# On Time Delay Margin Estimation for Adaptive Control and Optimal Control Modification

Nhan Nguyen*

*NASA Ames Research Center, Moffett Field, CA 94035*

### Abstract

This paper presents methods for estimating time delay margin for adaptive control of input delay systems with almost linear structured uncertainty. The bounded linear stability analysis method seeks to represent an adaptive law by a locally bounded linear approximation within a small time window. The time delay margin of this input delay system represents a local stability measure and is computed analytically by three methods: Pade approximation, Lyapunov-Krasovskii method, and the matrix measure method. These methods are applied to the standard model-reference adaptive control, $\sigma$-modification adaptive law, and optimal control modification adaptive law. The windowing analysis results in non-unique estimates of the time delay margin since it is dependent on the length of a time window and parameters which vary from one time window to the next. The optimal control modification adaptive law overcomes this limitation in that, as the adaptive gain tends to infinity and if the matched uncertainty is linear, then the closed-loop input delay system tends to a LTI system. A lower bound of the time delay margin of this system can then be estimated uniquely without the need for the windowing analysis. Simulation results demonstrates the feasibility of the bounded linear stability method for time delay margin estimation.

## 1 Introduction

Input delay systems are generally non-minimum phase. For linear input delay systems, feedback gain must be kept to a reasonable value to maintain stability. Input delay influences stability of adaptive control in a similar manner. Adaptive gain is used to control the rate of adaptation in adaptive control. For model-reference adaptive control, it is well-known that as the adaptive gain increases, the closed-loop system loses robustness, thereby rendering it susceptible to instability in the presence of unmodeled dynamics and or input time delay. Thus, to maintain stability of an input delay adaptive system, the adaptive gain must be carefully selected. For a given value of the adaptive gain, there exists a corresponding value of input time delay for which the adaptive system is on the verge of instability. This is known as a time delay margin. To maintain stability, the adaptive gain of the system must be kept below the value that corresponds to the time delay margin of the system.

Global stability analysis for input delay adaptive systems is a challenging problem. Lyapunov-Krasovskii method or Lyapunov-Razumikhin method are much more difficult to apply to an adaptive system. Even for a simple scalar LTI system, both the Lyapunov-Krasovskii method or Lyapunov-Razumikhin method can result in conservative estimates of the time delay margin [1]. So, even if a global stability analysis for input delay adaptive control is available, the conservatism in the estimation could render it impractical. The lack of available analytical methods for computing the time delay margin of adaptive control is a hurdle for certification of adaptive control [2].

While global stability analysis is challenging, several studies have recently been done to address local stability of input delay adaptive systems. One such method applies a Pade approximation to transform an input delay system into a delay-free higher-order system [3]. The transformed system is then analyzed using the standard Lyapunov method to estimate the time delay margin. However, this approach yields a highly conservative time delay margin even for a simple scalar adaptive system [3] thereby rendering the approach rather less practical. Another method, called bounded linear stability analysis, attempts to analyze the stability margins of an adaptive system in a local context [4, 5]. The method approximates an adaptive system as a series of bounded linear systems inside time windows. The windowing analysis allows the bounded linear systems to be analyzed using linear analytical tools. The method has been shown

---

*Research Scientist, AIAA Associate Fellow, nhan.t.nguyen@nasa.gov

to provide a less conservative estimate of the time delay margin. Subsequently, a similar method has been developed using the windowing approach to estimate the stability margin of an adaptive system [6]. LMI methods also has been used to analyze stability of adaptive control [7]. The Lyapunov-Razumikhin method has been used to estimate the time delay margin for a simple scalar adaptive system [8]. The method requires optimization of the candidate Lyapunov function in order to reduce the conservatism in the estimated time delay margin.

This paper extends the bounded linear stability analysis method for analyzing input delay adaptive control with almost linear structured uncertainty. Stability of an input delay adaptive system is analyzed by three methods: Pade approximation, Lyapunov-Krasovskii method, and the matrix measure method, to estimate the local time delay margin of a bounded linear system inside each time window. Three different adaptive laws are used: the standard MRAC, $\sigma$-modification adaptive law [9], and optimal control modification adaptive law [10]. Asymptotic analysis of the time delay margin as the adaptive gain tends to infinity is performed to study to effect of large adaptive gain on the time delay margin. Simulations are studied to demonstrate the feasibility of the time delay margin estimation.

## 2   Input Delay Adaptive Systems and Bounded Linearity Stability Analysis

Given an input delay nonlinear plant

$$\dot{x}(t) = Ax(t) + B\left[u(t - t_d) + \Theta^{*\top}\Phi(x(t))\right] \tag{1}$$

where $x(t) : [0, \infty) \to \mathbb{R}^n$ is a state vector, $u(t) : [0, \infty) \to \mathbb{R}^p$ is a control vector, $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times p}$ are known such that the pair $(A, B)$ is controllable, $\Theta^* \in \mathbb{R}^{m \times p}$ is an unknown constant weight matrix that represents a parametric uncertainty, $\Phi(x(t)) : \mathbb{R}^n \to \mathbb{R}^m$ is a vector of known functions, and $t_d$ is an input time delay.

The structure of the uncertainty is assumed to be linearly dominant. That is

$$\Phi(x(t)) = x(t) + \delta(x) \tag{2}$$

where $\|\delta(x)\| \ll \|x(t)\|$ is small.

The input delay $t_d$ could also be viewed as the time delay margin for robustness against unmodeled dynamics of the delay-free system

$$\dot{x}(t) = Ax(t) + B\left[u(t) + \Theta^{*\top}\Phi(x(t))\right] \tag{3}$$

The reference model is specified as

$$\dot{x}_m(t) = A_m x_m(t) + B_m r(t) \tag{4}$$

where $A_m \in \mathbb{R}^{n \times n}$ is Hurwitz and known, $B_m \in \mathbb{R}^{n \times p}$ is also known, and $r(t) : [0, \infty) \to \mathbb{R}^p \in \mathscr{L}_\infty$ is a bounded command vector.

Defining the tracking error as $e(t) = x_m(t) - x(t)$, then the controller $u(t)$ is specified by

$$u(t) = -K_x x(t) + K_r r(t) - u_{ad}(t) \tag{5}$$

where $K_x \in \mathbb{R}^{p \times n}$ and $K_r \in \mathbb{R}^{p \times p}$ are known nominal gain matrices, and $u_{ad}(t) \in \mathbb{R}^p$ is an adaptive signal given by

$$u_{ad}(t) = \Theta^\top(t)\Phi(x) \tag{6}$$

where $\Theta \in \mathbb{R}^{m \times p}$ is an estimate of $\Theta^*$.

Assuming that the model matching conditions can be satisfied, then

$$A_m = A - BK_x \tag{7}$$

$$B_m = BK_r \tag{8}$$

The standard model-reference adaptive control law is

$$\dot{\Theta}(t) = -\Gamma\Phi(x(t))e^\top(t)PB \tag{9}$$

where $\Gamma = \Gamma^\top \in \mathbb{R}^{m \times m} > 0$ and $P = P^\top \in \mathbb{R}^{n \times n} > 0$ solves the Lyapunov equation

$$PA_m + A_m^\top P = -Q \tag{10}$$

2

where $Q = Q^\top \in \mathbb{R}^{n \times n} > 0$.

The error equation corresponding to the input delay system (3) can be derived by substituting the time-delay version of the controller from Eq. (5), thus resulting in

$$\dot{e}(t) = Ae(t) - BK_x e(t - t_d) + Bu_{ad}(t - t_d) - B\Theta^{*\top}\Phi(x(t)) - BK_x[x_m(t) - x_m(t - t_d)] + BK_r[r(t) - r(t - t_d)] \quad (11)$$

To analyzing this system, the bounded linear stability analysis method has been proposed to approximate the adaptive system as a bounded linear system within a given time window [4, 5]. The windowing analysis then permits the use of linear tools to analyze stability of the approximated bounded linear system inside a given time window.

**Theorem 1:** The adaptive law (9) is bounded locally by a linear approximation as

$$\dot{\Theta}^\top(t)\Phi(x(t)) = -\gamma B^\top Pe(t) \quad (12)$$

where $\gamma$ is a constant defined locally and retrospectively as

$$\gamma = \frac{1}{T_0} \int_{t_i - T_0}^{t_i} \Phi^\top(x(\tau))\Gamma\Phi(x(\tau))\,d\tau \quad (13)$$

for $t \in [t_i - T_0, t_i)$, where $t_0 = 0$, $t_i = t_{i-1} + T_0$ and $i = 1, 2, \ldots, n \to \infty$.

**Proof:** Choose a Lyapunov candidate function

$$V(t) = e^\top(t)Pe(t) + \text{trace}\left[\tilde{\Theta}^\top(t)\Gamma^{-1}\tilde{\Theta}(t)\right] \quad (14)$$

The error equation of the delay-free system is

$$\dot{e}(t) = A_m e(t) + B\tilde{\Theta}^\top(t)\Phi(x(t)) \quad (15)$$

where $\tilde{\Theta}(t) = \Theta(t) - \Theta^*$.

Denoting $V_g(t)$ as the Lyaponov candidate function to be evaluated globally using the adaptive law (9) as follows:

$$\dot{V}_g(t) = -e^\top(t)Qe(t) + 2e^\top(t)PB\tilde{\Theta}^\top(t)\Phi(x(t)) - 2\text{trace}\left[\tilde{\Theta}^\top(t)\Phi(x(t))e^\top(t)PB\right] = -e^\top(t)Qe(t) \leq 0 \quad (16)$$

Denoting $V_l(t)$ as the Lyapunov candidate function to be evaluated locally with a time window using the locally bounded linear approximation (12) yields

$$\dot{V}_l(t) = -e^\top(t)Pe(t) + 2e^\top(t)PB\tilde{\Theta}^\top(t)\Phi(x(t)) - 2\text{trace}\left[\tilde{\Theta}^\top(t)\Gamma^{-1}\dot{\tilde{\Theta}}(t)\right]$$

$$= \dot{V}_g(t) + 2e^\top(t)PB\tilde{\Theta}^\top(t)\Phi(x(t)) - 2\text{trace}\left\{\tilde{\Theta}^\top(t)\Phi(x(t))\left[\Phi^\top(x(t))\Gamma\Phi(x(t))\right]^{-1}\Phi^\top(x(t))\dot{\tilde{\Theta}}(t)\right\}$$

$$= \dot{V}_g(t) + 2e^\top(t)PB\tilde{\Theta}^\top(t)\Phi(x(t))\left\{1 - \gamma\left[\Phi^\top(x(t))\Gamma\Phi(x(t))\right]^{-1}\right\} \quad (17)$$

for $t \in [t_i - T_0, t_i)$, where $t_0 = 0$, $t_i = t_{i-1} + T_0$ and $i = 1, 2, \ldots, n \to \infty$.

