h}{\partial q_2}\right|_o \left(\left.\frac{\partial h}{\partial q_1}\right|_o\right)^{-1}.$$

A low energy particle will be confined close to this curvilinear minimum of the pseudo-potential. This is the so-called *guiding center* of the particle's motion; more detail will be provided presently for the point multipole and current loop cases.

Using (9-1), it follows that in the case when $\partial h / \partial q_i = 0$, the *discriminant* D of the quadratic form on the right side of (9-3) is,

$$\text{(9-6)} \qquad D = 16 h_{\text{o}}^2 \left[\left(\left. \frac{\partial^2 h}{\partial q_1 \partial q_2} \right|_{\text{o}} \right)^2 - \left(\left. \frac{\partial^2 h}{\partial^2 q_1} \right|_{\text{o}} \right) \left(\left. \frac{\partial^2 h}{\partial^2 q_2} \right|_{\text{o}} \right) \right].$$

If $D > 0$, the point at which $\partial h / \partial q_i = 0$ is a saddle-point, since the quadratic form on the right side of (9-3) will be divided into regions of positive and negative values. On the other hand $D < 0$ signifies a local maximum or minimum, and $D = 0$ denotes a level line, similar to (9-4).

<u>Global structure – point multipoles</u>

Combining equation (8-6) with (6-19) yields h for the point multipole of degree n:

$$\text{(9-7)} \qquad h = \frac{P_\phi}{pr \sin\theta} + \frac{e}{pc} \frac{a_n}{n r^{n+1}} P_n^1(\cos\theta).$$

We now define the *Størmer radius* r_S and *constant* γ, and the dimensionless variable R:

$$\text{(9-8)} \qquad \text{a) } r_S^{n+1} = \frac{e a_n}{npc} \qquad \text{b) } \gamma = \frac{P_\phi}{(n+1) p r_S} \qquad \text{c) } R = \frac{r}{r_S}.$$

Using these (9-7) can be written in a clearly non-dimensional form:

$$\text{(9-9)} \qquad h = \frac{(n+1)\gamma}{R \sin\theta} + \frac{P_n^1(\cos\theta)}{R^{n+1}}.$$

Thus, the topology of the pseudo-potential h^2 is controlled by the dimensionless parameter γ.

The absolute minimum of h^2 occurs when $h = 0$. Although h 0 as R , using (9 -9) gives the relation between *finite* R and θ for an absolute minimum:

$$\text{(9-10)} \qquad \text{a) } (n+1)\gamma R_{\text{o}}^n = -\sin\theta \, P_n^1(\cos\theta) \qquad \text{b) } \gamma P_n^1(\cos\theta) < 0.$$

Let us define $L_{\text{o}}^{-n} = (n+1)|\gamma|$ and assume that (9-10b) holds; then (9-10a) takes the form

$$\text{(9-11)} \qquad R_{\text{o}}^n = L_{\text{o}}^n \sin\theta \left| P_n^1(\cos\theta) \right|$$

Using the table in Appendix A, the case $n = 1$ becomes $R_0 = L_0\sin^2\theta$. As is well known, this is the equation for a field line in a dipole magnetic field – the *guiding center* of a particle's motion in a dipole field. Only in the case $n = 1$ is it possible that (9-10b) holds (with $\gamma > 0$) over the entire range of angles: $0 < \theta < \pi$. For those cases with $n > 1$, there are intervals between 0 and π in which θ takes values such that the right side of (9-10b) does not hold for either $\gamma > 0$ or $\gamma < 0$; in these intervals, no absolute minimum exists for finite values of R.

Using (9-9) and (6-17), the derivatives of h are readily found to be

(9-12) $$\frac{\partial h}{\partial R} = -\frac{n+1}{R^2}\left[\frac{\gamma}{\sin\theta} + \frac{P_n^1(\cos\theta)}{R^n}\right]$$

(9-13) $$\frac{\partial h}{\partial \theta} = -\frac{(n+1)\gamma\cot\theta}{R\sin\theta} + \frac{1}{R^{n+1}}\left[P_n^2(\cos\theta) + \cot\theta P_n^1(\cos\theta)\right].$$

At extrema, the conditions $\partial h/\partial R = \partial h/\partial\theta = 0$ hold; these conditions define the coordinates R_p and θ_p of the extremal points. Using equations (9-12), (9-13), and (6-18), we have

(9-14) $$R_p^n = -\frac{\sin\theta_p P_n^1(\cos\theta_p)}{\gamma}$$

(9-15) $$\cos\theta_p P_n^1(\cos\theta_p) - (n+1)\sin\theta_p P_n(\cos\theta_p) = P_{n+1}^1(\cos\theta_p) = 0\ .$$

[In (9-15), #8.733-3 of Ref. 12 has been used.] Solving these equations yields R_p and θ_p; since (9-15) contains a factor that is an n^{th} order polynomial in $\cos\theta_p$, there will be, in general, n different sets of real values for R_p and θ_p with $0 < \theta_p < \pi$. If n is odd, $\theta_p = \pi/2$ is a solution; the other $n - 1$ solutions come in $(n - 1)/2$ pairs $\theta_p = \alpha_p$ and $\theta_p = \pi - \alpha_p$, with $0 < \alpha_p < \pi/2$. If n is even, then there are $n/2$ pairs α_p and $\pi - \alpha_p$, again with $0 < \alpha_p < \pi/2$. For $n > 1$, different values of θ_p will give the right side of (9-14) a positive value or negative value, so that for a fixed sign of γ, not all solutions of equation (9-15) lead to the location of an extremum, but only those for which the right side of (9-14) is positive.

