Translation of Bernstein Coefficients Under an Affine Mapping of the Unit Interval

J. A. Alford II
Langley Research Center, Hampton, Virginia

Since its founding, NASA has been dedicated to the advancement of aeronautics and space science. The NASA scientific and technical information (STI) program plays a key part in helping NASA maintain this important role.

The NASA STI Program operates under the auspices of the Agency Chief Information Officer. It collects, organizes, provides for archiving, and disseminates NASA's STI. The NASA STI Program provides access to the NASA Aeronautics and Space Database and its public interface, the NASA Technical Report Server, thus providing one of the largest collection of aeronautical and space science STI in the world. Results are published in both non-NASA channels and by NASA in the NASA STI Report Series, which includes the following report types:

- TECHNICAL PUBLICATION. Reports of completed research or a major significant phase of research that present the results of NASA programs and include extensive data or theoretical analysis. Includes compilations of significant scientific and technical data and information deemed to be of continuing reference value. NASA counterpart of peer-reviewed formal professional papers, but having less stringent limitations on manuscript length and extent of graphic presentations.

- TECHNICAL MEMORANDUM. Scientific and technical findings that are preliminary or of specialized interest, e.g., quick release reports, working papers, and bibliographies that contain minimal annotation. Does not contain extensive analysis.

- CONTRACTOR REPORT. Scientific and technical findings by NASA-sponsored contractors and grantees.

- CONFERENCE PUBLICATION. Collected papers from scientific and technical conferences, symposia, seminars, or other meetings sponsored or co-sponsored by NASA.

- SPECIAL PUBLICATION. Scientific, technical, or historical information from NASA programs, projects, and missions, often concerned with subjects having substantial public interest.

- TECHNICAL TRANSLATION. English- language translations of foreign scientific and technical material pertinent to NASA's mission.

Specialized services also include creating custom thesauri, building customized databases, and organizing and publishing research results.

For more information about the NASA STI Program, see the following:

- Access the NASA STI program home page at *http://www.sti.nasa.gov*

- E-mail your question via the Internet to help@sti.nasa.gov

- Fax your question to the NASA STI Help Desk at 443-757-5803

- Phone the NASA STI Help Desk at 443-757-5802

- Write to:
  NASA STI Help Desk
  NASA Center for AeroSpace
  Information
  7115 Standard Drive
  Hanover, MD 21076–1320

NASA/TM–2012–217557
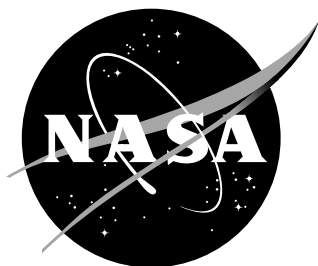
Translation of Bernstein Coefficients Under an Affine Mapping of the Unit Interval

J. A. Alford II
Langley Research Center, Hampton, Virginia

March 2012

Acknowledgments

I would like to thank César Muñoz and Anthony Narkowicz for introducing me to this subject and for their encouragement.

Abstract

We derive an expression connecting the coefficients of a polynomial expanded in the Bernstein basis to the coefficients of an equivalent expansion of the polynomial under an affine mapping of the domain. The expression may be useful in the calculation of bounds for multi-variate polynomials.

1   Introduction

The set of Bernstein basis polynomials of degree n can be used to form a basis for a vector space of polynomials of degree less than or equal to n. These polynomials take the form

$$B_{\nu,n}(x) = \left(\begin{array}{c} n \\ \nu \end{array} \right) x^{\nu}(1-x)^{n-\nu}, \nu = 0, \ldots, n \qquad (1)$$

where $\left(\begin{array}{c} n \\ \nu \end{array} \right)$ is a binomial coefficent. It is known that the coefficients of a given polynomial expressed using the Bernstein basis provide guaranteed bounds on the global minimum and maximum of the polynomial [1,2]. This result extends from the fact that these basis polynomials are non-negative over the unit interval and the values at the end points are simply

$$B_{\nu,n}(0) = \delta_{\nu 0} \quad \text{and} \quad B_{\nu,n}(1) = \delta_{\nu n} \qquad (2)$$

where δ is the Kronecker delta function. Furthermore, such bounds may be improved by calculating and comparing the various expansion, or Bernstein, coefficients for the polynomial over subdivisions of the domain interval, i.e. the coefficients that result from an affine mapping of a subdivision of the domain back to the original domain. The de Casteljau algorithm [3] is a method for calculating the Bernstein coefficients that are generated via a repeated halving of the domain. Muñoz and Narkawicz [4] have provided a simpler variant of this algorithm, by explicitly calculating the analytical expressions that relate the coefficients on the original and halved domains. We present here an expression connecting the Bernstein coefficients on the unit domain to those for an affine mapping of the domain. This is in a sense a generalization of the formulas given by Muñoz and Narkawicz [4]. The formula here reduces to their result in the special case of dividing the domain into halves. We also show how to apply the equation to a division of the domain into any number of intervals of equal size.

