Robustness-based Design Optimization under Data Uncertainty

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Abstract

This paper proposes formulations and algorithms for design optimization under both aleatory (i.e., natural or physical variability) and epistemic uncertainty (i.e., imprecise probabilistic information), from the perspective of system robustness. The proposed formulations deal with epistemic uncertainty arising from both sparse and interval data without any assumption about the probability distributions of the random variables. A decoupled approach is proposed in this paper to un-nest the robustness-based design from the analysis of non-design epistemic variables to achieve computational efficiency. The proposed methods are illustrated for the upper stage design problem of a two-stage-to-orbit (TSTO) vehicle, where the information on the random design inputs are only available as sparse point and/or interval data. As collecting more data reduces uncertainty but increases cost, the effect of sample size on the optimality and robustness of the solution is also studied. A method is developed to determine the optimal sample size for sparse point data that leads to the solutions of the design problem that are least sensitive to variations in the input random variables.

1. Introduction

In deterministic design optimization, it is generally assumed that all design variables and system variables are precisely known; the influence of natural variability and data uncertainty on the optimality and feasibility of the design is not explicitly considered. However, real-life engineering problems are non-deterministic, and a deterministic assumption about inputs may lead to infeasibility or poor performance (Sim, 2006). In recent years, many methods have been developed for design under uncertainty. Reliability-based design (e.g., Chiralaksanakul and Mahadevan, 2005; Ramu et al, 2006; Agarwal et al, 2007and Du and Huang, 2007) and robust design (e.g., Parkinson et al,

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1993; Du and Chen, 2000; Doltsinis and Kang, 2004 and Huang and Du, 2007) are two directions pursued by these methods. While reliability-based design aims to maintain design feasibility at desired reliability levels, robust design optimization attempts to minimize variability in the system performance due to variations in the inputs (Lee et al, 2008). In recent years, several methods have also been proposed to integrate these two paradigms of design under uncertainty (e.g., Du et al, 2004, Lee et al, 2008).

Taguchi proposed robust design methods for selecting design variables in a manner that makes the product performance insensitive to variations in the manufacturing process (Taguchi, 1993). Taguchi's methods have widespread applications in engineering; however, these methods are implemented through statistical design of experiments and cannot solve problems with multiple measures of performances and design constraints (Wei et al, 2009). With the introduction of nonlinear programming to robust design, it has become possible to achieve robustness in both performance and design constraints (Du and Chen, 2000).

The essential elements of robust design optimization are: (1) maintaining robustness in the objective function (objective robustness); (2) maintaining robustness in the constraints (feasibility robustness); (3) estimating mean and measure of variation (variance) of the performance function; and (4) multi-objective optimization. The rest of this section briefly reviews the literature with respect to these four elements and establishes the motivation for the current study.

<u>Objective robustness</u>

In robust optimization, the robustness of the objective function is usually achieved by *simultaneously optimizing its mean and minimizing its variance*. Two major robustness measures are available in the literature: one is the variance, which is extensively discussed in the literature (Du and Chen, 2000; Lee and Park, 2001 and Doltsinis and Kang, 2004) and the other is based on the percentile difference (Du et al, 2004). Although the percentile difference method has the advantage that it contains the information of probability in the tail regions of the performance distribution, this method is only applicable to unimodal distributions. Variance as a measure of variation of the performance function can be applied to any distribution (unimodal or multimodal), but it only characterizes the dispersion around the mean (Huang and Du, 2007).

Feasibility robustness

Feasibility robustness i.e., robustness in the constraints can be defined as *satisfying the constraints of the design in the presence of uncertainty*. Du and Chen (2000) classified the methods of maintaining feasibility robustness into two categories, methods that use probabilistic and statistical analysis, and methods that do not require them. Among the methods that require probabilistic and statistical analysis, a probabilistic feasibility formulation (Du and Chen, 2000 and Lee et al, 2008), and a moment matching formulation (Parkinson et al, 1993) have been proposed. Du and Chen (2000) used a most probable point (MPP)-based importance sampling method to reduce the computational burden associated with the probabilistic feasibility formulation. The moment matching formulation is a simplified approach which requires only the constraints on the first and second moments of the performance function to be satisfied, and assumes that the performance function is normally distributed. A variation of this approach, *the feasible region reduction method* has been described in Park et al (2006), which is more general and does not require the normality assumption. This is a tolerance design method, where width of the feasible space in each direction is reduced by the amount $k\sigma$, where k is a user-defined constant and σ is the standard deviation of the performance function. This method only requires the mean and variance of the performance function.

Methods that do not require probabilistic and statistical analysis are also available, for example, worst case analysis (Parkinson et al, 1993), corner space evaluation (Sundaresan et al, 1995), and manufacturing variation patterns (MVP) (Yu and Ishii, 1998). A comparison study of the different constraint feasibility methods can be found in Du and Chen (2000).

Estimating mean and variance of the performance function

Various methods have been reported in the literature to estimate the mean and standard deviation of the performance function. These methods can be divided into three major classes: (i) Taylor series expansion methods, (ii) sampling-based methods and (iii) point estimate methods (Huang and Du, 2007).

The Taylor series expansion method (Haldar and Mahadevan, 2000; Du and Chen, 2000; and Lee et al, 2001) is a simple approach. However, for a nonlinear performance function, if the variances of the random variables are large, this approximation may result in large errors (Du et al., 2004). Although a second-order Taylor series expansion is generally more accurate than the first-order approximation, it is also computationally more expensive.

Sampling-based methods require information on distributions of the random variables, and are expensive. Efficient sampling techniques such as importance sampling, Latin hypercube sampling, etc. (Robert and Cesalla, 2004) can be used to reduce the computational effort, but are still prohibitive in the context of optimization. Surrogate models (Ghanem and Spanos 1991; Bichon et al, 2008; Cheng and Sandu, 2009) may be used to further reduce computational effort.

In an attempt to overcome the difficulties associated with the computation of derivatives required in Taylor series expansion, Rosenlblueth (1975) proposed a point estimate method to compute the first few moments of the performance function. Different variations of this point estimate method (Hong, 1998; Zhao and Ono, 2000 and Zhao and Ang, 2003) have been studied. Although point estimate methods are easier to implement, the accuracy may be low and may generate points that lie outside the domain of the random variable.

