

# M-MRAC for Nonlinear Systems with Bounded Disturbances

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**Abstract**—This paper presents design and performance analysis of a modified reference model MRAC (M-MRAC) architecture for a class of multi-input multi-output uncertain nonlinear systems in the presence of bounded disturbances. M-MRAC incorporates an error feedback in the reference model definition, which allows for fast adaptation without generating high frequency oscillations in the control signal, which closely follows the certainty equivalent control signal. The benefits of the method are demonstrated via a simulation example of an aircraft's wing rock motion.

## I. INTRODUCTION

Controlling uncertain nonlinear systems is a challenging task, and remains one of the active research topics in the systems theory. There are several directions in this field, one of which is the adaptive control. The asymptotic behavior of adaptive systems has been a well researched topic during the last couple of decades. However, the transient behavior of the input and output signals is still a challenging problem. Since the transient of the adaptive signals can be very oscillatory with big excursions [18], there has been a great deal of effort to modify the control architecture and the adaptive laws from the perspective of improving it. The majority of these efforts led to nonadaptive high gain feedback [3], [4], [16], switching control law [9], [10] or to a parameter dependent persistent excitation condition [1], and addressed only the behavior of output signal.

First contribution to transient analysis of the the adaptive control signal can be found in [6], where it is shown that the bound on the control signal is proportional to the square root of the adaptation rate. This result is conservative, but it reflects the general observations about the control signal behavior of the MRAC system. In [2], an adaptive control architecture, called  $L_1$  adaptive control, has been introduced, which can achieve arbitrarily close tracking of a given reference command both in transient and steady state by increasing the adaptation gain.

In [15] we have introduced the concept of a M-MRAC architecture for linear systems that can achieve desired level of accuracy in tracking both input and output signals of a given reference model by a proper selection of design parameters. In this paper we apply the M-MRAC approach to a class of

multi-input multi-output uncertain nonlinear systems subject to bounded disturbances. The proposed algorithm guarantees tracking performance both in input and output signals similar to  $L_1$  adaptive control, but has a simpler structure and is easier to implement. Moreover, it requires the selection of only two control parameters, for which a design guideline is provided. This guideline is based on the results obtained for second order linear time variant systems, which is a contribution by itself.

The rest of the paper is organized as follows. In Section II we state the problem, and in Section III present the control design. The error signals are defined in Section IV. In Section V the controller's performance is analyzed and the design specifics are discussed. A simulation example is presented in Section VI and some concluding remarks are given in Section VII.

## II. PROBLEM FORMULATION

Consider a multi-input multi-output controllable uncertain system

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}[\mathbf{u}(t) + \mathbf{W}\mathbf{f}(\mathbf{x}(t)) + \mathbf{d}(t)] \quad (1)$$

with  $\mathbf{x}(0) = \mathbf{x}_0$ , where  $\mathbf{x} \in \mathbb{R}^n$  and  $\mathbf{u} \in \mathbb{R}^q$  are the state and input of the system,  $\mathbf{f} : \mathbb{R}^n \rightarrow \mathbb{R}^q$  is a known vector of regressor functions, assumed to satisfy the existence and uniqueness conditions,  $\mathbf{W} \in \mathbb{R}^{p \times q}$  is a matrix of unknown constant parameters, and  $\mathbf{d} : \mathbb{R} \rightarrow \mathbb{R}^p$  is a bounded but otherwise unknown external disturbance, and  $\mathbf{A} \in \mathbb{R}^{n \times n}$  and  $\mathbf{B} \in \mathbb{R}^{n \times p}$  are unknown constant matrices satisfying the following matching conditions.

*Assumption 2.1:* Given a Hurwitz matrix  $\mathbf{A}_m \in \mathbb{R}^{n \times n}$  and a matrix  $\mathbf{B}_m \in \mathbb{R}^{n \times p}$  of full column rank, there exists a matrix  $\mathbf{K}_1 \in \mathbb{R}^{p \times n}$  and a sign definite matrix  $\Lambda \in \mathbb{R}^{p \times p}$  such that the following equations hold

$$\begin{aligned} \mathbf{B} &= \mathbf{B}_m \Lambda \\ \mathbf{A} &= \mathbf{A}_m + \mathbf{B} \mathbf{K}_1. \end{aligned} \quad (2)$$

*Remark 2.1:* The sign definiteness of  $\Lambda$  corresponds to the conventional condition on the high frequency gain matrix of MIMO systems (see for example [11]). Without loss of generality we assume that  $\Lambda$  is positive definite. The rest of the conditions for the existence of an adaptive controller are given by the equations (2).