Placing (9-14) into (9-9) yields

(9-16) $$h_p = \frac{n\gamma}{R_p\sin\theta_p}.$$

At this point, we can take the second derivatives of h with respect to R and θ, and evaluate them at R_p and θ_p. Differentiating (9-3) and (9-13) results in

$$(9\text{-}17)\qquad \left.\frac{\partial^2 h}{\partial R^2}\right|_p = -\frac{n(n+1)\gamma}{R_p^3 \sin\theta_p}$$

$$(9\text{-}18)\qquad \left.\frac{\partial^2 h}{\partial \theta^2}\right|_p = \frac{\gamma n}{R_p \sin^3\theta_p}\left[1+(n+1)\sin^2\theta_p\right]$$

$$(9\text{-}19)\qquad \left.\frac{\partial^2 h}{\partial R\partial\theta}\right|_p = -\frac{n(n+1)\gamma\cot\theta_p}{R_p^2 \sin\theta_p}$$

It is evident that the 2nd derivatives in (9-17) and (9-18) are negative definite and positive definite, respectively, if $\gamma > 0$, and positive definite and negative definite, respectively, if $\gamma < 0$, and both zero if $\gamma = 0$. The discriminant D, as given by (9-6), is thus positive if $|\gamma| > 0$, and it has value $D = 0$ only if $\gamma = 0$. Using (9-16), (9-17), (9-18), and (9-19), along with (9-6), gives

$$(9\text{-}20)\qquad D = 16\frac{h_p^4}{n^2\gamma^2}(n+1)(n+2)\,.$$

Since $D > 0$ for $|\gamma| > 0$, R_p and θ_p are the locations of saddle points, if the right side of (9-14) is positive.

Setting $n = 1, \ldots, 5$ in (9-15) and using the $P_{n+1}^1(\cos\theta_p)$ from Appendix A, leads to the results in Table I (for $-1 < \cos\theta_p < 1$):

Table I. Values of $\cos\theta_p$ of possible saddle points for $n = 1$ to 5.

$n = 1$	$\cos\theta_p = 0$
$n = 2$	$\cos\theta_p = \pm\sqrt{5}$
$n = 3$	$\cos\theta_p = 0,\ \pm\sqrt{3/7}$
$n = 4$	$\cos\theta_p = \pm\frac{1}{\sqrt{3}}\sqrt{1\pm\frac{2}{\sqrt{7}}}$
$n = 5$	$\cos\theta_p = 0,\ \pm\sqrt{\frac{5\pm2\sqrt{5/3}}{11}}$

In those cases that have $n > 5$, equations higher than quadratic must be solved. For $n = 6$ and 7, the solution of a cubic equation is needed, and for $n = 8$ and 9, a quartic equation must be solved (although these will lead to closed form solutions). For $n > 9$, a numerical solution is generally required. Here, we stop at $n = 5$, as this is sufficient for our purpose.

In order that a particle be able to actually pass thru a saddle point, we must have the pseudo-potential at the saddle point satisfy $|h_p| < 1$. Setting $|h_p| = 1$ in (9-16), using (9-14) and the values of θ_p for $n = 1$ to 5 in Table I, allows the critical value γ_c to be determined:

(9-21) a) $$|\gamma_c| = \sin\theta_p \left|P_n^1(\cos\theta_p)\right|^{\frac{1}{n+1}} n^{-\frac{n}{n+1}}$$ b) $$\gamma_c P_n^1(\cos\theta_p) < 0 .$$

Once γ_c is found, equation (9-14) can be used to find R_p. Using this procedure for $n = 1$ to 5, gives the results in Table II. The meaning of γ_c is the following: If $|\gamma|$ $|\gamma_c|$, the particle can no longer pass through the associated saddle point.

Table II. Saddle point positions and critical values γ_c.

n	R_p	θ_p (degrees)	γ_c
1	1.0000	90.0000	1.0000
2	1.3389	63.4350	0.5988
	1.3389	116.5650	-0.5988
3	1.4042	49.1066	0.3538
	1.4565	90.0000	-0.4855
	1.4042	130.8934	0.3538
4	1.4014	40.0881	0.2256
	1.4605	73.4273	-0.3410
	1.4605	106.5727	0.3410
	1.4014	139.9119	-0.2256
5	1.3810	33.8783	0.1540
	1.4369	62.0404	-0.2538
	1.4521	90.0000	0.2904
	1.4369	117.9596	-0.2538
	1.3810	146.1217	0.1540

Notice that if equation (9-10a), with $\theta = \theta_p$, is divided by (9-14), the result is

(9-22) $$\frac{R_o^n}{R_p^n} = \frac{1}{n+1} .$$

Thus, if a saddle point exists, then a point of absolute minimum ($h = 0$) always exists between the saddle point and the origin. Furthermore, $R_o = 0$ when $P_n^1(\cos\theta) = 0$ and for $n > 1$, the values of θ where this occurs ($\theta = \theta_{o,n}$) are given by $\theta_{o,n} = \theta_{p,(n-1)}$, where $\theta_{p,(n-1)}$ signifies the θ_p for $n - 1$ in Table II. Since h^2 as R 0 unless $R = R_o$, and since the only saddle points of h^2 appear at the (R_p,θ_p) in Table II, for each n, then there is no way for a particle to get from the minimum associated with one value of θ_p to another, unless it first go out through one saddle point and then in through another.

To further illustrate these ideas and as an example of the use of Table II, consider $n = 3$. If $\gamma > 0$, saddle points exist at $\theta_p = 49°$ and $131°$. If $0 < \gamma < 0.3538$, the particle can pass through the saddle point at $\theta_p = 49°$ from a region where $R < R_p$ to the unbounded region where $R > R_p$; it could then enter through saddle point at $\theta_p = 131°$ into another region where $R < R_p$. However, if $\gamma > 0.3538$ and $R < R_p$, the particle cannot pass through either saddle point and is *trapped*. (Remember that the particle can only move in those regions where $h^2 < 1$, so that these are *allowed* regions, while those for which h^2 1 are *forbidden* regions. Also, once a particle leaves a saddle point, it tends to get farther and farther away from the point multipole.)

In order to plot h^2 we choose cylindrical coordinates; the conversion of h, given by (9-9), from spherical polar to cylindrical coordinates is given in Appendix B. Using these results, leads to Figures 1, 2, 3, 4, and 5, which illustrate allowed and forbidden regions for dipole, quadrupole, octupole fields, hecadecupole, and fifth-order pole for various values of γ. The values of γ chosen for Figure 1 correspond to the values used by Størmer[1] (in his Fig. 123). For an object in space possessing a magnetic field well approximated by a point multipole, particles in the local space environment can be trapped if the associated value of γ is outside the interval (γ_c^-, γ_c^+), where γ_c^- is the largest negative γ and γ_c^+ is the smallest positive γ for each n in Table II.