Multi-objective optimization

Robustness-based optimization considers two objectives: optimize the mean of the objective function and minimize its variation. An extensive survey of the multi-objective optimization methods can be found in Marler and Arora (2004). Among the available methods, the weighted sum approach is the most common approach to multi-objective optimization and has been extensively used in robust design optimization (Lee and Park, 2001; Doltsinis and Kang, 2004; Zou and Mahadevan, 2006). The designer can obtain alternative design points by varying the weights and can select the one that offers the best trade-off among multiple objectives. Despite its simplicity, the weighted sum method may not obtain potentially desirable solutions (Park et al, 2006). Another common approach is the ε-constraint method in which one of the objective functions is optimized while the other objective functions are used as constraints. Despite its advantages over weighted sum method in some cases, the ε-constraint method can be computationally expensive for more than two objective functions (Mavrotas, 2009).

Other methods include goal programming (Zou and Mahadevan, 2006), compromise decision support problem (Bras and Mistree, 1993, 1995; Chen et al, 1996), compromise programming (CP) (Zalney, 1973; Zhang, 2003; Chen et al, 1999) and physical programming (Messac, 1996; Messac et al, 2001; Messac and Ismail-Yahaya, 2002; Chen et al, 2000). Each of these methods has its own advantages and limitations.

Although there is now an extensive volume of literature for robust optimization methods and applications, all these methods have only been studied with respect to physical or natural variability represented by probability distributions. Uncertainty in system design also arises from other contributing factors. Sources of uncertainty may be divided into two types: aleatory and epistemic (Oberkampf et. al., 2004). Aleatory uncertainty is irreducible. Examples include phenomena that exhibit natural variation like operating conditions, material properties, geometric tolerances, etc. In contrast, epistemic uncertainty results from a lack of knowledge about the system, or due to approximations in the system behavior models, or due to limited or subjective (e.g., expert opinion) data; it can be reduced as more information about the system is obtained.

One type of data uncertainty involves having limited data to properly define the distribution parameters of the random variables. This type of uncertainty may be reduced by collecting more data. In some cases of data uncertainty, distribution information of a random variable may only be available as intervals given by experts. The objective of this paper is to develop an efficient robust optimization methodology that includes both aleatory and epistemic uncertainty described through sparse point data and interval data.

A few studies on robust design optimization are reported in the literature to deal with epistemic uncertainty arising from lack of information. Youn et al (2007) used a possibility-based method, and redefined the performance measure of robust design using the most likely values of fuzzy random variables. Dai and Mourelatos (2003) proposed two two-step methods for robust design optimization that can treat aleatory and epistemic uncertainty separately using a range method and a fuzzy sets approach. Most of the current methods of robust optimization for epistemic uncertainty need additional non-probabilistic formulations to incorporate epistemic uncertainty into the robust optimization framework, which may be computationally expensive. However, if the epistemic uncertainty can be converted to a probabilistic format, the need for these additional formulations is avoidable, and well-established probabilistic methods of robust design optimization can be used. Therefore, there is a need for an efficient robust design optimization methodology that deals with both aleatory and epistemic uncertainty.

In this paper, we propose robustness-based design optimization formulations that work under both aleatory and epistemic uncertainty using probabilistic representations of different types of uncertainty. Our proposed formulations deal with both sparse point and interval data without any assumption about probability distributions of the random variables.

The performance of robustness-based design can be defined by the mean and variation of the performance function. In our proposed formulations, we obtain the optimum mean value of the objective function (e.g., gross weight) while also minimizing its variation (e.g., standard deviation). Thus, the design will meet target values in terms of both design bounds and standard deviations of design objectives and design variables thereby ensure feasibility robustness.

A Taylor series expansion method is used in this paper to estimate the mean and standard deviation of the performance function, which requires means and standard deviations of the random variables. However, with sparse point data and interval data, it is impossible to know the true moments of the data, and there are many possible probability distributions that can represent these data (Zaman et al, 2009). In this paper, we propose methods for robustness-based design optimization that account for this uncertainty in the moments due to sparse point data and interval data and thereby include epistemic uncertainty into the robust design optimization framework. As collecting more data reduces uncertainty but increases cost, the effect of sample size on the optimality and the robustness of the solution is also studied. A method to determine the optimal sample size for sparse point data that will lead to the minimum scatter on solutions to the design problem is also presented in this paper.

In some existing methods for robust design under epistemic uncertainty, all the epistemic variables are considered as design variables (Youn et al, 2007). However, if the designer does not have any control on an epistemic variable (e.g., Young's modulus in beam design), considering that variable as a design variable might lead to a solution that could underestimate the design objectives. Therefore, in this paper, we propose a general formulation for robust design that considers some of the epistemic variables as non-design variables, which leads to a conservative design under epistemic uncertainty.

Note that the proposed robustness-based design optimization method is general and capable of handling a wide range of application problems under data uncertainty. The proposed methods are illustrated for the conceptual level design process of a two-stage-to-orbit (TSTO) vehicle, where the distributions of the random inputs are described by sparse point and/or interval data.

The rest of the paper is organized as follows. Section 2 proposes robustness-based design optimization framework for sparse point data and interval data. In Section 3, we illustrate the proposed methods for the conceptual level design process of a TSTO vehicle. Section 4 provides conclusions and suggestions for future work.

2. Proposed methodology

Deterministic design optimization

In a deterministic optimization formulation, all design variables and system variables are considered deterministic. No random variability or data uncertainty is taken into account. The deterministic optimization problem is formulated as follows:

$$\begin{aligned} &\min_x \ f(x) \\ &s.t. \quad LB \le g_i(x) \le UB \quad \text{for all } i \\ &\qquad lb \le x \le ub \end{aligned} \tag{1}$$

where $f(x)$ is the objective function, $\boldsymbol{x}$ is the vector of design variables, $g_i(x)$ is the ith constraint, LB and UB are the vectors of lower and upper bounds of constraints $g_i's$ and lb and ub are the vectors of lower and upper bounds of design variables.

In practice, the input variables might be uncertain and solutions of this deterministic formulation could be sensitive to the variations in the input variables. Robustness-based design optimization takes this uncertainty into account. The optimal design points obtained using the deterministic method could be used as initial guesses in robustness-based optimization.

