Two-point turbulence closure applied to variable resolution modeling

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Variable resolution methods have become frontline CFD tools, but in order to take full advantage of this promising new technology, more formal theoretical development is desirable. Two general classes of variable resolution methods can be identified: *hybrid* or *zonal* methods in which RANS and LES models are solved in different flow regions, and *bridging* or *seamless* models which interpolate smoothly between RANS and LES. This paper considers the formulation of bridging methods using methods of two-point closure theory. The fundamental problem is to derive a subgrid two-equation model. We compare and reconcile two different approaches to this goal: the Partially Integrated Transport Model, and the Partially Averaged Navier-Stokes method.

Nomenclature

k	turbulent kinetic energy
k_r	resolved turbulent kinetic energy
k_c	unresolved turbulent kinetic energy
ℓ	integral length scale
E	energy spectrum
T	energy transfer rate
$\mathcal{F}$	energy flux
F_c	modified energy flux defined in Eq. (29)
P	total energy production rate
P_r	energy production rate due to resolved scales
P_c	energy production rate due to unresolved scales
S_{ij}	mean strain rate tensor
U_i	mean velocity
S^2	second invariant of strain rate tensor
C	a constant
ϵ	dissipation rate
ϵ_r	resolved dissipation rate
ϵ_c	unresolved dissipation rate
ν	kinematic viscosity
Π	energy production spectrum
θ	time scale of turbulence
η	turbulent frequency
κ	wavenumber
κ_i	partition wavenumber
κ_c	filter wavenumber in LES
κ_d	Kolmogorov scale

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I. Introduction

In the context of turbulence modeling, a *variable resolution* method is one that resolves more or less of the turbulent fluctuations in different flow regions. This can be accomplished most simply by a *zonal* or *hybrid* method, in which RANS equations, which model all fluctuations, are solved in some flow regions, and LES, which models only small-scale fluctuations, are solved elsewhere. An alternative is a *bridging* (or *seamless*, or *continuous*) model, which interpolates smoothly between RANS and LES. Variable resolution methods for turbulent flows attempt to fill the gap between industrial RANS methods, which may sometimes be too coarsely resolved to be useful, and LES, which might resolve unnecessary details, thereby incurring excessive computational cost. At present, however, variable resolution methods cannot be said to have attained the level of theoretical rigor of either RANS or LES. To contribute to a more sound understanding of these methods, this paper outlines the introduction of spectral closure theories as a foundation for variable resolution methods.

It may be helpful to begin by considering what an ideal RANS/LES hybrid model would be. Suppose that we knew that some hypothetical RANS model was a limit of LES under well-defined analytical conditions. This would entail the possibility of statistical comparisons at the level at least of two-point correlations, so that we could state with confidence when RANS can replace LES; we would also know *a priori* conditions under which the RANS solution will become unrealistic and must be replaced by LES. Suppose further that the transition between RANS and LES can be gradual enough that overlap regions exist in which both models are valid, so that a smooth transition is possible, rather than a transition through a fixed sharp interface. Then it would be possible, at least in principle, to solve the RANS model wherever it is valid and the LES model elsewhere. This would be a zonal method in which LES and RANS are smoothly connected though an overlap region, the location of which is determined as part of the numerical solution.

Since these requirements might seem somewhat unrealistic, we describe a deterministic analog, in which a RANS model is connected, not to LES or DNS, but to a spectral closure.[1] Like the hybrid described above, this hybrid connects a statistically complex, more realistic description (the spectral closure) to a simpler abridged one (the RANS model). In this case, a multiple-scale perturbation analysis demonstrates that when

$$\dot{\epsilon} \ll \epsilon^2/k \quad \text{and} \quad \dot{\ell} \ll k^{1/2} \tag{1}$$

where ϵ is the dissipation rate, k is the turbulent kinetic energy, and ℓ is a length characteristic of the large scales, there is an approximate solution of the spectral closure of the form $E(\kappa, t) = E(\kappa; \epsilon(t), \ell(t))$ where $E(\kappa, t)$ is the time-dependent energy spectrum, and where ϵ and ℓ satisfy a RANS-type two-equation model obtained as part of the perturbation scheme. This model is, however, not applicable if the inequalities in Eq. (1) are not satisfied; in this case, the complete spectral closure must be solved. Since the conditions for an ideal hybrid are satisfied, a (temporal) hybrid model could be constructed that combines the spectral closure and the two-equation model, so that the complex spectral closure is solved only when required by the dynamics.