Actual trajectories can also be numerically produced and compared to the global solutions shown in Figures 1 through 5. We again choose to use cylindrical coordinates, since these are easier to plot. We take equations (8-1) to (8-3) and use (9-8) and (C-1) to turn these equations into the dimensionless form [where $s = vt$ and $R = (z^2 + \rho^2)^{1/2}$]:

(9-23) $$\frac{d^2\rho}{ds^2} = h_n\left[\frac{(n+1)\gamma}{\rho^2} - \frac{\partial}{\partial\rho}\left(\frac{P_n^1(zR^{-1})}{R^{n+1}}\right)\right]$$

(9-24) $$\frac{d^2 z}{ds^2} = -h_n\frac{\partial}{\partial z}\left(\frac{P_n^1(zR^{-1})}{R^{n+1}}\right)$$

(9-25) $$\frac{d\phi}{ds} = \frac{h_n}{\rho}.$$

The explicit form of the functions h_n is given in the table in Appendix B.

In the case of a dipole field, $n = 1$, and equations (9-23) to (9-25) take the form

(9-26) $$\frac{d^2\rho}{ds^2} = h_1\left[\frac{2\gamma}{\rho^2} - \frac{\left(2\rho^2 - z^2\right)}{R^5}\right]$$

(9-27) $$\frac{d^2\rho}{ds^2} = -h_1 \frac{3\rho z}{R^5}$$

(9-28) $$\frac{d\phi}{ds} = \frac{h_1}{\rho} = \frac{2\gamma}{\rho^2} - \frac{1}{R^3}.$$

These can be integrated to produce particle orbits for any value of γ. Here, this was done with a 3rd Adams-Bashforth method[13]. The results of doing so are shown in Figure 6 for $\gamma = 1.016$, and in Figure 7 for $\gamma = 0.999$. Allowed and forbidden regions are shown in Figure 1 for $\gamma = 1.016$, and in Figure 8 for $\gamma = 0.999$. In the case where $\gamma = 1.016$, there are two disjoint allowed regions, so that a particle may be in a trapped orbit, while in the case $\gamma = 0.999$, there is only one connected allowed region.

10. Conclusion

In this work, a number of novel developments have been presented. First, the magnetic vector potential equivalent to a magnetic scalar potential was derived. This allowed the use of Lagrangian dynamical methods, which require a magnetic vector potential in their formulation. Next, the global structure the allowed and forbidden regions of motion for an arbitrary axisymmetric multipole of order n was found. Explicit results were found up to order $n = 5$, while higher order results can be found by applying our general theory to $n > 5$, as desired. Finally, the topologies of the allowed and forbidden regions for n up to 5 were shown in graphical form.

These results may be of use in describing the radiation belt structure of astrophysical objects that have insignificant magnetic dipole fields but instead have some higher order axisymmetric multipole structure. In the design of plasma propulsion systems that utilize magnetically confined plasma, certain design requirements may constrain the net magnetic field to be an axisymmetric quadrupole (or higher order) field, rather than a dipole field (This constraint may arise due to the unacceptable levels of torque that occurs when a magnetic dipole is placed in a planetary magnetic field).

In the future, we plan to extend the analysis presented here for axisymmetric multipoles to the case of axisymmetric collections of current loops. This will allow a mapping of allowed and forbidden regions of motion in regions interior to a plasma propulsion device, for example. The magnetic field associated with a current loop is mathematically described in terms of elliptical integrals and functions derived from them, and the associated analysis is more difficult than the analysis associated with spherical harmonics. However, analysis of the motion interior to a set of current loops is an important one and the plan is to treat this problem as priorities, resources, and time allow.

Appendix A

Table A. Associated Legendre Functions

Some associated Legendre functions $P_n^m(x)$, with $x = \cos\theta$ and $s = \sin\theta = (1 - x^2)^{1/2}$, are

n	m	$P_n^m(x)$
0	0	1
1	0	x
1	1	$-s$
2	0	$(3x^2 - 1)/2$
2	1	$-3xs$
2	2	$3s^2$
3	0	$x(5x^2 - 3)/2$
3	1	$-3s(5x^2 - 1)/2$
3	2	$15xs^2$
4	0	$(35x^4 - 30x^2 + 3)/8$
4	1	$-5xs(7x^2 - 3)/2$
4	2	$15s^2(7x^2 - 1)/2$
5	0	$x(63x^4 - 70x^2 + 15)/8$
5	1	$-15s(21x^4 - 14x^2 + 1)/8$
6	0	$(231x^6 - 315x^4 + 105x^2 - 5)/16$
6	1	$-21xs(33x^4 - 30x^2 + 5)/8$

Appendix B

The function h of the multipole pseudo-potential h^2 in cylindrical coordinates

Equation (9-9): $$h = \frac{(n+1)\gamma}{R\sin\theta} + \frac{P_n^1(\cos\theta)}{R^{n+1}}$$

Coordinate transformation (z, ρ, R are measured in terms of r_s):

(B-1) a) $z = R\cos\theta$ b) $\rho = R\sin\theta$ c) $R = (z^2 + \rho^2)^{1/2}$

Table B. The multipole function $h = h_n$ in cylindrical coordinates

n	h_n
1	$\frac{2\gamma}{\rho} - \frac{\rho}{R^3}$
2	$\frac{3\gamma}{\rho} - \frac{3\rho z}{R^5}$
3	$\frac{4\gamma}{\rho} - \frac{3}{2}\frac{\rho\left(4z^2 - \rho^2\right)}{R^7}$
4	$\frac{5\gamma}{\rho} - \frac{5}{2}\frac{\rho z\left(4z^2 - 3\rho^2\right)}{R^9}$
5	$\frac{6\gamma}{\rho} - \frac{15}{8}\frac{\rho\left(8z^4 - 12\rho^2 z^2 + \rho^4\right)}{R^{11}}$