We notice that systems in the form of (1) frequently arise in aerospace applications (see for example [8], [17] for diagonal  $\Lambda$ ) and in robotics. Obviously, any fully actuated mechanical system can be described by equation (1).

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The objective is to design a control signal  $\mathbf{u}(t)$  such that the state of the system tracks the state  $\mathbf{x}^0(t)$  of a reference model

$$\dot{\mathbf{x}}^0(t) = A_m \mathbf{x}^0(t) + B_m \mathbf{r}(t), \quad \mathbf{x}^0(0) = \mathbf{x}_0, \quad (3)$$

where  $A_m, B_m$  are chosen according to performance specifications and satisfy Assumption 2.1, and  $\mathbf{r}(t)$  is a bounded and piecewise continuous external command. To achieve this objective we use the M-MRAC architecture introduced in [15], where the system (3) is called an ideal reference model. It is not a part of the control design and is only used for the analysis purposes.

Taking into account Assumption 2.1 we write

$$\dot{\mathbf{x}}(t) = A_m \mathbf{x}(t) + B_m \mathbf{r}(t) + B_m \Lambda [\mathbf{u}(t) + \theta \phi(t) + \mathbf{d}(t)] \quad (4)$$

where for the convenience we denote  $K_2 = -\Lambda^{-1}$ ,  $\phi(t) = [\mathbf{f}^\top(\mathbf{x}(t)) \quad \mathbf{x}^\top(t) \quad \mathbf{r}^\top(t)]^\top$ , and  $\theta = [W \quad K_1 \quad K_2]$ .

### III. CONTROL DESIGN

According to M-MRAC architecture, the design of the adaptive control is based on the modified reference model

$$\dot{\mathbf{x}}_m(t) = A_m \mathbf{x}_m(t) + B_m \mathbf{r}(t) + \lambda \mathbf{e}(t), \quad \mathbf{x}_m(0) = \mathbf{x}_0, \quad (5)$$

where  $\mathbf{e}(t) = \mathbf{x}(t) - \mathbf{x}_m(t)$  is the error between the system and the modified reference model,  $\lambda > 0$  is a feedback gain to be specified in the analysis. The adaptive control is given by the equation

$$\mathbf{u}(t) = -\hat{\theta}(t)\phi(t) - \hat{\mathbf{d}}(t),$$

where  $\hat{\theta}(t)$  is the estimate of the unknown matrix  $\theta$ , and  $\hat{\mathbf{d}}(t)$  is the estimate of a constant vector  $\bar{\mathbf{d}}$ , which is the constant part (or an average value) of  $\mathbf{d}(t)$ . The ideal version of this control signal that exactly cancels the uncertainties is

$$\mathbf{u}^0(t) = -\theta\phi(t) - \mathbf{d}(t). \quad (6)$$

Obviously, the ideal control  $\mathbf{u}^0(t)$  reduces the system (4) into the ideal reference model (3), which always can be specified from the performance perspective. However, the ideal control signal (6) cannot be implemented and is only used for the analysis purposes.

The adaptive laws for the estimates  $\hat{\theta}(t)$  and  $\hat{\mathbf{d}}(t)$  are defined using the projection based adaptive law

$$\begin{aligned} \dot{\hat{\theta}}(t) &= \gamma \text{Pr} \left( \hat{\theta}(t), B_m^\top P \mathbf{e}(t) \phi^\top(t) \right) \\ \dot{\hat{\mathbf{d}}}(t) &= \gamma \text{Pr} \left( \hat{\mathbf{d}}(t), B_m^\top P \mathbf{e}(t) \right), \end{aligned} \quad (7)$$

where  $\gamma > 0$  is the adaptation rate,  $P = P^\top > 0$  is the solution of the Lyapunov equation