Robustness-based design optimization

In the proposed methodology, we use *variance* as a measure of variation of the performance function in order to achieve *objective robustness*, *the feasible region reduction method* to achieve *feasibility robustness, a first-order Taylor series expansion* to estimate the mean and variance of the performance function, and *a weighted sum method* for the aggregation of multiple objectives. This combination of methods is only used for the sake of illustration. Other approaches can be easily substituted in the proposed methodology. The robustness-based design optimization problem can now be formulated as follows:

$$\begin{aligned}
&\min_{d} \; f(\mu,\sigma) = (w^* \mu_f + v^* \sigma_f) \\
&s.t. \qquad LB + k\sigma(g_i(d,z)) \le E(g_i(d,z)) \le UB - k\sigma(g_i(d,z)) \quad \text{for all } i \qquad (2) \\
&\qquad\quad lb + k\sigma(x_i) \le d_i \le ub - k\sigma(x_i) \quad \text{for } i = 1,2,\ldots,nrdv \\
&\qquad\quad lb \le d_i \le ub \quad \text{for } i = 1,2,\ldots,nddv
\end{aligned}$$

where ***d*** is the vector of deterministic design variables as well as the mean values of the uncertain design variables ***x***; *nrdv* and *nddv* are the numbers of the random design variables and deterministic design variables, respectively; and ***z*** is the vector of non-design input random variables, whose values are kept fixed at their mean values as a part of the design. The weighting coefficients $w \ge 0$ and $v \ge 0$ represent the relative importance of the objectives μ_f and σ_f in Eq. (2); $g_i(d,z)$ is the *i*th constraint; $E(g_i(d,z))$ is the mean and $\sigma(g_i(d,z))$ is the standard deviation of the *i*th constraint. *LB* and *UB* are the vectors of lower and upper bounds of constraints $g_i's$; *lb* and *ub* are the vectors of lower and upper bounds of the design variables; $\sigma(x)$ is the vector of standard deviations of the random variables and k is some constant. The role of the constant k is to adjust the robustness of the method against the level of conservatism of the solution. It reduces the feasible region by accounting for the variations in the design variables and is related to the probability of constraint satisfaction. For example, if a design variable or a constraint function is normally distributed, k =1 corresponds to the probability 0.8413, k =2 to the probability 0.9772, etc.

Note that the robust design formulation in Eq. (2) is a standard nonlinear multi-objective optimization formulation. The optimality conditions of such a formulation have been extensively described in the literature including Cagan and Williams (1993) and Marler and Arora (2004).

In the proposed formulation, the performance functions considered are in terms of the model outputs. The means and standard deviations of the objective and constraints are estimated by using a first-order Taylor series approximation as follows:

Performance function: $Y = g(X_1, X_2, ..., X_n)$ (3)

First-order approximate mean of y: $E(Y') \approx g(\mu_{X_1}, \mu_{X_2}, ..., \mu_{X_n})$ (4)

First-order variance of y: $Var(Y') \approx \sum_{i=1}^{n} \left(\frac{\partial g}{\partial X_i} \right)^2 Var(X_i) + \sum_{i=1}^{n} \sum_{\substack{j=1 \\ i \neq j}}^{n} \frac{\partial g}{\partial X_i} \frac{\partial g}{\partial X_j} Cov(X_i, X_j)$ (5)

The implementation of Eq. (2) requires that variances of the random design variables X_i and the means and variances of the random non-design variables Z_i be precisely known, which is possible only when a large number of data points are available. In practical situations, only a small number of data points may be available for the input variables. In other cases, information about random input variables may only be specified as intervals, as by expert opinion. This is input data uncertainty, causing uncertainty regarding the distribution parameters (e.g., mean and variance) of the inputs X_i and Z_i. Robustness-based optimization has to take this into account. In the following subsections, we propose a new methodology for robustness-based design optimization that accounts for data uncertainty.

2.1 Robustness-based design optimization under data uncertainty

The inclusion of epistemic uncertainty in robust design adds another level of complexity in the design methodology. The design variables ***d*** and/or the input random variables ***z*** in Eq. (2) might have epistemic uncertainty. Since the designer does not have any control on the non-design epistemic variables ***z***, the design methodology has to employ a search among the possible values of such epistemic variables in order to find an optimal solution. In such case, we get a conservative robust design. The robustness-based design optimization problem can now be formulated with the following generalized statement:

$$\begin{aligned} &\max_{\mu_z}\Big(\min_{d} \;\; f(\mu,\sigma) = (w*\mu_f + v*\sigma_f)\Big) \\ &s.t. \qquad LB + k\sigma(g_i(d,\mu_z)) \le E(g_i(d,z)) \le UB - k\sigma(g_i(d,\mu_z)) \;\; \text{for all } i \\ &\qquad\qquad lb + k\sigma(x) \le d \le ub - k\sigma(x) \\ &\qquad\qquad Z_l \le \mu_z \le Z_u \end{aligned} \tag{6}$$

where Z_l and Z_u are the vectors of lower and upper bounds of the decision variables μ_z of the outer loop optimization problem.

Note that in this formulation, the inner loop decision variables ***d*** may consist of stochastic design variables as well as epistemic design variables. The inner loop optimization is a design optimization problem, where a robust design optimization is carried out for a fixed set of non-design epistemic variables. The outer loop optimization is the analysis for the non-design epistemic variables, where the optimizer searches among the possible values of the non-design epistemic variables for a conservative solution of the robust design problem.

This nested optimization problem can be decoupled and expressed as:

$$\begin{aligned} d^* &= \arg\min_{d} \left(w * \mu_f\left(d, \mu_z^*\right) + v * \sigma_f\left(d, \mu_z^*\right)\right) \\ s.t. \quad & LB + k\sigma(g_i(d, \mu_z^*)) \le E(g_i(d, z)) \le UB - k\sigma(g_i(d, \mu_z^*)) \quad \text{for all } i \\ & lb + k\sigma(x) \le d \le ub - k\sigma(x) \end{aligned} \tag{7}$$

$$\begin{aligned} \mu_z^* &= \arg\max_{\mu_z} \left(w * \mu_f\left(d^*, \mu_z\right) + v * \sigma_f\left(d^*, \mu_z\right)\right) \\ s.t. \quad & LB + k\sigma(g_i(d^*, \mu_z)) \le E(g_i(d, z)) \le UB - k\sigma(g_i(d^*, \mu_z)) \quad \text{for all } i \\ & Z_l \le \mu_z \le Z_u \end{aligned} \tag{8}$$

The optimization problems in Eqs. (7) and (8) are solved iteratively until convergence. Note that the first constraint (i.e., the robustness constraint) in Eq. (8) is required to ensure that the optimization is driven by all non-design epistemic variables, because sometimes the objective function may not be a function of all non-design epistemic variables. In cases when the objective function is the function of all non-design epistemic variables, this constraint is not required.

2.1.1 Robustness-based design with sparse point data

In this section, we propose a methodology for robustness-based design optimization with sparse point data. It is assumed that only sparse point data are available for the uncertain design variables as well as non-design epistemic variables.

