Abstract

Michelson Interferometer
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The Michelson Interferometer is a device used in many applications, but here it was used to measure small differences in distance, in the milli-inch range, specifically for defects in the Orbiter windows. In this paper, the method of using the Michelson Interferometer for measuring small distances is explained as well as the mathematics of the system. The coherence length of several light sources was calculated in order to see just how small a defect could be measured. Since white light is a very broadband source, its coherence length is very short and thus can be used to measure small defects in glass. After finding the front and back reflections from a very thin glass slide with ease and calculating the thickness of it very accurately, it was concluded that this system could find and measure small defects on the Orbiter windows. This report also discusses a failed attempt for another use of this technology as well as describes an area of promise for further analysis. The latter of these areas has applications for finding possible defects in Orbiter windows without moving parts.
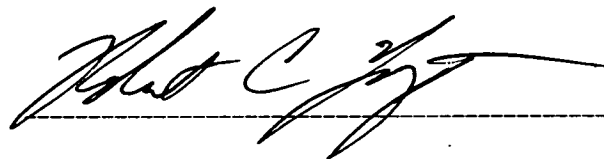
MICHELSON INTERFEROMETER

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Introduction

The Michelson Interferometer is implemented in many different fields, like medicine, detection of gravitational waves, sensing, and many others. The application that we are concerned with involves using the interference properties of light to find small differences in length. For this project, the Michelson Interferometer will be used to measure small defects in the Orbiter windows. As of now, there are instruments and tools that can detect defects to a few micro-inches, but it takes a large amount of time to do this. This project involves using the Michelson Interferometer so that scratches and defects on the surface of the Orbiter windows can be pinpointed and dealt with in a timely fashion. A defect in a window is of concern on the Orbiter windows when a defect is greater than .6 milli-inches (mils) deep. We want to be able to see defects this deep with great accuracy.

The Michelson Interferometer consists of a light source that travels to a beam splitter and evenly splits the power of the source. Half the power of the source travels to a mirror that sometimes can be made to oscillate, called the reference arm, and the other half travels to the sample surface, called the sample arm. After the source is reflected off of the reference mirror and the sample, the light waves recombine at the beam splitter and are sent to a detector (see Fig. 1).[1]
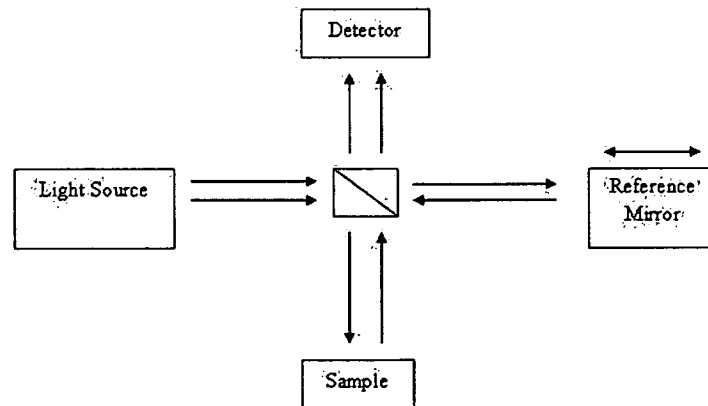

Figure 1: Michelson Interferometer

Method

When light travels through the Michelson Interferometer, the light goes to both the reference arm and the sample arm where each of the beams contains the same frequencies as the original light source. Assuming that the two paths are of equal distance and the two mirrors in the system reflect 100% of the light, the light beams will recombine at the beam splitter and all the light from the source will go to the detector. When the mirror in the reference arm moves at a given frequency and amplitude through the point of equal distance, the two light paths will have total destructive and constructive interference with each other. This interference can be displayed on an oscilloscope where the peak to peak voltage amplitude that is displayed represents the total alternating light intensity that is hitting the detector. When the oscilloscope displays an oscillating voltage, rather than a constant voltage, we know that the waves from both paths are interfering and are thus coherent at that point.[2] When the oscilloscope is displaying a constant voltage, the light from both the paths are not interfering, meaning they are not coherent. To find coherence length, we can start where the amplitude of the light hitting the detector is at a maximum, and then move back the reference arm until the peak to peak amplitude has dropped by a half. Now, move the reference arm back through the maximum amplitude and stop where the amplitude has again dropped by a half. This total distance from half the maximum amplitude one way to half the maximum amplitude the other way (also called the full width at half maximum method, FWHM) is the coherence length.[2] Since light waves can be added up linearly, the mathematics of the Michelson Interferometer will be the same if we send one frequency or many frequencies of light. Thus, if we pick a source that includes many frequencies of light, also known as a broadband light source, then the coherence length will be very small because each frequency in the source must be coherent and that only happens when the two paths of the Interferometer are lined up within a few microns.[3]

