Error and Uncertainty Quantification in the Numerical Simulation of Complex Fluid Flows

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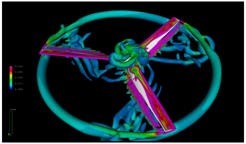
Time Dependent Flow Problems

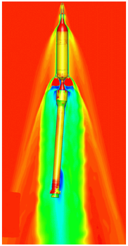
The growth in computer hardware performance and capacity has enabled large scale computations of complex physical models.

These calculations raise several questions:

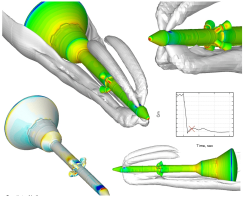
- How accurate is the simulation?
- Can predictions be trusted?
- Can differences between computation and experiment be rigorously reconciled?



Helicopter Aerodynamics[3]



Launch Vehicle Analysis[2]



Abort Systems Analysis[1]

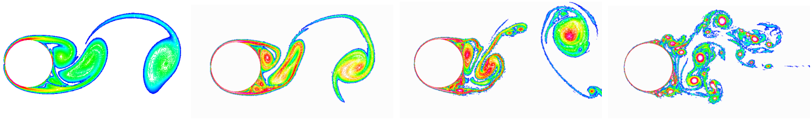
[1] POC: S. Rogers, [2] POC: G. Klopfer, [3] POC: N. Chaderjian (NASA)

Overview

- The main topic of discussion is error representation and error control of functional outputs via dual problems (Erickson *et. al.*,1995), (Becker and Rannacher, 1997).
- Particular attention is given to the time-dependent calculation of the compressible Navier-Stokes flow. Specifically, we examine the backwards-in-time dual problem and issues associated with
  - the deterioration (blowup) of dual problems with increasing Reynolds number,
  - the loss of sharpness in error bounds over long time integrations.
- In the remainder of the presentation, we briefly examine a novel uncertaintly quantification technique proposed by Estep and Neckels (2006) for the quantification of uncertain functional outputs given aleatoric (statistical) random variable inputs.
- Surprisingly, the dual problems in the Estep and Neckels technique are identical to those arising in *a posteriori* error estimation (!!) but now the dual problem is used to construct a piecewise linear approximation of the random variable response surface.

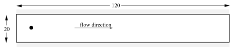
Motivating Computational Challenge #1: Cylinder Flow

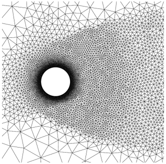
Cylinder flow at Mach = 0.10, logarithm of |vorticity| contours



Re=1000 **Re=3900** **Re=10000** **Re=50000**

- Quartic space-time elements
- 25K element mesh
- Viscous walls only imposed on cylinder surface
- Reynolds number based on cylinder diameter





Question: How is the ability to estimate and control numerical error effected by increasing Reynolds number?

Conservation law system in $\mathbf{R}^{d\times 1}$

$$\mathbf{u}_{,t} + \operatorname{div}\mathbf{f} = 0, \quad \mathbf{u}, \mathbf{f}_i \in \mathbf{R}^m \quad i = 1, \ldots, d$$

Convex entropy extension

$$U_{,t} + \operatorname{div} F \le 0, \quad U, F_i \in \mathbf{R}$$

Piecewise polynomial approximation space:

$$\mathcal{V}^h = \left\{ \mathbf{v}_h \mid \mathbf{v}_h|_{K \times I^n} \in \left(\mathcal{P}_k(K \times I^n) \right)^m \right\}$$

Find $\mathbf{v}_h \in \mathcal{V}^h$ such that for all $\mathbf{w}_h \in \mathcal{V}^h$

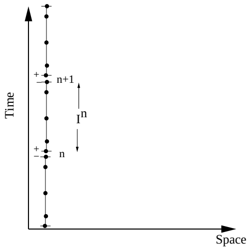
$$B(\mathbf{v}_h, \mathbf{w}_h)_{\mathrm{DG}} = \sum_{n=0}^{N-1} B^n(\mathbf{v}_h, \mathbf{w}_h)_{\mathrm{DG}} = 0 \ ,$$

$$\begin{aligned} B^n(\mathbf{v}, \mathbf{w})_{\mathrm{DG}} \quad &= \quad \int_{I^n} \sum_{K \in \mathcal{T}} \int_K -(\mathbf{u}(\mathbf{v}) \cdot \mathbf{w}_{,t} + \mathbf{f}^i(\mathbf{v}) \cdot \mathbf{w}_{,x_i})\, dx\, dt \\ &+ \quad \int_{I^n} \sum_{K \in \mathcal{T}} \int_{\partial K} \mathbf{w}(x_-) \cdot \mathbf{h}(\mathbf{v}(x_-), \mathbf{v}(x_+); \mathbf{n})\, ds\, dt \\ &+ \quad \int_\Omega \left(\mathbf{w}(t_-^{n+1}) \cdot \mathbf{u}(\mathbf{v}(t_-^{n+1})) - \mathbf{w}(t_+^n) \cdot \mathbf{u}(\mathbf{v}(t_-^n)) \right) dx \end{aligned}$$