This description of a hypothetical hybrid is intended only to motivate the theoretical possibility of such a model. Of course, in practice, or at least in current practice, the requirements of the ideal hybrid model are far from being satisfied: we do not know precise conditions under which LES can be replaced by RANS, nor are there known *a priori* conditions under which the RANS description must become inadequate. Instead, the adequacy of RANS is assessed after the fact, by comparison with numerical or experimental data. There is no connection between RANS and even partial statistical information like two-point correlations; thus, the more complete agreement at a refined statistical level needed for a meaningful hybrid method is not available. Interface conditions are typically prescribed according to various empirical heuristics: there is neither theorical guidance, nor even any general concensus, on how to switch between RANS and LES, or on how to treat the buffer region between them. All of these limitations mean that the validity of hybrid models cannot be insured in advance and problems must be fixed as they arise.

Bridging models offer a practical way to overcome at least some of these limitations. In the simplest terms, the idea is that RANS can be connected to LES or DNS by reducing the RANS eddy viscosity according to grid size or some other numerical consideration.[2] The expectation is that reducing the eddy viscosity can "free up" fluctuations since in the limit of zero eddy viscosity and, of course, sufficient numerical resolution, DNS is certainly recovered.

Although this idea is attractive, later work[3] showed that better control over the consistency of the RANS and LES models is necessary: for example, it should be possible to demonstrate that the total energy is

independent of the resolution. Such requirements impose much tighter connection between the RANS and subgrid model formulation than is possible by a simple damping of the eddy viscosity. Yet this requirement poses a difficulty, because the most popular subgrid formulation, Smagorinsky-based LES, is in RANS terminology a zero-equation model; thus, it seems difficult to connect such subgrid models smoothly, say to a two-equation RANS model.

The simplest way to enforce consistency between RANS and LES is to develop a subgrid two-equation model that reduces to a standard RANS model in the limit that all fluctuations are modeled. Two current systematic approaches to this goal are the Partially Integrated Transport Model (PITM) of Schiestel[4] and the Partially Averaged Navier-Stokes (PANS) of Girimaji.[3] The purpose of this paper is to clarify the formulation of PITM with particular reference to the role of spectral closure, which appears to be a natural theoretical framework for this type of modeling. The arguments and conclusions of PANS will be compared with PITM as appropriate. Our immediate concern is to reconcile the derivations of the subgrid dissipation rate equation, because the same conclusion is reached in both PITM and PANS, yet the arguments are quite different. We conclude with some observations on the role of spectral closure and of self-similarity in formulating bridging models.

II. Background from Spectral Closure Theory

PITM originates in Schiestel's multiple-scale model,[5] which we briefly summarize. The foundation is *spectral closure*, the Fourier form of two-point closure for homogeneous turbulence. In the usual formulation, the basic unknown is the energy spectrum $E(\kappa,t)$, the Fourier transform of the two-point correlation function, under the kinematic assumption of isotropy of fluctuations. Important single-point quantities can be expressed as moments of the energy spectrum, for example, the kinetic energy and dissipation rate are

$$k = \int_0^\infty E(\kappa)d\kappa \quad \epsilon = 2\nu \int_0^\infty \kappa^2 E(\kappa)d\kappa \tag{2}$$

The wavenumber variable κ corresponds to the intuitive idea of a "scale of motion."

The *spectral evolution equation*

$$\dot{E}(\kappa,t) = \Pi(\kappa,t) - T(\kappa,t) - 2\nu\kappa^2 E(\kappa,t) \tag{3}$$

can be derived from the Navier-Stokes equations. In Eq. (3), Π is a spectrum of energy input, T is the nonlinear transfer, a third-order moment of the velocity field, and the last term is the dissipation spectrum.[6] Although the time variable is appears in Eq. (3), subsequently it will be assumed rather than written explicitly. The spectral formulation is convenient because it can model the interaction between different scales of motion more or less faithfully, depending on what is used for T. This makes possible a systematic formulation of the interactions between resolved and unresolved scales of motion.