$$A_m^\top P + P A_m = -Q \quad (8)$$

for some  $Q = Q^\top > 0$ , and  $\text{Pr}(\cdot, \cdot)$  denotes the projection operator [12] defined as  $\text{Pr}(\hat{\theta}, \mathbf{y}) = [\mathbb{I} - G(\hat{\theta})]\mathbf{y}$ , where

$$G(\hat{\theta}) = \begin{cases} 0, & \text{if } \varphi(\hat{\theta}) < 0 \\ 0, & \text{if } \varphi(\hat{\theta}) \geq 0, \nabla \varphi^\top(\hat{\theta})\mathbf{y} \leq 0 \\ \frac{\nabla \varphi(\hat{\theta}) \nabla \varphi^\top(\hat{\theta})}{\|\nabla \varphi(\hat{\theta})\|^2} \varphi(\hat{\theta}), & \text{if } \varphi(\hat{\theta}) \geq 0, \nabla \varphi^\top(\hat{\theta})\mathbf{y} > 0 \end{cases}$$

with the notation  $\nabla \varphi(\hat{\theta}) = \frac{\partial \varphi(\hat{\theta})}{\partial \hat{\theta}}$ , and the smooth convex functions  $\varphi(\hat{\theta})$  is given by

$$\varphi(\hat{\theta}) = \frac{\text{tr}(\hat{\theta}^\top \hat{\theta}) - \theta_{\max}^2}{\epsilon_\theta \theta_{\max}^2}, \quad (9)$$

with  $\theta_{\max}$  denoting the norm bound imposed on the parameter matrix  $\hat{\theta}$  and  $\epsilon_\theta$  denoting the convergence tolerance. The projection operator has the following properties

*Lemma 3.1:* [12] Let  $\theta_0 \in \Omega_0 = \{\hat{\theta} \in \mathbb{R}^n \mid \varphi(\hat{\theta}) \leq 0\}$ , and let the parameter  $\hat{\theta}(t)$  evolve according to the following dynamics

$$\dot{\hat{\theta}}(t) = \text{Pr}(\hat{\theta}(t), \mathbf{y}), \quad \hat{\theta}(t_0) \in \Omega. \quad (10)$$

Then 1)  $\hat{\theta}(t) \in \Omega_1 = \{\hat{\theta} \in \mathbb{R}^n \mid \varphi(\hat{\theta}) \leq 1\}$  or  $\|\hat{\theta}(t)\| \leq \theta^*$  for all  $t \geq t_0$ , where  $\theta^* = \sqrt{1 + \epsilon_\theta} \theta_{\max}$ , 2)  $[\hat{\theta}(t) - \theta_0]^\top [\text{Pr}(\hat{\theta}(t), \mathbf{y}) - \mathbf{y}] \leq 0$  for all  $t \geq t_0$ .

### IV. ERROR DYNAMICS

Introducing the parameter estimation error as  $\tilde{\theta}(t) = \hat{\theta}(t) - \theta$ , the dynamics of the tracking error  $\mathbf{e}(t)$  can be written in the form

$$\dot{\mathbf{e}}(t) = (A_m - \lambda \mathbb{I}_n) \mathbf{e}(t) + B_m \Lambda [-\tilde{\theta}(t)\phi(t) + \mathbf{d}(t) - \hat{\mathbf{d}}(t)],$$

where  $\mathbb{I}_n$  denotes  $n$ -dimensional identity matrix. The error signal  $\mathbf{e}(t)$  is used in stability analysis, but for the performance analysis we also need the actual tracking error  $\mathbf{e}^0(t) = \mathbf{x}(t) - \mathbf{x}^0(t)$ , which satisfies the equation

$$\dot{\mathbf{e}}^0(t) = A_m \mathbf{e}^0(t) + B_m \Lambda [-\tilde{\theta}(t)\phi(t) + \mathbf{d}(t) - \hat{\mathbf{d}}(t)].$$

These two error signals are related via the linear equation

$$\frac{d}{dt}[\mathbf{e}(t) - \mathbf{e}^0(t)] = A_m[\mathbf{e}(t) - \mathbf{e}^0(t)] - \lambda \mathbf{e}(t). \quad (11)$$