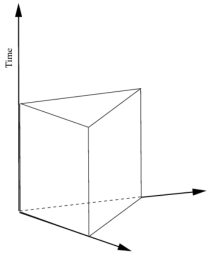
- Proposed by Reed and Hill (1973), LeSaint and Raviart (1974) and further developed for conservation laws by Cockburn and Shu (1990)
- $\mathbf{u}$ the conservation variables, $\mathbf{v}$ the symmetrization variables
- $\mathbf{h}$ a numerical flux function, $\mathbf{h}(\mathbf{v}_-, \mathbf{v}_+; \mathbf{n}) = -\mathbf{h}(\mathbf{v}_+, \mathbf{v}_-; -\mathbf{n})$, $\mathbf{h}(\mathbf{v}, \mathbf{v}; \mathbf{n}) = \mathbf{f}(\mathbf{v}) \cdot \mathbf{n}$

The Discontinuous in Time Approximation Space

- Natural setting for the discontinuous Galerkin (DG) method for hyperbolic problems
- Utilized in the space continuous Galerkin least-squares method (Hughes and Shakib, 1988)
- Often used in the discretization of parabolic problems (Douglas and Dupont, 1976)
- Requires solving the implicit slab equations.



Discontinuous timeslab intervals



Space-time prism element

Theorem E: Global space-time entropy inequality (Cauchy IVP):

$$\int_\Omega U(\mathbf{u}^*(t_-^0))\,dx \le \int_\Omega U(\mathbf{u}(\mathbf{v}_h(x,t_-^N)))\,dx \le \int_\Omega U(\mathbf{u}(\mathbf{v}_h(x,t_-^0)))\,dx$$

$$\mathbf{u}^*(t_-^0) = \frac{1}{\mathrm{meas}(\Omega)} \int_\Omega \mathbf{u}(\mathbf{v}_h(x,t_-^0))\,dx$$

whenever the numerical flux satisfies the system extension of Osher's famous "E-flux" condition

$$\boxed{[\mathbf{v}]_{x_-}^{x^+} \cdot (\mathbf{h}(\mathbf{v}_-,\mathbf{v}_+;\mathbf{n}) - \mathbf{f}(\mathbf{v}(\theta))\cdot\mathbf{n}) \le 0 \;\;,\quad \forall\theta\in[0,1] \;\;, \mathbf{v}(\theta) = \mathbf{v}_- + \theta[\mathbf{v}]_-^+}$$

- Several flux functions satisfy this technical condition when recast in entropy variables, e.g. Lax-Friedrichs, HLLE, Roe with modifications, etc.

Suppose $\mathbf{u}_{,\mathbf{v}}$ remains bounded in the sense

$$0 < c_0 \le \frac{\mathbf{z} \cdot \mathbf{u}_{,\mathbf{v}}(\mathbf{v}_h(x,t))\,\mathbf{z}}{\|\mathbf{z}\|^2} \le C_0 \quad , \qquad \forall \mathbf{z} \neq 0$$

and Theorem E is satisfied for the Cauchy IVP, then following L_2 stability result is readily obtained

L_2 Stability:

$$\|\mathbf{u}(\mathbf{v}_h(\cdot, t_-^N) - \mathbf{u}^*(t_-^0)\|_{L_2(\Omega)} \le (c_0^{-1} C_0)^{1/2} \, \|\mathbf{u}(\mathbf{v}_h(\cdot, t_-^0)) - \mathbf{u}^*(t_-^0)\|_{L_2(\Omega)} \quad .$$

Given a system of PDEs with exact solution $u \in \mathbf{R}^m$ in a domain Ω, the overall objective is to construct a locally adapted discretization with numerical solution u_h that achieves

