Robust Fault Detection and Isolation for Stochastic Systems

Jemin George and Irene M. Gregory

Abstract—This paper outlines the formulation of a robust fault detection and isolation scheme that can precisely detect and isolate simultaneous actuator and sensor faults for uncertain linear stochastic systems. The given robust fault detection scheme based on the discontinuous robust observer approach would be able to distinguish between model uncertainties and actuator failures and therefore eliminate the problem of false alarms. Since the proposed approach involves precise reconstruction of sensor faults, it can also be used for sensor fault identification and the reconstruction of true outputs from faulty sensor outputs. Simulation results presented here validate the effectiveness of the robust fault detection and isolation system.

I. INTRODUCTION

Faults are deviations from the normal behavior of the plant or its instrumentation and they can be categorized into: i) additive process faults, ii) multiplicative process faults, iii) sensor faults, and iv) actuator faults. There exist several fault monitoring procedures which can be used to recognize and distinguish different types of faults [1]. These fault monitoring procedures can be categorized into: i) fault detection, ii) fault isolation, and iii) fault identification. A survey on design methods for fault detection is given in [2]. Most of the existing FDI (fault detection and isolation) schemes are based on measurement residual generation. Generated residual is used to facilitate the decision making procedures involved in FDI. The basic difference between most FDI schemes is the underlying instrumentation and they can be categorized into: i) additive uncertainties, which can be used to recognize and distinguish different types of faults [1]. These fault monitoring procedures can be categorized into: i) fault detection, ii) fault isolation, and iii) fault identification.

There are no constraints on system uncertainties, both matched and mismatched uncertainties are considered. Present scheme can be easily extended to nonlinear systems by considering Lipschitz continuous affine nonlinear terms with known Lipschitz constant [13], [15]. The structure of this paper is as follows. A detailed formulation of the observer based FDI scheme is first given. Afterwards, the results from numerical simulations and the concluding remarks are given in sections III and IV, respectively.

II. OBSERVER-BASED FAULT DETECTION FILTER

Let \( (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}) \) denote a complete filtered probability space. Consider an \( n \times n \)-order stochastic system of the following form:

\[
X_1(t) = A_{11}X_1(t) + A_{12}X_2(t) + W_1(t)
\]

\[
X_2(t) = A_{21}X_1(t) + A_{22}X_2(t) + Bu(t) + W_2(t)
\]

\[
Y_1(t) = C_{11}X_1(t) + C_{12}X_2(t) + \nu_1(t)
\]

\[
Y_2(t) = C_{21}X_1(t) + C_{22}X_2(t) + \nu_2(t) + \nu_3(t)
\]

where \( W_1(t) \) and \( W_2(t) \) denote stochastic disturbances and \( \nu_1(t) \) and \( \nu_2(t) \) indicate measurement noises. The state vectors, \( X_1(t) \) and \( X_2(t) \), are of dimensions, \( X_1(t) \in \mathbb{R}^{m_1} \) and \( X_2(t) \in \mathbb{R}^{m_2} \), respectively. The true state matrices, \( A_{11} \in \mathbb{R}^{(n-r) \times (n-r)} \), \( A_{12} \in \mathbb{R}^{(n-r) \times r} \), \( A_{21} \in \mathbb{R}^{r \times (n-r)} \), \( A_{22} \in \mathbb{R}^{r \times r} \), and the control distribution matrix \( B \in \mathbb{R}^{r \times m_u} \) are assumed to be unknown. The desired input signal is denoted as \( u_d(t) \) and \( u_c(t) \) indicates the error in applied control, \( u(t) \), due to actuator faults, i.e.,

\[
u_c(t) = u_d(t) + u_c(t)
\]

The stochastic measurement vectors, \( Y_1(t) \) and \( Y_2(t) \), are of dimensions, \( Y_1(t) \in \mathbb{R}^{m_1} \) and \( Y_2(t) \in \mathbb{R}^{m_2} \), respectively. The output matrices, \( C_{11}, C_{12}, C_{21} \) and \( C_{22} \), are assumed to be known. The measurement noise, \( [\nu_1^T(t) \ \nu_2^T(t)]^T = \nu(t) \in \mathbb{R}^m \), is assumed to be zero-mean Gaussian white noise process, i.e., \( \mathbb{V}(\nu(t) \sim \mathcal{N}(0, \mathbb{R}^m \mathbb{R}^m)) \). The vector, \( \nu_c(t) \), indicates sensor failures and is modeled as