In studies of homogeneous isotropic turbulence, it is usual to assume that the energy input is concentrated at some large scale, so that $\Pi(\kappa)$ is nonzero only inside some interval $|\kappa - \kappa_0| \le \Delta\kappa$. The total production (energy input) rate is

$$P = \int_0^\infty d\kappa\ \Pi(\kappa) \tag{4}$$

A spectrum for production by mean shear can be constructed by using Lumley's[7] inertial range spectrum for shear stress. The result is

$$\Pi(\kappa) = \mathsf{C} S^2 \epsilon^{1/3} \kappa^{-7/3} \tag{5}$$

where $S^2 = S_{ij}S_{ij}$ and $S_{ij} = \partial U_i/\partial x_j + \partial U_j/\partial x_i$ is the mean strain rate, U_i is the mean velocity, and C is a model constant. Eq. (5) is an inertial range result only; some modification near $\kappa = 0$ is assumed to insure finiteness. The crucial difference between the large-scale forcing used in studies of homogeneous isotropic turbulence and production by mean shear is that in the latter case, energy production occurs at all scales, not only at large scales. A simple way to motivate Eq. (5) is to note that the total production due to scales with wavenumbers larger than κ is

$$P(\kappa) = \mathsf{C} S^2 \int_\kappa^\infty dp\ \epsilon^{1/3} p^{-7/3} \propto S^2 \epsilon^{1/3} \kappa^{-4/3} \tag{6}$$

where $\epsilon^{1/3}\kappa^{-4/3}$ is the scale-dependent eddy viscosity corresponding to Kolmogorov inertial range scaling.

Energy conservation by nonlinear interaction implies

$$\int_0^\infty d\kappa\ T(\kappa) = 0 \tag{7}$$

This fact permits the definition

$$T(\kappa) = \frac{\partial}{\partial \kappa}\mathcal{F}(\kappa) \tag{8}$$

where $\mathcal{F}$ is the energy flux satisfying $\mathcal{F}(0) = \mathcal{F}(\infty) = 0$.

It will be useful to note some elementary models for the energy flux. The simplest is the *Kovaznay model*[8]

$$\mathcal{F}(\kappa) = \mathsf{C}\eta(\kappa)E(\kappa) \text{ where } \eta = \sqrt{\kappa^3 E(\kappa)} \tag{9}$$

Here and subsequently, we adopt the convention that C denotes a constant, not necessarily the same one each time it appears. A more realistic model is the *Heisenberg model*[6,8]

$$\mathcal{F}(\kappa) = \mathsf{C}\int_0^\kappa d\mu\ \mu^2 E(\mu)\int_\kappa^\infty dp\ E(p)\theta(p) \text{ where } \theta(p) = 1/\sqrt{p^3 E(p)} \tag{10}$$

The Kovaznay model describes energy transfer as a *stepwise* cascade in wavenumber space; the Heisenberg model treats energy flux as local energy production: the product of a spectral eddy viscosity (the second integral on the right side of Eq. (10)) and the square of a local strain (the first integral). Some refinements are offered by a generalized Heisenberg model,[9]

$$\mathcal{F}(\kappa) = \mathsf{C}\left\{\int_0^\kappa d\mu\ \mu^2 E(\mu)\int_\kappa^\infty dp\ E(p)\theta(p) - \int_0^\kappa d\mu\ \mu^4\int_\kappa^\infty dp\ \frac{E(p)^2\theta(p)}{p^2}\right\} \tag{11}$$

Whereas the Kovaznay and Heisenberg model treat transfer as purely "forward," that is, from large scales to small scales, this model allows transfer from small to large scales, often referred to as energy backscatter. This feature is crucial to a correct formulation of the problem of isotropic decay.