- Norm control [Babuska and Miller, 1984]

$$\|\mathbf{u} - \mathbf{u}_h\| < \text{tolerance}$$

- Functional output control [Erickson *et. al.* (1995), Becker and Rannacher, 1997]

$$|J(\mathbf{u}) - J(\mathbf{u}_h)| < \text{tolerance} \quad , \quad J(\mathbf{u}) : \mathbf{R}^m \mapsto \mathbf{R}$$

Example functional outputs:

- Time-averaged lift force, drag force, pitching moments
- Average flux rates through specified surfaces
- Weighted-average functionals of the form

$$J_\Psi(\mathbf{u}) = \int_{T_0}^{T_1} \int_\Omega \Psi(x,t) \cdot N(\mathbf{u}) dx\, dt$$

for some user-specified weighting $\Psi(x,t)$ and nonlinear function $N(u)$

Assume $\mathcal{B}(\cdot,\cdot)$ bilinear and $J(\cdot)$ linear.

Primal Numerical Problem: Find $\mathbf{u}_h \in \mathcal{V}_h^{\mathrm{B}}$ such that

$$B(\mathbf{u}_h, \mathbf{w}) = F(\mathbf{w}) \quad \forall\, \mathbf{w} \in \mathcal{V}_h^{\mathrm{B}}.$$

Auxiliary Dual Problem: Find $\Phi \in \mathcal{V}^{\mathrm{B}}$ such that

$$B(\mathbf{w}, \Phi) = J(\mathbf{w}) \quad \forall\, \mathbf{w} \in \mathcal{V}^{\mathrm{B}}.$$

$$\begin{aligned}
J(\mathbf{u}) - J(\mathbf{u}_h) &= J(\mathbf{u} - \mathbf{u}_h) && \text{(linearity of } J\text{)}\\
&= B(\mathbf{u} - \mathbf{u}_h, \Phi) && \text{(dual problem)}\\
&= B(\mathbf{u} - \mathbf{u}_h, \Phi - \pi_h\Phi) && \text{(Galerkin orthogonality)}\\
&= B(\mathbf{u}, \Phi - \pi_h\Phi) - B(\mathbf{u}_h, \Phi - \pi_h\Phi) && \text{(linearity of } B\text{)}\\
&= F(\Phi - \pi_h\Phi) - B(\mathbf{u}_h, \Phi - \pi_h\Phi) && \text{(primal problem)}
\end{aligned}$$

Final error representation formula:

$$J(\mathbf{u}) - J(\mathbf{u}_h) = F(\Phi - \pi_h\Phi) - B(\mathbf{u}_h, \Phi - \pi_h\Phi)$$

Estimating $\Phi - \pi_h \Phi$:

Various techniques in use for estimating $\Phi - \pi_h \Phi$:

- Higher order solves [Becker and Rannacher, 1998],[B. and Larson, 1999], [Süli and Houston, 2002], [Houston and Hartman, 2002]
- Patch postprocessing techniques [Cockburn, Luskin, Shu, and S uli, 2003]
- Extrapolation from coarse grids

Mean-value linearized forms:

$$\begin{aligned} \mathcal{B}(\mathbf{u},\mathbf{v}) &= \mathcal{B}(\mathbf{u}_h,\mathbf{v}) + \overline{\mathcal{B}}(\mathbf{u}-\mathbf{u}_h,\mathbf{v}) \quad \forall\, \mathbf{v} \in \mathcal{V}^{\mathrm{B}} \\ J(\mathbf{u}) &= J(\mathbf{u}_h) + \overline{J}(\mathbf{u}-\mathbf{u}_h), \end{aligned}$$

Example: $\mathcal{B}(u,v) = (L(u),v)$ with $L(u)$ differentiable

$$\begin{aligned} L(u_B) - L(u_A) &= \int_{u_A}^{u_B} dL = \int_{u_A}^{u_B} \frac{dL}{du} du \\ &= \int_0^1 \frac{dL}{du}(\tilde{u}(\theta))\, d\theta \cdot (u_B - u_A) = \overline{L}_{,u} \cdot (u_B - u_A) \end{aligned}$$

with $\tilde{u}(\theta) \equiv u_A + (u_B - u_A)\,\theta$.

$$\begin{aligned} \mathcal{B}(\mathbf{u},\mathbf{w}) &= \mathcal{B}(\mathbf{u}_h,\mathbf{w}) + (\overline{L}_{,\mathbf{u}} \cdot (\mathbf{u}-\mathbf{u}_h),\mathbf{w}) \\ &= \mathcal{B}(\mathbf{u}_h,\mathbf{w}) + \overline{\mathcal{B}}(\mathbf{u}-\mathbf{u}_h,\mathbf{w}) \quad \forall v \in \mathcal{V}^{\mathrm{B}} \end{aligned}$$

Semilinear form $\mathcal{B}(\cdot,\cdot)$ and nonlinear $J(\cdot)$.