\[
\nu_c(t) = f(\nu_c(t), t), \quad \nu_c(t_0) = \mathbf{0}
\]

where \( f(\cdot) \) is an unknown operator. Stochastic external disturbance \([W_1^T(t) \ W_2^T(t)]^T = W(t) \in \mathbb{R}^n\) is modeled as a linear system driven by a Gaussian white noise process, i.e.,

\[
W(t) = L(W(t), t) + \mathbb{V}(t), \quad W(t_0) = \mathbf{0}
\]
where \( L(\cdot) \) is an unknown linear operator and \( W(t) \in \mathbb{R}^n \), is a zero-mean Gaussian white noise process, i.e., \( W(t) \sim \mathcal{N}(0, Q_0(\tau)) \).

**Assumption 1.** Assume the sensor faults are not instantaneous and therefore there exist a known conservative upper bound on \( f(y_e(t), t) \) such that
\[
|f(y_e(t), t)| \leq \sigma(t), \quad \forall t \geq t_0
\]
where \( \cdot \) denotes the Euclidean norm.

The external disturbance, \( W(t) \), is mean square bounded \([16],[17]\), i.e.,
\[
\sup_{t \geq t_0} E\left[W(t)W^T(t)\right] \leq K
\]
where \( K \) is a constant matrix whose elements are finite. The assumed (known) model of the plant in (1) is given as
\[
\begin{align*}
\dot{x}_m(t) &= A_{m11}x_m(t) + A_{m12}x_m(t) + B_m u_d(t) \\
\end{align*}
\]
Define the model-error vectors \( D_1(t) \in \mathbb{R}^{n-m} \) and \( D_2(t) \in \mathbb{R}^m \) as
\[
\begin{align*}
D_1(t) &= \Delta A_{11}x_1(t) + \Delta A_{12}x_2(t) + W_1(t) \\
D_2(t) &= \Delta A_{21}x_1(t) + \Delta A_{22}x_2(t) + \Delta B_u u_d(t) + W_2(t)
\end{align*}
\]
where \( \Delta A_{11} = A_{11} - A_{m11}, \Delta A_{12} = A_{12} - A_{m12}, \Delta A_{21} = A_{21} - A_{m21}, \Delta A_{22} = A_{22} - A_{m22}, \) and \( \Delta B = B - B_m \).

**Assumption 2.** Given the system parameter uncertainties are bounded and the system states are bounded in mean square sense, an upper bound on the model error vector \( D(t) \) can be obtained as
\[
\mathbb{P}(\|D(t)\| \leq \bar{\mu}(t)) = 1, \quad \forall t \geq t_0
\]
That is, \( |D(t)| \) is almost surely (a.s.) upper bounded by \( \bar{\mu}(t) \) for all \( t \geq t_0 \).

Now the plant dynamics in (1) can be written in-terms of known parameters as
\[
\begin{align*}
\dot{X}_1(t) &= A_{m11}X_1(t) + A_{m12}X_2(t) + D_1(t) \\
\dot{X}_2(t) &= A_{m21}X_1(t) + A_{m22}X_2(t) + B_m u_d(t) + D_2(t) + B_u u(t)
\end{align*}
\]
Re-parameterize \( X_2(t) \) as
\[
\dot{X}_2(t) = \alpha X_{2\alpha}(t) + \beta X_{2\beta}(t)
\]
where \( X_{2\alpha}(t) \in \mathbb{R}^{n}, X_{2\beta}(t) \in \mathbb{R}^m, \alpha \) and \( \beta \) are user selected scalar parameters. Now \( X_2(t) \) can be written as \( \dot{X}_2(t) = \alpha X_{2\alpha}(t) + \beta X_{2\beta}(t) \).
Select \( X_{2\alpha}(t) \) and \( X_{2\beta}(t) \) as
\[
\begin{align*}
\dot{X}_{2\alpha}(t) &= \frac{1}{\alpha}A_{m21}X_1(t) + A_{m22}X_{2\alpha}(t) + \frac{1}{\alpha}D_2(t) \\
\dot{X}_{2\beta}(t) &= A_{m22}X_{2\beta}(t) + \frac{1}{\beta}B_m u_d(t) + \frac{1}{\beta}B_u u(t)
\end{align*}
\]
**Remark 1.** One of the main challenges in the design of observer based FDI scheme is the presence of coupled system uncertainties and actuator faults \([15]\). In the presence of coupled system uncertainties and actuator faults, it is difficult to design an observer that yields measurement residual which is only sensitive to the actuator faults. Notice that the re-parametrization of \( X_2(t) \) given in (7) allows decoupling of system uncertainties and actuator faults as shown in (8).