Since the data size is small, there is uncertainty about the mean and variance calculated from the samples. The chi-square distribution is a good assumption for the distribution of the variance, if the underlying population is normal. The two-sided (1-α) confidence interval for the population variance σ^2 can be expressed as (Haldar and Mahadevan, 2000):

$$\left[\frac{(n-1)s^2}{c_{1-\alpha/2,n-1}};\frac{(n-1)s^2}{c_{\alpha/2,n-1}}\right] \tag{9}$$

where n is the sample size, s is the sample standard deviation of sparse point data, and $c_{\alpha/2,n-1}$ is obtained from the chi-square distribution at (n-1) degrees of freedom and α significance level. Note that Eq. (9) can still be used to obtain approximate confidence bounds for variance if the underlying population is not normal. However, in such cases, other approximation methods (Bonett, 2006; Cojbasic and Tomovic, 2007) can be used to obtain more reliable estimates of confidence bounds. In robustness-based design optimization, we are interested in obtaining a solution that is least sensitive to the variations in the input random variables; therefore we use the upper bound variances for the input random variables x_i and z_i to solve the formulations in Eqs. (7)-(8) for sparse point data.

For non-design epistemic variables described by sparse point data, the constraints on the decision variables in Eq. (8) are implemented through the construction of confidence intervals about means. As the design variables are described by the sparse point data, it is possible that the underlying distributions of the design variables might have major deviations from normality. Therefore, we have used the Johnson's modified t

statistic (Johnson, 1978) to construct the confidence bounds on means of the design variables as follows:

$$
\begin{aligned}
Z_l &= \bar{z} - t_{\alpha/2,n-1}\frac{s}{\sqrt{n}} + \frac{\mu_3}{6s^2 n} \\
Z_u &= \bar{z} + t_{\alpha/2,n-1}\frac{s}{\sqrt{n}} + \frac{\mu_3}{6s^2 n}
\end{aligned}
\qquad (10)
$$

where $\bar{z}$ is the vector of means of the epistemic variables, s is the vector of standard deviations, n is the sample size of the sparse point data, μ_3 is the third central moment and $t_{\alpha/2,n-1}$ is obtained from the Student t distribution at (n-1) degrees of freedom and α significance level. This modified statistic takes into account the skewness of the distribution and thus provides a better estimate of the confidence bound in the presence of limited data.

The optimization formulation shown in Eqs. (7)-(8) involves aggregation of multiple objectives. In the proposed formulations, the aggregate objective function consists of two types of objectives, expectation and standard deviation of model outputs. Since different objectives have different magnitudes, a scaling factor has to be used in the formulation.

2.1.2 Determination of optimal sample size for sparse point data

The optimal solutions depend on the sample size of the sparse data as will be discussed in Section 3.1. Therefore, it is of interest to determine the optimal sample size of the sparse data that leads to the solution of the design problem that is least sensitive to the variations of design variables. This will facilitate resource allocation decisions for data collection. The following two optimization formulations are solved iteratively until convergence for the optimal sample sizes of the epistemic design variables (n_d^*) and epistemic non-design variables(n_e^*).The formulations in Eq. (11)-(12) are the weighted sum formulations of a three-objective optimization problem, where the first and second objectives are the mean and standard deviation of GW respectively and the third objective is the total cost of obtaining samples for all the random variables.

$$
\begin{aligned}
\left[d^*, n_d^*\right] = \arg\min_{d, n_d} \quad & \left(w * E(g_i(d, \mu_z^*)) + v * \sigma(g_i(d, \mu_z^*), n_d, n_e^*) + (1 - w - v) * \left(\sum_{j=1}^{m} n_{d_j} c_{d\,j} + \sum_{j=1}^{q} n_{e_j}^* c_{e\,j} \right) \right) \\
s.t. \quad & LB + k\sigma(g_i(d, \mu_z^*), n_d, n_e^*) \le E(g_i(d, z)) \le UB - k\sigma(g_i(d, \mu_z^*), n_d, n_e^*) \quad \text{for all } i \qquad (11) \\
& lb + k\sigma(x, n_d) \le d \le ub - k\sigma(x, n_d) \\
& \sum_{j=1}^{m} n_{d_j} c_{d\,j} + \sum_{j=1}^{q} n_{e_j}^* c_{e\,j} \le C \\
& n_{d\,j} \le b_{d\,j} \quad \text{for all j}
\end{aligned}
$$

$$
\begin{aligned}
\left[\mu_z^*, n_e^*\right] = \arg\max_{\mu_z, n_e} \quad & \left(w * E(g_i(d^*, \mu_z)) + v * \sigma(g_i(d^*, \mu_z), n_d^*, n_e) + (1 - w - v) * \left(\sum_{j=1}^{m} n_{d_j}^* c_{d\,j} + \sum_{j=1}^{q} n_{e_j} c_{e\,j} \right) \right) \\
s.t. \quad & LB + k\sigma(g_i(d^*, \mu_z), n_d^*, n_e) \le E(g_i(d, z)) \le UB - k\sigma(g_i(d^*, \mu_z), n_d^*, n_e) \quad \text{for all } i \qquad (12) \\
& Z_l(n_e) \le \mu_z \le Z_u(n_e) \\
& \sum_{j=1}^{m} n_{d_j}^* c_{d\,j} + \sum_{j=1}^{q} n_{e_j} c_{e\,j} \le C \\
& n_{e_j} \le b_{e\,\mathrm{j}} \quad \text{for all j}
\end{aligned}
$$

where $w \ge 0$ and $v \ge 0$ are the weighting coefficients that represent the relative importance of the objectives; n_{d_j} and n_{e_j} are the sample sizes and b_{d_j} and b_{e_j} are the maximum sample size possible for the *j*th design and non-design random variables, respectively. The number of design and non-design random variables are denoted by *m* and *q*, respectively. Terms c_{d_j} and c_{e_j} are the cost of obtaining one sample for the *j*th random design and non-design variables, respectively and C is the total cost allocated for obtaining samples for all the random variables. Note that as in Eq. (8), the robustness constraint in Eq. (12) is only required if the objective function is not a function of all non-design epistemic variables. The optimization formulation presented above is a mixed-integer nonlinear problem. A relaxed problem is solved in Section 3.

2.1.3 Robustness-based design with interval data

This section proposes a methodology for robustness-based design optimization with interval data. In this case, the only information available for one or more input random variables is in the form of single interval or multiple interval data. The following discussion develops a methodology to solve the formulations in Eq. (7)-(8) for uncertainty represented through interval data.