Since  $A_m$  is Hurwitz, the  $\mathcal{L}_1$  norm of the the state transition matrix  $\Phi(t) = e^{A_m t}$  is bounded. That is, there exists a positive constants  $k_m$  such that  $\|\Phi(t)\|_{\mathcal{L}_1} \leq k_m$ . Therefore, it follows from the equation (11) that

$$\|\mathbf{e}_\tau^0(t)\|_{\mathcal{L}_\infty} \leq (1 + \lambda k_m) \|\mathbf{e}_\tau(t)\|_{\mathcal{L}_\infty}, \quad (12)$$

where the subscript  $\tau$  indicate the extended  $\mathcal{L}_\infty$  norm on the interval  $0 \leq t \leq \tau$  (see [7], p. 200 for details). Moreover, if  $\mathbf{e}(t) \in \mathcal{L}_\infty$ , then

$$\|\mathbf{e}^0(t)\|_{\mathcal{L}_\infty} \leq (1 + \lambda k_m) \|\mathbf{e}(t)\|_{\mathcal{L}_\infty}, \quad (13)$$

In the following analysis we will also need the control error signal that is defined as  $\tilde{\mathbf{u}}(t) = \mathbf{u}(t) - \mathbf{u}^0(t)$ . From M-MRAC architecture it follows that

$$\tilde{\mathbf{u}}(t) = \mathbf{d}(t) - \hat{\mathbf{d}}(t) - \tilde{\theta}(t)\phi(t). \quad (14)$$

Therefore, the error dynamics (11) can be also represented as

$$\dot{\mathbf{e}}(t) = (A_m - \lambda \mathbb{I}_n) \mathbf{e}(t) + B_m \Lambda \tilde{\mathbf{u}}(t), \quad (15)$$

Since the ideal control signal is the best achievable signal, we are interested in minimizing the control error  $\tilde{\mathbf{u}}(t)$ , as well as the tracking error signals  $\mathbf{e}(t)$  and  $\mathbf{e}^0(t)$  by selecting proper values for the adaptation rate  $\gamma$  and feedback parameter  $\lambda$ .

## V. ANALYSIS OF THE M-MRAC PERFORMANCE

### A. Boundedness

*Theorem 5.1:* Let the system (1) be controlled by the M-MRAC scheme given by (5), (6), and (7). Then all closed-loop signals are ultimately bounded.

*Proof:* Consider the following candidate Lyapunov function

$$V(t) = \mathbf{e}^\top(t) P \mathbf{e}(t) + \frac{1}{\gamma} \text{tr}([\tilde{\boldsymbol{\theta}}(t) \tilde{\boldsymbol{\theta}}^\top(t) + \tilde{\mathbf{d}}(t) \tilde{\mathbf{d}}^\top(t)] \Lambda), \quad (16)$$

where  $\tilde{\mathbf{d}}(t) = \hat{\mathbf{d}}(t) - \bar{\mathbf{d}}$ . Computing its derivative along the trajectories of the systems (11) and (7), and using the properties of the projection operator, it is straightforward to obtain the inequality

$$\begin{aligned} \dot{V}(t) &\leq -\mathbf{e}^\top(t) (Q + 2\lambda P) \mathbf{e}(t) + 2\mathbf{e}^\top(t) P B [\mathbf{d}(t) - \bar{\mathbf{d}}] \\ &\leq -\|\mathbf{e}(t)\| [a \|\mathbf{e}(t)\| - 2d_*], \end{aligned} \quad (17)$$

where we denote  $a = \lambda_{\min}(Q) + 2\lambda_{\min}(P)\lambda$  and  $d_* = \|PB_m \Lambda(\mathbf{d}(t) - \bar{\mathbf{d}})\|_{\mathcal{L}_\infty}$  with  $\lambda_{\min}(S)$  being the minimum eigenvalue of the matrix  $S$ . It follows that  $V(t)$  is negative semi-definite whenever  $a\|\mathbf{e}\| \geq 2d_*$ , which along with the properties of the projection operator imply that the closed-loop error signals  $\mathbf{e}(t)$ ,  $\hat{\boldsymbol{\theta}}(t)$ ,  $\hat{\mathbf{h}}(t)$  are uniformly ultimately bounded. The boundedness of  $\mathbf{e}^0(t)$  follows from the inequality (13). Since  $A_m$  is Hurwitz, the bounded signals  $\mathbf{r}(t)$  and  $\mathbf{e}(t)$  produce a bounded signal  $\mathbf{x}_m(t)$ . Therefore,  $\mathbf{x}(t)$  is bounded. It follows that  $\mathbf{u}(t)$  and  $\hat{\mathbf{u}}(t)$  are bounded as well. ■