**Assumption 3.** Assume there exists a bounded vector \( \zeta(t) \in \mathbb{R}^r \) such that \( B_u u(t) = B_m \zeta(t) \), i.e.,
\[
\zeta(t) = B_m^{-1}B_u u(t) \quad \text{and} \quad |\zeta(t)| \leq \xi(t), \quad \forall t \geq t_0
\]
After appending the sensor error dynamics given in (3), the extended system can be written as
\[
\begin{bmatrix}
X_1(t) \\
X_{2\alpha}(t) \\
X_{2\beta}(t) \\
y_e(t)
\end{bmatrix} =
\begin{bmatrix}
A_{m11} & \alpha A_{m12} & \beta A_{m12} & 0 \\
A_{m21} & A_{m22} & 0 & 0 \\
0 & 0 & A_{m22} & 0 \\
0 & 0 & 0 & A_{ym}
\end{bmatrix}\begin{bmatrix}
X_1(t) \\
X_{2\alpha}(t) \\
X_{2\beta}(t) \\
y_e(t)
\end{bmatrix} + \begin{bmatrix}
0 \\
\frac{1}{B_m} u_d(t) + I \\
0 & 0 & \frac{1}{\beta}B_m \\
0 & 0 & 0 & I
\end{bmatrix}\begin{bmatrix}
D_1(t) \\
D_2(t) \\
\zeta(t) \\
h(t)
\end{bmatrix}
\]
where \( h(\cdot) = f(\cdot) - A_y y_e \) and \( A_y \in \mathbb{R}^{n_y \times m_z} \) is a user selected Hurwitz matrix. Let \( Z(t) = [X_1^T(t) \ X_{2\alpha}^T(t) \ X_{2\beta}^T(t) \ y_e^T(t)]^T \), now the above extended system can be rewritten as
\[
\dot{Z}(t) = FZ(t) + G_1 u_d(t) + G_1 D_1(t) + G_2 D_2(t) + G_3 \zeta(t) + G_4 h(t)
\]
where \( F = \begin{bmatrix} A_{m11} & \alpha A_{m12} & \beta A_{m12} & 0 \\
0 & 0 & A_{m22} & 0 \\
0 & 0 & 0 & A_{ym}
\end{bmatrix} \) and
\[
G = \begin{bmatrix} G_1 \\
G_2 \\
G_3 \\
G_4
\end{bmatrix} \begin{bmatrix} I & 0 & 0 & 0 \\
0 & \frac{1}{\beta}I & 0 & 0 \\
0 & 0 & \frac{1}{B_m}I & 0 \\
0 & 0 & 0 & I
\end{bmatrix}
\]
Let \( H = \begin{bmatrix} C_{11} & \alpha C_{12} & \beta C_{12} & 0 \\
C_{21} & \alpha C_{22} & \beta C_{22} & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} \), the measurement equations can be rewritten as \( Y(t) = HZ(t) + V(t) \). Now the system in (1) can be written as the following dynamically equivalent form
\[
\dot{Z}(t) = FZ(t) + G_1 u_d(t) + G_1 D_1(t) + G_2 D_2(t) + G_3 \zeta(t) + G_4 h(t)
\]
\[
Y(t) = HZ(t) + V(t)
\]
**Remark 2.** Even though the above representation of the plant is a non-minimal realization, the observability of the extended system may be obtained by making appropriate changes to the state matrix, \( F \), and the corresponding changes to \( D_1(t), D_2(t), \) and \( h(\cdot) \).