Primal numerical problem: Find $\mathbf{u}_h \in \mathcal{V}_h^{\mathrm{B}}$ such that

$$\mathcal{B}(\mathbf{u}_h, \mathbf{w}) = F(\mathbf{w}) \quad \forall\, \mathbf{w} \in \mathcal{V}^{\mathrm{B}}.$$

Linearized auxiliary dual problem: Find $\Phi \in \mathcal{V}^{\mathrm{B}}$ such that

$$\overline{\mathcal{B}}(\mathbf{w}, \Phi) = \overline{J}(\mathbf{w}) \quad \forall\, \mathbf{w} \in \mathcal{V}^{\mathrm{B}}.$$

$$\begin{aligned}
J(\mathbf{u}) - J(\mathbf{u}_h) &= \overline{J}(\mathbf{u} - \mathbf{u}_h) && \text{(mean value } J\text{)}\\
&= \overline{\mathcal{B}}(\mathbf{u} - \mathbf{u}_h, \Phi) && \text{(dual problem)}\\
&= \overline{\mathcal{B}}(\mathbf{u} - \mathbf{u}_h, \Phi - \pi_h\Phi) && \text{(Galerkin orthogonality)}\\
&= \mathcal{B}(\mathbf{u}, \Phi - \pi_h\Phi) - \mathcal{B}(\mathbf{u}_h, \Phi - \pi_h\Phi) && \text{(mean value } \mathcal{B}\text{)}\\
&= F(\Phi - \pi_h\Phi) - \mathcal{B}(\mathbf{u}_h, \Phi - \pi_h\Phi), && \text{(primal problem)}
\end{aligned}$$

Final error representation formula:

$$J(\mathbf{u}) - J(\mathbf{u}_h) = F(\Phi - \pi_h\Phi) - \mathcal{B}(\mathbf{u}_h, \Phi - \pi_h\Phi)$$

Refinement Indicators

Space-time error representation formula

$$B_{\mathrm{DG}}(\mathbf{v}_h, w) - F_{\mathrm{DG}}(\Phi - \pi_h \Phi) = \sum_{n=0}^{N-1} \sum_{Q^n} B_{\mathrm{DG},Q^n}(\mathbf{v}_h, \Phi - \pi_h \Phi) - F_{\mathrm{DG},Q^n}(\Phi - \pi_h \Phi)$$

Stopping Criteria:

$$|J(\mathbf{u}) - J(\mathbf{u}_h)| = \left| \sum_{n=0}^{N-1} \sum_{Q^n} B_{\mathrm{DG},Q^n}(\mathbf{v}_h, \Phi - \pi_h \Phi) - F_{\mathrm{DG},Q^n}(\Phi - \pi_h \Phi) \right|$$

Refinement/Coarsening Indicator:

$$|J(\mathbf{u}) - J(\mathbf{u}_h)| \le \sum_{n=0}^{N-1} \sum_{Q^n} \underbrace{\left| B_{\mathrm{DG},Q^n}(\mathbf{v}_h, \Phi - \pi_h \Phi) - F_{\mathrm{DG},Q^n}(\Phi - \pi_h \Phi) \right|}_{\text{refinement indicator}, \eta_{Q^n}}$$

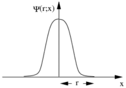
Fixed fraction mesh adaptation:

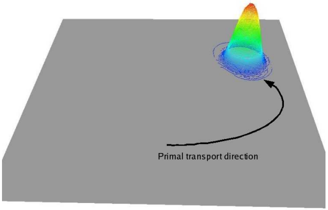
- Refine a fixed fraction of element indicators, η_{Q^n}, that are too large and coarsen a fixed fraction of element indications that are too small.