Consider the integral form of Eq. (17)

$$\int_{t_i - T_0}^{t_i} \dot{V}_l(t)\,dt = \int_{t_i - T_0}^{t_i} \dot{V}_g(t)\,dt + \int_{t_i - T_0}^{t_i} 2e^\top(t)PB\tilde{\Theta}^\top(t)\Phi(x(t))\left\{1 - \gamma\left[\Phi^\top(x(t))\Gamma\Phi(x(t))\right]^{-1}\right\}dt \quad (18)$$

Then for $V_g(t)$ and $V_l(t)$ to be equal in a time window

$$\int_{t_i - T_0}^{t_i} \dot{V}_l(t)\,dt = \int_{t_i - T_0}^{t_i} \dot{V}_g(t)\,dt \quad (19)$$

which implies

$$\int_{t_i - T_0}^{t_i} 2e^\top(t)PB\tilde{\Theta}^\top(t)\Phi(x(t))\left\{1 - \gamma\left[\Phi^\top(x(t))\Gamma\Phi(x(t))\right]^{-1}\right\}dt = 0 \quad (20)$$

3

It is important to note that this is a definite integral equation for which a valid solution can include a constant solution of $\gamma$. Such a solution is called a "weak-form" or integral-form solution which is valid only over a finite time interval. In contrast, the "strong-form" solution is a global solution that satisfies for all time. In the windowing analysis, the weak-form solution is used.

The mean value theorem for integration states that

$$\int_a^b F(t)G(t)\,dt = F(c)\int_a^b G(t)\,dt \tag{21}$$

where $c \in [a,b]$ and $G(t) \geq 0$.

Let $\bar{t} \in [t_i - T_0, t_i)$, then applying the mean value theorem for integration to Eq. (20) yields

$$\int_{t_i-T_0}^{t_i} 2e^\top(t)PB\tilde{\Theta}^\top(t)\Phi(x(t))\left[\Phi^\top(x(t))\Gamma\Phi(x(t))\right]^{-1}\left[\Phi^\top(x(t))\Gamma\Phi(x(t)) - \gamma\right]dt =$$

$$+2e^\top(\bar{t})PB\tilde{\Theta}^\top(\bar{t})\Phi(x(\bar{t}))\left[\Phi^\top(x(\bar{t}))\Gamma\Phi(x(\bar{t}))\right]^{-1}\left[\int_{t_i-T_0}^{t_i}\Phi^\top(x(t))\Gamma\Phi(x(t))\,dt - \gamma T_0\right] = 0 \tag{22}$$

Hence, (13) is thus obtained. Then it follows that

$$V_l(t_i) - V_l(t_i - T_0) = V_g(t_i) - V_g(t_i - T_0) \leq 0 \tag{23}$$

Thus, the local Lyapunov candidate function $V_l(t)$ is a piecewise approximation of the global Lyapunov candidate function $V_g(t)$ where their values are equal at the beginning and end points of a time window.

Using the bounded linear approximation of the adaptive law (9), one gets a piecewise locally bounded linear approximation of the standard MRAC adaptive law (9)

$$\dot{u}_{ad}(t) = \dot{\Theta}^\top(t)\Phi(x(t)) + \Theta^\top(t)\dot{\Phi}(x(t)) \approx -\gamma B^\top Pe(t) + \Theta^\top(t)\dot{\Phi}(x(t)) \tag{24}$$

for $t \in [t_i - T_0, t_i)$, where $t_0 = 0$, $t_i = t_{i-1} + T_0$, and $i = 1, 2, \ldots, n \to \infty$.

The second term in the right hand side can be locally approximated by a first-order Taylor's series as

$$\Theta^\top(t)\dot{\Phi}(x(t)) = \Theta_i^\top\sum_{j=1}^n\frac{\partial\Phi(x(t_i))}{\partial x_j}\dot{x}_j(t_i) + \Theta_i^\top\sum_{j=1}^n\sum_{k=1}^n\frac{\partial^2\Phi(x(t_i))}{\partial x_j\partial x_k}\dot{x}_j(t_i)[x_k(t) - x_k(t_i)]$$

$$+\Theta_i^\top\sum_{j=1}^n\frac{\partial\Phi(x(t_i))}{\partial x_j}[\dot{x}_j(t) - \dot{x}_j(t_i)] + \ldots$$

$$= \Theta_i^\top\Phi_i'[\dot{x}_m(t) - \dot{e}(t)] + \Theta_i^\top\Phi_i''\dot{x}_i[x_m(t) - e(t) - x_i] + \ldots \tag{25}$$

where $\left(\Phi_i'\right)_j = \partial\Phi(x(t_i))/\partial x_j$ and $\left(\Phi_i''\dot{x}_i\right)_j = \sum_{k=1}^n\dot{x}_k(t_i)\partial^2\Phi(x(t_i))/\partial x_j\partial x_k$, $j = 1, \ldots, n$.

Thus, the locally bounded linear approximation of the error equation and the standard MRAC adaptive law are expressed as

$$\dot{e}(t) = Ae(t) - BK_xe(t - t_d) + Bu_{ad}(t - t_d) - B\Theta^{*\top}\Phi_i - B\Theta^{*\top}\Phi_i'[x_m(t) - e(t) - x_i] - BK_x[x_m(t) - x_m(t - t_d)]$$

$$+ BK_r[r(t) - r(t - t_d)] \tag{26}$$

$$\dot{u}_{ad}(t) = -\gamma B^\top Pe(t) + \Theta_i^\top\Phi_i'[\dot{x}_m(t) - \dot{e}(t)] \tag{27}$$

for $t \in [t_i - T_0, t_i)$, where $t_0 = 0$, $t_i = t_{i-1} + T_0$ and $i = 1, 2, \ldots, n \to \infty$.

Equations (26) and (27) show that the stability of the locally bounded linear approximation depends on several factors:

- The initial condition $x(0)$

- The structure of the matched uncertainty $\Phi(x(t))$

- The parametric uncertainty $\Theta^*$

4

- The plant and reference model matrices $A$, $B$, $A_m$, and $B_m$

- The adaptive gain parameter $\gamma$ which includes the adaptive gain $\Gamma$ as well as the square of the amplitude of $\Phi(x(t))$.

Thus, it can be seen that this bounded linear approximation does capture the complex nature of stability of a nonlinear adaptive control system, at least in a local sense.

In a special case when $\Phi(x(t)) = x(t)$, then the bounded linear approximation of the error equation and the standard MRAC adaptive law become

$$\dot{e}(t) = Ae(t) - BK_x e(t - t_d) + Bu_{ad}(t - t_d) - B\Theta^{*\top}[x_m(t) - e(t)] - BK_x[x_m(t) - x_m(t - t_d)] + BK_r[r(t) - r(t - t_d)] \tag{28}$$

$$\dot{u}_{ad}(t) = -\gamma B^\top Pe(t) + \Theta_i^\top[\dot{x}_m(t) - \dot{e}(t)] \tag{29}$$

# 3   Time Delay Margin Estimation of LTI Systems

Consider an input delay closed-loop LTI system

$$\dot{x}(t) = Ax(t) - BKx(t - t_d) \tag{30}$$

where $x(t): [0, \infty) \to \mathbb{R}^n$ and $\lambda(A - BK) \in \mathbb{C}^-$, i.e., $A - BK$ is Hurwitz.

The time delay margin is defined by the following characteristic equation

$$\det\left(j\omega I - A + BKe^{-j\omega t_d}\right) = 0 \tag{31}$$

For simple systems, analytical solutions of $t_d$ can be computed, but in general such solutions are not easily obtained. We present three methods for estimating the time delay margin.

## 3.1   Pade Approximation

The Laplace transform of the input delay LTI system is

$$sX(s) = AX(s) - BKX(s)e^{-t_d s} \tag{32}$$

Consider the following first-order Pade approximation

$$e^{-t_d s} = \frac{2 - t_d s}{2 + t_d s} \tag{33}$$

Then the approximate input delay system becomes

$$(2 + t_d s)sX(s) = (2 + t_d s)AX(s) - (2 - t_d s)BKX(s) \tag{34}$$

In the time domain, this is expressed as

$$t_d \ddot{x}(t) = (-2I + t_d A + t_d BK)\dot{x}(t) + 2(A - BK)x(t) \tag{35}$$

The time delay margin is then found by

$$\det\left[\omega^2 I + j\omega\left(-\frac{2}{t_d}I + A + BK\right) + \frac{2}{t_d}(A - BK)\right] = 0 \tag{36}$$

Alternatively, the time delay margin can also be obtained as

$$\det\begin{bmatrix} j\omega I & -I \\ -\frac{2}{t_d}(A - BK) & j\omega + \frac{2}{t_d}I - A - BK \end{bmatrix} = 0 \tag{37}$$

5

Example: Given

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}, BK = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$$

The time delay margin using the first-order Pade approximation is estimated to be $t_d = 0.528$ sec.
The exact results can be determined from

$$\det \begin{bmatrix} j\omega & -1 \\ 1 & j\omega - 1 + 2\left(\cos \omega t_d^* - j \sin \omega t_d^*\right) \end{bmatrix} = -\omega^2 + 2\omega \sin \omega t_d^* + 1 - j\omega\left(1 - 2\cos \omega t_d^*\right) = 0$$

$$\omega = \frac{\sqrt{3} + \sqrt{7}}{2} = 2.189 \text{ rad/sec}$$

$$t_d = \frac{2\pi}{3\left(\sqrt{3} + \sqrt{7}\right)} = 0.478 \text{ sec}$$

So the time delay margin by Pade approximation is non-conservative.

## 3.2  Lyapunov-Krasovskii Method

Stability of time-delay differential equations based on the Lyapunov-Krasovskii method using Lyapunov-Krasovskii functionals have been studied exhaustively [11, 12]. The negative-definiteness of the time derivative of a Lyapunov-Krasovskii functional results in a linear matrix inequality that can be solved for a time delay margin. The solution is generally non-unique since it depends on the choice of a Lyapunov-Krasovskii functional. As a result, the time delay margin obtained by the Lyapunov-Krasovskii method can be conservative.