For interval data, the moments (e.g., mean and variance) are not a single value, rather only bounds can be given (Zaman et al, 2009). We have proposed methods to compute the bounds of moments for both single and multiple interval data in Zaman et al (2009).The methods for computing bounds of the first two moments for interval data are given later in this section. Once the bounds on the mean and variance of interval data are estimated, we use the upper bounds of sample variance to solve the formulations of robust design under uncertainty represented through single interval or multiple interval data. Therefore, the resulting solution becomes least sensitive to the variations in the uncertain variables.

For non-design epistemic variables described by interval data, the constraints on the decision variables in Eqs. (8) and (12) are implemented through estimating the bounds of the means by the methods as described later in this section.

The following discussions briefly summarize the methods to estimate the bounds on the first two moments for single interval and multiple interval data, respectively.

Bounds on moments with single interval data

The methods for calculating bounds on the first two moments for single interval data are summarized in Table 1 below.

Table 1: Methods for calculating moment bounds for single interval data

Moment	**Condition**		**Formula**
	Lower bound	**Upper bound**	
1	PMF = 1 at lower endpoint = 0 elsewhere	PMF = 1 at upper endpoint = 0 elsewhere	$M_1 = E(x)$
2	PMF = 1 at any point = 0 elsewhere	PMF = 0.5 at each endpoint	$M_2 = E(x^2) - (E(x))^2$

Note: $E(x) = \sum_{i=1}^{2} x_i p(x_i)$ $E(x^2) = \sum_{i=1}^{2} x_i^2 p(x_i)$

where $p(x_i)$ = Probability Mass Function (PMF)

In Table 1, the formulas lead to the lower and upper endpoints of the interval as the lower and upper bounds for the first moment, respectively. The formulas also imply that the lower bound for the second moments is zero.

Bounds on moments with multiple interval data

The methods for calculating bounds on the first two moments for multiple interval data are summarized in Table 2 below.

Table 2: Methods for calculating moment bounds for multiple interval data

Moment	Formula
1	$\left[\underline{M} \quad \overline{M}\right] = \left[\frac{1}{n}\sum_{i}^{n} lb_i, \frac{1}{n}\sum_{i}^{n} ub_i\right]$
2	$\min/\max_{x_1,\ldots,x_n} \quad M_2 = \frac{1}{n}\sum_{i=1}^{n}\left(x_i - \frac{1}{n}\sum_{j=1}^{n} x_j\right)^2$ s.t. $\quad lb_i \le x_i \le ub_i \quad i = \{1,\ldots,n\}$

Note: $[lb_i \quad ub_i]$ = Set of intervals n = Number of intervals

Once the bounds on the mean and variance of interval data are estimated by the methods described above, we can now use these bounds to solve the formulations of robustness-based design optimization under uncertainty represented through single interval or multiple interval data. In the following section, we illustrate our proposed formulations for robustness-based design optimization with both sparse point and interval data.

3. Example Problem

In this section, the proposed methods are illustrated for the conceptual level design process of a TSTO vehicle. The multidisciplinary system analysis consists of geometric modeling, aerodynamics, aerothermodynamics, engine performance analysis, trajectory analysis, mass property analysis and cost modeling (Stevenson et al, 2002). In this paper, a simplified version of the upper stage design process of a TSTO vehicle is used to illustrate the proposed methods. High fidelity codes of individual disciplinary analysis are replaced by inexpensive surrogate models. Figure 1 illustrates the analysis process of a TSTO vehicle.