Example: A Scalar Time-Dependent PDE

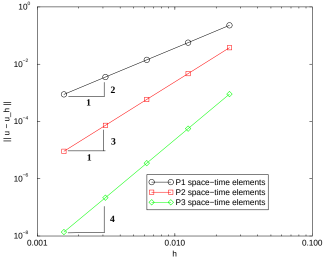
Circular transport, $\lambda = (y, -x)$, of bump data

$$u_t + \lambda \cdot \nabla u = 0 \ , \qquad x \in [-1,1]^2$$
$$u(x,0) = \Psi(1/10; x - x_0) \ , \qquad x_0 = (7/10, 0, 0)$$





Primal numerical problem

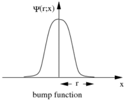


Convergence, $\|u - u_h\|_{L_2(\Omega \times [0,T])}$

Example: A Scalar Time-Dependent PDE

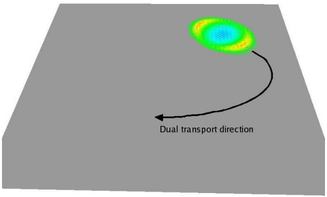
A functional is chosen that averages the solution data in the space-time ball of radius 1/10 located at $x_c = (1/2, 1/2, 1.05)$ in space-time

$$J(\mathbf{u}) = \int_0^{1.15} \int_\Omega \Psi(1/10; x - x_c)\, \mathbf{u}\, dxdt$$

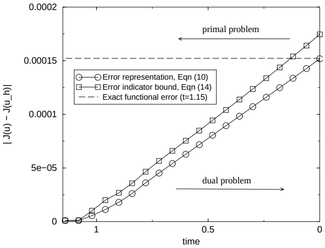


$$J(\mathbf{u}) - J(\mathbf{u}_h) = \sum_{n=N-1}^{0} \sum_K F_{\mathrm{DG},Q^n}(\Phi - \pi_h \Phi) - B_{\mathrm{DG},Q^n}(\mathbf{v}_h, \Phi - \pi_h \Phi)$$

$$|J(\mathbf{u}) - J(\mathbf{u}_h)| \le \sum_{n=N-1}^{0} \sum_K \left| F_{\mathrm{DG},Q^n}(\Phi - \pi_h \Phi) - B_{\mathrm{DG},Q^n}(\mathbf{v}_h, \Phi - \pi_h \Phi) \right|$$



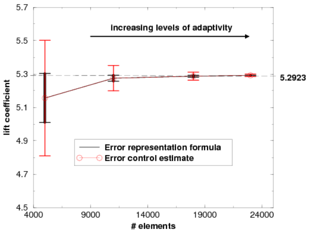
Dual defect, $\Phi - \pi\Phi$



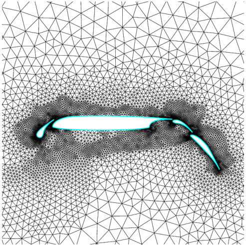
Error estimate buildup

Example: Euler flow past multi-element airfoil geometry. $M = .1,\ 5^\circ$ AOA.

lift coefficient (error representation)	lift coefficient (error control)	refinement level	# elements	equivalent uniform refinement # elements
$5.156 \pm .147$	$5.156 \pm .346$	0	5000	5000
$5.275 \pm .018$	$5.275 \pm .076$	1	11000	20000
$5.287 \pm .006$	$5.287 \pm .024$	2	18000	80000
$5.291 \pm .002$	$5.291 \pm .007$	3	27000	320000



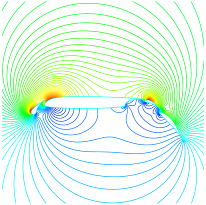
Error reduction during mesh adaptivity



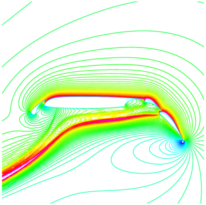
Adapted mesh (18000 elements)

Primal-Dual Problems in Fluid Mechanics

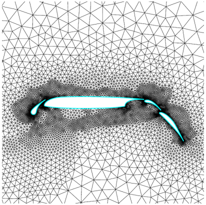
Subsonic Euler flow, $M = .10,\ 5^\circ$ AOA, Lift force functional.



Primal Mach number

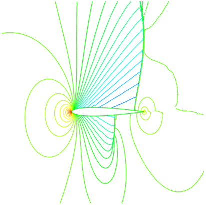


Dual x-momentum

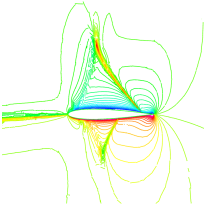


Adapted Mesh

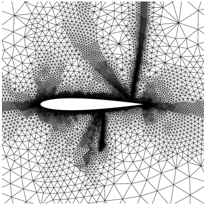
Transonic Euler flow, $M = .85,\ 2^\circ$ AOA, Lift force functional.