If we replace a sample with a defect in the sample arm, we can scan across the surface of the sample with our light source, and when the intensity of the light hitting the detector decreases, we know that there is a defect. Then we can focus into the defect by pulling the reference arm back which makes the light travel further into the defect. The problem is that in order to even see an amplitude change when scanning across a sample, the coherence

length of the source must be smaller than the defect depth. Otherwise, the source will be coherent with the surface of the sample, and when it passes over the defect, it will remain coherent with the bottom of the defect. Thus, the smaller the coherence length, the more accurately we can measure the depth of a defect. Because white light is very broadband and has a small coherence length, we will use it as our source.

Mathematics of the Interferometer

The wave equation denotes the propagation of light and is given by:[4]

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Where

$$c = \text{speed of light}$$

Let $u(t)$ be a function that describes the amplitude, or electrical field of a given light source. The solution of the wave equation, $u(t)$, can be made into a complex function, but for this project we will treat it as a real valued function.

Power, $p(t)$, is related to amplitude with the following proportionality:

$$p(t) = \beta u^2(t)$$

Where

$$\beta \text{ is the proportionality constant}$$

Now, when we are dealing with a Michelson Interferometer, the power of a light source is evenly distributed to each arm of the Interferometer. Thus, the electrical field down each arm, the reference arm $u_r(t)$ and the sample arm $u_s(t)$, is related to the source's electric field, $u(t)$ by:

$$u_r(t) = \frac{1}{\sqrt{2}} u(t)$$

$$u_s(t) = \frac{1}{\sqrt{2}} u(t)$$

However, the light must travel from the arms back to the beam splitter,

so the amount of amplitude that makes it to the detector, $u_d(t)$, if both arms are the same distance from the beam splitter, is:

$$u_d(t) = \frac{1}{2}\epsilon_r u(t) + \frac{1}{2}\epsilon_s u(t)$$

Where

$\epsilon_r^2 = $ fractional amount of energy transmitted through the reference arm

$\epsilon_s^2 = $ fractional amount of energy transmitted through the sample arm

The detector cannot respond to the actual optical frequencies and instead takes a time average from an arbitrary starting position T_0 to $T+T_0$. In other words, $T \gg \frac{2\pi}{\omega}$. Also, the arms of the Interferometer are not the same distance away from the beam splitter because the reference arm is translating back and forth through the point of equal distance, so there is a time delay, τ, between the two waves. The detector takes the power, $p(t)$, from the light source and converts it to a current. This current is then converted to voltage using a known resistance by Ohm's Law. Thus, the detector is reading voltage, $V(T_0)$, which can be modeled by the following equation, assuming that the sample has a single reflection:

$$V(T) = \frac{1}{T_0}\int_T^{T+T_0} \beta(\frac{1}{2}\epsilon_r u(t - \tau_r - \Delta\tau \sin{(\omega t + \phi)}) + \frac{1}{2}\epsilon_s u(t - \tau_s))^2 dt$$

Where

$t = $ continuous time that the detector is integrating over

$T_0 = $ time response of the detector

$\tau_r = $ time delay from the source to the detector through the reference arm

$\tau_s = $ time delay from the source to the detector through the sample arm

$\Delta\tau = $ magnitude of the mirror oscillations in the reference arm

$\omega = $ angular frequency of the mirror in the reference arm

$\phi = $ phase of the mirror in the reference arm

$T = $ perceived time

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The magnitude is squared so that the integrand we are taking is power. But, what if the sample was replaced with a sample with multiple reflections? The new formula would be the following:[5]

$$V(T) = \frac{1}{T_0} \int_T^{T+T_0} \beta \left[\frac{1}{2} \epsilon_r u(t - \tau_r - \Delta\tau \sin(\omega t + \phi)) + \sum_{n=1}^{N} \frac{1}{2} \epsilon_{s_n} u(t - \tau_{s_n}) \right]^2 dt$$