Integrate the spectral evolution equation over all wavenumbers κ using the relations (some of which have already been noted)

$$\begin{aligned}\int_0^\infty d\kappa\ \dot{E}(\kappa) &= \dot{k} \qquad & \int_0^\infty d\kappa\ \Pi(\kappa) &= P\\ \int_0^\infty d\kappa\ T(\kappa) &= 0 \qquad & \int_0^\infty d\kappa\ 2\nu\kappa^2 E(\kappa) &= \epsilon\end{aligned} \tag{12}$$

to obtain the energy balance

$$\dot{k} = P - \epsilon \tag{13}$$

We note that although this equation provides an exact relation, the nonlinear transfer $\mathcal{F}$, the heart of the turbulence problem, plays no role in it.

III. Formulation of PITM

Schiestel[5] refined the energy balance of Eq. (13) by introducing partition wavenumbers $\kappa_1, \cdots \kappa_n$, corresponding spectral "bins" or "slices" $\kappa_{i-1} \le \kappa \le \kappa_i$, and *partial energies*

$$k_i = \int_{\kappa_{i-1}}^{\kappa_i} d\kappa\ E(\kappa) \tag{14}$$

Heuristic arguments were used to obtain coupled sets of two-equation models for the k_i and fluxes $\mathcal{F}(\kappa_i)$. The result is an alternative to numerical integration of Eq. (3) with some flux model like Eq. (10). However, this model is not a consequence of spectral closure, but is instead a heuristic modification specifically designed to connect to the usual RANS models. Our suggestion is that this multiple-scale model could be formulated as a perturbation theory in which the energy flux is a slowly varying function of wavenumber, so that the spectrum is approximately of Kolmogorov form at each wavenumber, but a global Kolmogorov spectrum over the entire inertial range does not necessarily exist.

Chaouat and Schiestel[10] applied this formalism to two-equation subgrid modeling by taking only one subgrid wavenumber shell. In their words, *"In this framework, a cutoff wavenumber κ_c is introduced in the medium range of eddies while the wave number κ_d is located at the end of the inertial range of the spectrum after the transfer zone."* We assume then that their κ_d is the inverse Kolmogorov scale,

$$\kappa_d \sim \left(\frac{\epsilon}{\nu^3}\right)^{1/4} \tag{15}$$

where ν is the kinematic viscosity. With this understanding, the unresolved scales are $\kappa_c \le \kappa \le \kappa_d$ and the resolved scales are $\kappa \le \kappa_c$; the RANS limit is $\kappa_c \to 0$ and the DNS limit is $\kappa_c \to \infty$.

Chaouat and Schiestel[10] define the resolved energy

$$k_r = \int_0^{\kappa_c} d\kappa\ E(\kappa) \tag{16}$$

and subgrid (or unresolved) energy

$$k_c = \int_{\kappa_c}^{\infty} d\kappa\ E(\kappa) \tag{17}$$

so that the sum of the resolved and unresolved energy is the total energy. The *partially* integrated spectral evolution equation contains the terms

$$\begin{aligned} &\int_0^{\kappa_c} d\kappa\ \dot{E}(\kappa) = \dot{k}_r \qquad \int_0^{\kappa_c} d\kappa\ \Pi(\kappa) = P_r \\ &\int_0^{\kappa_c} d\kappa\ T(\kappa) = \int_0^{\kappa_c} d\kappa\ \frac{\partial \mathcal{F}}{\partial \kappa} = \mathcal{F}(\kappa_c) \qquad \int_0^{\kappa_c} d\kappa\ 2\nu\kappa^2 E(\kappa) = \epsilon_r \end{aligned} \tag{18}$$

so we obtain the resolved energy equation

$$\dot{k}_r = P_r - \mathcal{F}(\kappa_c) - \epsilon_r \tag{19}$$

For production by forcing concentrated near $\kappa_0 \ll \kappa_c$, $P_r = P$. We expect dissipation to occur at small scales; then $\epsilon_r \approx 0$, so consequently Eq. (19) becomes

$$\dot{k}_r = P - \mathcal{F}(\kappa_c) \tag{20}$$

In the Heisenberg model Eq. (10),

$$\mathcal{F}(\kappa_c) = \mathsf{C} \int_0^{\kappa_c} d\mu\ \mu^2 E(\mu) \int_{\kappa_c}^{\infty} dp\ E(p)\theta(p) \tag{21}$$

Although we will not solve the Heisenberg model, it suggests the usual heuristic that the subgrid scales act on the resolved scales as an eddy viscosity. We might model this subgrid eddy viscosity by a two-equation *ansatz*

$$\int_{\kappa_c}^{\infty} dp\ E(p)\theta(p) = \mathsf{C}\frac{k_c^2}{\epsilon_c} \tag{22}$$

In this case, we need equations for k_c and $\epsilon_c \approx \epsilon$.