Primal density



Dual density



Adapted Mesh

Space-Time FEM:

- DG extension to the compressible Navier-Stokes equations using the symmetric interior penalty method of Douglas and Dupont, 1976) as described in Hartmann and Houston (2006)
- Implements the discontinuous Galerkin discretization in entropy variables.
- Unconditionally stable for all time step sizes
- Solves both the primal numerical problem and the jacobian linearized dual problem arising in space-time error estimation.
- High-order accuracy demonstrated in both space and space-time

Dual Problems for Time Dependent Problems

Computing dual (backwards in time) problems looks expensive in terms of both storage and computation

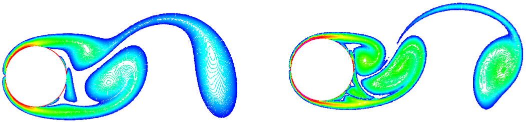
- Storage of the primal time slices for use in the locally linearized dual problem.
- Approximation of the infinite-dimensional dual problem for the backwards in time dual problem.

Tremendous simplification arising for periodic flow problems with period P when phase-independent functionals are utilized, e.g. mean drag

- Functional independent of the startup transient
- Only a small number of periods of the primal problem need be stored or recreated.

Periodic Cylinder Flow

Cylinder flow at Mach = 0.10, logarithm of |vorticity| contours



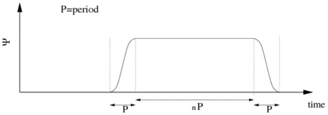
Re=300 **Re=1000**

Task: Represent and estimate the error in the mean drag force coefficient

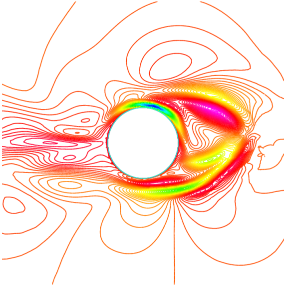
- Solve the primal problem using linear space-time elements
- Construct a smooth phase invariant functional measuring the mean drag force coefficient
- Solve the dual (backwards in time) problem using quadratic space-time elements
- Calculated the estimated functional error and compare with a reference calculation using cubic elements

Mean Drag for Cylinder Flow

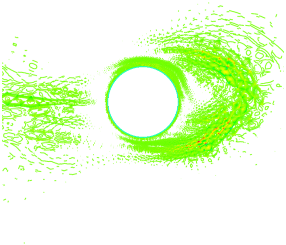
$$J_{\text{drag}}(u) = \int_0^T \int_{\Gamma_{\text{wall}}} (\text{Force} \cdot \hat{t}_{\text{drag}})\, \Psi(t)\, dx\, dt$$



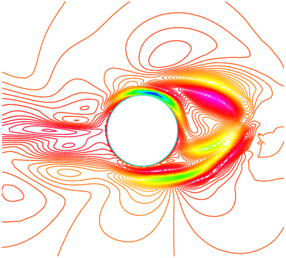
Example: Cylinder flow at Re=300



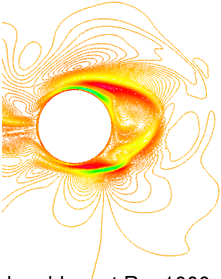
Dual problem, $\phi^{(x-mom)}$



Dual defect, $\phi^{(x-mom)} - \pi_h \phi^{(x-mom)}$.



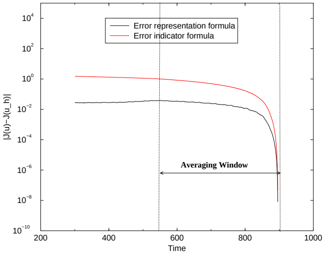
Dual problem at Re=300

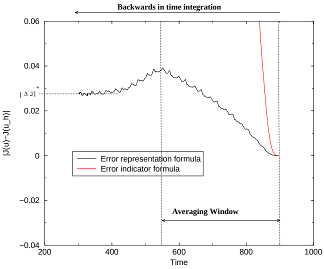


Dual problem at Re=1000

Mean Drag for Cylinder Flow at Re=1000

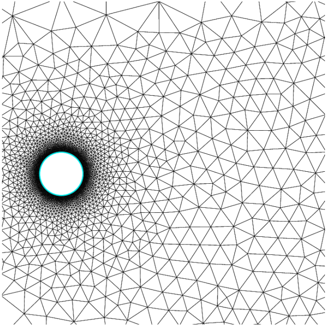
Error representation buildup during the backward in time dual integration