Where

$$N = \text{number of discrete reflections in the sample}$$

Now, expand the integrand and factor out the β to get:

$$V(T) = \frac{\beta}{T_0} \int_T^{T+T_0} \left[\frac{\epsilon_r^2}{4} u^2(t - \tau_r - \Delta\tau \sin(\omega t + \phi)) + \right.$$

$$\frac{\epsilon_r}{4} u(t - \tau_r - \Delta\tau \sin(\omega t + \phi)) \sum_{n=1}^{N} \epsilon_{s_n} u(t - \tau_{s_n}) + \sum_{i=1}^{N} \sum_{j=a}^{N} \frac{\epsilon_{s_i} \epsilon_{s_j}}{4} u(t - \tau_{s_i}) u(t - \tau_{s_j}) \bigg] dt$$

Since $\beta u^2(t) = p(t)$ and $\Delta\tau$ is very small, we can treat the first term of the integral as a constant:

$$P = \frac{\beta}{T_0} \int_T^{T+T_0} \frac{\epsilon_r^2}{4} u^2(t - \tau_r - \Delta\tau \sin(\omega t + \phi)) dt$$

Which gives us:

$$V(T) = P + \frac{\beta}{T_0} \int_T^{T+T_0} \frac{\epsilon_r}{4} u(t - \tau_r - \Delta\tau \sin(\omega t + \phi)) \sum_{n=1}^{N} \epsilon_{s_n} u(t - \tau_{s_n}) +$$

$$\sum_{i=1}^{N} \sum_{j=1}^{N} \frac{\epsilon_{s_i} \epsilon_{s_j}}{4} u(t - \tau_{s_i}) u(t - \tau_{s_j}) dt$$

The last term of the integrand is an autocorrelation function. When the reflections from the sample are far apart, meaning they are several coherence lengths away, then the autocorrelation function is zero. Thus the only terms that survive from this double sum are when $i = j$:

$$V(T) = P + \frac{\beta}{T_0} \int_T^{T+T_0} \frac{\epsilon_r}{4} u(t - \tau_r - \Delta\tau \sin(\omega t + \phi)) \sum_{n=1}^{N} \epsilon_{s_n} u(t - \tau_{s_n}) +$$

$$\sum_{i=1}^{N} \frac{\epsilon_{s_i}^2}{4} u^2(t - \tau_{s_i}) dt$$

Here, the last term is a sum of different powers, which can be taken out of the integral and combined with the constant power P to make a new constant, P':

$$V(T) = P' + \frac{\beta}{T_0} \int_T^{T+T_0} \frac{\epsilon_r}{4} u(t - \tau_r - \Delta\tau \sin{(\omega t + \phi)}) \sum_{n=1}^{N} \epsilon_{s_n} u(t - \tau_{s_n}) dt$$

The sum can be moved to the outside of the integral along with the constants. We can assume that the time interval that we are averaging over is very large since the time interval that the oscilloscope is detecting is much larger than the actual time that a single wave of light takes to cross the detector, so we can let $T \to \infty$.

$$V(T) = P' + \frac{\epsilon_r}{4} \sum_{n=1}^{N} \epsilon_{s_n} \lim_{T_0 \to \infty} \frac{\beta}{T_0} \int_{T_0}^{T+T_0} u(t - \tau_r - \Delta\tau \sin{(\omega t + \phi)}) u(t - \tau_{s_n}) dt$$

Now, we can treat the resulting integral as the autocorrelation of an ergodic process, $\Gamma(\tau_{s_n})$.

$$V(T) = P' + \frac{\beta\epsilon_r}{4} \sum_{n=1}^{N} \epsilon_{s_n} \Gamma(\tau_{s_n} - \tau_r - \Delta\tau \sin{(\omega t + \phi)})$$

We know from the Wiener-Khinchin theorem that if we take the Fourier Transform of the autocorrelation function, we get the power spectrum (the proof is in the appendix). Since the Fourier transform is a linear transform, we get the following:
Let $\tau_n = \tau_{s_n} - \tau_r - \Delta\tau \sin{(\omega t + \phi)}$.

$$\mathcal{F}[V(T)] = \int_{-\infty}^{\infty} [P' + \frac{\beta\epsilon_r}{4} \sum_{n=1}^{N} \epsilon_{s_n} \Gamma(\tau_n)] e^{-i\omega\tau_n} d\tau_n$$

$$= \sqrt{2\pi} P' \delta(\omega_n) + \frac{\beta\epsilon_r}{4} \sum_{n=1}^{N} \epsilon_{s_n} p(\omega_n)$$

Where

$$\delta(\omega_n) = \text{ the Dirac Delta function}$$

However, the power spectrum $p(\omega_n)$ is centered about the center frequency of the light source. We must center the power spectrum over the origin so that we can single out the envelope of the autocorrelation. We can

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also use the power spectrum to see if our results make sense with our light source.