### B. Transient behavior of the tracking error

The projection operator in the adaptive laws (7) guarantees the inequalities

$$\|\hat{\boldsymbol{\theta}}(t)\| \leq \theta^*, \quad \|\hat{\mathbf{d}}(t)\| \leq d^*, \quad (18)$$

therefore

$$\text{tr}([\tilde{\boldsymbol{\theta}}(t) \tilde{\boldsymbol{\theta}}^\top(t) + \tilde{\mathbf{d}}(t) \tilde{\mathbf{d}}^\top(t)] \Lambda) \leq \beta,$$

where  $\beta = 4\theta^{*2} + 4h^{*2}$ . From Theorem 5.1 it follows that  $\dot{V}(t) \leq 0$  whenever

$$V(t) > V_* = \lambda_{\max}(P) \frac{4d_*^2}{a^2} + \frac{\beta}{\gamma}. \quad (19)$$

Therefore, it follows that the trajectories stay inside the Lyapunov level set

$$L = \left\{ (\mathbf{e}, \tilde{\boldsymbol{\theta}}, \tilde{\mathbf{d}}) : V(\mathbf{e}, \tilde{\boldsymbol{\theta}}, \tilde{\mathbf{d}}) = V_* \right\}. \quad (20)$$

From the definition of  $V(t)$  we have

$$\lambda_{\min}(P) \|\mathbf{e}(t)\|^2 \leq \mathbf{e}^\top(t) P \mathbf{e}(t) \leq V(t) \leq V_*. \quad (21)$$

Hence, the following conservative bound can be derived

$$\|\mathbf{e}(t)\|_{\mathcal{L}_\infty} \leq c = \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)} \frac{4d_*^2}{a^2} + \frac{\beta}{\gamma \lambda_{\min}(P)}}. \quad (22)$$

Since the inequality (22) holds uniformly in  $t$ , the bound  $\|\mathbf{e}(t)\|_{\mathcal{L}_\infty} \leq c$  follows. We notice that the second term in

the square root can be arbitrarily decreased by increasing the adaptation rate  $\gamma$ . The first term is independent of  $\gamma$ , but can be arbitrarily decreased by increasing  $\lambda$ . This is not the case for the conventional MRAC design. Indeed, when  $\lambda = 0$ , the only design parameter that affects that term is  $\lambda_{\min}(Q)$ . However, increasing  $\lambda_{\min}(Q)$  scales also  $P$ , hence increases  $d_*$ .

The bound on  $\|\mathbf{e}^0(t)\|_{\mathcal{L}_\infty}$  follows from the inequality (13) and has the form

$$\|\mathbf{e}^0(t)\|_{\mathcal{L}_\infty} \leq c(1 + \lambda k_m) \quad (23)$$

We notice that the derived bound on  $\|\mathbf{e}^0(t)\|_{\mathcal{L}_\infty}$  cannot be arbitrarily decreased by increasing the design parameters  $\lambda$  and  $\gamma$ . If we set  $\lambda = c_0 \sqrt{\gamma}$ , where the proportionality coefficient  $c_0$  will be defined shortly, the following asymptotic bound can be written

$$\lim_{\gamma \rightarrow \infty} \|\mathbf{e}^0(t)\|_{\mathcal{L}_\infty} \leq k_m \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}^3(P)} d_*^2 + \frac{c_0^2 \beta}{\lambda_{\min}(P)}}, \quad (24)$$

which can be decreased by increasing  $\lambda_{\min}(P)$ .