**Theorem 2:** For the closed-loop input delay system (30), the system is asymptotically stable if the following linear matrix inequality is satisfied:

$$(A - BK)^\top P + P(A - BK) + (\alpha + \beta) t_d PBKK^\top B^\top P + \frac{t_d}{\alpha} A^\top A + \frac{t_d}{\beta} K^\top B^\top BK < 0 \tag{38}$$

The time delay margin is the largest value that renders the LMI feasible.
**Proof:** For the input delay LTI system (30), we write

$$\int_{t-t_d}^{t} \dot{x}(\tau) d\tau = x(t) - x(t - t_d) \tag{39}$$

Then

$$x(t - t_d) = x(t) - \int_{t-t_d}^{t} Ax(\tau) d\tau + \int_{t-t_d}^{t} BKx(\tau - t_d) d\tau \tag{40}$$

The input delay system now becomes

$$\dot{x}(t) = (A - BK)x(t) + BK \int_{t-t_d}^{t} Ax(\tau) d\tau - BK \int_{t-t_d}^{t} BKx(\tau - t_d) d\tau \tag{41}$$

Consider the following Lyapunov-Krasovskii functional [11]

$$V(t) = x^\top(t) Px(t) + \int_{t-t_d}^{t} \int_{\tau}^{t} x^\top(\theta) Qx(\theta) d\theta d\tau + \int_{t-t_d}^{t} \int_{\tau-t_d}^{t} x^\top(\theta) Sx(\theta) d\theta d\tau > 0 \tag{42}$$

where $P = P^\top > 0$.
Evaluating $\dot{V}(t)$ yields

$$\dot{V}(t) = \dot{x}^\top(t) Px(t) + x^\top(t) P\dot{x}(t) + \int_{t-t_d}^{t} x^\top(t) Qx(t) d\tau - \int_{t-t_d}^{t} x^\top(\tau) Qx(\tau) d\tau + \int_{t-t_d}^{t} x^\top(t - t_d) Sx(t - t_d) d\tau$$

$$- \int_{t-t_d}^{t} x^\top(\tau - t_d) Sx(\tau - t_d) d\tau \tag{43}$$

6

This becomes

$$\dot{V}(t) = x^\top(t)\left[(A-BK)^\top P + P(A-BK)\right]x(t) + 2x^\top(t)PBK\int_{t-t_d}^t Ax(\tau)d\tau - 2x^\top(t)PBK\int_{t-t_d}^t BKx(\tau-t_d)d\tau$$

$$+ t_d x^\top(t)Qx(t) - \int_{t-t_d}^t x^\top(\tau)Qx(\tau)d\tau + t_d x^\top(t)Sx(t) - \int_{t-t_d}^t x^\top(\tau-t_d)Sx(\tau-t_d)d\tau \quad (44)$$

From the following quadratic expressions

$$\alpha t_d x^\top(t)PBKK^\top B^\top Px(t) - 2x^\top(t)PBK\int_{t-t_d}^t Ax(\tau)d\tau + \frac{1}{\alpha t_d}\left(\int_{t-t_d}^t x^\top(\tau)A^\top d\tau\right)\left(\int_{t-t_d}^t Ax(\tau)d\tau\right) \geq 0 \quad (45)$$

$$\beta t_d x^\top(t)PBKK^\top B^\top Px(t) + 2x^\top(t)PBK\int_{t-t_d}^t BKx(\tau-t_d)d\tau$$

$$+ \frac{1}{\beta t_d}\left(\int_{t-t_d}^t x^\top(\tau-t_d)K^\top B^\top d\tau\right)\left(\int_{t-t_d}^t BKx(\tau-t_d)d\tau\right) \geq 0 \quad (46)$$

we obtain

$$2x^\top(t)PBK\int_{t-t_d}^t Ax(\tau)d\tau \leq \alpha t_d x^\top(t)PBKK^\top B^\top Px(t) + \frac{1}{\alpha t_d}\left(\int_{t-t_d}^t x^\top(\tau)A^\top d\tau\right)\left(\int_{t-t_d}^t Ax(\tau)d\tau\right)$$

$$\leq \alpha t_d x^\top(t)PBKK^\top B^\top Px(t) + \frac{1}{\alpha}\int_{t-t_d}^t x^\top(\tau)A^\top Ax(\tau)d\tau \quad (47)$$

$$-2x^\top(t)PBK\int_{t-t_d}^t BKx(\tau-t_d)d\tau \leq \beta t_d x^\top(t)PBKK^\top B^\top Px(t)$$

$$+ \frac{1}{\beta t_d}\left(\int_{t-t_d}^t x^\top(\tau-t_d)K^\top B^\top d\tau\right)\left(\int_{t-t_d}^t BKx(\tau-t_d)d\tau\right)$$

$$\leq \beta t_d x^\top(t)PBKK^\top B^\top Px(t) + \frac{1}{\beta}\int_{t-t_d}^t x^\top(\tau-t_d)K^\top B^\top BKx(\tau-t_d)d\tau \quad (48)$$

for some $\alpha > 0$ and $\beta > 0$.

Choose $Q = A^\top A/\alpha$ and $S = K^\top B^\top BK/\beta$. Then

$$\dot{V}(t) \leq x^\top(t)\left[(A-BK)^\top P + P(A-BK)\right]x(t) + \alpha t_d x^\top(t)PBKK^\top B^\top Px(t) + \beta t_d x^\top(t)PBKK^\top B^\top Px(t)$$

$$+ \frac{t_d}{\alpha}x^\top(t)A^\top Ax(t) + \frac{t_d}{\beta}x^\top(t)K^\top B^\top BKx(t) \quad (49)$$

For stability, $\dot{V}(t) < 0$. Thus we obtain the LMI (38).

Example: Given

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}, \ BK = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$$

Choose $\alpha = 1$ and $\beta = 1$. Let

$$(A-BK)^\top P + P(A-BK) = -Q$$

and choose $Q = I$ where $I$ is the identity matrix. Then the time delay margin is estimated to be $t_d = 0.068$ sec.

$t_d$ can be maximized by a suitable selection of $\alpha$ and $\beta$. Figure 1 is a plot of $t_d$ as a function of $\alpha$ and $\beta$. The maximum value of $t_d$ is 0.072 sec corresponding to $\alpha = 0.51$ and $\beta = 0.89$. As expected, the Lyapunov-Krasovskii method produces a very conservative estimation of the time delay margin.
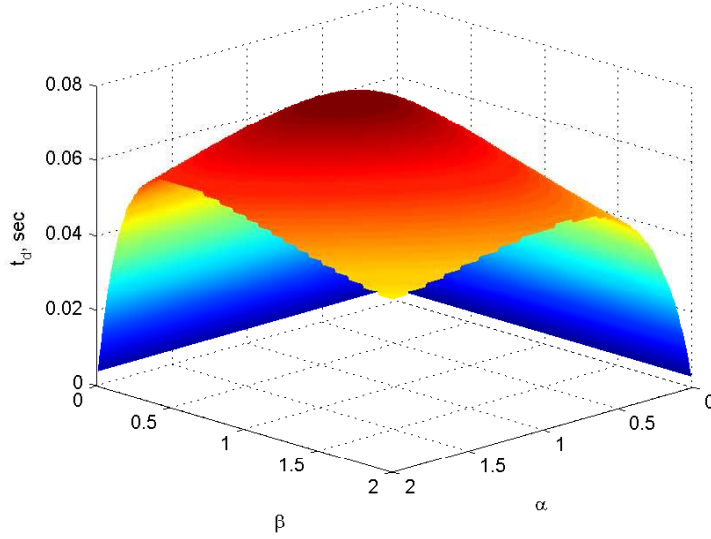
Fig. 1 - Time Delay Margin by Lyapunov-Krasovskii Method

## 3.3 Matrix Measure Method

The matrix measure method has been developed recently and affords a simple way to estimate the time delay margin and the effective phase margin for the MIMO LTI system [1]. The matrix measure $\mu$ is defined as an eigenvalue of a symmetric part of a complex matrix [11] such that

$$\mu_i(C) = \lambda_i \left( \frac{C + C^*}{2} \right) \tag{50}$$

where $C \in \mathbb{C}$ is a complex matrix and $C^*$ is its complex conjugate transpose, then $\mu$ has the following properties

$$\mu_i(C) \in \mathbb{R} \tag{51}$$

$$\overline{\mu}(C) = \max_{1 \leq i \leq n} \lambda_i \left( \frac{C + C^*}{2} \right) = \lim_{\varepsilon \to 0} \frac{\|I + \varepsilon C\| - 1}{\varepsilon} \tag{52}$$

$$\underline{\mu}(C) = \min_{1 \leq i \leq n} \lambda_i \left( \frac{C + C^*}{2} \right) = \lim_{\varepsilon \to 0} \frac{1 - \|I - \varepsilon C\|}{\varepsilon} \tag{53}$$

$$\underline{\mu}(C) \leq \operatorname{Re}\lambda_i(C) \leq \overline{\mu}(C) \tag{54}$$

$$\operatorname{Im}\lambda(C) \leq \overline{\mu}(-jC) \tag{55}$$

**Theorem 3:** The input delay LTI system (30) is asymptotically stable if the following inequalities hold

$$t_d \leq \frac{1}{\omega} \cos^{-1} \left[ \frac{\overline{\mu}(A) + \overline{\mu}(jBK)}{\|BK\|} \right] \tag{56}$$

$$\omega \leq \overline{\mu}(-jA) + \|BK\| \tag{57}$$

where $\|.\| = \|.\|_2$ is the $\mathcal{L}_2$-norm.

   **Proof:** The real parts of the system poles are bounded from above by

$$\sigma = \operatorname{Re}\lambda_i \left( A - BKe^{-j\omega t_d} \right) \leq \overline{\mu}(A) + \overline{\mu}\left( -BKe^{-j\omega t_d} \right) \leq \overline{\mu}(A) + \overline{\mu}(-BK)|\cos \omega t_d| + \overline{\mu}(jBK)|\sin \omega t_d| \tag{58}$$

Let $0 \leq \omega t_d \leq \frac{\pi}{2}$, then the input delay system is stable if $\sigma < 0$ which implies

$$\overline{\mu}(A) \leq -\overline{\mu}(-BK)\cos \omega t_d - \overline{\mu}(jBK)\sin \omega t_d = \underline{\mu}(BK)\cos \omega t_d - \overline{\mu}(jBK)\sin \omega t_d$$

$$\leq \overline{\mu}(BK)\cos \omega t_d - \overline{\mu}(jBK)\sin \omega t_d \tag{59}$$

8

Upon some algebra, this can be expressed as

$$\left[\overline{\mu}^2\left(BK\right)+\overline{\mu}^2\left(jBK\right)\right]\cos^2\omega t_d - 2\overline{\mu}\left(A\right)\overline{\mu}\left(BK\right)\cos\omega t_d + \overline{\mu}^2\left(A\right) - \overline{\mu}^2\left(jBK\right) \geq 0 \tag{60}$$

The solution yields a bound on time delay margin $t_d$ as

$$t_d \leq \frac{1}{\omega}\cos^{-1}\frac{\overline{\mu}\left(A\right)\overline{\mu}\left(BK\right)+\overline{\mu}\left(jBK\right)\sqrt{\overline{\mu}^2\left(BK\right)+\overline{\mu}^2\left(jBK\right)-\overline{\mu}^2\left(A\right)}}{\overline{\mu}^2\left(BK\right)+\overline{\mu}^2\left(jBK\right)} \tag{61}$$

But

$$\overline{\mu}^2\left(BK\right) \leq \overline{\mu}^2\left(BK\right)+\overline{\mu}^2\left(jBK\right) \leq \|BK\|^2 \tag{62}$$

So

$$t_d \leq \frac{1}{\omega}\left[\cos^{-1}\frac{\overline{\mu}\left(A\right)\|BK\|+\overline{\mu}\left(jBK\right)\sqrt{\|BK\|^2-\overline{\mu}^2\left(A\right)}}{\|BK\|^2}\right] \leq \frac{1}{\omega}\cos^{-1}\left[\frac{\overline{\mu}\left(A\right)+\overline{\mu}\left(jBK\right)}{\|BK\|}\right] \tag{63}$$

The imaginary parts of the system poles are bounded from above by

$$\omega = \mathrm{Im}\lambda_i\left(-jA+jBKe^{-j\omega t_d}\right) \leq \overline{\mu}\left(-jA\right)+\overline{\mu}\left(jBKe^{-j\omega t_d}\right) \leq \overline{\mu}\left(-jA\right)+\overline{\mu}\left(jBK\right)\left|\cos\omega t_d\right|+\overline{\mu}\left(BK\right)\left|\sin\omega t_d\right| \tag{64}$$

which can be expressed as

$$\omega \leq \overline{\mu}\left(-jA\right)+\sqrt{\overline{\mu}^2\left(BK\right)+\overline{\mu}^2\left(jBK\right)} \leq \overline{\mu}\left(-jA\right)+\|BK\| \tag{65}$$

Without loss of generality, we replace the inequality sign with equality sign for the purpose of estimation of the time delay margin.