Let $\omega_0 =$ the center frequency.

$$= \sqrt{2\pi}P'\delta(\omega_n) + \frac{\beta\epsilon_r}{4}\sum_{n=1}^{N}\epsilon_{s_n}p((\omega_n - \omega_0) + \omega_0)$$

The shift by $+ \omega_0$ makes the high frequency part of the autocorrelation function. Thus, we will take the Fourier transform again and use the shifting property of the transform to get:[4]

$$= \mathcal{F}[\sqrt{2\pi}P'\delta(\omega_n)] + \frac{\beta\epsilon_r}{4}\sum_{n=1}^{N}\epsilon_{s_n}e^{-i\omega_0\tau_n}\mathcal{F}[p(\omega_n - \omega_0)]$$

$$= P' + \frac{\beta\epsilon_r}{4}\sum_{n=1}^{N}\epsilon_{s_n}e^{-i\omega_0\tau_n}\Gamma_E(\tau_n)$$

Where $\Gamma_E(\tau_n)$ is the envelope of the autocorrelation function. Substitute in what τ_n is and expand the exponent:

$$= P' + \frac{\beta\epsilon_r}{4}\sum_{n=1}^{N}\epsilon_{s_n}e^{-i\omega_0(\tau_r - \tau_{s_n})}e^{i\omega_0\Delta\tau\sin(\omega t + \phi)}\Gamma_E(\tau_{s_n} - \tau_r - \Delta\tau\sin(\omega t + \phi))$$

Since the modulation $\Delta\tau$ does not greatly affect the envelope of the autocorrelation function, we can treat it as zero which gives us:

$$= P' + \frac{\beta\epsilon_r}{4}\sum_{n=1}^{N}\epsilon_{s_n}e^{-i\omega_0(\tau_r - \tau_{s_n})}e^{i\omega_0\Delta\tau\sin(\omega t + \phi)}\Gamma_E(\tau_{s_n} - \tau_r)$$

Because we are using real valued functions, we will take the real part of this function to get:

$$= P' + \frac{\beta\epsilon_r}{4}\sum_{n=1}^{N}\epsilon_{s_n}\cos(\omega_0(\tau_r - \tau_{s_n}))\cos(\omega_0\Delta\tau\sin(\omega t + \phi))\Gamma_E(\tau_{s_n} - \tau_r)$$

Where

$$\cos(\omega_0(\tau_r - \tau_{s_n})) = \text{ the high frequency term of the autocorrelation function}$$

$$\cos(\omega_0\Delta\tau\sin(2\pi ft + \phi)) = \text{ the modulation of the mirror}$$

The setup that we have here at Kennedy Space Center involves a oscilloscope that displays the voltage of the light source that is hitting the detector versus time. This oscilloscope is then connected to a spectrum analyzer which takes the Fourier transform of the voltage versus time graph and shows the voltage versus frequency graph. This enables us to look at the harmonics of the light source of a sample. The spectrum analyzer is then connected to a lock-in-amplifier which allows us to take the voltage off of a single harmonic, rather than averaging over a series of harmonics.

Results

We started with a superluminiscent diode (SLD) with a center frequency of 799 nm and a spectral width of 41 nm. We calculated the coherence length to be close to 11 microns. We placed a sample with a known defect into the sample arm and focused into the defect by pulling back the reference arm. Then we recorded the oscillating portion of the voltage on the oscilloscope in 10 micron increments, and obtained the graph in Fig. 2.
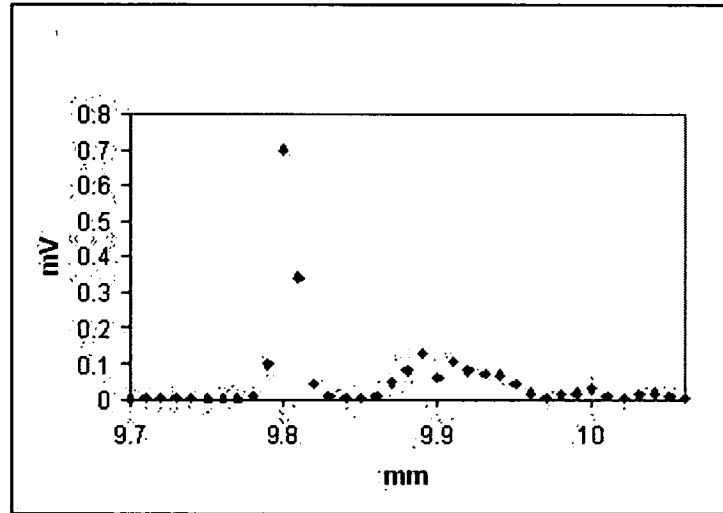