### C. Transient behavior of the control signal

To investigate the behavior of the control signal with respect to design parameters we recall that  $\mathbf{u}(t)$  does not explicitly depend on design parameters  $\lambda$  and  $\gamma$ . Instead,  $\dot{\mathbf{u}}(t)$  depends on  $\gamma$  through the adaptive laws, and  $\ddot{\mathbf{u}}(t)$  depends on  $\lambda$  through the tracking error dynamics. Therefore, for the purpose of this subsection we assume that  $\mathbf{r}(t)$  has bounded time derivatives. This assumption is only needed for the analysis purposes and is conditioned on the way the bound on the control signal is derived. Alternatively one could use the integral representation of the parameter estimates and the error signal  $\mathbf{e}(t)$  to derive an integral equation for the  $\mathbf{u}(t)$  without assuming the differentiability of  $\mathbf{r}(t)$ . However, for more transparency we use the differential form of the equations.

Differentiating  $\mathbf{u}(t)$  and substituting the adaptive laws we obtain

$$\dot{\mathbf{u}}(t) = -\gamma[\rho(t)\mathbb{I}_q - H(t)]B_m^\top P \mathbf{e}(t) - \mathbf{r}_a(t), \quad (25)$$

where we denote  $\rho(t) = \boldsymbol{\phi}^\top(t) \boldsymbol{\phi}(t) + 1$ ,  $\mathbf{r}_a(t) = \hat{\boldsymbol{\theta}}^\top(t) \dot{\boldsymbol{\phi}}(t)$ , and  $H(t) = G(\hat{\boldsymbol{\theta}}) \boldsymbol{\phi}^\top(t) \boldsymbol{\phi}(t) + G(\hat{\mathbf{d}})$ . We notice that the terms  $\rho(t)$ ,  $\mathbf{r}_a(t)$  and  $H(t)$  do not explicitly depend on the design parameters  $\lambda$  and  $\gamma$ . Moreover, from the results of the previous subsection it follows that all signals involved in the equation (25) are bounded. In particular, there exist positive constants  $\alpha_1$ ,  $\alpha_2$ ,  $\alpha_3$  such that  $\|\rho(t)\|_{\mathcal{L}_\infty} \leq \alpha_1$ ,  $\|\dot{\rho}(t)\|_{\mathcal{L}_\infty} \leq \alpha_2$  and  $\|\mathbf{r}_a(t)\|_{\mathcal{L}_\infty} \leq \alpha_3$ . We also notice that the matrix  $F(t) = \rho(t)\mathbb{I}_p - H(t)$  is symmetric and positive semi-definite, since  $\|G(\boldsymbol{\theta})\| \leq 1$ , which follows from the properties of the projection operator.

Differentiating the equation (25) with respect to time and using the equation (15) we obtain the following second order differential equation

$$\begin{aligned} \ddot{\mathbf{u}}(t) + \lambda \dot{\mathbf{u}}(t) + \gamma F(t) L \mathbf{u}(t) &= -\gamma F(t) B_m^\top P A_m \mathbf{e}(t) \\ &\quad - \gamma \dot{F}(t) B_m^\top P \mathbf{e}(t) + \gamma F(t) L \mathbf{u}^0(t) - \lambda \mathbf{r}_a(t) - \dot{\mathbf{r}}_a(t), \end{aligned} \quad (26)$$

where  $L = B_m^\top P B_m \Lambda$ . It follows from the definition of the projection operator that and Theorem 5.1 that  $\dot{F}(t)$  is piecewise continuous and bounded. Since all the terms in equation (26) are bounded, it can be considered as a second order linear equation with time varying coefficients in  $\mathbf{u}(t)$ . Although the equation (26) is non-autonomous, it can be still inferred that the adaptation rate  $\gamma$  determines the frequency of the control signal  $\mathbf{u}(t)$ . Therefore, increasing  $\gamma$  increases the oscillations in  $\mathbf{u}(t)$  as it is the case for the conventional MRAC design. On the other hand  $\lambda$  determines the damping ratio. Therefore increasing  $\lambda$  suppresses the oscillations in the control signal  $\mathbf{u}(t)$ . That is, by selecting a proper value for  $\lambda$  the desired performance can be achieved. This is the main difference from the MRAC design, which results when  $\lambda = 0$ .