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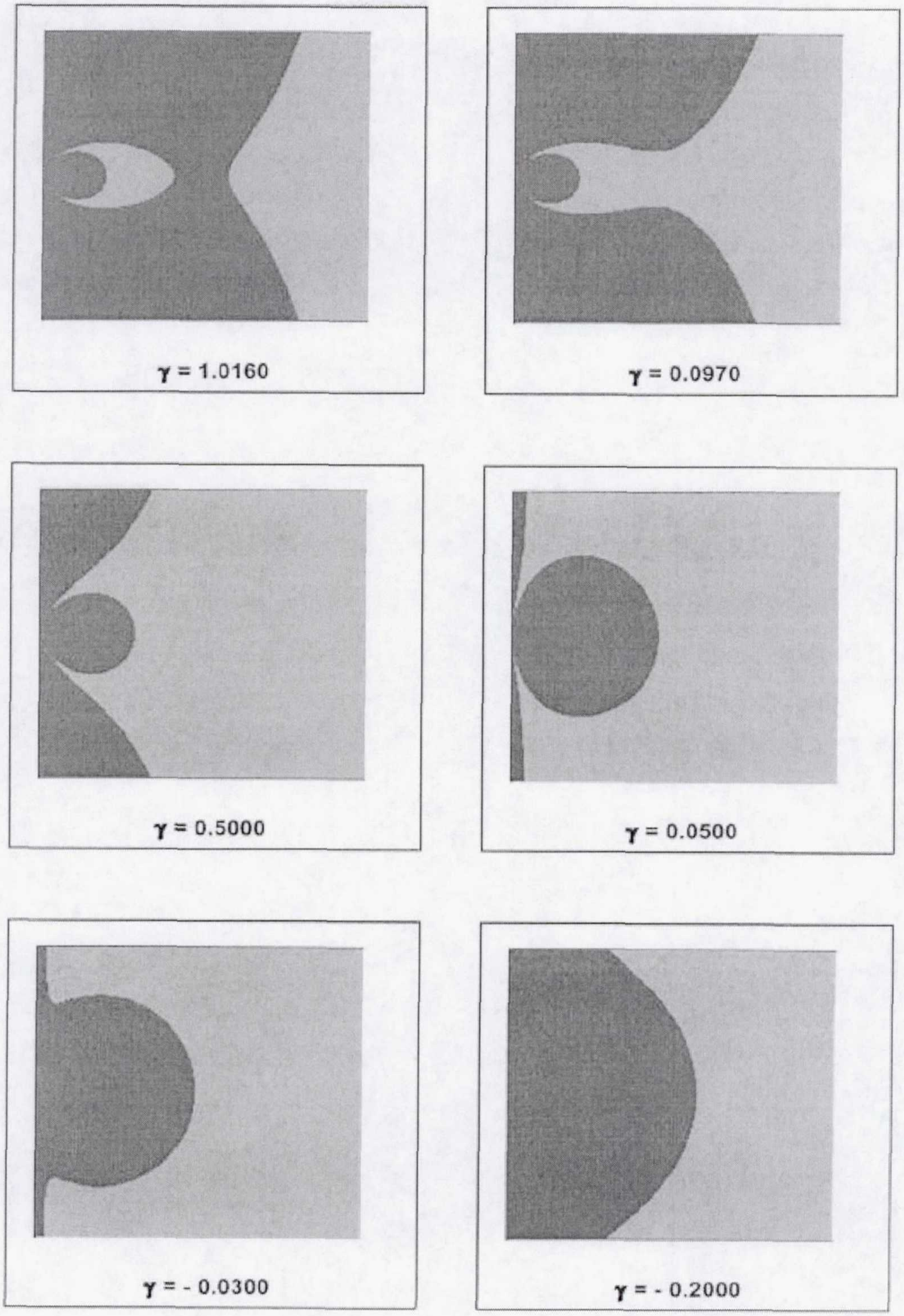

Figure 1. Allowed (white) and forbidden (gray) regions for an axisymmetric 2nd order pole (dipole); the polar axis is aligned with and centered on the left vertical edge of each figure.