Consider the following partition of \( G_1 \) as \( n - r \) column vectors, \( G_2 \) as \( r \) column vectors, \( G_3 \) as \( r \) column vectors, and \( G_4 \) as \( m_z \) column vectors as shown below
\[
\begin{bmatrix}
G_1 \\
G_2 \\
G_3 \\
G_4
\end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} & \hdots & g_{1(n-r)} \\
g_{21} & g_{22} & \hdots & g_{2r} \\
g_{31} & g_{32} & \hdots & g_{3r} \\
g_{41} & g_{42} & \hdots & g_{4m_z}
\end{bmatrix}
\]
Also consider the individual elements of the vectors \( \zeta(t), D(t), \) and \( h(\cdot), \) i.e.,
\[
\begin{bmatrix}
\zeta_1(t) \\
\zeta_r(t) \\
D_1(t) \\
D_2(t)
\end{bmatrix} \quad \text{and} \quad h(\cdot) = \begin{bmatrix} h_1(\cdot) \\
h_2(\cdot) \\
h_3(\cdot) \\
h_4(\cdot)
\end{bmatrix}
\]
Now the extended system in (9) can be written in summation form as
\[
\dot{Z}(t) = FZ(t) + \sum_{i=1}^{n-r} g_{1i} D_1(t) + \sum_{j=1}^{r} g_{2j} D_{n-r+j}(t) + \sum_{k=1}^{m_z} g_{4k} \zeta_k(t) + \sum_{i=1}^{m_z} g_{4i} u_i(\cdot) + G_3 u_d(t)
\]

Define $G_1 \triangleq \begin{bmatrix} G_1 & G_2 \end{bmatrix}$, $G_2 \triangleq \begin{bmatrix} G_3 & G_4 \end{bmatrix}$, and $\eta^T(t) \triangleq \begin{bmatrix} \zeta^T(t) & h^T(t) \end{bmatrix}^T$. Now (11) may be rewritten as
\[
\dot{Z}(t) = FZ(t) + G_1u_d(t) + \sum_{i=1}^{n} G_{1i}D_i(t) + \sum_{j=1}^{r+m_2} G_{2j}\eta_j(t) \quad (12)
\]
where $G_{1k}$ and $G_{2k}$ are the $k$th column vectors of $G_1$ and $G_2$ matrices, respectively. Now $\ell$, $1$, $2$, $\ldots$, $r + 1$ of the following from are considered

If $\ell \leq r$
\[
\dot{Z}(t) = F\dot{Z}(t) + L^T\begin{bmatrix} Y(t) - H\dot{Z}(t) \end{bmatrix} + G_1u_d(t) + \sum_{i=1}^{n} G_{1i}\mu_i(t) + \sum_{j=1}^{r+m_2} G_{2j}\nu_j(t) \quad (13)
\]

If $\ell = r + 1$
\[
\dot{Z}(t) = F\dot{Z}(t) + L^T\begin{bmatrix} Y(t) - H\dot{Z}(t) \end{bmatrix} + G_1u_d(t) + \sum_{i=1}^{n} G_{1i}\mu_i(t) + \sum_{j=1}^{r+m_2} G_{2j}\nu_j(t) \quad (14)
\]

where $L^T \in \mathbb{R}^{(n + r + m_2) \times m}$ is the observer gain corresponding to the $\ell$th observer. The observer inputs are denoted as, $[\nu_1(t) \ldots \nu_{r+m_2}(t)]^T \triangleq \nu(t) \in \mathbb{R}^{r+m_2}$, and $[\mu_1(t) \ldots \mu_{r+m_2}(t)]^T \triangleq \mu(t) \in \mathbb{R}^m$, $\forall \ell = 1$, $2$, $\ldots$, $r + 1$

Equations (13) and (14) correspond to the typical observer model. The observer gain $L^T$ and the observer inputs $\mu(t)$ and $\nu(t)$ corresponding to the $\ell$th observer are selected so that the generated residual of a sensor or fault occurrence obtained from observers given in (13) is asymptotically stable if there is no fault in the $\ell$th actuator and the residual obtained from the observer given in (14) is asymptotically stable despite any actuator or sensor fault occurrences.

Define the observer error as $\tilde{Z}(t) = Z(t) - \hat{Z}(t)$. After subtracting (14) from (11), the observer error dynamics can be written as
\[
\dot{\tilde{Z}}(t) = \begin{bmatrix} F - L^TH \end{bmatrix} \tilde{Z}(t) - L^TY(t) + \sum_{i=1}^{n} G_{1i}\left[D_i(t) - \mu_i(t)\right] + \sum_{j=1}^{r+m_2} G_{2j}\left[\eta_j(t) - \nu_j(t)\right] \quad (15)
\]