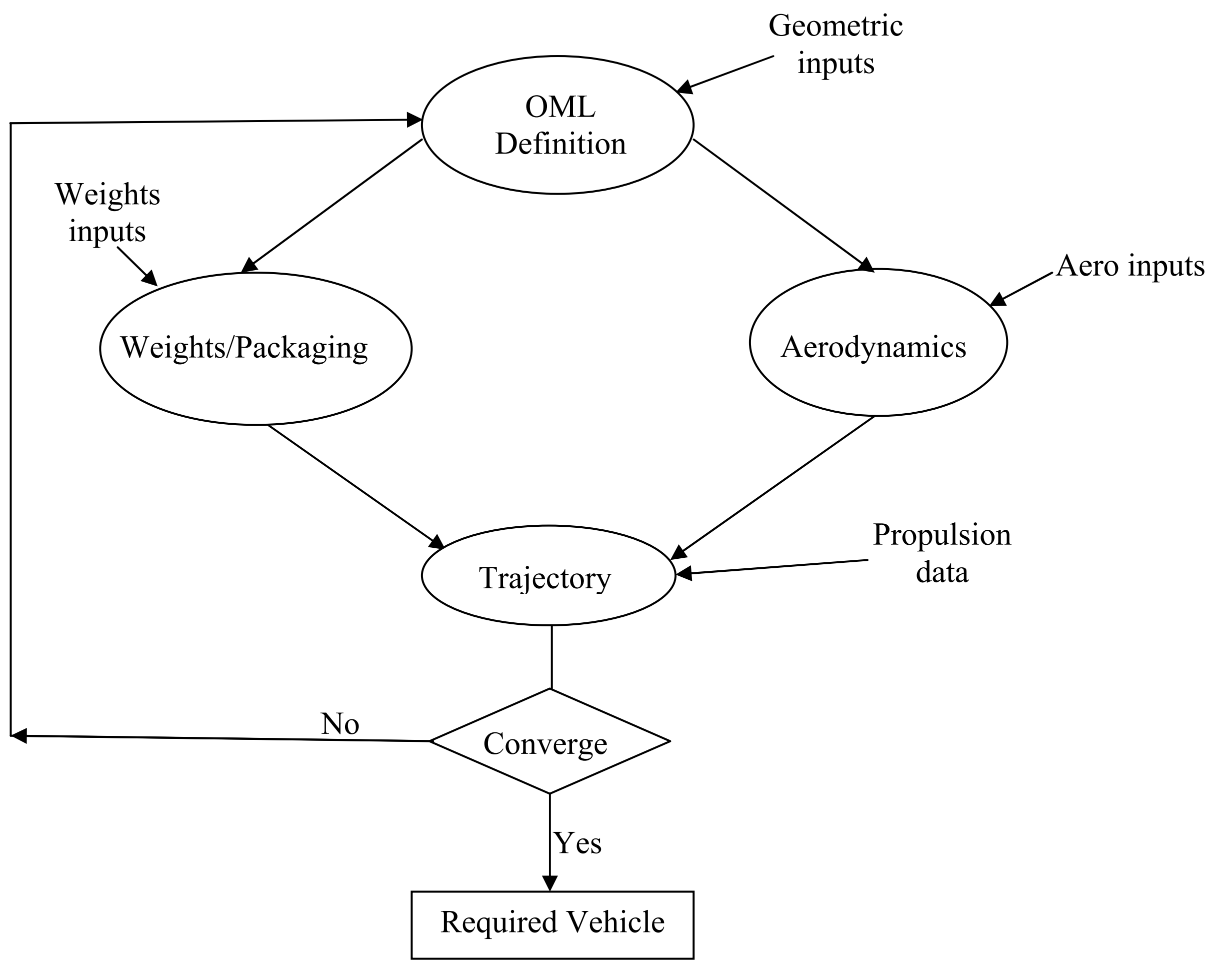

Figure 1: TSTO vehicle concept

The analysis outputs (performance functions) are Gross Weight (GW), Engine Weight (EW), Propellant Fraction Required (PFR), Vehicle Length (VL), Vehicle Volume (VV) and Body Wetted Area (BWA). Each of the analysis outputs is approximated by a second-order response surface and is a function of the random design variables Nozzle Expansion Ratio (ExpRatio), Payload Weight (Payload), Separation Mach (SepMach), Separation Dynamic Pressure (SepQ), Separation Flight Path Angle (SepAngle), and

Body Fineness Ratio (Fineness). Each of the random variables is described by either sparse point data or interval data.

The objective is to optimize an individual analysis output (e.g., Gross Weight) while satisfying the constraints imposed by each of the design variables as well as all the analysis outputs. We note here that we have assumed independence among the uncertain input variables and thereby ignored the covariance terms in Eq. (5) to estimate the variance of the performance function in each of the following examples. The numerical values of the design bounds for the design variables and analysis outputs are given in Tables 3 and 4, respectively.

Table 3: Design bounds for the design variables

Design Variable	lb	ub
ExpRatio	40	150
Payload	8000	40000
SepMach	7	12
SepQ	40	200
SepAngle	7	12
Fineness	4	6

Table 4: Design bounds for the analysis outputs

Analysis output	LB	UB
GW	0	100e+005
EW	0	100e+005
PFR	0.4	0.95
VL	0	100e+002
VV	0	100e+003
BWA	0	100e+005

3.1. Robustness-based design optimization with sparse point data

The methodology proposed in Section 2.1.1 is illustrated here for the TSTO problem. It is assumed that all the input variables $\boldsymbol{x}$ are described by sparse point data as given in Table 5. For this example, the input variable SepQ is assumed to be a non-design epistemic variable and all the remaining variables are assumed to be design variables. The design bounds for the respective design variables and the analysis outputs are given in Tables 3 and 4.

Table 5: Sparse Point Data for the random design variables

Sample	ExpRatio	Payload	SepMach	SepQ	SepAngle	Fineness
01	85.23	2.8952e+004	10.85	115.38	9.12	4.07
02	82.25	2.9747e+004	10.56	111.63	9.49	4.02
03	88.79	2.6638e+004	10.93	118.57	9.85	4.47
04	83.93	2.8356e+004	10.70	111.60	9.87	4.15
05	80.67	2.7193e+004	10.58	100.34	9.27	4.15
06	91.32	2.9168e+004	10.82	102.42	9.21	4.17
07	83.64	2.8844e+004	10.88	117.25	9.57	4.23
08	86.64	2.5836e+004	10.99	109.69	9.64	4.32
09	90.32	2.9310e+004	10.00	116.90	9.42	4.01
10	85.39	2.9949e+004	10.87	104.19	9.21	4.42

The design problem becomes:

$$
\begin{aligned}
& d^* = \arg\min_d \left(w * E(GW) + (1-w) * \sigma(GW) \right) \\
s.t. \quad & LB_1 + k\sigma(GW) \le E(GW) \le UB_1 - k\sigma(GW) \\
& LB_2 + k\sigma(EW) \le E(EW) \le UB_2 - k\sigma(EW) \\
& LB_3 + k\sigma(PFR) \le E(PFR) \le UB_3 - k\sigma(PFR) \\
& LB_4 + k\sigma(VL) \le E(VL) \le UB_4 - k\sigma(VL) \\
& LB_5 + k\sigma(VV) \le E(VV) \le UB_5 - k\sigma(VV) \\
& LB_6 + k\sigma(BWA) \le E(BWA) \le UB_6 - k\sigma(BWA) \\
& lb + k\sigma(x) \le d_i \le ub - k\sigma(x) \qquad \text{for } i = 1,2,\ldots,5
\end{aligned}
\tag{13}
$$

$$\mu_z^* = \arg\max_{\mu_z} \left(w * E(GW) + (1-w) * \sigma(GW) \right) \tag{14}$$
$$s.t. \qquad Z_l \le \mu_{z_i} \le Z_u \qquad \text{for } i = 1$$

where the bounds Z_l and Z_u for the mean of the non-design epistemic variable SepQ are calculated by Eq. (10) as given in Section 2.1.1. Note that in Eq. (14), we do not use the robust design constraints, since the objective function in this case is a function of all non-design epistemic variables.

As mentioned earlier in Section 2, $w \ge 0$ is the weight parameter that represents the relative importance of the objectives and k is a constant that adjusts the robustness of the method against the level of conservatism of the solution. In this paper, k is assumed to be unity.

Variances of the random variables $\boldsymbol{x}$ and $\boldsymbol{z}$ are estimated as single point values. Confidence intervals for the variances are estimated for each random variable described by the sparse point data. The weight parameter w is varied (from 0 to 1) and the optimization problem in Eqs. (13)-(14) are solved iteratively until convergence by the Matlab solver 'fmincon' for different sample sizes (n) of the sparse point data. In each case, the optimization problems converged in less than 5 iterations. Here, ‘fmincon’ uses a sequential quadratic programming (SQP) algorithm. The estimate of the Hessian of the Lagrangian is updated using the BFGS formula at each iteration. The convergence properties of SQP have been discussed by many authors including Fletcher (1987) and Panier and Tits (1993).

The solutions are obtained by solving the problem using the upper confidence bound for the variances of the random variables ***x*** and ***z***. The solutions are presented in Figure 2.