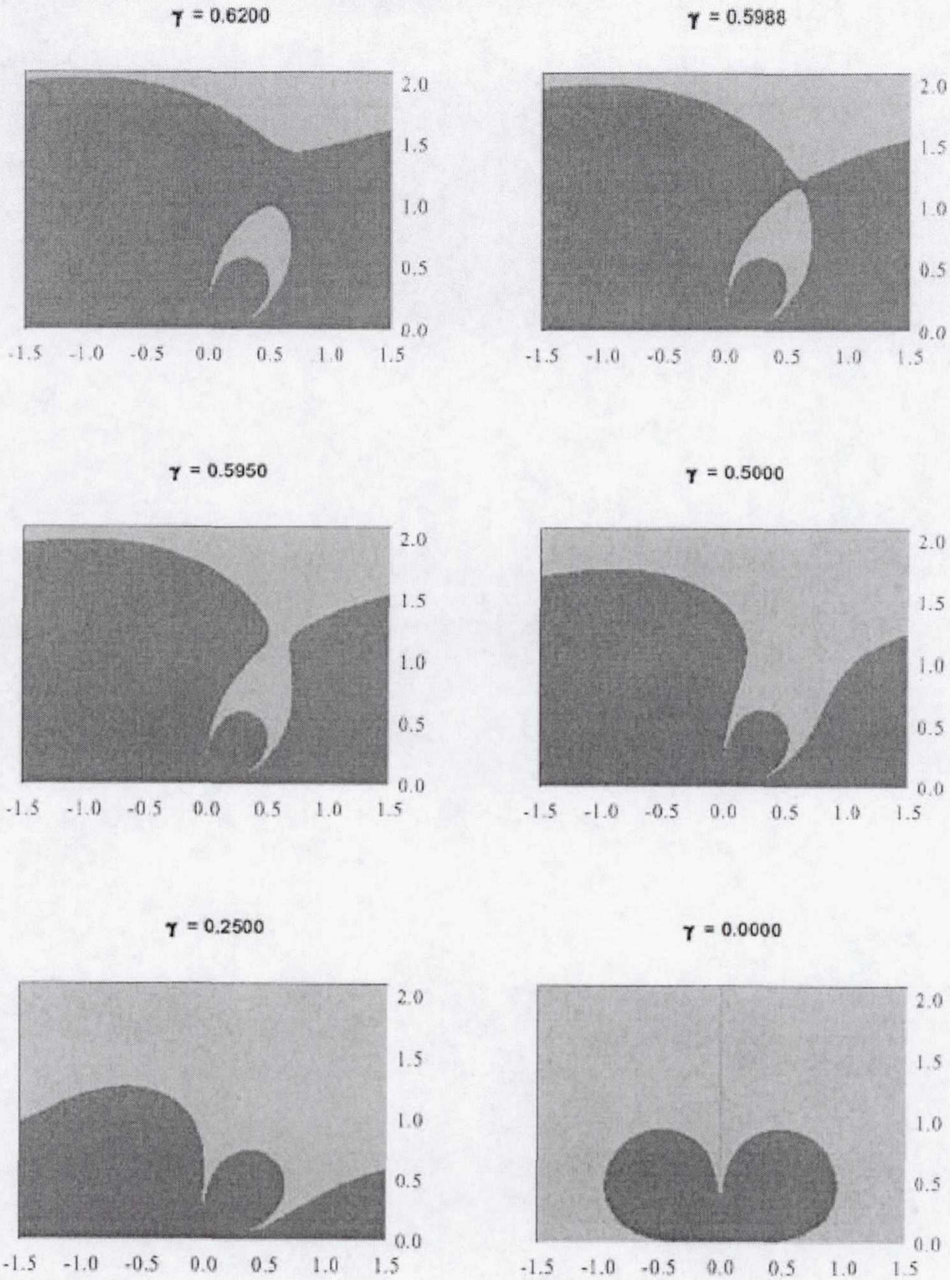

Figure 2. Allowed (light) and forbidden (dark) regions for an axisymmetric 2nd order pole (quadrupole); the polar axis is aligned with and centered on the lower horizontal edge of each figure. The vertical axis is ρ and horizontal one is z. For $-\gamma$, figures reflect about $z = 0$.

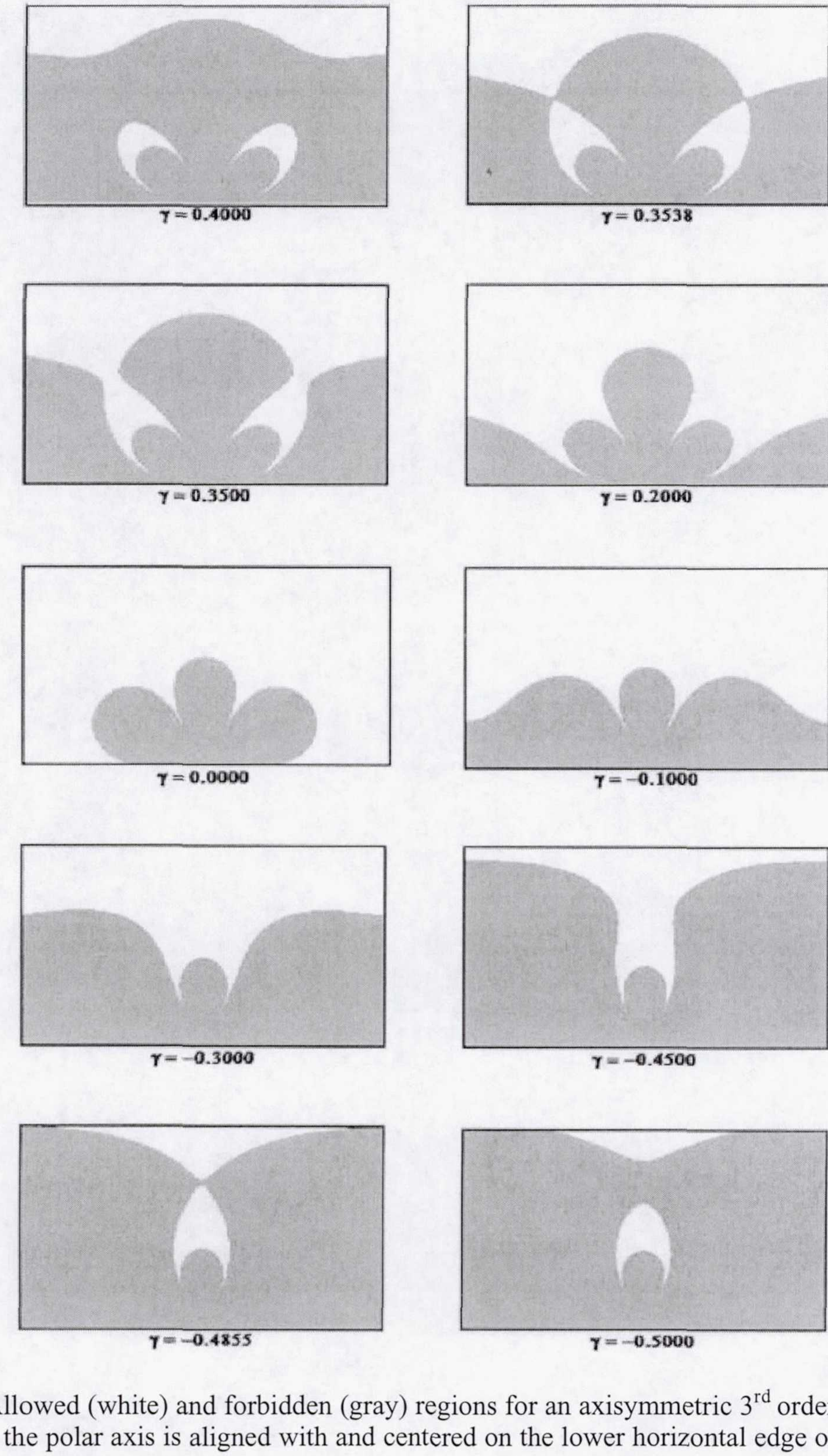

Figure 3. Allowed (white) and forbidden (gray) regions for an axisymmetric 3rd order pole (octupole); the polar axis is aligned with and centered on the lower horizontal edge of each figure.