It is important to note that the solution to the stochastic differential equation given in (15) cannot be based on the ordinary mean square calculus because the integral involved in the solution depends on $Y(t)$, which is of unbounded variation, i.e., $E\left[Y(t)\dot{Y}(t + \tau)\right] = R\delta(\tau)$. For the treatment of this class of problems, the stochastic differential equation can be rewritten in Ito form as [18], [19]
\[
d\tilde{Z}(t) = \begin{bmatrix} F - L^TH \end{bmatrix} \tilde{Z}(t) + \sum_{i=1}^{n} G_{1i}\left[D_i(t) - \mu_i(t)\right] + \sum_{j=1}^{r+m_2} G_{2j}\left[\eta_j(t) - \nu_j(t)\right] dt - L^Td\mathcal{B}(t) \quad (16)
\]

where the zero-mean Gaussian white noise $\mathcal{Y}(t)$ is written as the increments of stationary Wiener process with zero-mean and the correlation of increments
\[
E\left[\{(B(\tau) - B(\zeta)) (B(\tau) - B(\zeta))^T\} \right] = R|\tau - \zeta|
\]

Details on stochastic Ito calculus can be found in [19]. The observer error corresponding to the $\ell$th observer, $\tilde{Z}(t)$, is a stochastic process and therefore the stability of the observer error dynamics given in (16) is depicted either as moment stability or stability in probabilistic sense. The stability in probabilistic sense is usually known as almost sure (a.s.) stability and it is defined as follows [16]:

**Definition 1.** The stochastic process $\tilde{Z}(t)$ is asymptotically stable with probability 1, or almost surely asymptotically stable, if
\[
\mathbb{P}\left(\tilde{Z}(t) \rightarrow 0 \text{ as } t \rightarrow \infty\right) = 1
\]

Notice that the almost sure stability of the observer error is impossible due to the persistently acting measurement noise $\mathcal{B}(t)$. Therefore it is desirable for the observer error corresponding to the $\ell$th observer, $\tilde{Z}(t)$, to have a dynamics that follows
\[
d\tilde{Z}_m(t) = \begin{bmatrix} F - L^TH \end{bmatrix} \tilde{Z}_m(t) dt + L^td\mathcal{B}(t) \quad (18)
\]

Let $\tilde{Z}(t) = \tilde{Z}(t) - \tilde{Z}_m(t)$, now subtracting (18) from (16) yields
\[
d\tilde{Z}(t) = \begin{bmatrix} F - L^TH \end{bmatrix} \tilde{Z}(t) + \sum_{i=1}^{n} G_{1i}\left[D_i(t) - \mu_i(t)\right] + \sum_{j=1}^{r+m_2} G_{2j}\left[\eta_j(t) - \nu_j(t)\right] dt \quad (19)
\]

Given next is an approach for the selection of the observer gain $L^T$ and the observer inputs $\mu(t)$ and $\nu(t)$ corresponding to the $\ell$th observer based on the stochastic Lyapunov approach. Since the only information regarding the true observer error is in the form of measurement residual, one do not have full access to the true $\tilde{Y}(t)$, i.e., one only has access to $\hat{Y}(t) = H\tilde{Z}(t)$. Based on (19), $d\hat{Y}(t)$ can be written as
\[
d\hat{Y}(t) = \begin{bmatrix} H [F - L^TH] \hat{Y}(t) + \sum_{i=1}^{n} HG_{1i}\left[D_i(t) - \mu_i(t)\right] + \sum_{j=1}^{r+m_2} HG_{2j}\left[\eta_j(t) - \nu_j(t)\right] \right] dt
\]

Based on Assumptions 1 and 3, define an upper bound on $\eta(t)$ as
\[
|\eta(t)| \leq \bar{\mu}(t), \quad \forall \ell \geq t_0
\]

**Theorem 1.** Given the Assumptions 1, 2, and 3, the individual fault detection filters given in Eq. (13) guarantee almost sure asymptotic stability of $\hat{Y}(t)$ if there is no fault occurrence in the $\ell$th actuator and the fault detection filter given in Eq. (14) guarantees almost sure asymptotic stability of $\tilde{Y}_t+1(t)$ despite any actuator or sensor fault occurrences, if the observer gain $L^T$ corresponding to the $\ell$th observer is selected so that the following matrix Lyapunov inequality is satisfied
\[
\begin{bmatrix} [F - L^TH]^T H^TH P^T H^TH + H^TH P^T H^TH [F - L^TH] + Q^T \end{bmatrix} \leq 0
\]

and the observer inputs corresponding to the $\ell$th observer are selected as
\[
\mu_i(t) = \text{sgn}\left(\hat{Y}(t)H^TH G_{1i}\right) \bar{\mu}(t), \quad \forall i = 1, \ldots, n
\]
\[
\nu_j(t) = \text{sgn}\left(\hat{Y}(t)H^TH G_{2j}\right) \bar{\nu}(t), \quad \forall j = 1, \ldots, r + m_2
\]
where \( P^\ell \in \mathbb{R}^{(n+r+m_2) \times (n+r+m_2)} \) and \( Q^\ell \in \mathbb{R}^{(n+r+m_2) \times (n+r+m_2)} \) are positive definite symmetric matrices and \( \text{sgn}\{\cdot\} \) denotes the signum function or the sign function.