Example: Given

$$A = \left[\begin{array}{cc} 0 & 1 \\ -1 & 1 \end{array}\right], \, BK = \left[\begin{array}{cc} 0 & 0 \\ 0 & 2 \end{array}\right]$$

$\omega$ and $t_d$ are computed as follows:

$$\overline{\mu}\left(A\right) = 1$$

$$\overline{\mu}\left(jBK\right) = 0$$

$$\|BK\| = 2$$

$$\omega = \overline{\mu}\left(-jA\right)+\|BK\| = 3 \text{ rad/sec}$$

$$t_d = \frac{1}{\omega}\cos^{-1}\left[\frac{\overline{\mu}\left(A\right)}{\|BK\|}\right] = \frac{\pi}{9} = 0.349 \text{ sec}$$

Comparing the result with the exact time delay margin, the matrix measure method produces the least conservative estimation of the time delay margin. Moreover, the solution is much simpler to compute than both the Pade approximation and Lyapunov-Krasovskii method.

The matrix measure method can also estimate the effective phase margin of a MIMO system. For this example, the phase margin estimate is

$$\phi = \omega t_d = \frac{\pi}{3}$$

# 4 Time Delay Margin Estimation of Adaptive Control

## 4.1 Standard MRAC

Using the bounded linear stability analysis method to approximate the standard MRAC as a bounded linear approximation within a given time window, the time delay margin of adaptive control can be estimated by any of the methods previously presented.

1. Pade approximation:

   The first-order Pade approximation of the bounded linear approximation of the error equations and adaptive law are expressed as

   $$sE\left(s\right) = \left(A + B\Theta^{*\top}\Phi_i'\right)E\left(s\right) - BK_xE\left(s\right)\frac{2 - t_d s}{2 + t_d s} + BU_{ad}\left(s\right)\frac{2 - t_d s}{2 + t_d s} - B\Theta^{*\top}\Phi_i - B\Theta^{*\top}\Phi_i'\left[X_m\left(s\right) - x_i\right]$$
   $$- BK_x\left[X_m\left(s\right) - X_m\left(s\right)\frac{2 - t_d s}{2 + t_d s}\right] + BK_r\left[R\left(s\right) - R\left(s\right)\frac{2 - t_d s}{2 + t_d s}\right] \quad (66)$$

   $$U_{ad}\left(s\right) = -\gamma B^\top P\frac{E\left(s\right)}{s} + \Theta_i^\top \Phi_i'\left[X_m\left(s\right) - E\left(s\right)\right] \quad (67)$$

   Then the time delay margin can be computed from the following characteristic equation:

   $$\det\left[j\omega_i I - \left(A + B\Theta^{*\top}\Phi_i'\right) + \left(BK_x + B\Theta_i^\top\Phi_i' + \frac{\gamma BB^\top P}{j\omega_i}\right)\frac{2 - j\omega_i t_{d_i}}{2 + j\omega_i t_{d_i}}\right] = 0 \quad (68)$$

   Another approach is to express the error equation in another form by differentiating the error equation and substituting in the adaptive law which yield

   $$\ddot{e}\left(t\right) = \left(A + B\Theta^{*\top}\Phi_i'\right)\dot{e}\left(t\right) - \left(BK_x + B\Theta_i^\top\Phi_i'\right)\dot{e}\left(t - t_d\right) - \gamma BB^\top Pe\left(t - t_d\right) + B\Theta_i^\top\Phi_i'\dot{x}_m\left(t - t_d\right)$$
   $$- B\Theta^{*\top}\Phi_i'\dot{x}_m\left(t\right) - BK_x\left[\dot{x}_m\left(t\right) - \dot{x}_m\left(t - t_d\right)\right] + BK_r\left[\dot{r}\left(t\right) - \dot{r}\left(t - t_d\right)\right] \quad (69)$$

   for $t \in [t_i - T_0, t_i)$, where $t_0 = 0$, $t_i = t_{i-1} + T_0$ and $i = 1, 2, \ldots, n \rightarrow \infty$.

   Then the error equation can be recast as

   $$\dot{z}\left(t\right) = C_i z\left(t\right) - D_i z\left(t - t_d\right) + E_i\left(t\right) \quad (70)$$

   where $z\left(t\right) = \begin{bmatrix} e\left(t\right) & \dot{e}\left(t\right) \end{bmatrix}$ and

   $$C_i = \begin{bmatrix} 0 & I \\ 0 & A + B\Theta^{*\top}\Phi_i' \end{bmatrix} \quad (71)$$

   $$D_i = \begin{bmatrix} 0 & 0 \\ \gamma BB^\top P & BK_x + B\Theta_i^\top\Phi_i' \end{bmatrix} \quad (72)$$

   $$E\left(t\right) = \begin{bmatrix} 0 \\ B\Theta_i^\top\Phi_i'\dot{x}_m\left(t - t_d\right) - B\Theta^{*\top}\Phi_i'\dot{x}_m\left(t\right) - BK_x\left[\dot{x}_m\left(t\right) - \dot{x}_m\left(t - t_d\right)\right] + BK_r\left[\dot{r}\left(t\right) - \dot{r}\left(t - t_d\right)\right] \end{bmatrix} \quad (73)$$

   Then using the result in the previous section, the time delay margin can be found by solving the following characteristic equation

   $$\det\begin{bmatrix} j\omega_i I & -I \\ -\frac{2}{t_{d_i}}\left(C_i - D_i\right) & j\omega_i + \frac{2}{t_d}I - C_i - D_i \end{bmatrix} = 0 \quad (74)$$

2. Lyapunov-Krasovskii method:

   Applying the result of the Lyapunov-Krasovskii method from the previous section, the time delay margin can be computed from the following LMI:

   $$\left(C_i - D_i\right)^\top P + P\left(C_i - D_i\right) + \left(\alpha + \beta\right)t_{d_i}PD_iD_i^\top P + \frac{t_{d_i}}{\alpha}C_i^\top C_i + \frac{t_{d_i}}{\beta}D_i^\top D_i < 0 \quad (75)$$

3. Matrix measure method:

   Using the matrix measure method, the time delay margin is estimated as

   $$t_{d_i} = \frac{1}{\omega_i}\cos^{-1}\left[\frac{\overline{\mu}\left(C_i\right) + \overline{\mu}\left(jD_i\right)}{\|D_i\|}\right] \quad (76)$$

10

$$\omega_i \leq \overline{\mu}\left(-jC_i\right)+\|D_i\| \tag{77}$$

Another approach is to derive the time delay margin from the original system matrices. The characteristic equation of the bounded linear approximation of the error equation and adaptive law is given by

$$\det\left[j\omega_i I-\left(A+B\Theta^{*\top}\Phi'_i\right)+\left(BK_x+B\Theta^{\top}_i\Phi'_i-\frac{j\gamma BB^{\top}P}{\omega_i}\right)e^{-j\omega_i t_{d_i}}\right]=0 \tag{78}$$

Then, applying the matrix measure method, the time delay margin is estimated as

$$t_{d_i} \leq \frac{1}{\omega_i}\cos^{-1}\left[\frac{\overline{\mu}\left(A+B\Theta^{*\top}\Phi'_i\right)\left[\omega_i^2\,\|BK_x+B\Theta^{\top}_i\|+\omega_i\overline{\mu}\left(-j\gamma BB^{\top}P\right)\right]}{\omega_i^2\,\|BK_x+B\Theta^{\top}_i\|^2+\|\gamma BB^{\top}P\|^2}\right.$$
$$\left.+\frac{\left[\omega_i\overline{\mu}\left[j\left(BK_x+B\Theta^{\top}_i\Phi'_i\right)\right]+\|\gamma BB^{\top}P\|\right]\sqrt{\omega_i^2\,\|BK_x+B\Theta^{\top}_i\|^2+\|\gamma BB^{\top}P\|^2}}{\omega_i^2\,\|BK_x+B\Theta^{\top}_i\|^2+\|\gamma BB^{\top}P\|^2}\right] \tag{79}$$

The frequency is computed as

$$\omega_i \leq \overline{\mu}\left[-j\left(A+B\Theta^{*\top}\Phi'_i\right)\right]+\overline{\mu}\left[j\left(BK_x+B\Theta^{\top}_i\right)e^{-j\omega_i t_{d_i}}\right]+\frac{1}{\omega_i}\overline{\mu}\left(\gamma BB^{\top}Pe^{-j\omega_i t_{d_i}}\right)$$
$$\leq \overline{\mu}\left[-j\left(A+B\Theta^{*\top}\Phi'_i\right)\right]+\left\|BK_x+B\Theta^{\top}_i\right\|+\frac{1}{\omega_i}\left\|\gamma BB^{\top}P\right\| \tag{80}$$

$$\omega \leq \frac{\overline{\mu}\left[-j\left(A+B\Theta^{*\top}\Phi'_i\right)\right]+\|BK_x+B\Theta^{\top}_i\|}{2}\left(1+\sqrt{1+\frac{4\left\|\gamma BB^{\top}P\right\|}{\left\{\overline{\mu}\left[-j\left(A+B\Theta^{*\top}\Phi'_i\right)\right]+\|BK_x+B\Theta^{\top}_i\|\right\}^2}}\right) \tag{81}$$

It is noted that $\omega_i \to \infty$ as $\gamma \to \infty$. Consequently, the time delay margin tends to zero as $\Gamma \to \infty$. This is consistent with the behavior of the standard MRAC.