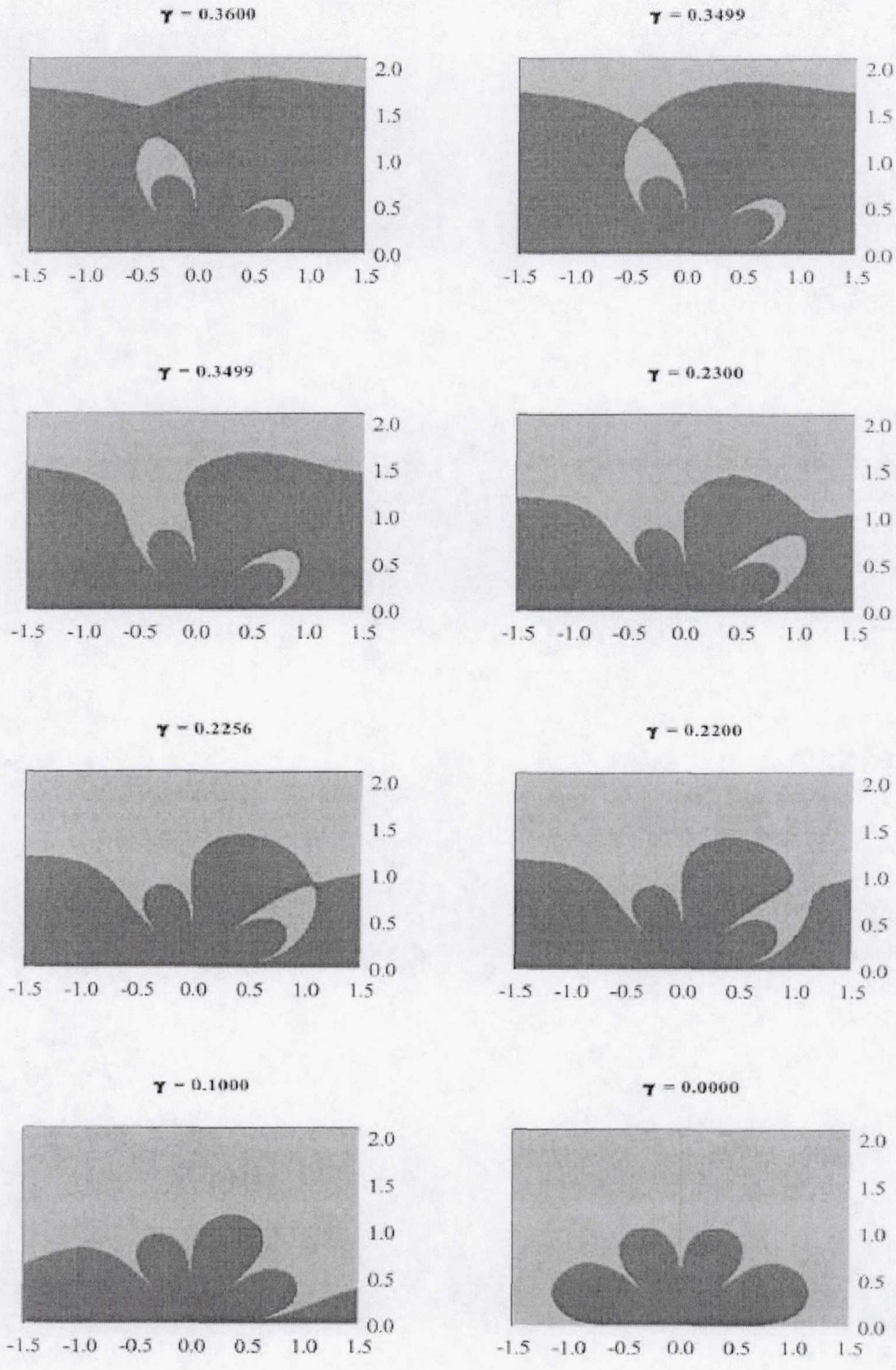

Figure 4. Allowed (light) and forbidden (dark) regions for axisymmetric 4th order pole (hexadecupole); polar axis is aligned with and centered on the lower horizontal edge of each figure. The vertical axis is ρ and horizontal one is *z*. For $-\gamma$, figures reflect about $z = 0$.