We select a proper value of  $\lambda$  from the perspective of minimizing the norm bound on the control signal  $\mathbf{u}(t)$  in transient. To this end we notice that selection of the initial parameter estimates inside the convex sets defined by the projection operator results in  $H(t) = 0$  on some initial interval  $[0, t_1]$  by continuity. Therefore,  $F(t) = \rho(t)\mathbb{I}_p$  on  $[0, t_1]$ . To simplify computations we notice that the matrix  $L$  is symmetric and positive definite, therefore there exists an orthogonal matrix  $T$  such that  $D = TLT^\top$  is diagonal with positive entries  $d_{ii}$ ,  $i = 1, \dots, p$ . That is, introducing a new variable  $\mathbf{v} = T\mathbf{u}$  ( $\mathbf{v}^0 = T\mathbf{u}^0$ ), we can write the equation (26) in the following form

$$\ddot{\mathbf{v}}(t) + \lambda \dot{\mathbf{v}}(t) + \gamma D \rho(t) \mathbf{v}(t) = \gamma \rho(t) D \mathbf{v}^0(t) + \gamma \mathbf{z}_1(t) - \lambda \mathbf{z}_2(t) - \dot{\mathbf{z}}_2(t). \quad (27)$$

where  $\mathbf{z}_1(t) = T[-\rho(t)B_m^\top P A_m - \dot{\rho}(t)B_m^\top P]e(t)$  and  $\mathbf{z}_2(t) = T\mathbf{r}_a(t)$ . Let  $\rho_0 = \frac{1+\alpha_1}{2}$ . Then for each component of vector  $\mathbf{v}(t)$  we can write the following equation

$$\ddot{v}_i(t) + \lambda \dot{v}_i(t) + \gamma d_{ii} \rho_0 v_i(t) = \gamma \rho(t) d_{ii} v_i^0(t) + \gamma z_{1i}(t) - \lambda z_{2i}(t) - \dot{z}_{2i}(t) + \gamma [d_0 \rho_0 - d_{ii} \rho(t)] v_i(t). \quad (28)$$

Since  $\rho(t) \geq 1$ , this equation is in the form of equation (38) from Appendix with  $k(t) = d_{ii} \rho(t)$ ,  $2a = \lambda$ ,  $\omega_i^2 = \gamma d_{ii} \rho_0$  and three external inputs

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \gamma \rho(t) d_{ii} v_i^0(t), \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix} \gamma z_{1i}(t), \quad \begin{bmatrix} 1 \\ \lambda \end{bmatrix} z_{2i}(t).$$

Applying the inequality (48) we obtain

$$|v_i(t)| \leq c_3 \sqrt{v_i^2(0) + \dot{v}_i^2(0)} e^{-\nu t} + \alpha_1 \|v_i^0(t)\|_{\mathcal{L}_\infty} + \frac{1}{d_{ii}} \|z_{1i}(t)\|_{\mathcal{L}_\infty} + \frac{(c_1 + 2)\sqrt{\rho_0}}{\sqrt{d_{ii}\gamma}} \|z_{2i}(t)\|_{\mathcal{L}_\infty} \quad (29)$$

for each  $i = 1, \dots, p$ , if  $\lambda \geq 2\sqrt{\omega_i} = \sqrt{\gamma d_{ii} \rho_0}$ . Therefore selecting

$$\lambda = \sqrt{2\gamma d^0(\alpha_1 + 1)}, \quad (30)$$

where  $d^0 = \max\{d_{ii}, i = 1, \dots, p\}$  results in the inequality

$$|v_i(t)| \leq c_3 \sqrt{v_i^2(0) + \dot{v}_i^2(0)} e^{-\nu t} + \alpha_1 \|v_i^0(t)\|_{\mathcal{L}_\infty} + \frac{1}{d_0} \|z_{1i}(t)\|_{\mathcal{L}_\infty} + \frac{(c_1 + 2)\sqrt{\rho_0}}{\sqrt{d_0\gamma}} \|z_{2i}(t)\|_{\mathcal{L}_\infty} \quad (31)$$