Figure 2: Defect Measurement with a SLD

The high voltage at 9.8 mm shows the reflection off the front surface, but as the reference mirror is pushed back, we start to see the smaller inner

reflections from the defect at about 9.9 mm. From this data, we can deduce that the defect is around .1 mm deep.

Now that we know we can see a defect, let us try to find smaller defects. The best way to do this is to use a light source with a small coherence length. White light has a broad bandwidth, and thus has a small coherence length.[3] When we replace a SLD with a white light source, the Interferometer becomes much more fragile in that it is very hard to find interference. Once interference is found, the coherence length can be calculated, again we will use the FWHM method again. The data that we obtained from a white light source by reading an oscilloscope and moving the reference arm in half micron increments away from the beam splitter is in Fig. 3.
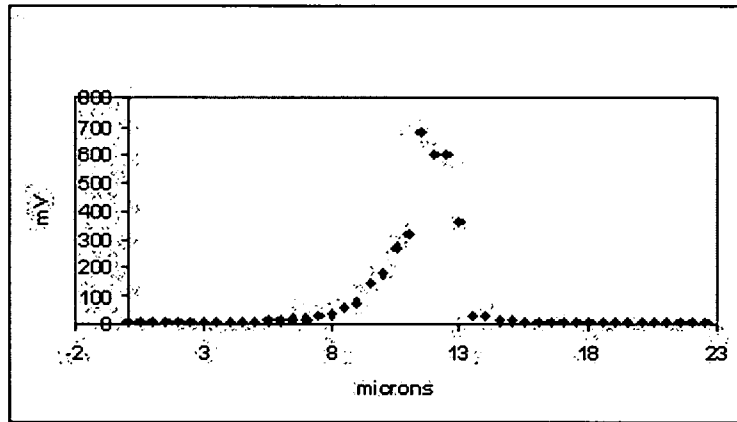

Figure 3: White Light Coherence

After fitting a gaussian to this data, we can use the FWHM method to find the coherence length. This data shows that we have a source with a coherence length of about 2.3 microns which is about .09 mils. Having a coherence length that is much shorter than our .6 mil cutoff point makes it easier to distinguish between the front reflection off the main surface of the sample and the defect reflections.

With all the mathematics of the system laid out, let's try to find the width of a thin glass cover slip by finding the point where the reference arm is coherent with the front surface, moving the reference arm back to find the coherent reflection off the back surface. We first measured the glass cover to be about 7 mils thick. This is still a factor of ten greater than the minimum depth a problem defect can be, but it will show us how easily this back reflec-

tion can be seen on the graph. Next, we replaced the 100% reflection mirror with the 4% reflection glass cover slip in the sample arm. Fig. 4 shows the data.
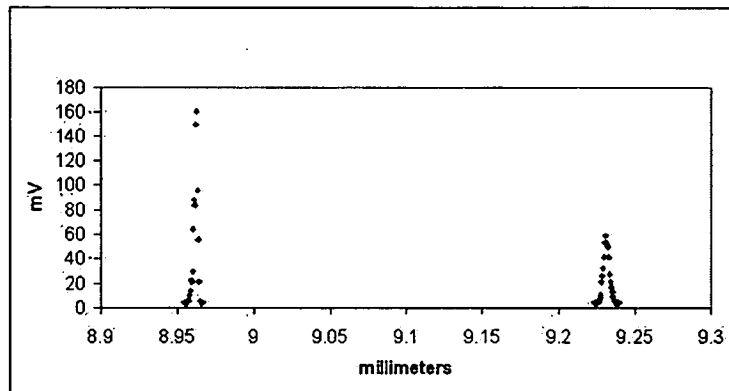

Figure 4: White Light Coherence with Two Reflection Sites

The peak at around 8.96 mm shows the reflection off the front surface. The peak at around 9.23 mm is the reflection off the back surface. This shows that the front and back reflections are about 268 microns apart. When we convert 7 mils to microns, we get 177 microns. We also have to take into account the index of refraction in glass, which is about 1.5. Multiplying this index of refraction by the width of the glass, we get 265.5 microns, which is extremely close to our measured width of 268 microns, but that was not exact. The amplitude of the back reflection should only be 96% of the amplitude of the front reflection because the glass cover slip only has a 4% reflection. The probable reason for why the back reflection's amplitude is much smaller than the front reflection is that the two surfaces were not exactly parallel because cover slips are not manufactured for this kind of experiment.