## 4.2 Scalar MRAC

Consider an input delay scalar MRAC system with linear structured uncertainty

$$\dot{x}(t)=ax(t)+b\left[u\left(t-t_d\right)+\theta^*x(t)\right] \tag{82}$$

The reference model is given by
$$\dot{x}_m(t)=a_m x_m(t)+b_m r(t) \tag{83}$$

The controller is given by
$$u(t)=-k_x x(t)+k_r r(t)-\theta(t)x(t) \tag{84}$$
$$\dot{\theta}(t)=-\Gamma x(t)pbe(t) \tag{85}$$

1. The time delay margin is estimated from the matrices $C_i$ and $D_i$

$$C_i=\begin{bmatrix}0 & 1\\ 0 & a+b\theta^*\end{bmatrix} \tag{86}$$

$$D_i=\begin{bmatrix}0 & 0\\ \gamma b^2 p & bk_x+b\theta_i\end{bmatrix} \tag{87}$$

where

$$\gamma=\frac{\Gamma}{T_0}\int_{t_i-T_0}^{t_i}x^2(\tau)d\tau \tag{88}$$

11

Using the matrix measure method, the following parameters are computed analytically as

$$\overline{\mu}\left(C_i\right)=\frac{a+b\theta^*+\sqrt{(a+b\theta^*)^2+1}}{2} \tag{89}$$

$$\overline{\mu}\left(-jC_i\right)=\frac{1}{2} \tag{90}$$

$$\overline{\mu}\left(jD_i\right)=\frac{\gamma b^2 p}{2} \tag{91}$$

$$\|D_i\|=\sqrt{(bk_x+b\theta_i)^2+\gamma^2 b^4 p^2} \tag{92}$$

$\omega_i$ and $t_{d_i}$ are then estimated as

$$\omega_i=\overline{\mu}\left(-jC_i\right)+\|D_i\|=\frac{1}{2}+\sqrt{(bk_x+b\theta_i)^2+\gamma^2 b^4 p^2} \tag{93}$$

$$t_{d_i}=\frac{1}{\omega_i}\cos^{-1}\frac{\overline{\mu}\left(C_i\right)+\overline{\mu}\left(jD_i\right)}{\|D_i\|}=\frac{2}{1+2\sqrt{(bk_x+b\theta_i)^2+\gamma^2 b^4 p^2}}\cos^{-1}\left[\frac{a+b\theta^*+\sqrt{(a+b\theta^*)^2+1}+\gamma b^2 p}{2\sqrt{(bk_x+b\theta_i)^2+\gamma^2 b^4 p^2}}\right] \tag{94}$$

2. Using the original system parameters, the time delay margin is estimated as

$$\omega_i=\frac{|bk_x+b\theta_i|}{2}+\frac{1}{2}\sqrt{(bk_x+b\theta_i)^2+4\gamma b^2 p} \tag{95}$$

$$t_{d_i}=\frac{1}{\omega_i}\cos^{-1}\left[\frac{\omega_i^2\left(a+b\theta^*\right)|bk_x+b\theta_i|+\gamma b^2 p\sqrt{\omega_i^2\left(bk_x+b\theta_i\right)^2+\gamma^2 b^4 p^2}}{\omega_i^2\left(bk_x+b\theta_i\right)^2+\gamma^2 b^4 p^2}\right] \tag{96}$$

Both approaches yield somewhat different results. The "exact" values of $\omega_i$ and $t_{d_i}$ for the locally bounded linear approximation of the error equation can be determined as follows:

$$\det\left[j\omega-\left(a+b\theta^*\right)+\left(bk_x+b\theta_i\right)e^{-j\omega t_d}+\gamma b^2 p\frac{e^{-j\omega t_d}}{j\omega}\right]=0 \tag{97}$$

This results in two equations

$$-\omega_i^2+\left(bk_x+b\theta_i\right)\omega_i\sin\omega_i t_{d_i}+\gamma b^2 p\cos\omega_i t_{d_i}=0 \tag{98}$$

$$-\left(a+b\theta^*\right)\omega_i+\left(bk_x+b\theta_i\right)\omega_i\cos\omega_i t_{d_i}-\gamma b^2 p\sin\omega_i t_{d_i}=0 \tag{99}$$

The frequency equation is obtained as

$$\omega_i^4+\left[\left(a+b\theta^*\right)^2-\left(bk_x+b\theta_i\right)^2\right]\omega_i^2-\gamma^2 b^4 p^2=0 \tag{100}$$

The "exact" solution gives

$$\omega_i=\sqrt{\frac{(bk_x+b\theta_i)^2-(a+b\theta^*)^2+\sqrt{\left[(a+b\theta^*)^2-(bk_x+b\theta_i)^2\right]^2+4\gamma^2 b^4 p^2}}{2}} \tag{101}$$

$$t_{d_i}=\frac{1}{\omega_i}\cos^{-1}\left[\frac{\omega_i^2\left(a+b\theta^*\right)\left(bk_x+b\theta_i\right)+\omega_i^2\gamma b^2 p}{\omega_i^2\left(bk_x+b\theta_i\right)^2+\gamma^2 b^4 p^2}\right] \tag{102}$$

12

It is well-known that $t_d \to 0$ as as $\Gamma \to \infty$ for the standard MRAC. This behavior is exhibited in the time delay margin estimation by the matrix measure method and the "exact" solution since

$$\lim_{\gamma \to \infty} \frac{1}{\omega_i} = \lim_{\gamma \to \infty} \frac{1}{\sqrt{\gamma b^2 p}} = 0 \tag{103}$$

For $\gamma = 0$, the system is non-adaptive and the time delay margin estimation by the matrix measure method using the $C_i$ and $D_i$ yields

$$t_d = \frac{2}{1 + 2bk_x} \cos^{-1} \left[ \frac{a + b\theta^* + \sqrt{(a + b\theta^*)^2 + 1}}{2bk_x} \right] \tag{104}$$

The exact time delay margin is computed to be

$$t_d = \frac{1}{\sqrt{(bk_x)^2 - (a + b\theta^*)^2}} \cos^{-1} \left( \frac{a + b\theta^*}{bk_x} \right) \tag{105}$$

Example: Given $a = 1$, $b = 1$, $\theta^* = 0.1$, $a_m = -1$, $b_m = 1$, $p = 1$, $\theta(0) = 0$, $r(t) = \sin(t)$. The control gains are computed to be $k_x = 2$ and $k_r = 1$. The adaptive gain is selected as $\Gamma = 1$.

For the non-adaptive LTI system for which $\theta(t) = 0$ for all $t$, the time delay margin estimates and the exact value are shown in Table 1.

| Method | $t_d$, sec |
|---|---|
| Pade Approximation | 0.646 |
| Lyapunov-Krasovskii, $\alpha = \beta = 1$ | 0.137 |
| Matrix Measure in Part 1 | 0.347 |
| Matrix Measure in Part 2 | 0.494 |
| Exact Value | 0.592 |

Table 1 - Time Delay Margin Estimation of Non-Adaptive System

Thus, it can be seen that the time delay margin computed by the matrix measure method is the least conservative lower bound estimate of the true time delay margin among the present approaches. The difference in both approaches using the matrix measure method is noted. The approach using the matrices $C_i$ and $D_i$ are more conservative.

Figure 2 is a plot of the variation of the local time delay margin estimates computed by the matrix measure method using the matrices $C_i$ and $D_i$ within three different time windows with $T_0 = 1$ sec, $T_0 = 5$ sec, and $T_0 = 10$ sec based on the bounded linear stability analysis method. It is noted that as the window size increases, the variation in the local time delay margin decreases. The longer time window allows a more uniform average value of the parameter $\gamma$ to be computed, thus reducing the local variation of time delay margin from one time window to the next. It appears that the mean value of the computed local time delay margins is relatively insensitive to the window size. In fact, the mean estimate of the time delay margin for the three time window sizes $T_0 = 1$ sec, $T_0 = 5$ sec, and $T_0 = 10$ sec are 0.434 sec, 0.432 sec, and 0.432 sec, respectively.

Figure 3 is a plot of the mean value of the time delay margin estimates by all the different methods as a function of the unknown parameter $-1 \le \theta^* \le 1$ for $T_0 = 1$ sec. Generally, $\theta^*$ is not known, so in a verification setting, the time delay margin should be computed over all possible parameter variations within their physical bounds. Also plotted is the numerical evidence of the time delay margin from simulations. Comparing to the numerical evidence, the mean value of the time delay margin estimates by the matrix measure method using the $C_i$ and $D_i$ matrices is the most reasonable and non-conservative. The difference between the numerical evidence and the estimate is relatively uniform by about 20% to 30%. On the other hand, the second matrix measure approach over-estimates the time delay margin for $\theta^* < 0$ but produces a very good agreement with the numerical evidence for $\theta^* > 0$. The Pade approximation under-estimates for a small range of $\theta^*$ between -1 and -0.5. Otherwise, it uniformly over-estimates the time delay margin by the largest margin among the three methods. The Lyapunov-Krasovskii method, as expected, yields the most conservative estimates of the time delay margin. The difference is significant. For example, for $\theta^* = -1$, it estimates a value of $t_d = 0.0356$ sec as compared to the numerical value of $t_d = 0.719$ sec which results in a difference of 20 times. At the other end, for $\theta^* = -1$, the Lyapunov-Krasovskii estimates $t_d = 0.0051$ sec as compared to the numerical evidence of $t_d = 0.250$ sec which is 50 times larger. Thus, in summary, the matrix measure method appears to give a better prediction of the time delay margin among all the three methods.
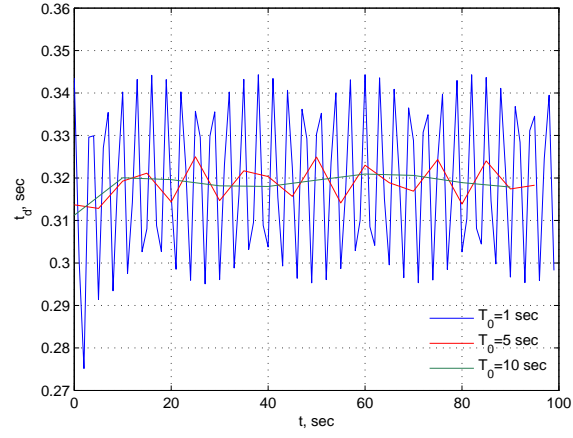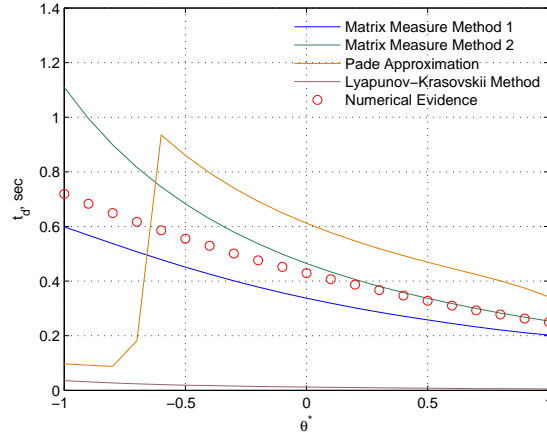
13

Fig. 2 - Time Delay Margin Estimates



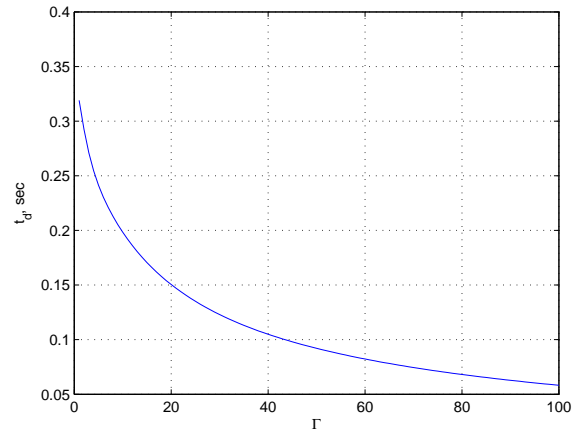Fig. 3 - Time Delay Margin Variation with Unknown Parameter $\theta^*$



Fig. 4 - Time Delay Margin Variation with Adaptive Gain

Figure 4 is a plot of the mean value of the time delay margin estimates by the matrix measure method as a function of the adaptive gain $1 \leq \Gamma \leq 100$ for $T_0 = 1$ sec. It can be seen that as the adaptive gain $\Gamma$ increases, the time delay margin of the adaptive system decreases as expected for the standard MRAC.