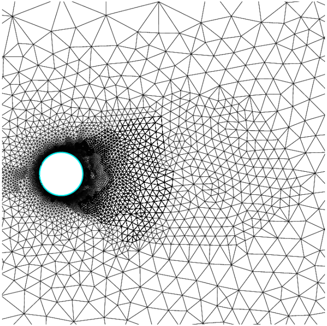


Mean Drag for Cylinder Flow at Re=1000

Adapted mesh from element indicators averaged over a period P



Coarse mesh (12K elements)

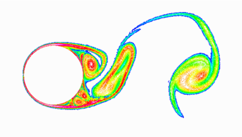


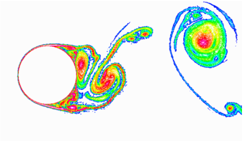
2 level refined mesh (20K elements)

Non-Periodic Cylinder

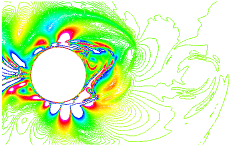
Cylinder flow at Re=3900 and Re=10000 using quartic ($p = 4$) space-time elements.

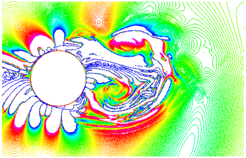
- Choosing measurement problems that are not genuinely stationary produces rapidly growing dual problems and dependency on the initial data.





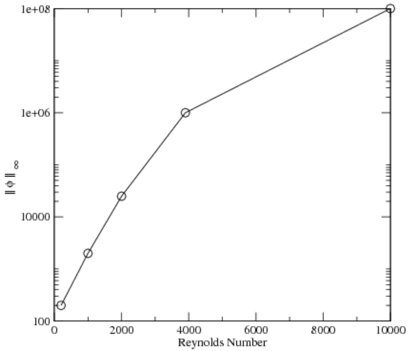
Dual solution corresponds to the average drag force over 3 approximate "periods".





Re=3900 **Re=10000**

Growth of Dual Problems



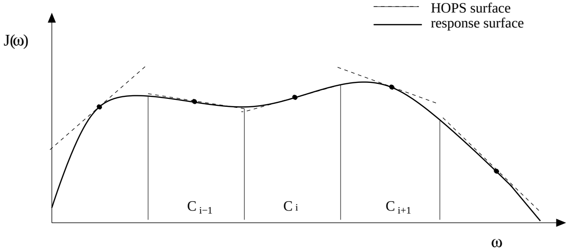
Growth of drag functional dual solution Φ with increasing Reynolds number

A Closing Note on the Use of Dual Problems in Uncertainty Quantification

Developing a capability to numerically compute primal/dual problems for compressible Navier-Stokes is a major undertaking.

Can this capability be reused in uncertainty quantification?

Estep and Neckels (2006) observed that dual problems can be used to build a piecewise linear response surface for use in Monte Carlo (MC) and Quasi Monte Carlo (QMC) sampling of uncertain outputs when the output of interest is a functional.



Given a nonlinear PDE system with solution $\mathbf{u} \in \mathbf{R}^m$ depending on n-dimensional random vector, $\omega \in \mathcal{P} \subset \mathbf{R}^n$

$$L\mathbf{u}(x;\omega) = \mathbf{f}$$

and output functional

$$J(\mathbf{u};\omega) : \mathbf{R}^m \times \mathbf{R}^n \to \mathbf{R}$$

calculate statistics of the functional such as expectation

$$E[J] = \int_{\mathcal{P}} J(\mathbf{u};\omega)\; pdf(\omega)\, d\omega = \int_0^1 J(\mathbf{u};\omega(\mu))\, d\,\mu$$

and variance

$$V[J] = E[J^2] - E[J]^2$$

using Monte Carlo (MC) or Quasi Monte Carlo (QMC) sampling.

Higher Order Parameter Sampling (HOPS), Estep and Neckels (2006)

1. Convert the statistics integration problem to uniform MC sampling on a unit hypercube.
2. Partition the unit hypercube into smaller hypercube subdomains with size determined from accuracy of the linearized sampling formula.
3. In each hypercube subdomain C_i center, calculate the primal solution u_i and adjoint solution ϕ_i

$$\left(\frac{\partial L}{\partial \mathbf{u}}(x,\omega_i)\right)^T \phi_i = \left(\frac{\partial J}{\partial \mathbf{u}}(x,\omega_i)\right)^T \rightarrow \overline{\mathcal{B}}(\mathbf{w},\phi_i;\omega_i) = \overline{J}(\mathbf{w};\omega_i)$$

and the reduced sensitivity gradients (cf. A. Jameson, 1988)