The obvious starting point is to integrate the spectral evolution equation over the unresolved scales. This generates the terms

$$\begin{aligned} &\int_{\kappa_c}^{\infty} d\kappa\ \dot{E}(\kappa) = \dot{k}_c \qquad \int_{\kappa_c}^{\infty} d\kappa\ \Pi(\kappa) = P_c \\ &\int_{\kappa_c}^{\infty} d\kappa\ T(\kappa) = \int_{\kappa_c}^{\infty} d\kappa\ \frac{\partial \mathcal{F}}{\partial \kappa} = -\mathcal{F}(\kappa_c) \qquad \int_{\kappa_c}^{\infty} d\kappa\ 2\nu\kappa^2 E(\kappa) = \epsilon_c \end{aligned} \tag{23}$$

so we obtain

$$\dot{k}_c = P_c + \mathcal{F}(\kappa_c) - \epsilon_c \tag{24}$$

Note that the sum of Eqs. (24) and (19) is the equation for total energy, Eq. (13). For production by forcing concentrated near $\kappa_0 \ll \kappa_c$, $P_c = 0$. For production by shear,

$$P_c = \frac{3}{4}\mathsf{C}S^2\epsilon^{1/3}\kappa_c^{-4/3} \tag{25}$$

is small when κ_c is large, but since the decay is only algebraic, the unresolved production P_c can be significant. Again, since dissipation will occur at very small scales, we expect $\epsilon_c \approx \epsilon$, so

$$\dot{k}_c = P_c + \mathcal{F}(\kappa_c) - \epsilon \tag{26}$$

A key point for Schiestel is that κ_c can be a function of time. Allowing $\kappa_c = \kappa_c(t)$, we have

$$\frac{d}{dt}\int_{\kappa_c(t)}^{\infty} d\kappa\ E(\kappa,t) = \int_{\kappa_c(t)}^{\infty} d\kappa\ \dot{E}(\kappa,t) - \dot{\kappa}_c(t)E(\kappa,t) \tag{27}$$

which adds a term to Eq. (26):

$$\dot{k}_c = P_c + \mathcal{F}(\kappa_c) - E(\kappa_c)\dot{\kappa}_c - \epsilon_c \tag{28}$$

Define a new quantity

$$F_c = \mathcal{F}(\kappa_c) - E(\kappa_c)\dot{\kappa}_c \tag{29}$$

so that

$$\dot{k}_c = P_c + F(\kappa_c) - \epsilon \tag{30}$$

Then solve Eq. (29) for $\dot{\kappa}_c$,

$$\dot{\kappa}_c = \frac{\mathcal{F}(\kappa_c) - F_c}{E(\kappa_c)} \tag{31}$$

Here, some caution seems indicated, since what is happening is a reshuffling of definitions, not the introduction of new relations.

Attempts to find a dissipation rate equation from "equilibrium" considerations alone introduce the Kolmogorov relation

$$\kappa = \mathsf{C}\frac{\epsilon}{k^{3/2}} \tag{32}$$

which is just a restatement of the Kolmogorov energy spectrum $E(\kappa) = \mathsf{C}\epsilon^{2/3}\kappa^{-5/3}$. Chaouat and Schiestel[10] also use the differential form of this relation

$$\frac{\dot{\epsilon}}{\epsilon} = \frac{3}{2}\frac{\dot{k}}{k} + \frac{\dot{\kappa}}{\kappa} \tag{33}$$