14

## 4.3 $\sigma$-Modification

The $\sigma$-modification adaptive law [9] is given by

$$\dot{\Theta}(t) = -\Gamma\Phi(x(t))\left[e^\top(t)PB + \sigma\Theta(t)\right] \tag{106}$$

Using the bounded linear stability analysis method, the adaptive law is approximated as

$$\dot{u}_{ad}(t) = -\gamma B^\top Pe(t) - \underline{\Gamma}\sigma u_{ad}(t) + \Theta_i^\top \Phi_i'[\dot{x}_m(t) - \dot{e}(t)] \tag{107}$$

where $\underline{\Gamma} = \lambda_{min}(\Gamma)$, for $t \in [t_i - T_0, t_i)$, where $t_0 = 0$, $t_i = t_{i-1} + T_0$ and $i = 1, 2, \ldots, n \to \infty$.

Then the error equation is obtained as

$$\begin{aligned}
\ddot{e}(t) &= \left(A + B\Theta^{*\top}\Phi_i' - \underline{\Gamma}\sigma\right)\dot{e}(t) - \left(BK_x + B\Theta_i^\top\Phi_i'\right)\dot{e}(t - t_d) + \underline{\Gamma}\sigma\left(A + B\Theta^{*\top}\Phi_i'\right)e(t) \\
&\quad - \left(\gamma BB^\top P + \underline{\Gamma}\sigma BK_x\right)e(t - t_d) - \underline{\Gamma}\sigma B\Theta^{*\top}\left[\Phi_i + \Phi_i'x_m(t) - \Phi_i'x_i\right] - \underline{\Gamma}\sigma BK_x[x_m(t) - x_m(t - t_d)] \\
&\quad + \underline{\Gamma}\sigma BK_r[r(t) - r(t - t_d)] + B\Theta_i^\top\Phi_i'\dot{x}_m(t - t_d) - B\Theta^{*\top}\Phi_i'\dot{x}_m(t) - BK_x[\dot{x}_m(t) - \dot{x}_m(t - t_d)] \\
&\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad + BK_r[\dot{r}(t) - \dot{r}(t - t_d)] \tag{108}
\end{aligned}$$

The time margin for the $\sigma$-modification adaptive law can be estimated by the following methods:

1. Pade Approximation:

   The time delay margin can be found from the following characteristic equation:

   $$\det\left[j\omega_i I - \left(A + B\Theta^{*\top}\Phi_i'\right) + \left(BK_x + B\Theta_i^\top\Phi_i'\frac{j\omega_i}{j\omega_i + \underline{\Gamma}\sigma} + \frac{\gamma BB^\top P}{j\omega_i + \underline{\Gamma}\sigma}\right)\frac{2 - j\omega_i t_{d_i}}{2 + j\omega_i t_{d_i}}\right] = 0 \tag{109}$$

2. Lyapunov-Krasovskii method:

   The time delay margin for the $\sigma$-modification adaptive law can be estimated using Eq. (75) with the matrices $C_i$ and $D_i$ given by

   $$C_i = \begin{bmatrix} 0 & I \\ \underline{\Gamma}\sigma\left(A + B\Theta^{*\top}\Phi_i'\right) & A + B\Theta^{*\top}\Phi_i' - \underline{\Gamma}\sigma \end{bmatrix} \tag{110}$$

   $$D_i = \begin{bmatrix} 0 & 0 \\ \gamma BB^\top P + \underline{\Gamma}\sigma BK_x & BK_x + B\Theta_i^\top\Phi_i' \end{bmatrix} \tag{111}$$

3. Matrix measure method:

   The characteristic equation is

   $$\det\left[j\omega_i I - \left(A + B\Theta^{*\top}\Phi_i'\right) + \left(BK_x + B\Theta_i^\top\Phi_i'\frac{j\omega_i}{j\omega_i + \underline{\Gamma}\sigma} + \frac{\gamma BB^\top P}{j\omega_i + \underline{\Gamma}\sigma}\right)e^{-j\omega_i t_{d_i}}\right] = 0 \tag{112}$$

   which can also be expressed as

   $$\det\left[j\omega_i I - \left(A + B\Theta^{*\top}\Phi_i'\right) + \left(BK_x + \frac{B\Theta_i^\top\Phi_i'\omega_i^2 + \underline{\Gamma}\sigma\gamma BB^\top P + j\omega_i\left(\underline{\Gamma}\sigma B\Theta_i^\top\Phi_i' - \gamma BB^\top P\right)}{\omega_i^2 + \underline{\Gamma}^2\sigma^2}\right)e^{-j\omega_i t_{d_i}}\right] = 0 \tag{113}$$

   The same approach can be applied to find the expressions for the time delay margin and frequency by working with the following inequalities

   $$0 \leq \overline{\mu}\left(A + B\Theta^{*\top}\Phi_i'\right) + \overline{\mu}\left[-\left(BK_x + \frac{B\Theta_i^\top\Phi_i'\omega_i^2}{\omega_i^2 + \underline{\Gamma}^2\sigma^2} + \frac{j\omega_i\left(\underline{\Gamma}\sigma B\Theta_i^\top\Phi_i' - \gamma BB^\top P\right)}{\omega_i^2 + \underline{\Gamma}^2\sigma^2}\right)e^{-j\omega_i t_d}\right] \tag{114}$$

15

$$\omega_i \leq \overline{\mu}\left[-j\left(A+B\Theta^{*\top}\Phi_i'\right)\right] + \overline{\mu}\left(jBK_x e^{-j\omega_i t_{d_i}}\right) + \frac{\omega_i^2}{\omega_i^2 + \underline{\Gamma}^2\sigma^2}\overline{\mu}\left(jB\Theta_i^\top\Phi_i' e^{-j\omega_i t_{d_i}}\right)$$
$$+ \frac{1}{\omega_i^2 + \underline{\Gamma}^2\sigma^2}\overline{\mu}\left(j\underline{\Gamma}\sigma\gamma BB^\top Pe^{-j\omega_i t_{d_i}}\right) + \frac{\omega_i}{\omega_i^2 + \underline{\Gamma}^2\sigma^2}\left[\overline{\mu}\left(\gamma BB^\top P - \underline{\Gamma}\sigma B\Theta_i^\top\Phi_i'\right)e^{-j\omega_i t_{d_i}}\right] \quad (115)$$

The frequency equation, however, results in a cubic polynomial which does not have an analytical solution.

Consider the case when $\Gamma \to \infty$. The time delay margin for the standard MRAC is known to be zero but remains finite for the $\sigma$-modification adaptive law. To see this, consider the characteristic equation in the limit as $\Gamma \to \infty$

$$\det\left[j\omega_i I - \left(A + B\Theta^{*\top}\Phi_i'\right) + \left(BK_x + \frac{\gamma BB^\top P}{\underline{\Gamma}\sigma}\right)e^{-j\omega_i t_{d_i}}\right] = 0 \quad (116)$$

If $\Gamma = cI$ where $c > 0$ is a constant, then

$$\frac{\gamma}{\underline{\Gamma}\sigma} = \frac{1}{\sigma T_0}\int_{t_i-T_0}^{t_i} \Phi^\top\left(x(\tau)\right)\Phi\left(x(\tau)\right)d\tau \quad (117)$$

is finite.

Using the matrix measure method, the time delay margin for the $\sigma$-modification as $\Gamma \to \infty$ can then be estimated as

$$\omega_i = \overline{\mu}\left[-j\left(A+B\Theta^{*\top}\Phi_i'\right)\right] + \left\|BK_x + \frac{\gamma BB^\top P}{\underline{\Gamma}\sigma}\right\| \quad (118)$$

$$t_{d_i} = \frac{1}{\omega_i}\cos^{-1}\frac{\overline{\mu}\left(A+B\Theta^{*\top}\Phi_i'\right) + \overline{\mu}\left(j\left[BK_x + \frac{\gamma BB^\top P}{\underline{\Gamma}\sigma}\right]\right)}{\left\|BK_x + \frac{\gamma BB^\top P}{\underline{\Gamma}\sigma}\right\|} \quad (119)$$

It is clear that the time delay margin for the $\sigma$-modification remains finite even as $\Gamma \to \infty$. It is noted that the frequency and time delay margin are dependent on time windows since $\gamma$ varies with different time windows.

## 4.4 Optimal Control Modification

The lack of robustness to unmodeled dynamics [13] and input time delay of the standard MRAC is well-known, as illustrated in the previous section. Increasing the adaptive gain allows the adaptation to attain a better tracking performance, but usually at the expense of the ability to maintain stability in the presence of unmodeled dynamics or time delay. To improve robustness, the two well-known robust modification methods; namely, the $\sigma$-modification [9] and $\varepsilon$-modification [14], have been used extensively in adaptive control. The optimal control modification method is a recently developed adaptive law to address robustness due to fast adaptation [10]. This adaptive law is given by

$$\dot{\Theta}(t) = -\Gamma\left[\Phi(x(t))e^\top(t)PB - \nu\Phi(x(t))\Phi^\top(x(t))\Theta(t)B^\top PA_m^{-1}B\right] \quad (120)$$

where $\nu > 0$ is a weighting constant.

In can be shown that in the limiting case as $\Gamma \to \infty$ and for a linear matched uncertainty where $\Phi(x(t)) = x(t)$, the optimal control modification exhibits an asymptotic linear behavior [16, 15]. This behavior can be explained using the analysis techniques developed herein.

The bounded linear stability analysis provides a method for analyzing an adaptive system with input time delay using a time windowing approach. Using the bounded linear approximation of an adaptive system, the matrix measure method can provide a non-conservative lower bound local estimate of the time delay margin. Unfortunately, the local estimation is non-unique and is dependent on the length of a time window, as is evident in Figure 2. The optimal control modification adaptive law has a unique property that enables it to be analyzed for stability without the use of the windowing approach of the bounded linear stability analysis as $\Gamma \to \infty$ for $\Phi(x(t)) = x(t)$. Thus, the time delay margin can be uniquely estimated for this adaptive law.

The bounded linear approximation of the optimal control modification adaptive law is

$$\dot{u}_{ad}(t) = -\gamma B^\top Pe(t) + \gamma\nu B^\top A_m^{-\top}PBu_{ad}(t) + \Theta_i^\top\Phi_i'\left[\dot{x}_m(t) - \dot{e}(t)\right] \quad (121)$$

for $t \in [t_i - T_0, t_i)$, where $t_0 = 0$, $t_i = t_{i-1} + T_0$ and $i = 1, 2, \ldots, n \to \infty$.