Conclusion

The short coherence length of white light enables us to distinguish very easily from reflections from the front and back of a thin glass cover slip. Looking at the last graph in the results section, the coherence from the front reflection falls very fast and allows for a wide gap between the end of the first reflection and the beginning of the back reflection. It would be very easy to

see a peak in the graph that is 15 microns from the peak of the front surface, which would enable us to see defects to the depth of the orbiter window's minimum problematic defect depth with great accuracy. To calculate the depth of a defect, find the front reflection surface and then move the reference arm back slowly to find the reflection from the bottom of the defect and see how far the reference arm moved back. Thus, this approach allows for an accurate and time-efficient way of measuring small defects in the orbiter windows as opposed to just detecting defects.

Other Projects with the Interferometer

Implementing the linearity of the Michelson Interferometer, we wanted to see if the coherence length would shorten when we added sources to the Interferometer. We thought that since white light is broadband, we could add SLD sources to get a broadband source that could theoretically be better than white light to find small defects. The next step was to see if the mathematics worked in our favor. Assuming that the power spectrum of a light source is a gaussian or a sum of gaussians, we attempted to add several known power spectrums together. We noticed that if we added different spectrum together that overlapped each other, this would add to the spectral width of the spectrum and thus make a shorter coherence length. For example, we took a SLD with a center wavelength of 799 nm and a spectral width of 41 nm and calculated a coherence length of about 13.4 microns. We then added another SLD with a center wavelength of 837 nm and a spectral width of 41 and a coherence length of 23.41 microns. Fig. 5 shows the addition of these two spectrum which generates a wider spectral width.

After shifting the gaussian to the origin and then taking the Fourier transform, we get the coherence function in Fig. 6.

Using the FWHM method, we get a coherence length of 9.5 microns which is shorter than the shortest coherence length source. However, this still is much greater than the white light coherence length and it appears that we would have to add several sources to get the coherence length down to the coherence length of a white light source. Theoretically, we could add enough sources to the Michelson Interferometer to get a shorter coherence length than white light, but there are further problems. First, the power spectrums
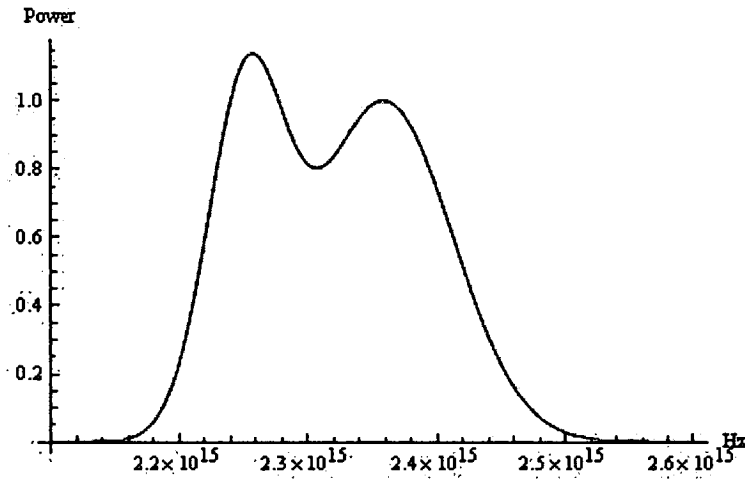
Power
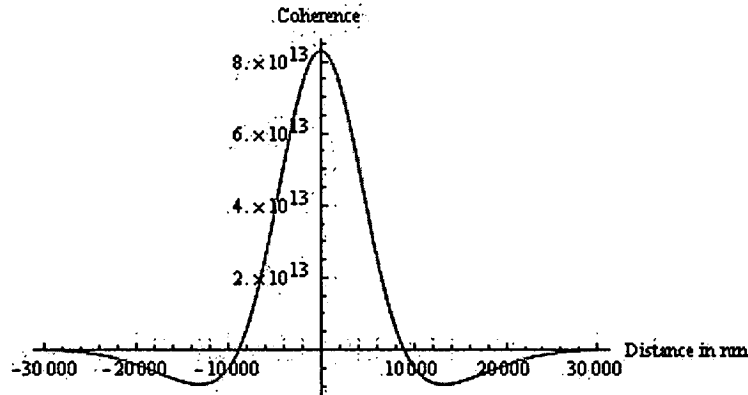