Here however we come to a difficulty, because if we integrate the Kolmogorov spectrum from κ_c to the Kolmogorov scale κ_d, we obtain

$$k_c - k_d = \mathsf{C}\int_{\kappa_c}^{\kappa_d} \epsilon^{2/3}\kappa^{-5/3} = \frac{3}{2}\mathsf{C}\epsilon^{2/3}\left(\kappa_d^{-2/3} - \kappa_c^{-2/3}\right) \tag{34}$$

Therefore, after some rearrangement,

$$\left[\frac{1}{\kappa_c^{2/3}} - \frac{1}{\kappa_d^{2/3}}\right]^{-3/2} = \mathsf{C}\frac{\epsilon}{(k_c - k_d)^{3/2}} \tag{35}$$

(recall the convention on the use of the constant C), whereas Chaouat and Schiestel[10] state a dimensionally consistent alternative

$$\kappa_d - \kappa_c = \mathsf{C}\frac{\epsilon}{k_c^{3/2}} \tag{36}$$

That Eqs. (35) and (36) are not equivalent is shown by the limits as $\kappa_d \to \infty$ of their left sides:

$$\left[\frac{1}{\kappa_c^{2/3}} - \frac{1}{\kappa_d^{2/3}}\right]^{-3/2} \to \kappa_c \tag{37}$$

but

$$\kappa_d - \kappa_c \to \infty \tag{38}$$

At this point, a basic principle, or perhaps belief, about turbulence becomes relevant: it is that limits $\kappa_d \to \infty$, or equivalently, in view of Eq. (15), $\nu \to 0$, are always finite. The idea is that letting $\nu \to 0$ creates more small scales but leaves large scales unchanged. This idea can be stated as the existence of a high Re "fixed point;" it permits the usual modeling heuristic that Reynolds number appears only in *low* Reynolds number models. This principle would seem to favor Eq. (35) over Eq. (36); but perhaps it is uncertain that this principle applies to the interval $\kappa_c \le \kappa \le \kappa_d$.

Meanwhile, completing the model requires two more relations. Schiestel[5] used the Kovaznay closure for the (static) flux $\mathcal{F}$, given by Eq. (9). This might be open to the objection that it uses both a model (Kovaznay's) and its solution (the Kolmogorov spectrum), but Chaouat and Schiestel[10] do not use any particular closure for the flux; instead, they set

$$F_c = \epsilon \tag{39}$$

to eliminate F_c from the equations.

We complete the derivation using what we think are correct equations, but following Chaouat and Schiestel[10] by leaving the flux model unspecified. Return to Eq. (31) and substitute Eq. (39),

$$\dot{\kappa}_c = \frac{\mathcal{F}(\kappa_c) - F_c}{E(\kappa_c)} = \frac{\mathcal{F}(\kappa_c) - \epsilon}{E(\kappa_c)} = \sigma \frac{\kappa_c}{k_c}(\mathcal{F}(\kappa_c) - \epsilon) \tag{40}$$

where

$$\sigma = \frac{k_c}{\kappa_c E(\kappa_c)} \tag{41}$$

is a (possibly) scale-dependent factor, although in an inertial range, $\sigma = \frac{3}{2}$ is a constant.

Next, return to Eq. (33) to obtain

$$\dot{\epsilon} = \frac{3}{2}\frac{\dot{k}_c}{k_c}\epsilon + \frac{\dot{\kappa}_c}{\kappa_c}\epsilon = \frac{3}{2}\frac{\epsilon}{k_c}(P_c + F_c) - \frac{3}{2}\frac{\epsilon^2}{k_c} + \sigma\frac{\epsilon^2}{k_c}\left(\frac{\mathcal{F}(\kappa_c)}{\epsilon} - 1\right) \tag{42}$$

This equation states a subgrid dissipation rate equation

$$\dot{\epsilon} = C_{\epsilon 1}^{sg}\frac{\epsilon}{k_c}(P_c + F_c) - C_{\epsilon 2}^{sg}\frac{\epsilon^2}{k_c} \tag{43}$$

with

$$C_{\epsilon 1}^{sg} = \frac{3}{2} \quad C_{\epsilon 2}^{sg} = \frac{3}{2} - \sigma\left(\frac{\mathcal{F}(\kappa_c)}{\epsilon} - 1\right) \tag{44}$$