2   Derivation of the General Formula

We derive here the relationship between the coefficients of a single variable polynomial in Bernstein form on the unit interval $x \in [0,1]$, $B_{\nu,n}(x)$,

and those of the corresponding Bernstein polynomial generated by the mapping $x \longrightarrow \alpha x + \beta$. This result holds for the multivariate case as can be easily shown using Smith's representation [5]. The derivation makes use of two results: 1) the connection between the coefficients of an expansion in $B_{\nu,n}(\alpha x)$ to those of an equivalent expansion in $B_{\nu,n}(x)$ and 2) the identity

$$B_{\nu,n}(1 - x) = B_{n-\nu,n}(x) \tag{3}$$

The three steps shown below create the mapping $x \longrightarrow \alpha x + \beta$. The first result is used in the first and third steps, and the second result is used for the second step.

$$x_1 = \alpha_1 x \tag{4}$$

$$x_2 = 1 - x_1 = 1 - \alpha_1 x \tag{5}$$

$$x_3 = \alpha_2 x_2 = \alpha_2 (1 - \alpha_1 x) \tag{6}$$

where $\beta = \alpha_2 \neq 0$ and $\alpha = -\alpha_1 \beta$. The derivation of the relationship proceeds similarly to the procedure in Section 2.3 of Ref. [4], and begins by determining the appropriate partition of the argument $1 - \alpha x$.

$$
\begin{aligned}
B_{\nu,n}(\alpha x) &= \binom{n}{\nu} \alpha^\nu x^\nu (1 - x + (1 - \alpha)x)^{n-\nu} \\
&= \binom{n}{\nu} \alpha^\nu x^\nu \sum_{k=0}^{n-\nu} \binom{n-\nu}{k} (1-\alpha)^k x^k (1-x)^{n-\nu-k} \quad (7) \\
&= \binom{n}{\nu} \alpha^\nu \sum_{k=\nu}^{n} \binom{n-\nu}{k-\nu} (1-\alpha)^{k-\nu} x^k (1-x)^{n-k} \quad (8)
\end{aligned}
$$

Using the trinomial revision formula

$$\binom{k}{i}\binom{n}{k} = \binom{n}{i}\binom{n-i}{k-i} \tag{9}$$

we have that

$$B_{\nu,n}(\alpha x) = \alpha^\nu \sum_{k=\nu}^{n} \binom{k}{\nu} (1-\alpha)^{k-\nu} B_{k,n}(x) \tag{10}$$

Equating two equivalent expansions for the same polynomial, one in the Bernstein basis with argument x, and one in the Bernstein basis with argument αx,

$$\sum_{\nu=0}^{n} c_{\nu,n}^\alpha B_{\nu,n}(\alpha x) = \sum_{k=0}^{n} c_{k,n} B_{k,n}(x) \tag{11}$$

gives the following identity involving the Bernstein coefficients $c_{\nu,n}^\alpha$ and $c_{k,n}$

$$c_{k,n} = \sum_{\nu=0}^{k} c_{\nu,n}^\alpha \binom{k}{\nu} \alpha^\nu (1-\alpha)^{k-\nu} = \sum_{\nu=0}^{k} c_{\nu,n}^\alpha B_{\nu,k}(\alpha) \tag{12}$$

2

where the α in $c_{k,n}^{\alpha}$ is not an exponent but a notational superscript. Using this result, the identity in Eq. (3), and representing now the Bernstein coefficients of an expansion in the arguments x, x_2, and x_3 as $b_{k,n}$, $b_{k,n}^2$, and $b_{k,n}^3$ respectively, the coefficients corresponding to Eqs. (4),(5), and (6) have the following relationships:

$$\sum_{\nu=0}^{n} b_{\nu,n}^3 B_{\nu,n}(\alpha_2 x_2) = \sum_{k=0}^{n} b_{k,n}^2 B_{k,n}(x_2) \tag{13}$$

$$b_{k,n}^2 = \sum_{\nu=0}^{k} b_{\nu,n}^3 B_{\nu,k}(\alpha_2) \tag{14}$$

$$\sum_{\nu=0}^{n} b_{\nu,n}^2 B_{\nu,n}(x_2) = \sum_{\nu=0}^{n} b_{\nu,n}^2 B_{\nu,n}(1 - x_1) \tag{15}$$

$$= \sum_{\nu=0}^{n} b_{\nu,n}^2 B_{n-\nu,n}(x_1) \tag{16}$$

$$= \sum_{\nu=0}^{n} b_{n-\nu,n}^2 B_{\nu,n}(x_1) \tag{17}$$

$$= \sum_{k=0}^{n} b_{k,n} B_{k,n}(x) \tag{18}$$

The coefficients in the last expression can thus be written as

$$b_{k,n} = \sum_{\nu=0}^{k} b_{n-\nu,n}^2 B_{\nu,k}(\alpha_1)$$

$$= \sum_{\nu=0}^{k} \sum_{\nu'=0}^{n-\nu} b_{\nu',n}^3 B_{\nu,k}(-\frac{\alpha}{\beta}) B_{\nu',n-\nu}(\beta) \tag{19}$$

This equation obviously holds only for $\beta \neq 0$. When $\beta = 0$, we can simply apply Eq. (12) instead.