Let $G = -BB^\top A_m^{-\top} P > 0$, then the error equation is obtained as

$$
\ddot{e}(t) = \left(A + B\Theta^{*\top}\Phi'_i - \gamma\nu G\right)\dot{e}(t) - \left(BK_x + B\Theta_i^\top\Phi'_i\right)\dot{e}(t - t_d) + \gamma\nu G\left(A + B\Theta^{*\top}\Phi'_i\right)e(t)
$$
$$
- \left(\gamma BB^\top P + \gamma\nu GBK_x\right)e(t - t_d) - \gamma\nu GB\Theta^{*\top}\left[\Phi_i + \Phi'_i x_m(t) - \Phi'_i x_i\right] - \gamma\nu GBK_x\left[x_m(t) - x_m(t - t_d)\right]
$$
$$
+ \gamma\nu GBK_r\left[r(t) - r(t - t_d)\right] + B\Theta_i^\top\Phi'_i \dot{x}_m(t - t_d) - B\Theta^{*\top}\Phi'_i \dot{x}_m(t) - BK_x\left[\dot{x}_m(t) - \dot{x}_m(t - t_d)\right]
$$
$$
+ BK_r\left[\dot{r}(t) - \dot{r}(t - t_d)\right] \quad (122)
$$

The time margin for the optimal control modification adaptive law can be estimated by the following methods:

1. Pade Approximation:

   The time delay margin can be found from the following characteristic equation:

$$
\det\left\{j\omega_i I - \left(A + B\Theta^{*\top}\Phi'_i\right) + \left[BK_x + (j\omega_i I + \gamma\nu G)^{-1}\left(j\omega_i B\Theta_i^\top\Phi'_i + \gamma BB^\top P\right)\right]\frac{2 - j\omega_i t_{d_i}}{2 + j\omega_i t_{d_i}}\right\} = 0 \quad (123)
$$

2. Lyapunov-Krasovskii method:

   The time delay margin for the $\sigma$-modification adaptive law can be estimated using Eq. (75) with the matrices $C_i$ and $D_i$ given by

$$
C_i = \begin{bmatrix} 0 & I \\ \gamma\nu G\left(A + B\Theta^{*\top}\Phi'_i\right) & A + B\Theta^{*\top}\Phi'_i - \gamma\nu G \end{bmatrix} \quad (124)
$$

$$
D_i = \begin{bmatrix} 0 & 0 \\ \gamma BB^\top P + \gamma\nu GBK_x & BK_x + B\Theta_i^\top\Phi'_i \end{bmatrix} \quad (125)
$$

3. Matrix measure method:

   The characteristic equation for determining the time delay margin is

$$
\det\left\{j\omega_i I - \left(A + B\Theta^{*\top}\Phi'_i\right) + \left[BK_x + (j\omega_i I + \gamma\nu G)^{-1}\left(j\omega_i B\Theta_i^\top\Phi'_i + \gamma BB^\top P\right)\right]e^{-j\omega_i t_{d_i}}\right\} = 0 \quad (126)
$$

   Since $(j\omega_i I + \gamma\nu G)^{-1}$ is a matrix inverse, it is difficult to apply the matrix measure operator. To handle this, $(j\omega_i I + \gamma\nu G)^{-1}$ can be replaced by $(j\omega_i I + \gamma\nu \underline{G})^{-1}$ where $\underline{G} = \sigma_{min}(G)$ and $\sigma_{min}$ is the minimum singular value. Then the characteristic equation becomes

$$
\det\left[j\omega_i I - \left(A + B\Theta^{*\top}\Phi'_i\right) + \left(BK_x + \frac{B\Theta_i^\top\Phi'_i\omega_i^2 + \gamma\nu\underline{G}\gamma BB^\top P + j\omega_i\left(\gamma\nu\underline{G}B\Theta_i^\top\Phi'_i - \gamma BB^\top P\right)}{\omega_i^2 + \gamma^2\nu^2\underline{G}^2}\right)e^{-j\omega_i t_{d_i}}\right] = 0
$$
$$
(127)
$$

   The solution approach is similar to that for the $\sigma$-modification. The frequency $\omega_i$ does not have an analytical expression.

   From the fact that $t_d \to 0$ as $\Gamma \to \infty$ for the standard MRAC, so a lower bound estimate of the time delay margin for which an adaptive law is stable can be estimated by the value of $t_d$ that corresponds to $\Gamma \to \infty$ or equivalently $\gamma \to \infty$. Consider the case when $\gamma \to \infty$ and the matched uncertainty is linear with $\Phi(x(t)) = x(t)$. The asymptotic solution of the bounded linear approximation of the optimal control modification is obtained by taking the limit as $\gamma \to \infty$

$$
Bu_{ad}(t) = \frac{1}{\nu}P^{-1}A_m^\top Pe(t) \quad (128)
$$

   Then the asymptotic error equation as $\gamma \to \infty$ becomes

$$
\dot{e}(t) = \left(A + B\Theta^{*\top}\right)e(t) - \left(BK_x - \frac{1}{\nu}P^{-1}A_m^\top P\right)e(t - t_d) - B\Theta^{*\top}x_m(t) - BK_x\left[x_m(t) - x_m(t - t_d)\right]
$$
$$
+ BK_r\left[r(t) - r(t - t_d)\right] \quad (129)
$$

17

which is a LTI input delay equation independent of any time windowing parameters such as $\gamma$.

The characteristic equation of the asymptotic error equation can be obtained by letting $\gamma \to \infty$ which yields

$$\det\left[j\omega I - \left(A + B\Theta^{*\top}\right) + \left(BK_x + \frac{1}{v}G^{-1}BB^\top P\right)e^{-j\omega_i t_{d_i}}\right] = 0 \tag{130}$$

where $G^{-1}BB^\top P = -P^{-1}A_m^\top P$.

The time delay margin and frequency of the optimal control modification adaptive law as $\Gamma \to \infty$ can then be estimated by the matrix measure method as

$$t_d = \frac{1}{\overline{\mu}\left(-jA - jB\Theta^{*\top}\right) + \left\|BK_x - \frac{1}{v}P^{-1}A_m^\top P\right\|} \cos^{-1}\left[\frac{\overline{\mu}\left(A + B\Theta^{*\top}\right) + \overline{\mu}\left(j\left[BK_x - \frac{1}{v}P^{-1}A_m^\top P\right]\right)}{\left\|BK_x - \frac{1}{v}P^{-1}A_m^\top P\right\|}\right] \tag{131}$$

$$\omega = \overline{\mu}\left(-jA - jB\Theta^{*\top}\right) + \left\|BK_x - \frac{1}{v}P^{-1}A_m^\top P\right\| \tag{132}$$

The effective phase margin is estimated as

$$\phi = \cos^{-1}\frac{\overline{\mu}\left(A + B\Theta^{*\top}\right) + \overline{\mu}\left(j\left[BK_x - \frac{1}{v}P^{-1}A_m^\top P\right]\right)}{\left\|BK_x - \frac{1}{v}P^{-1}A_m^\top P\right\|} \tag{133}$$

Both the time delay and phase margins are independent of the time windows in the limit as $\Gamma \to \infty$ so the subscript $i$ is dropped from the expressions. Because the time delay margin is a minimum as $\Gamma \to \infty$, the time delay margin estimate for the asymptotic solution of the optimal control modification adaptive law establishes a lower bound of the time delay margin for any adaptive gain $\Gamma < \infty$. Thus, to maintain stability of the input delay adaptive system using the optimal control modification adaptive law, a suitable selection of the weighting constant $v$ can be chosen to satisfy the time delay margin requirement and or phase margin requirement in an adaptive control design. In order to compute this estimate, the knowledge of the unknown parametric uncertainty $\Theta^*$ must be available.

# 5  Simulations

Consider a longitudinal pitch dynamic model of an aircraft

$$\begin{bmatrix} mV + \frac{C_{L_{\dot{\alpha}}}\bar{q}S\bar{c}}{2V} & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{C_{m_{\dot{\alpha}}}\bar{q}S\bar{c}^2}{2V} & 0 & I_{yy} \end{bmatrix} \begin{bmatrix} \dot{\alpha}(t) \\ \dot{\theta}(t) \\ \dot{q}(t) \end{bmatrix} = \begin{bmatrix} mg\gamma - C_{L_\alpha}\bar{q}S & -mg\gamma & mV - \frac{C_{L_q}\bar{q}S\bar{c}}{2V} \\ 0 & 0 & 1 \\ C_{m_\alpha} & 0 & \frac{C_{m_q}\bar{q}S\bar{c}^2}{2V} \end{bmatrix} \begin{bmatrix} \alpha(t) \\ \theta(t) \\ q(t) \end{bmatrix}$$

$$+ \begin{bmatrix} -C_{L_{\delta_e}} \\ 0 \\ C_{m_{\delta_e}} \end{bmatrix} \left( \delta_e(t - t_d) + \begin{bmatrix} \theta_\alpha^* & 0 & \theta_q^* \end{bmatrix} \begin{bmatrix} \alpha(t) \\ \theta(t) \\ q(t) \end{bmatrix} \right) \tag{134}$$

where $t_d$ is a time delay.

A numerical model for a full-scale generic transport model (GTM) at Mach 0.8 and 30,000 ft with the flight path angle $\gamma = 0$ is given by

$$\begin{bmatrix} \dot{\alpha}(t) \\ \dot{\theta}(t) \\ \dot{q}(t) \end{bmatrix} = \underbrace{\begin{bmatrix} -0.7018 & 0 & 0.9761 \\ 0 & 0 & 1 \\ -2.6923 & 0 & -0.7322 \end{bmatrix}}_{A} \begin{bmatrix} \alpha(t) \\ \theta(t) \\ q(t) \end{bmatrix}$$

$$+ \underbrace{\begin{bmatrix} -0.0573 \\ 0 \\ -3.5352 \end{bmatrix}}_{B} \left( \delta_e(t - t_d) + \begin{bmatrix} 0.1393 & 0 & -0.2071 \end{bmatrix} \begin{bmatrix} \alpha(t) \\ \theta(t) \\ q(t) \end{bmatrix} \right)$$

A desired reference model of the pitch attitude is given by

$$\ddot{\theta}_m(t) + 2\zeta\omega_n\dot{\theta}_m(t) + \omega_n^2\theta_m(t) = \omega_n^2 r(t) \tag{135}$$

where $\zeta = 0.85$ and $\omega_n = 1.5$ rad/sec are chosen to give a desired handling characteristic.

Let $x(t) = \begin{bmatrix} \alpha(t) & \theta(t) & q(t) \end{bmatrix}^\top$, $u(t) = \delta_e(t)$, and $\Theta^{*\top} = \begin{bmatrix} \theta_\alpha^* & 0 & \theta_q^* \end{bmatrix} = \begin{bmatrix} 0.4 & 0 & -0.3071 \end{bmatrix}$ which represents a parametric uncertainty that is equivalent to an 80% reduction in the pitch damping coefficient $C_{m_q}$. A nominal controller is designed as $u_{nom}(t) = -K_x x(t) + k_r r(t)$ where $K_x = \frac{1}{b_3}\begin{bmatrix} a_{31} & \omega_n^2 & 2\zeta\omega_n + a_{33} \end{bmatrix} = \begin{bmatrix} 0.7616 & -0.6365 & -0.5142 \end{bmatrix}$ and $k_r = \frac{1}{b_3}\omega_n^2 = -0.6365$. The closed-loop eigenvalues are $-0.6582$ and $-1.2750 \pm 0.7902i$. The nominal closed-loop plant is then chosen to be the reference model as

$$\underbrace{\begin{bmatrix} \dot{\alpha}_m(t) \\ \dot{\theta}_m(t) \\ \dot{q}_m(t) \end{bmatrix}}_{\dot{x}_m} = \underbrace{\begin{bmatrix} -0.6582 & -0.0365 & 0.9466 \\ 0 & 0 & 1 \\ 0 & -2.2500 & -2.5500 \end{bmatrix}}_{A_m} \underbrace{\begin{bmatrix} \alpha_m(t) \\ \theta_m(t) \\ q_m(t) \end{bmatrix}}_{x_m} + \underbrace{\begin{bmatrix} 0.0365 \\ 0 \\ 2.2500 \end{bmatrix}}_{B_m} r(t)$$

The control input is given by

$$u(t) = -K_x x(t) + k_r r(t) - \Theta^\top(t) x(t) \tag{136}$$

Figures 5 and 6 are plots of the estimates of phase and time delay margins of the optimal control modification adaptive law for $\Gamma \to \infty$ computed by the matrix measure method from Eqs. (133) and (131) as a function of $v$ with and without the parametric uncertainty $\Theta^*$. Note that the phase margin generally decreases as $v$ increases and reaches a steady state value, while the time delay margin reaches a maximum at about $v = 1$. Thus, for practical design purposes, $v$ should be kept between 0 and 1. A large value of $v$ produces a better time delay margin, but also results in a poorer tracking. For the specified uncertainty $\Theta^*$, the maximum time delay margin is estimated to be 78 msec. Therefore, the input delay adaptive system will be stable with the optimal control modification adaptive law for any $t_d < 78$ msec.