Because of using Eq. (36) instead of Eq. (35), a much different result for the constants is obtained in Chaouat and Schiestel:[10]

$$C_{\epsilon 1}^{sg} = \frac{3}{2} \tag{45}$$

but

$$C_{\epsilon 2}^{sg} = \frac{3}{2} - \frac{k_c}{\kappa_d E(\kappa_d)}\left(\frac{\mathcal{F}(\kappa_c)}{\epsilon} - 1\right) \tag{46}$$

so that when $k_c = k$ and all of the kinetic energy is modeled, we obtain the RANS model coefficient

$$C_{\epsilon 2} = \frac{3}{2} - \frac{k}{\kappa_d E(\kappa_d)}\left(\frac{\mathcal{F}(\kappa_c)}{\epsilon} - 1\right) \tag{47}$$

In either case, the value $C_{\epsilon 1}^{sg} = \frac{3}{2}$, obtained here because of the Kolmogorov relation Eq. (33), is in reasonable agreement, perhaps coincidentally, with the value obtained by calibrating to homogeneous shear flow. However, since $\kappa_d E(\kappa_d) \propto \kappa_d^{-2/3}$ (recall that κ_d is assumed in Chaouat and Schiestel[10] to be at the end of the inertial range), the model constants in Eqs. (46) and (47) are divergent as $\kappa_d \to \infty$.

That this is unsatisfactory is of course recognized in Chaouat and Schiestel,[10] and some arguments are given to lead to a finite result. One possible approach is to evaluate the ratio of the last terms on the right sides of Eqs. (46) and (47); this (purely formal) step cancels the divergences to give

$$\frac{C_{\epsilon 2}^{sg} - \frac{3}{2}}{C_{\epsilon 2} - \frac{3}{2}} = \frac{k_c}{k} \tag{48}$$

and so permits Chaouat and Schiestel[10] Eq. (46),

$$C_{\epsilon 2}^{sg} - \frac{3}{2} = \frac{k_c}{k}(C_{\epsilon 2} - \frac{3}{2}) \tag{49}$$

where now we can set $C_{\epsilon 2}$ to some empirical value. We stress that although this formal manipulation produces a finite answer, this theory does not predict a definite value of $C_{\epsilon 2}$.

To conclude, we have obtained the subgrid dissipation rate equation in the form

$$\begin{aligned} \dot{\epsilon} &= C_{\epsilon 1}\frac{\epsilon}{k_c}(P_c + F_c) - \left(C_{\epsilon 1} + \frac{k_c}{k}(C_{\epsilon 2} - C_{\epsilon 1})\right)\frac{\epsilon^2}{k_c} \\ &= C_{\epsilon 1}\frac{\epsilon}{k_c}(P_c + F_c - \epsilon) - (C_{\epsilon 2} - C_{\epsilon 1})\frac{\epsilon^2}{k} \end{aligned} \tag{50}$$

although the argument cannot inspire complete confidence. It is therefore important that the same subgrid dissipation rate equation is obtained in PANS by a simpler, more straightforward argument:

1. *Postulate* the standard equation

$$\dot{\epsilon} = C_{\epsilon 1}\frac{\epsilon}{k}P - C_{\epsilon 2}\frac{\epsilon^2}{k} \tag{51}$$

2. Remark that in any self-similar flow, k_c/k is constant, so $\frac{\dot{k}_c}{k_c} = \frac{\dot{k}}{k}$. Then

$$P - \epsilon = \frac{k}{k_c}(P_c + F_c - \epsilon) \tag{52}$$

The appeal to self-similarity is justified by the fact that the dissipation rate equation is calibrated to self-similar flows; we will return to this point shortly.

3. Substituting in the standard equation,

$$\dot{\epsilon} = C_{\epsilon 1}\frac{\epsilon^2}{k} + C_{\epsilon 1}\frac{\epsilon}{k_c}(P_c + F_c - \epsilon) - C_{\epsilon 2}\frac{\epsilon^2}{k} \tag{53}$$

in agreement with Eq. (50) after simple regrouping of terms. This alternate derivation suggests that although we can question some aspects of the argument of Chaouat and Schiestel,[10] incuding Eq. (36) and the consequent need to cancel divergent terms (although we stress that Eq. (48) is our own interpretation of Chaouat and Schiestel[10]), the final result is consistent with an alternative derivation which does not raise these questions.