**Proof:** Construct a Lyapunov function candidate of the form
\[
V(\tilde{y}^\ell(t)) = \left( \tilde{y}^\ell(t) \right)^T H P^\ell H^T \tilde{y}^\ell(t).
\]
Now using the Itô formula [19], \( dV(\tilde{y}^\ell(t)) \) can be calculated as
\[
dV(\tilde{y}^\ell(t)) = \left\{ \left( \tilde{y}^\ell(t) \right)^T \left[ \left[ F - L^T H^T H \right]^T H^T H P^\ell H^T H + H^T H P^\ell H^T H \right] \right\} \tilde{y}^\ell(t)
\]
\[
+ 2 \left( \tilde{y}^\ell(t) \right)^T H P^\ell H^T H \sum_{i=1}^{n} \left[ D_i(t) - \mu_i^\ell(t) \right]
\]
\[
+ 2 \left( \tilde{y}^\ell(t) \right)^T H P^\ell H^T H \sum_{j=1}^{r} \left[ G_j(t) - \nu_j^\ell(t) \right]
\]
dt.

After substituting (20), \( \mathcal{L}V(\tilde{y}^\ell) \) can be written as
\[
\mathcal{L}V(\tilde{y}^\ell) \leq -\left( \tilde{y}^\ell(t) \right)^T Q^\ell \tilde{y}^\ell(t)
\]
\[
+ 2 \sum_{i=1}^{n} \left\{ \left( \tilde{y}^\ell(t) \right)^T H P^\ell H^T H \mathcal{G}_{1i} \left[ D_i(t) - \mu_i^\ell(t) \right] \right\}
\]
\[
+ 2 \sum_{j=1}^{r} \left\{ \left( \tilde{y}^\ell(t) \right)^T H P^\ell H^T H \mathcal{G}_{2j} \left[ \eta_j(t) - \nu_j^\ell(t) \right] \right\}
\]
where the operator \( \mathcal{L}\{\cdot\} \) acting on \( V(x, t) \) is defined as
\[
\mathcal{L}V(x, t) = \lim_{dt \to 0} \frac{1}{dt} E[dV(X(t), t)|X(t) = x]
\]
(23)

Substituting (21) and (22) yields
\[
\mathcal{L}V(\tilde{y}^\ell) \leq -\left( \tilde{y}^\ell(t) \right)^T Q^\ell \tilde{y}^\ell(t)
\]
\[
+ 2 \sum_{i=1}^{n} \left\{ \left( \tilde{y}^\ell(t) \right)^T H P^\ell H^T H \mathcal{G}_{1i} \left[ D_i(t) - \mu_i^\ell(t) \right] \right\}
\]
\[
+ 2 \sum_{j=1}^{r} \left\{ \left( \tilde{y}^\ell(t) \right)^T H P^\ell H^T H \mathcal{G}_{2j} \left[ \eta_j(t) - \nu_j^\ell(t) \right] \right\}
\]
Thus
\[
\mathcal{L}V(\tilde{y}^\ell(t)) \leq -\left( \tilde{y}^\ell(t) \right)^T Q^\ell \tilde{y}^\ell(t)
\]
(24)

Therefore the \( \ell = (r + 1) \)th observer in (14) is almost surely asymptotically stable despite the occurrence of any actuator or sensor faults. Based on the given proof one could easily make the argument that if there is no fault occurrence in the \( \ell \)th actuator, where \( 1 \leq \ell \leq r \), then the \( \ell \)th observer given in (13) is almost surely asymptotically stable. Thus any observed residual \( \tilde{y}^\ell(t) \), will indicate a fault occurrence in the \( \ell \)th actuator.