Suppose the input time delay of the system is $t_d = 50$ msec. For this input time delay, the optimal control modification adaptive law produces a stable adaptation for $v \geq 0.244$. The controller is then implemented with $v = 0.25$ and $\Gamma = 3000I$ selected. Figures 7 and 8 illustrate the pitch angle and pitch rate responses due to the standard MRAC and optimal control modification adaptive law. The MRAC with the adaptive gain $\Gamma = 3000I$ does not track the reference pitch angle very well. High frequency oscillations are discernible in the pitch rate response. On the other hand, the optimal control modification adaptive law produces good tracking of the reference pitch angle and pitch rate. The high frequency oscillations in the pitch rate response with the standard MRAC is substantially reduced by the optimal control modification adaptive law.

The time delay margin of the closed-loop system is estimated by numerical evidence to verify the lower bound estimate of the time delay margin for the optimal control modification. The results are shown in Table 1.

|  | Time Delay Margin, msec |
| --- | --- |
| No Adaptation | 550 |
| MRAC | 50 |
| OCM | 100 |

Table 1 - Time Delay Margin Estimates

The numerical evidence of the time delay margin for the optimal control modification adaptive law is estimated to be 100 msec. This is a factor of two larger than the time delay margin of 51 msec as estimated by the matrix measure method for $v = 0.25$.
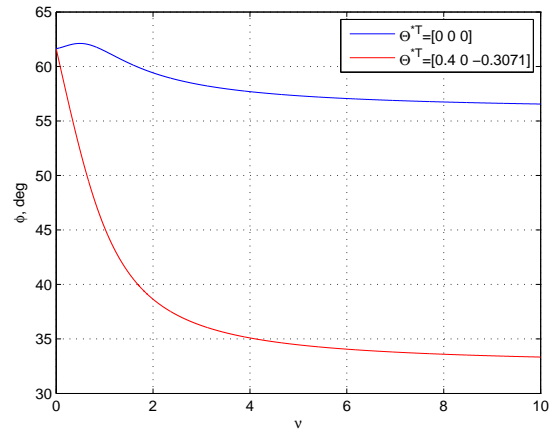
19

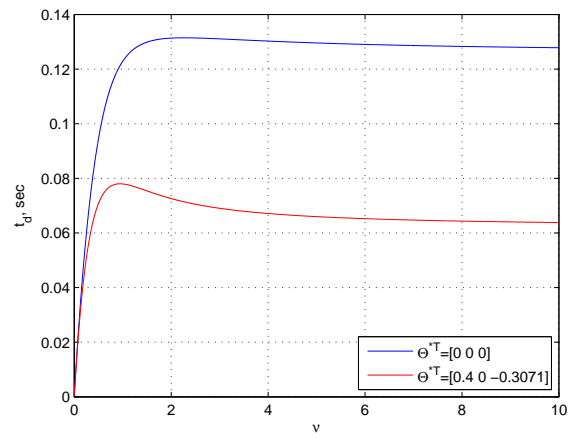Fig. 5 - Phase Margin Estimation of Optimal Control Modification



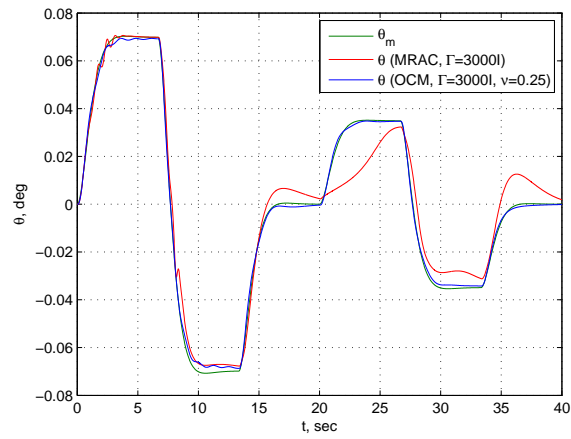Fig. 6 - Time Delay Margin Estimation of Optimal Control Modification
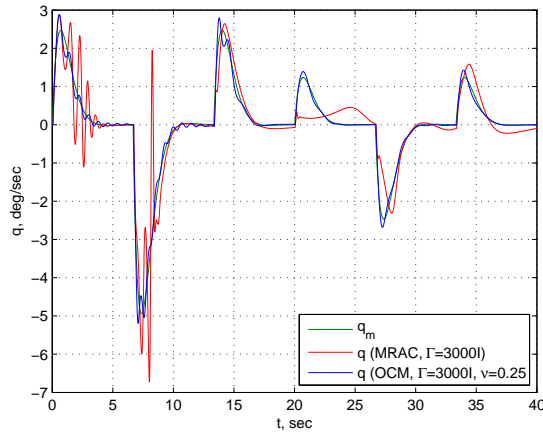


Fig. 7 - Pitch Angle Response

20

Fig. 8 - Pitch Rate Response

# 6  Conclusions

A method for analyzing input delay adaptive systems is presented. The bounded linear stability analysis approximates a nonlinear adaptive system as a bounded linear approximation within a time window. The windowing analysis is used to analyze local stability of the bounded linear approximation to estimate local stability behavior of the original nonlinear system within a given time window. Analytical approaches for computing the time delay margin are presented for three different methods: Pade approximation, Lyapunov-Krasovskii method, and matrix measure method. The windowing analysis results in non-unique estimates of the time delay margin of the bounded linear system since the results are dependent on the time window size. The non-uniqueness is a drawback of the windowing analysis. It is found, however, the mean value of the local time delay margins are relatively insensitive to the time window size. Among the three different methods, the Pade approximation using a first-order Pade rational polynomial generally tends to over-predict the time delay margin, while the Lyapunov-Krasovskii tends to be highly conservative in the time delay margin estimation. Neither of these estimations may be deemed practical for adaptive control design purposes. The matrix measure method seems to be able to predict the time delay margin with a reasonable agreement with the numerical evidence. Moreover, the method is much simpler to use and does not require solving a linear matrix inequality as in the case of the Pade approximation or Lyapunov-Krasovskii method.

The asymptotic behavior of the time delay margin as the adaptive gain tends to infinity is studied. The standard model-reference adaptive control has zero time delay margin as the adaptive gain tends to infinity, as expected. The time delay margins for both the $\sigma$-modification adaptive law and optimal control modification adaptive law remain finite as the adaptive gain tends to infinity. The optimal control modification adaptive law also exhibits another useful feature in that the asymptotic value of the time delay margin is independent of the time window and the closed-loop input delay adaptive system tends to a LTI system. This behavior enables a lower bound of the time delay margin to be estimated with ease using the matrix measure method to guarantee stability for the input delay adaptive system. Flight control simulations demonstrate that the time delay margin estimation by the matrix measure method provides a good lower bound.

# References

[1] Nguyen, N., Bakhtiari-Nejad, M., Krishnakumar, K., Ishihara, A., "Bounded Linear Stability Analysis - A Time Delay Margin Estimation Approach," AIAA Guidance, Navigation, and Control Conference, AIAA-2009-5968, August 2009.

[2] Nguyen, N., Jacklin, S., "Stability, Convergence, and Verification and Validation Challenges of Neural Net Adaptive Flight Control," Applications of Neural Networks in High Assurance Systems Studies in Computational Intelligence, Springer-Verlag, 2010, Vol. 268/2010, 77-110

[3] Annaswamy, A., Jang, J., Lavretsky, E., "Stability Margins for Adaptive Controllers in the Presence of Time-Delay," AIAA Guidance, Navigation, and Control Conference, AIAA 2008-6659, August 2008.

[4] Nguyen, N., Bakhtiari-Nejad, M., Huang, Y., "Hybrid Adaptive Flight Control with Bounded Linear Stability Analysis," AIAA Guidance, Navigation, and Control Conference, AIAA 2007-6422, August 2007.

[5] Nguyen, N. T. and Boskovic, J. D., "Bounded Linear Stability Margin Analysis of Nonlinear Hybrid Adaptive Control," American Control Conference, June 2008.

[6] Lavretsky, E., Annaswamy, A., Dydek, Z., Vega-Brown, W., "On the Computation of Stability Margins for Adaptive Controllers Using Linear System Tools," AIAA Guidance, Navigation, and Control Conference, AIAA-2009-5964, August 2009.

[7] Yang, B.-J. , Yucelen, T. , Calise, A. J. , Shin, J.-Y. , "LMI-based Analysis of Adaptive Controller," American Control Conference, June 2009.

[8] Ishihara, A., Ben-Menahem, S., and Nguyen, N., "Time Delay Margin Computation via the Razumikhin Method for an Adaptive Control System," AIAA Guidance, Navigation, and Control Conference, AIAA 2009-5969, August 2009.

[9] Ioannou, P. and Kokotovic, P., "Instability Analysis and Improvement of Robustness of Adaptive Control," Automatica, Vol. 20, No. 5, 1984, pp. 583-594.

[10] Nguyen, N., Krishnakumar, K., Boskovic, J., "An Optimal Control Modification to Model-Reference Adaptive Control for Fast Adaptation", AIAA Guidance, Navigation, and Control Conference, AIAA 2008-7283, August 2008.

[11] Mahmoud, M. S., *Robust Control and Filtering for Time-Delay Systems*, Marcel-Dekker, Inc., New York, 2000.

[12] Hale, J. K., *Functional Differential Equations*, Springer-Verlag, New York, 1971.

[13] Rohrs, C.E., Valavani, L., Athans, M., and Stein, G., "Robustness of Continuous-Time Adaptive Control Algorithms in the Presence of Unmodeled Dynamics," IEEE Transactions on Automatic Control, Vol AC-30, No. 9, pp. 881-889, 1985.

[14] Narendra, K. S. and Annaswamy, A. M., "A New Adaptive Law for Robust Adaptation Without Persistent Excitation", IEEE Transactions on Automatic Control, Vol. AC-32, No. 2, pp. 134-145, 1987.

[15] Nguyen, N., "Asymptotic Linearity of Optimal Control Modification Adaptive Law with Analytical Stability Margins," AIAA Infotech@Aerospace Conference, AIAA-2010-3301, April 2010.

[16] Nguyen, N., "Verifiable Adaptive Control with Analytical Stability Margins by Optimal Control Modification," AIAA Guidance, Navigation, and Control Conference, AIAA-2010-7770, August 2010.