For the bridging model to succeed, it must indeed connect RANS to LES. In the limit $\kappa_c \to 0$, Eq. (28) reduces to Eq. (13) because κ_c being constant, $\dot{\kappa}_c = 0$; the production becomes the total production, and the dissipation rate is unchanged. The reduction of the resolved dissipation rate equation to its RANS counterpart is obvious from Eq. (50) on setting $F_c = 0$, $P_c = P$ and $k_c = k$.

What is perhaps less trivial is the DNS limit $\kappa_c \gg \kappa_d$. In this limit, $k_c \to 0$; substituting Eq. (49) in Eq. (43) shows that in this limit, we can have a steady state in which $P_c + F_c = \epsilon$. Then the entire two-equation subgrid model is replaced by a local equilibrium condition and the two-equation model effectively disappears in the limit of high resolution.

Recall that for production by a mean shear, the subgrid production is given by Eq. (25). For very large κ_c corresponding to highly resolved LES, approaching DNS, $P_c \propto \kappa_c^{-4/3} \ll F_c \propto \kappa_c^0$: direct production by shear is dominated by local nonlinear transfer. Thus, homogeneous shear is not an appropriate calibration case for large κ_c; consistency with a forced steady state is more plausible. This change of calibration case is implied by the possibility, noted previously, of a steady state in the subgrid scales in the DNS limit. This consistency check gives some theoretical support to the subgrid dissipation rate equation common to PITM and PANS.

IV. Comments on the Subgrid Dissipation Rate Equation

In a self-similar flow, quantities with the same units are constant multiples of each other. Thus, in self-similar decay, the equation

$$\frac{\dot{\epsilon}}{\epsilon} = -\mathsf{C}\frac{\epsilon}{k} \tag{54}$$

is not a "model;" it is rather a consequence of the assumption of self-similarity. Since different self-similar flows can give different values of the proportionality constant, such a relation is not at all a general principle, and there is therefore no reason to believe that Eq. (54) can be deduced from any theory. The standard dissipation rate equation Eq. (51) is simply a more general statement of self-similarity that can apply to two self-similar flows, usually chosen as homogeneous shear and decay: data from these flows is used to determine the model constants.

As derived in Eqs. (51)–(53), the subgrid dissipation rate equation follows from the usual self-similarity assumptions and the additional observation that k_c/k is constant in a self-similar flow. Since we do not consider that this equation can be derived from anything more basic, the elaborate explanation in Chaouat and Schiestel[10] may not add anything significant to the very elementary PANS argument that is based openly and explicitly on self-similarity alone.

We have stressed the role of self-similarity in formulating the standard and the subgrid dissipation rate equations. These equations describe certain calibration cases; their extension to other problems comes with no guarantees. A different viewpoint[1] is that the length-scale determining equation of the two-equation model explicitly must describe *not* self-similarity or "equilibrium" but *departures from them.* Woodruff and Rubinstein[1] propose a two-equation model based on the classical Heisenberg model Eq. (10) that is consistent with this idea.

V. Conclusions

The spectral formulation of PITM is a useful and convenient way to discuss variable-resolution modeling. Deriving the subgrid energy equation is straightforward; the new element in the role of nonlinear transfer in addition to direct energy production by shear. In PANS, the subgrid dissipation rate equation is formulated in terms of self-similarity (or 'fixed-point' analysis); the agreement with the PITM formulation, obtained by an apparently different argument, perhaps illustrates the extent to which the assumption of a (self-similar) Kolmogorov spectrum limits the range of possible end results. An interesting new feature of the subgrid dissipation rate equation is that one calibration case changes from homogeneous shear to steady-state forced turbulence as the resolution increases. Finally, the subgrid two-equation becomes nugatory in the DNS limit; this fact insures the consistency of the bridging model with this limit.

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