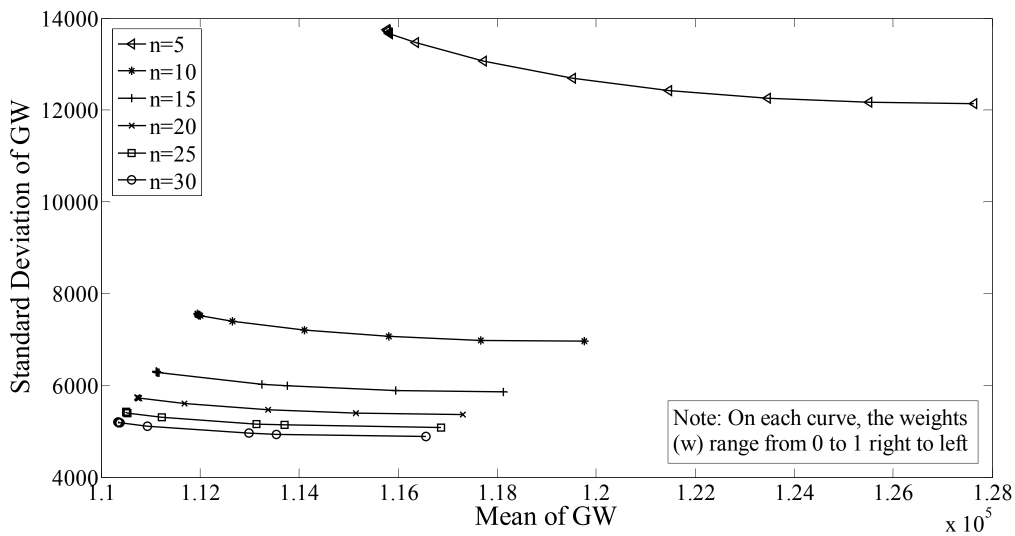

Figure 2: Robustness-based design optimization with sparse data for different sample sizes (*n*)

It is seen in Figure 2 that the solutions become more conservative (i.e., the mean and standard deviation of GW assume higher values) as we add uncertainty to the design problem. It is also seen from Figure 2 that as the sample size (*n*) increases, both the standard deviation and mean of GW decrease. As gathering more data reduces data uncertainty, the solutions become less sensitive (i.e., the standard deviation of GW assumes lower value) to the variations of the input random variables as the sample size (*n*) increases. Also, looking at the mean of GW, it is seen that as the uncertainty decreases with sample size, the optimum mean weight required is less.

3. 2 Determination of optimal sample size for sparse point data

The optimal sample size formulations are illustrated here for the TSTO design problem. The formulations are relaxed by assuming that standard deviations of the data do not change significantly as sample size changes. To make the problem simpler, we first relax the integer requirement on the optimal sample size *n* and then round off the solution for *n* to the nearest integer value. The input variable SepQ is assumed to be a non-design epistemic variable and all the remaining variables are assumed to be design variables. The design bounds for the respective design variables and the analysis outputs remain the same as in Tables 3 and 4.

Therefore, the design problem becomes as follows:

$$
\begin{aligned}
\left[d^*, n_d^*\right] = \arg\min_{d, n_d} \; & \left(w * E(GW) + v * \sigma(GW) + (1-w-v) * \left(5n_{d1} + 10n_{d2} + 5n_{d3} + 5n_{d4} + 4n_{d5} + 6n_e^*\right)\right) \\
s.t. \quad & LB_1 + \sigma(GW) \le E(GW) \le UB_1 - \sigma(GW) \\
& LB_2 + \sigma(EW) \le E(EW) \le UB_2 - \sigma(EW) \\
& LB_3 + \sigma(PFR) \le E(PFR) \le UB_3 - \sigma(PFR) \\
& LB_4 + \sigma(VL) \le E(VL) \le UB_4 - \sigma(VL) \\
& LB_5 + \sigma(VV) \le E(VV) \le UB_5 - \sigma(VV) \\
& LB_6 + \sigma(BWA) \le E(BWA) \le UB_6 - \sigma(BWA) \\
& lb + k\sigma(\underline{x}) \le d_i \le ub - k\sigma(\underline{x}) \qquad \text{for all } i = 1,2,\ldots,5 \\
& 5n_{d1} + 10n_{d2} + 5n_3 + 5n_{d3} + 4n_{d5} + 6n_e^* \le 1050 \\
& n_{d_j} \le 30 \quad \text{for } \; \mathrm{j} = 1,2,\ldots,5
\end{aligned}
\tag{15}
$$

$$
\begin{aligned}
\left[\mu_z^*, n_e^*\right] = \arg\max_{\mu_z, n_e} \; & \left(w * E(GW) + v * \sigma(GW) + (1-w-v) * \left(5n_{d_1}^* + 10n_{d_2}^* + 5n_{d_3}^* + 5n_{d_3}^* + 4n_{d_5}^* + 6n^e\right)\right) \\
s.t. \quad & Z_l(n_e) \le \mu_z \le Z_u(n_e) \\
& 5n_{d_1}^* + 10n_{d_2}^* + 5n_{d_3}^* + 5n_{d_3}^* + 4n_{d_5}^* + 6n^e \le 1050 \\
& n_{e_j} \le 30 \quad \text{for } \; \mathrm{j} = 1
\end{aligned}
\tag{16}
$$

We have solved this problem for different combinations of weights *w* and *v* and the optimal solutions are presented in Table 6. In each case, the optimization problems converged in less than 4 iterations.

Table 6: Objective function values at optimal solutions and optimal sample sizes

Weights			Objective function Value			Optimal Sample Sizes					
w	v	$1-w-v$	Mean GW	Std GW	Total Cost	n_{d1}	n_{d2}	n_{d3}	n_{d4}	n_{d5}	n_e
0	0	1	1.6118e+005	6.3732e+004	455.3008	5	10	15	8	9	30
0.6	0.2	0.2	1.4684e+005	5.3219e+004	539.8948	6	10	30	8	10	30
0.5	0.4	0.1	1.4878e+005	5.0526e+004	593.6961	7	10	30	14	15	30
0.5	0.5	0	1.5143e+005	4.7604e+004	886.9363	25	25	30	30	30	15

It is seen in Table 6 that the total cost incurred in obtaining samples is the minimum when we solve the problem giving the maximum importance on the total cost. In this case, we get the most conservative robust design i.e., the mean and the standard deviation of GW assume the maximum of all possible values. Note that the optimal sample size required is also the minimum in this case. As we give more importance on the mean and standard deviation of GW, the total cost and also the optimal sample size increase with a decrease in both the mean and standard deviation of GW.

3. 3 Robustness-based design optimization with sparse point and interval data

The methodology proposed in Section 2.1 is illustrated here for the same TSTO problem. Here, it is assumed that the design variable ExpRatio is described by sparse point data as given in Table 5, the design variable Payload is described by multiple interval data as given in Table 7 and the design variables SepMach and SepQ are described by single interval data as given in Table 8. The non-design epistemic variables

SepAngle and Fineness are described by the sparse point data (as given in Table 5) and the single interval data (as given in Table 7), respectively. The design bounds for the respective design variables and the analysis outputs remain the same as in Tables 3 and 4.