Any observed residual in the \( \ell \)th observer given in (13), where \( 1 \leq \ell \leq r \), indicates a fault occurrence in the \( \ell \)th actuator. Based on the observability condition one could easily show that the estimated or the observed generated sensor error terms, \( \hat{y}_s(t) \), asymptotically approaches the true sensor error, \( y_s(t) \). Therefore \( \hat{y}_s(t) \) obtained from the observer given in (14) can be directly used for sensor fault detection. That is, if \( \hat{y}_s(t) = 0 \), then there is no sensor fault and if \( \hat{y}_s(t) \neq 0 \), then the nonzero \( \hat{y}_s(t) \) indicates a fault occurrence in the \( \ell \)th sensor. Moreover, by subtracting \( \hat{y}_s(t) \) from the measured output yields the true system output.

### III. Numerical Simulations

Numerical simulation results are presented in this section to validate the efficiency of the proposed FDI scheme. Consider a stochastic system of the form given in (1) where the true system matrices are given as
\[
A_{11} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_{21} = \begin{bmatrix} -1.3 & 0.1 \\ -0.12 & -1.8 \end{bmatrix},
\]
\[
A_{22} = \begin{bmatrix} -0.9 & -0.011 \\ 0.3 & -3.4 \end{bmatrix}, \quad B = \begin{bmatrix} 2.4 & -0.23 \\ -0.11 & 1.5 \end{bmatrix}
\]
and the assumed system matrices are
\[
A_{m11} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_{m12} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B_m = \begin{bmatrix} 3.19 & -0.31 \\ -0.1 & 2.01 \end{bmatrix}
\]
The system output matrices are given as
\[
C_{11} = \begin{bmatrix} I_{2 \times 2} & 0_{2 \times 2} \end{bmatrix}^T, \quad C_{12} = \begin{bmatrix} 0_{2 \times 2} & I_{2 \times 2} \end{bmatrix}^T, \quad C_{21} = \begin{bmatrix} 0.95 & 1.7 \\ -2.4 & 1.54 \end{bmatrix}, \quad C_{22} = \begin{bmatrix} 0.43 & -2.31 \\ 1.3 & 0.43 \end{bmatrix}
\]
For simulation purposes, the external disturbance is modeled as \( W_1 = 0 \) and \( W_2(t) = [W_{21}(t) \ W_{22}(t)]^T \) is given as
\[
W_{21}(t) = -W_{21}(t) + W_1(t), \quad W_{22}(t) = -W_{22}(t) + W_2(t)
\]
where \( [W_3(t) \ W_2(t)]^T = W(t) \) is zero-mean Gaussian white noise process with
\[
E\left[ W(t)W^T(t + \tau) \right] = 10^{-2} \times I_{2 \times 2} \delta(\tau)
\]
The measurement noise, \( \nu(t) \in \mathbb{R}^6 \), is assumed to be zero-mean Gaussian white noise process with
\[
E\left[ \nu(t)\nu^T(t + \tau) \right] = 10^{-2} \times I_{6 \times 6} \delta(\tau)
\]
Note that \( \mathcal{D}_1 = 0 \) and \( \mathcal{D}_2(t) \) is given as
\[
\mathcal{D}_2(t) = \Delta A_{21}X_1(t) + \Delta A_{22}X_2(t) + \Delta Bu_0(t) + W_2(t)
\]
where the desired control input \( u_0(t) \) is given in Fig. 1. For the re-parametrization of the system states, the constants \( \alpha \) and \( \beta \) are selected as \( \alpha = \beta = 1 \). Note that the two possible sensors faults are associated with the fifth and sixth outputs. Two fault scenarios are considered here and the details on the fault scenarios are...
1) **Fault Scenario I**: For the first fault scenario, the faults are associated with the second actuator and the fifth output sensor. The actuator fault occurs at thirty seconds (sec) and the sensor fault occurs at sixty-five sec. Given in Fig. 2 are the \( u_e(t) \) and \( y_e(t) \) corresponding to the first fault scenario.

2) **Fault Scenario II**: For the second fault scenario considered, it is assumed that the faults are associated with the first actuator, the fifth and the sixth output sensors. The fault associated with the first actuator occurs at thirty sec. The fifth and sixth output sensor faults occur at sixty-five sec and eighty-five sec, respectively. Given in Fig. 3 are the \( u_e(t) \) and \( y_e(t) \) corresponding to the second fault scenario.