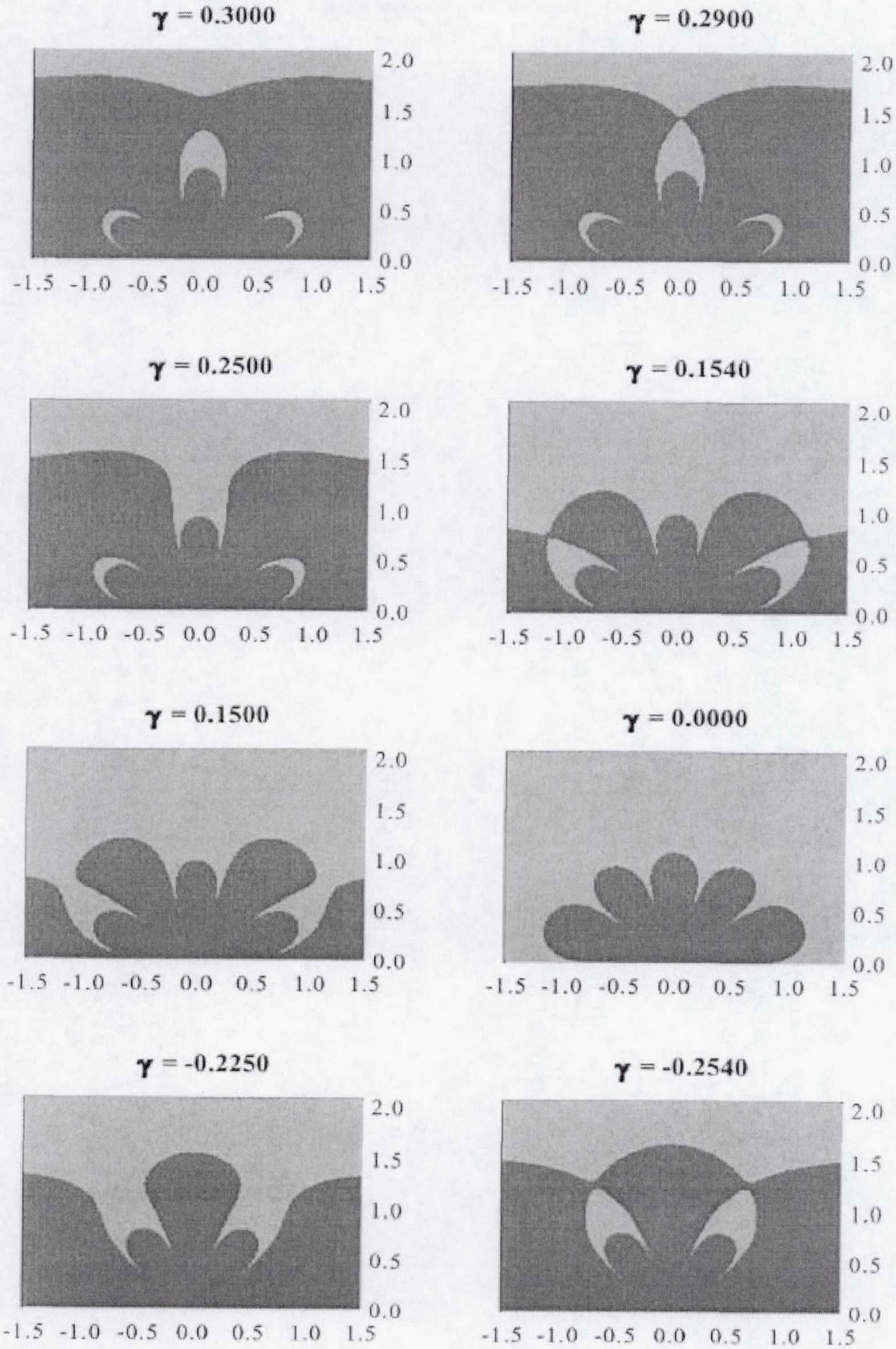

Figure 5. Allowed (light) and forbidden (dark) regions for axisymmetric 5th order pole; the polar axis is aligned with and centered on the lower horizontal edge of each figure.

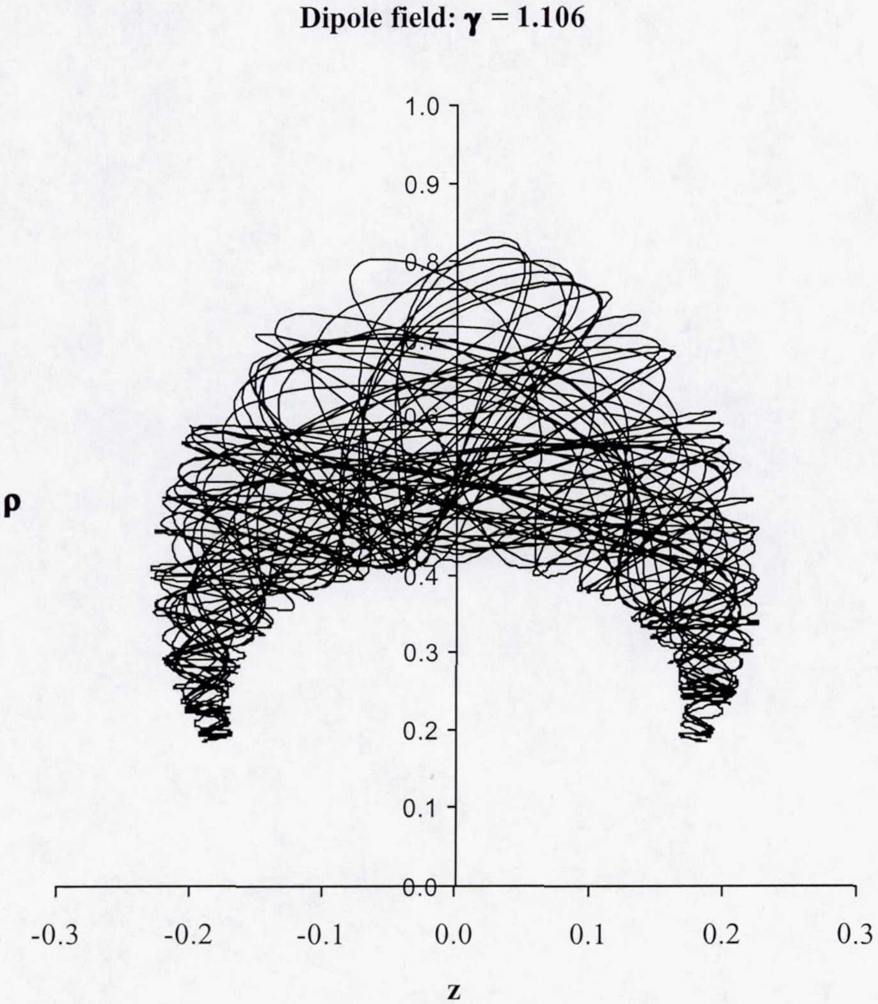

Figure 6. Particle motion in a dipole field for $\gamma = 1.016$.