Figure 5: Added SLD's Frequency Spectrum

Figure 6: Coherence Function of Added Sources

of each source must overlap the other power spectrums. If the power spectrums do not overlap, then the sum of them will not affect the spectral width of either spectrum. Thus, we must have SLD's with enough difference in central wavelength and spectral width to still overlap, but not lie right over each other (If they had similar central wavelength, then the spectral width would be the average of the two). Another problem involves the number of beam splitters required for a summation of light sources and the amount of power lost in each beam splitter. In order to add sources, Fig. 7 shows the modification to the Michelson Interferometer that could be applied.

A new beam splitter would have to be added for every additional source. This would cause half the power of each source to be wasted. Thus, if we add
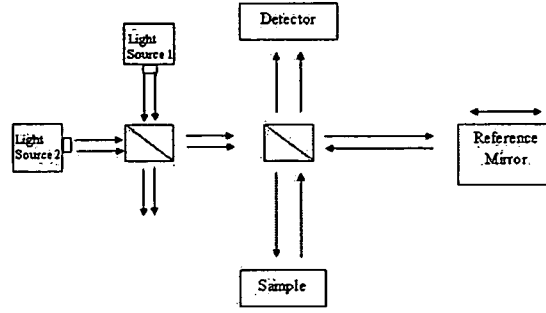
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Figure 7: Michelson Interferometer with Two Sources

5 sources we would lose 2^5 of the power from the furthest source from the detector. There are ways to prevent this huge loss in power, but the number of beam splitters are still a problem. Ultimately, we want to be able to manufacture a hand-held device, and more beam splitters will only make it bigger. We concluded from this that adding sources would not be as efficient as simply using white light interferometry.

The last thing that we tried, but did not finish entirely, was changing the detector of the Interferometer to a spectrometer. Spectrometers have thousands of different sensors where each sensor corresponds to a small range of wavelengths and the spectrometer displays the intensity from a source hitting it. The benefit of this detector is that it does not require the reference arm of the Interferometer to oscillate. Therefore, there are no moving parts in this setup, which is a big plus. When we block one of the arms of the spectrometer with white light as our source, we obtain the spectrum in Fig. 8.

Also, when the two arms of the Interferometer are equal, we obtain a similar spectrum with a little more power. When we change the reference arm's distance, we get oscillation in the spectrum. Assume that a single wavelength of light travels through the Interferometer, which can be modeled by:

$$A \cos{(kz - \nu t)}$$

Where

$$A = \text{amplitude of the wave}$$

$$k = \frac{2\pi}{\lambda} \text{ and } \lambda = \text{wavelength}$$

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$$z = \text{distance the light wave traveled}$$

$$\nu = \text{angular frequency of the light wave}$$
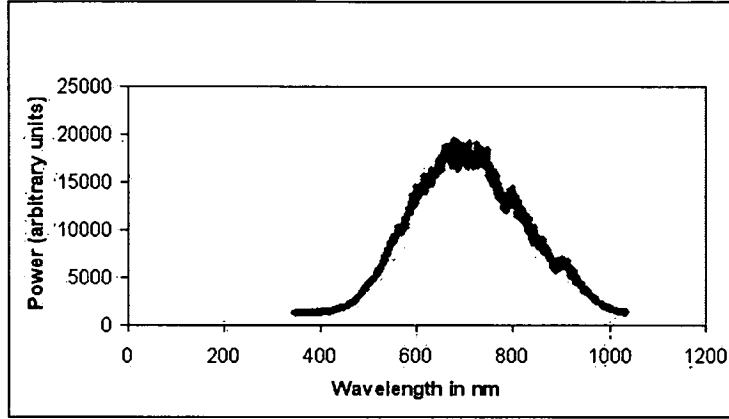
$$t = \text{time}$$