For simulation purposes the upper bounds on \( D_2(t), \xi(t), \) and \( h(\cdot) \) are selected as
\[
|D_2(t)| \leq 10 \quad |D_2(t)| \leq 20 \quad |\xi(t)| \leq 4 \quad |\xi(t)| \leq 3 \quad |h_1(t)| \leq 2 \quad |h_2(t)| \leq 2
\]

![Image of Fault Scenario I](image1)

![Image of Fault Scenario II](image2)

![Image of Fault Scenario III](image3)

Note that for the system considered here, there are two actuators and therefore three different observers are designed. For both fault scenarios considered, \( A_0 = 0_{2\times2}, P^f \) is selected as \( P^f = 10^{-2} \times I, \forall t \in \{1, 2, 3\} \). The observer gain is calculated as \( L^1 = L^2 = L^3 =
\[
\begin{bmatrix}
49.9299 & -0.2986 & -0.2402 & 0.7390 & -2.5331 & -0.3157 \\
0.3753 & 49.8876 & -0.3033 & -0.2856 & 0.7403 & -1.7170 \\
0.1847 & 0.0195 & 24.9291 & -0.3715 & 1.0916 & -0.7319 \\
-0.3075 & 0.3204 & 0.5508 & 24.6784 & 1.2925 & 1.9531 \\
0.1847 & 0.0195 & 24.9291 & -0.3715 & 1.0916 & -0.7319 \\
-0.3075 & 0.3204 & 0.5508 & 24.6784 & 1.2925 & 1.9531 \\
-17.3614 & -85.1196 & -21.7003 & 115.5758 & 49.6501 & -0.7843 \\
120.8081 & -76.9340 & -64.9756 & -21.7627 & 0.8362 & 49.8110
\end{bmatrix}
\]

The extended output matrix \( H \) and the matrix \( G \) can be calculated as
\[
H = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0_{4 \times 2} \\ C_{21} & C_{22} & C_{22} & I_{2 \times 2} \end{bmatrix}, 
G = \begin{bmatrix} 0_{2 \times 2} & 0_{2 \times 2} & 0_{2 \times 2} \\ I_{2 \times 2} & 0_{2 \times 2} & 0_{2 \times 2} \\ 0_{2 \times 2} & B_m & 0_{2 \times 2} \\ 0_{2 \times 2} & 0_{2 \times 2} & I_{2 \times 2} \end{bmatrix}
\]

Details on the results obtained for both fault scenarios are given next.

1) **Fault Scenario I**: Given in Fig. 4 are the true error vectors, \( D_2(t) = \Delta A_{21} X_1(t) + \Delta A_{22} X_2(t) + \Delta B u_1(t) + W_2(t), \xi(t) = B_m B u_1(t), \) and \( h(\cdot) = I(\cdot) \) corresponding to the first fault scenario.

![Image of Fault Scenario III](image4)

Given in Fig. 5 are the generated residual and the estimated sensor errors corresponding to the first fault scenario. Figure 5(a) contains the measurement residual generated for observer one and two. The first two kinks in the residual are due to the start of input application that occurs around five seconds and the leveling-off the input to its steady state value around twenty seconds. Notice the jump in observer residual around thirty seconds due to the fault occurrence in the second actuator. Figure 5(b) contains the estimated sensor errors obtained from the third observer. Note that the estimated sensor error is similar to the true sensor error given in Fig. 2(b).

2) **Fault Scenario II**: Given in Fig. 6 are the true error vectors, \( D_2(t), \xi(t), \) and \( h(\cdot) \) corresponding to the second fault scenario. The upper bounds on error vectors used here are the same upper bounds used for the first fault scenario.

Given in Fig. 7 are the generated residual and the estimated sensor errors corresponding to the second fault scenario. Figure 7(a) contains the measurement residual generated for observer one and two. Notice the sudden increase in observer one residual around thirty seconds due to the fault occurrence in the first actuator. Figure 7(b) contains the estimated sensor errors obtained from the third observer. Note that the estimated sensor errors are similar to the true sensor errors given in Fig. 3(b).
A bank of discontinuous observers is designed for fault detection and isolation scheme where a discontinuous observer is used for alarms. The presented approach involves precise reconstruction of outputs. The proposed approach is an observer based fault detection identification and the reconstruction of true outputs from faulty sensor faults and therefore this approach can be used for sensor fault detection, isolation and identification. Moreover, by subtracting the estimated sensor errors from the measured outputs, true system outputs can be generated. The simulation results reveal clear indication of actuator faults despite the presence of matched system uncertainties and external disturbances. Moreover, the estimated sensor errors are identical to the true sensor error regardless of the measurement noise present.

**REFERENCES**