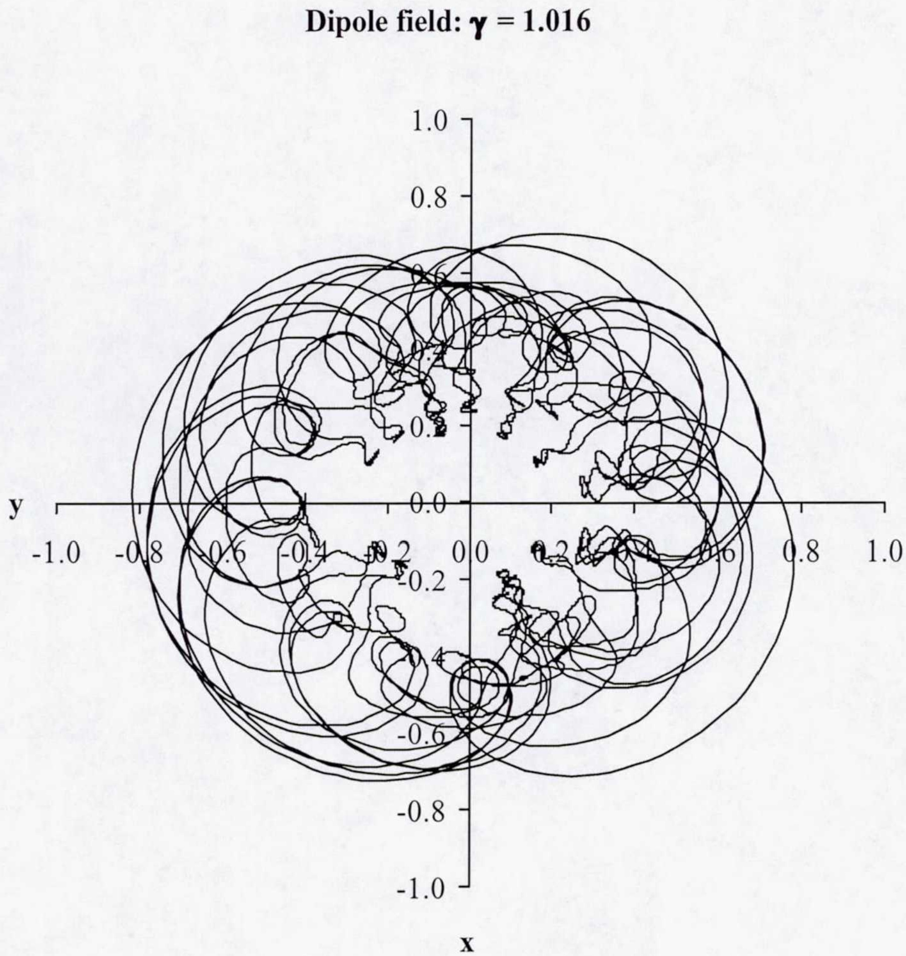

Figure 7. Particle motion in *x*-*y* plane for a dipole field with $\gamma = 1.016$.

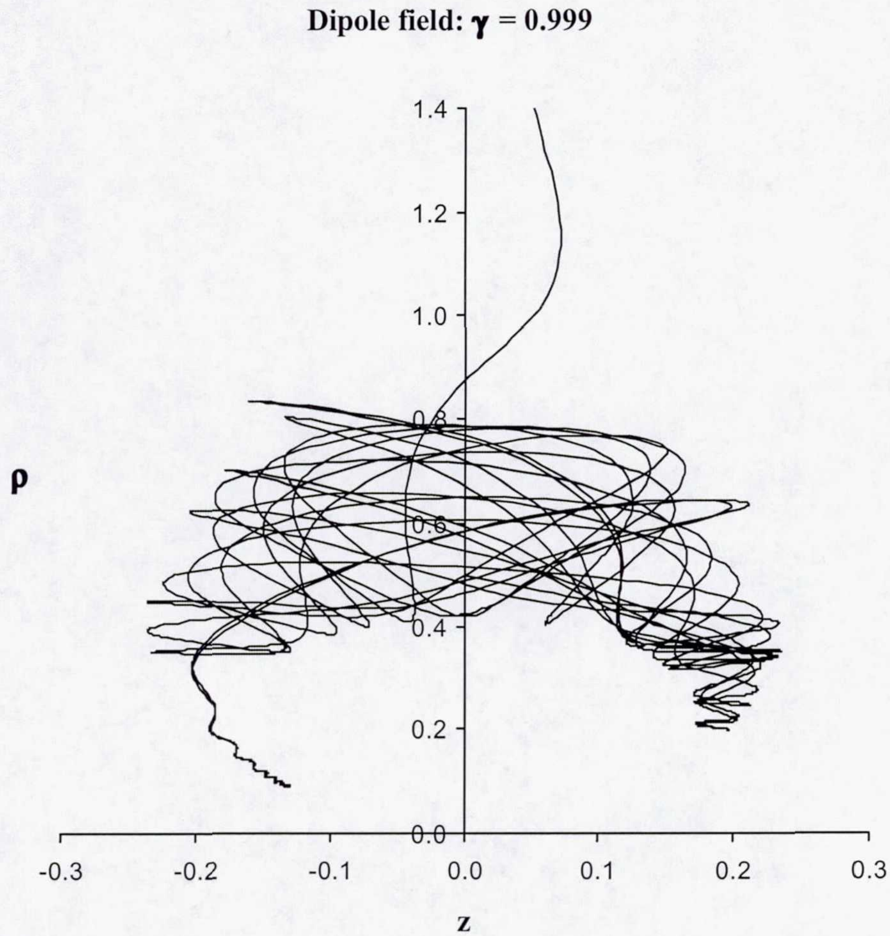

Figure 8. Particle motion in a dipole field for $\gamma = 0.999$.

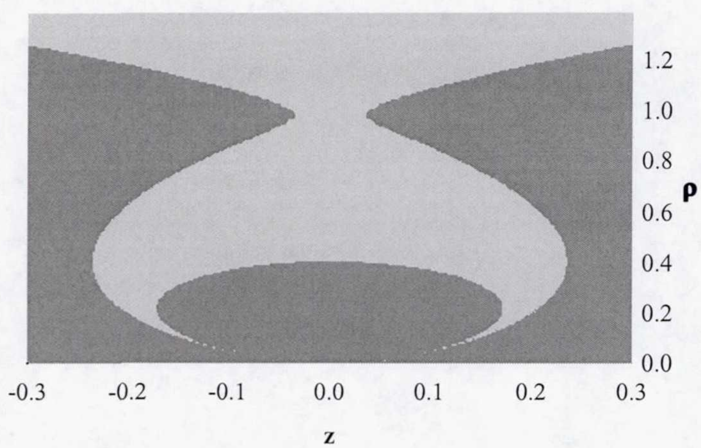

Figure 9. Dipole, allowed (light) and forbidden (dark) regions for $\gamma = 0.999$.