Table 7: Multiple Interval Data for the random input variables

Payload	[25000, 28000], [26000, 29000], [25000, 29000], [26000, 30000], [25000, 30000]

Table 8: Single Interval Data for the random input variables

SepMach	[9, 10]
SepQ	[100, 120]

The design problem is now formulated as follows:

$$\begin{aligned}
& d^* = \arg\min_{d} \left(w * E(GW) + (1-w) * \sigma(GW) \right) \\
& s.t. \quad LB_1 + k\sigma(GW) \le E(GW) \le UB_1 - k\sigma(GW) \\
& \qquad LB_2 + k\sigma(EW) \le E(EW) \le UB_2 - k\sigma(EW) \\
& \qquad LB_3 + k\sigma(PFR) \le E(PFR) \le UB_3 - k\sigma(PFR) \\
& \qquad LB_4 + k\sigma(VL) \le E(VL) \le UB_4 - k\sigma(VL) \\
& \qquad LB_5 + k\sigma(VV) \le E(VV) \le UB_5 - k\sigma(VV) \\
& \qquad LB_6 + k\sigma(BWA) \le E(BWA) \le UB_6 - k\sigma(BWA) \\
& \qquad lb + k\sigma(x) \le d_i \le ub - k\sigma(x) \qquad \text{for } i = 1,2,3,4
\end{aligned} \tag{15}$$

$$\begin{aligned}
& \mu_z^* = \arg\max_{\mu_z} \left(w * E(GW) + (1-w) * \sigma(GW) \right) \\
& s.t. \qquad Z_l \le \mu_{z_i} \le Z_u \qquad \text{for } i = 1,2
\end{aligned} \tag{16}$$

where the bounds Z_l and Z_u for the mean value of the non-design epistemic variable SepAngle are calculated by Eq. (10) as given in Section 2.1.1 and those for the epistemic variable Fineness are calculated by the method described in Section 2.1.3. Note that in

Eq. (16), we do not use the robust design constraints, since the objective function in this case is a function of all non-design epistemic variables.

Variances of the random variables ExpRatio and SepAngle are estimated as single point values. Confidence intervals for the variances are estimated for each random variable described by sparse point data. Bounds on the variances of the random variables SepMach, SepQ, Fineness, and Payload are estimated by the methods described in Sections 2.1.3. The free parameter *w* is varied (from 0 to 1) and the optimization problems in Eqs. (15) and (16) are solved iteratively until convergence. In each case, the optimization problems converged in less than 5 iterations. The solutions are obtained by solving the problems using the upper confidence bound on sample variance for the random variables ExpRatio and SepAngle, and the upper bound on sample variances for the random variables Payload, SepMach, SepQ and Fineness. The solutions are presented in Figure 3.

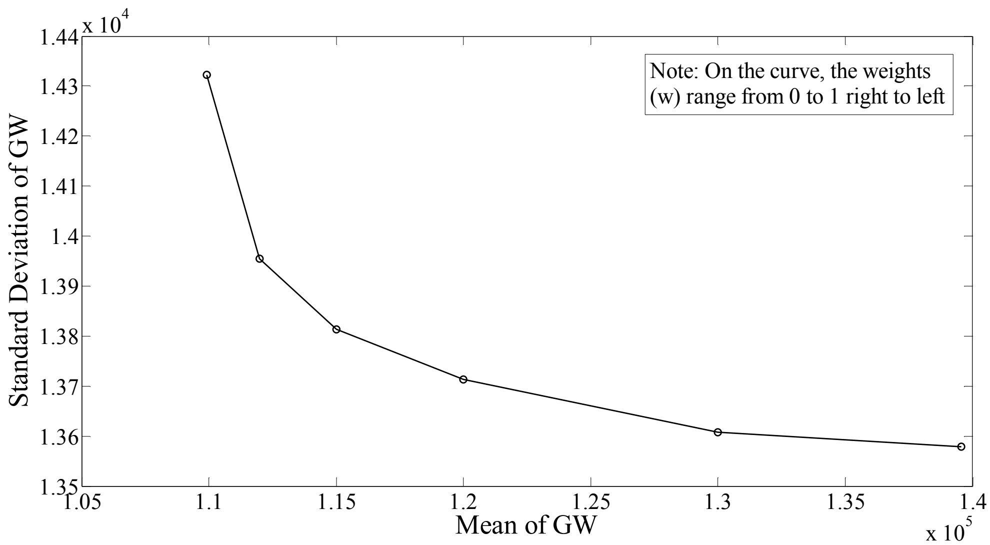

Figure 3: Robustness-based design optimization with non-design epistemic variables

Figure 3 shows the solutions of the conservative robust design in presence of uncontrollable epistemic uncertainty described through mixed data i.e., both sparse point data and interval data, which is seen frequently in many engineering applications.

4. Summary and Conclusion

This paper proposed several formulations for robustness-based design optimization under data uncertainty. Two types of data uncertainty – sparse point data and interval data – are considered. The proposed formulations are illustrated for the upper stage design problem of a TSTO space vehicle. A decoupled approach is proposed in this paper to un-nest the robustness-based design from the analysis of non-design epistemic variables to achieve computational efficiency. As gathering more data reduces uncertainty but increases cost, the effect of sample size on the optimality and the robustness of the solution is also studied. This is demonstrated by numerical examples, which suggest that as the uncertainty decreases with sample size, the resulting solutions become more robust. We have also proposed a formulation to determine the optimal sample size for sparse point data that leads to the solution of the design problem that is least sensitive (i.e., robust) to the variations of design variables. In this paper, we have used the weighted sum approach for the aggregation of multiple objectives and to examine the trade-offs among multiple objectives. Other multi-objective optimization techniques can also be explored within the proposed formulations.

The major advantage of the proposed methodology is that unlike existing methods, it does not use separate representations for aleatory and epistemic uncertainties and does not require nested analysis. Both types of uncertainty are treated in a unified manner using a probabilistic format, thus reducing the computational effort and simplifying the optimization problem. The results regarding robustness of the design versus data size are valuable to the decision maker. The design optimization procedure also optimizes the sample size, thus facilitating resource allocation for data collection efforts. Due to the use of a probabilistic format to represent all the uncertain variables, the proposed robustness-based design optimization methodology facilitates the implementation of multidisciplinary robustness-based design optimization, which is a challenging problem in presence of epistemic uncertainty.

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