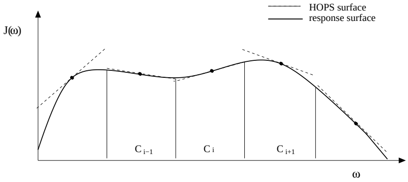
$$\mathbf{g}_i^T = \frac{\partial J}{\partial \omega}(x,\omega_i) - \phi^T \frac{\partial L}{\partial \omega}(x,\omega_i)$$

4. Apply MC or QMC integration in each C_i using the linearized sampling formula for *fixed* values of $J(\mathbf{u}_i,\omega_i)$ and $\mathbf{g}_i^T$

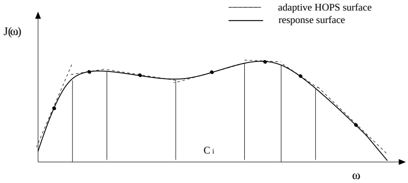
$$J(\mathbf{u},\omega) \approx J(\mathbf{u}_i,\omega_i) + \mathbf{g}_i^T(\omega - \omega_i)$$

Estep and Neckels then consider adaptive refinement to improve approximation properties of the HOPS surface.

Original HOPS surface



Adaptively refined HOPS surface

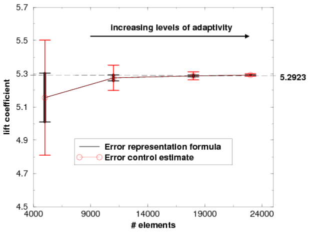


Concluding Remarks

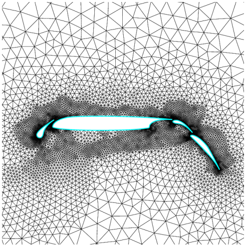
- Estimating and controlling numerical error in time-dependent calculations is fraught with difficulties
  - growth in backward-in-time dual problems,
  - loss of sharpness in error bounds.
- The calculation of dual problems is computationally demanding
  - storage of primal time slices,
  - higher order solves of dual problem
- Error representation/estimation results presented today barely scratch the surface
  - error control for general transient problems,
  - dual problems in the presence of flow bifurcations.

Example: Euler flow past multi-element airfoil geometry. $M = .1,\ 5^\circ$ AOA.

lift coefficient (error representation)	lift coefficient (error control)	refinement level	# elements	equivalent uniform refinement # elements
$5.156 \pm .147$	$5.156 \pm .346$	0	5000	5000
$5.275 \pm .018$	$5.275 \pm .076$	1	11000	20000
$5.287 \pm .006$	$5.287 \pm .024$	2	18000	80000
$5.291 \pm .002$	$5.291 \pm .007$	3	27000	320000



Error reduction during mesh adaptivity



Adapted mesh (18000 elements)

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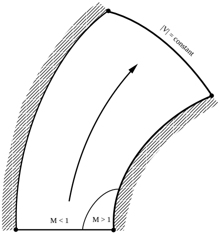
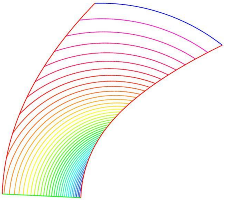
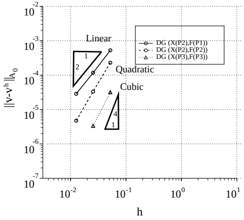
Example: Ringleb Flow



Schematic of Ringleb flow



Iso-Density contours

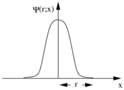


Discontinuous Galerkin

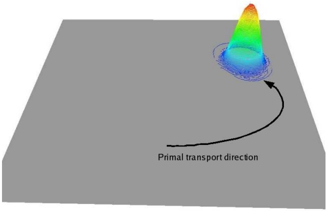
Example: A Scalar Time-Dependent PDE

Circular transport, $\lambda = (y, -x)$, of bump data

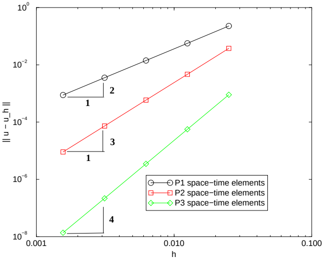
$$u_t + \lambda \cdot \nabla u = 0 \ , \qquad x \in [-1,1]^2$$
$$u(x,0) = \Psi(1/10; x - x_0) \ , \qquad x_0 = (7/10, 0, 0)$$



bump function



Primal numerical problem



Convergence, $\|u - u_h\|_{L_2(\Omega \times [0,T])}$

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