Figure 8: Spectrum of White Light

When we place this wavelength in the Michelson Interferometer, we get two power terms which is the DC offset, but what we are concerned with is the cross term:

$$A^2 \cos\left(kz_{\text{reference}} - \nu t\right) \cos\left(kz_{\text{sample}} - \nu t\right)$$

Where the subscripts on the z's denote which path they travel down (refer back to the Mathematics of the Interferometer section). Using a trigonometric identity, we can replace the cross term with:

$$\frac{A^2}{2} \cos\left(k(z_{\text{reference}} - z_{\text{sample}})\right) + \frac{A^2}{2} \cos\left(k(z_{\text{sample}} + z_{\text{reference}}) - 2\nu t\right)$$

The last term is a high frequency term, which we can ignore. This leaves us with a difference in distances from each arm:

$$\frac{A^2}{2} \cos\left(k\Delta z)\right)$$

Where

$$\Delta z = z_{\text{reference}} - z_{\text{sample}}$$

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This shows that when the path lengths in the Interferometer are different, the number of peaks shown on the spectrum is about double the number of microns that the path lengths are actually different. It is double because the round trip of the light is taken into account. However, we can get more accurate measurements than just counting peaks because we can count fractions of peaks, thus we can see fractions of microns.

We moved the reference arm back 5 microns from where the two paths of the Interferometer were equal and we obtained the spectrum in Fig. 9.
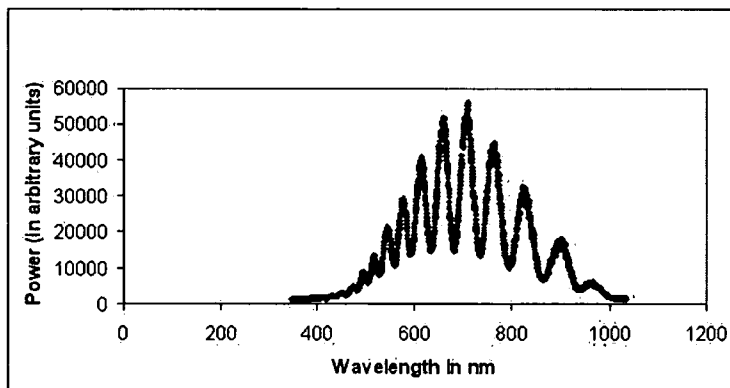

Figure 9: Spectrum of White Light with different path lengths in the Interferometer

Notice that the basic curve of the graph is still the power spectrum of the source. By dividing out the original spectrum of the source, we are left with the oscillation. What this comes down to is that when a wave is split in an Interferometer, the phase of each split wave is related to the difference in distance each wave travels.

If we had more time we would be able to pursue this project further, but unfortunately, we ran out of time. Hopefully this idea can be developed further and actually tested to see if it can actually find small defects in windows.

Appendix

Proof of the Wiener-Khinchin Theorem

The autocorrelation function $\Gamma(\tau_n)$ is defined as

$$\Gamma_E(\tau_n) = \int_{-\infty}^{\infty} \bar{u}(t)u(\tau_n + t)dt$$

The Fourier transform of $u(t)$ is

$$u(t) = \int_{-\infty}^{\infty} u(\omega)e^{-i\omega t}d\omega$$

The Fourier transform of the complex conjugate is

$$\bar{u}(t) = \int_{-\infty}^{\infty} \bar{u}_{\omega}e^{i\omega t}d\omega$$

Plug these into the autocorrelation function to get

$$\Gamma_E(\tau_n) = \int_{-\infty}^{\infty} [\int_{-\infty}^{\infty} \bar{u}(\omega)'e^{i\omega' t}d\omega'][\int_{-\infty}^{\infty} u(\omega)e^{-i\omega(\tau_n+t)}d\omega]dt$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \bar{u}(\omega')u(\omega)e^{-it(\omega-\omega')}e^{-i\omega\tau_n}dtd\omega d\omega'$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \bar{u}(\omega')u(\omega)\delta(\omega' - \omega)e^{-i\omega\tau_n}d\omega d\omega'$$

$$= \int_{-\infty}^{\infty} \bar{u}(\omega)u(\omega)e^{-i\omega\tau_n}d\omega$$

$$= \int_{-\infty}^{\infty} |u(\omega)|^2 e^{-i\omega\tau_n}d\omega$$

$$= \mathcal{F}[|u(\omega)|^2](\tau_n).$$

Therefore, the autocorrelation function is the Fourier transform of the power spectrum.[2]

References

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