3 Specific Cases

It is straightforward to compute the coefficients for the case where the unit domain is divided into halves. This division is equivalent to the mapping $x \longrightarrow \alpha x + \beta$, where $\alpha = 1/2$, $\beta = 0$, for the left half, and $\alpha = -1/2$, $\beta = 1$ for the right half. In the former case, we use Eq.(12)

$$b_{k,n} = \sum_{\nu=0}^{k} b_{\nu,n}^3 B_{\nu,k}(\frac{1}{2}) = \frac{1}{2^k} \sum_{\nu=0}^{k} \binom{k}{\nu} b_{\nu,n}^3 \tag{20}$$

and in the latter, Eq.(19) yields

$$b_{k,n} = \sum_{\nu=0}^{k} \sum_{\nu'=0}^{n-\nu} b_{\nu',n}^{3} B_{\nu,k}(\tfrac{1}{2}) B_{\nu',n-\nu}(1) \qquad (21)$$

$$= \sum_{\nu=0}^{k} b_{n-\nu,n}^{3} B_{\nu,k}(\tfrac{1}{2}) \qquad (22)$$

$$= \frac{1}{2^k} \sum_{\nu=0}^{k} \binom{k}{\nu} b_{n-\nu,n}^{3} \qquad (23)$$

which are the same formulas given in Eq.(14) of Ref. [4].

These equations can be used to calculate the Bernstein coefficients corresponding to domains that are created by repeated halving of the unit domain. Alternatively, we may from the outset define a linear division of the domain into M equal size windows by simply setting

$$\alpha = -1/M, \quad \beta = m/M, \qquad m = 1, \ldots, M \qquad (24)$$

Such a scheme allows one to repeatedly divide the unit domain into smaller pieces faster and perhaps in a non-even fashion if one so chooses.

4   Conclusions

We have derived an expression relating the Bernstein coefficients of a polynomial on the unit domain $x \in [0,1]$ to those for the mapped domain $x \longrightarrow \alpha x + \beta$. The expression provides a generalization of the method implemented by Muñoz and Narkawicz [4]. The full implications of this work are still being determined; for example, one may study the aforementioned linear division of the domain in the limit of infinitely small divisions, $M \to \infty$.

References

1. G. Lorentz, *Bernstein Polynomials*, Chelsea Publishing Company, New York N.Y., Second Edition (1986)

2. J. Garloff, *Application of Bernstein Expansion to the Solution of Control Problems*, Reliable Computing **6**, 303-320 (2000)

3. J. Garloff, *The Bernstein Algorithm*, Interval Computations **4**, 154-168 (1993)

4. C. Muñoz and A. Narkawicz, *Formalization of an Efficient Representation of Bernstein Polynomials and Applications to Global Optimization*, submitted to Journal of Automated Reasoning (2011)

5. A. P. Smith, *Fast Construction of Constant Bound Functions for Sparse Polynomials*, J. of Global Optimization **43**, 445-458 (2009)

1. REPORT DATE *(DD-MM-YYYY)* 01-03-2012	2. REPORT TYPE Technical Memorandum	3. DATES COVERED *(From - To)* 12/2011–03/2012

4. TITLE AND SUBTITLE Translation of Bernstein Coefficients Under an Affine Mapping of the Unit Interval	5a. CONTRACT NUMBER
	5b. GRANT NUMBER
	5c. PROGRAM ELEMENT NUMBER
6. AUTHOR(S) J. A. Alford II	5d. PROJECT NUMBER
	5e. TASK NUMBER
	5f. WORK UNIT NUMBER 031102.02.07.05.9D65.12

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) NASA Langley Research Center Hampton, Virginia 23681-2199	8. PERFORMING ORGANIZATION REPORT NUMBER L–20124
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES) National Aeronautics and Space Administration Washington, DC 20546-0001	10. SPONSOR/MONITOR'S ACRONYM(S) NASA
	11. SPONSOR/MONITOR'S REPORT NUMBER(S) NASA/TM–2012–217557

13. SUPPLEMENTARY NOTES

An electronic version can be found at http://ntrs.nasa.gov.

14. ABSTRACT

We derive an expression connecting the coefficients of a polynomial expanded in the Bernstein basis to the coefficients of an equivalent expansion of the polynomial under an affine mapping of the domain. The expression may be useful in the calculation of bounds for multi-variate polynomials.

15. SUBJECT TERMS

Bernstein Polynomials, Global Optimization, Polynomial Bounds, LFM

16. SECURITY CLASSIFICATION OF:			17. LIMITATION OF ABSTRACT	18. NUMBER OF PAGES	19a. NAME OF RESPONSIBLE PERSON STI Help Desk (email: help@sti.nasa.gov)
a. REPORT	b. ABSTRACT	c. THIS PAGE			
U	U	U	UU	12	19b. TELEPHONE NUMBER *(Include area code)* (443) 757-5802