for all components simultaneously, where  $d_0 = \min\{d_{ii}, i = 1, \dots, p\}$ . A similar inequality holds (as it is derived in the appendix) for the vector  $\mathbf{v}(t)$

$$\|\mathbf{v}(t)\| \leq c_4 e^{-\nu t} + \alpha_1 \|\mathbf{v}^0(t)\|_{\mathcal{L}_\infty} + \frac{1}{d_0} \|\mathbf{z}_1(t)\|_{\mathcal{L}_\infty} + \frac{(c_1 + 2)\sqrt{\rho_0}}{\sqrt{d_0\gamma}} \|\mathbf{z}_2(t)\|_{\mathcal{L}_\infty} \quad (32)$$

Since  $T$  is orthogonal with  $\|T\|_2 = 1$ , we have  $\|\mathbf{u}(t)\| = \|\mathbf{v}(t)\|$ ,  $\|\mathbf{z}_2(t)\| = \|\mathbf{r}_a(t)\|$  and

$$\|\mathbf{z}_1(t)\|_{\mathcal{L}_\infty} \leq [\alpha_1 \|B_m^\top P A_m\| + \alpha_2 \|B_m^\top P\|] \|\mathbf{e}(t)\|_{\mathcal{L}_\infty}.$$

Combining the above relationships we obtain

$$\|\mathbf{u}(t)\| \leq c_4 e^{-\nu t} + \alpha_1 \|\mathbf{u}^0(t)\|_{\mathcal{L}_\infty} + c_5 \|\mathbf{e}(t)\|_{\mathcal{L}_\infty} + \frac{(c_1 + 2)\sqrt{\rho_0}}{\sqrt{d_0\gamma}} \|\mathbf{r}_a(t)\|_{\mathcal{L}_\infty}, \quad (33)$$

where  $c_5 = \frac{1}{d_0} [\alpha_1 \|B_m^\top P A_m\| + \alpha_2 \|B_m^\top P\|]$ .

It is easy to see that the bound on  $\|\mathbf{u}(t)\|$  has the form

$$\|\mathbf{u}(t)\|_{\mathcal{L}_\infty} \leq c_4 e^{-\nu t} + \alpha_1 \|\mathbf{u}^0(t)\|_{\mathcal{L}_\infty} + \frac{c_6}{\sqrt{\gamma}}, \quad (34)$$

where

$$c_6 = c_5 \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}^3(P)} d_*^2 + \frac{\beta}{\lambda_{\min}(P)}}.$$

and the constants  $\alpha_1, c_4, c_6$  do not depend on the adaptation rate  $\gamma$ . We notice that the first term on the right hand side of (34) represents the exponentially decaying effect of the initial conditions  $\mathbf{u}(0)$  and  $\dot{\mathbf{u}}(0)$ , and the last term can be decreased by increasing  $\gamma$ . Therefore we can conclude that for large  $\gamma$ , the adaptive control  $\mathbf{u}(t)$  behaves similar to the ideal control  $\mathbf{u}^0(t)$ . That is unlike the conventional MRAC design, the fast adaptation in the M-MRAC design does not generate high gain effect.

The transient behavior of the error signals is summarized as follows.

**Theorem 5.2:** Let the system (1) be controlled by the M-MRAC scheme given by (5), (6) and (7), and the design parameters are chosen according to equation  $\lambda = c_0 \sqrt{\gamma}$  with  $c_0 = \sqrt{2d^0(\alpha_1 + 1)}$ . Then, the inequalities (22), (23) and (34) are true.

## VI. SIMULATION RESULTS

For the simulation we consider a wing rock motion of a slender delta wing given by the equation [14]

$$\ddot{\phi}(t) = f(\phi, \dot{\phi}) + b u(t) + d(t).$$

where  $\phi(t)$  is the roll angle,

$$f(\phi, \dot{\phi}) = a_1 \phi + a_2 \dot{\phi} + a_3 |\dot{\phi}|^2 \dot{\phi} + a_4 |\dot{\phi}| \dot{\phi} + a_3 \dot{\phi}^3. \quad (35)$$

with  $a_1 = -0.0186$ ,  $a_2 = 0.0152$ ,  $a_3 = -0.0625$ ,  $a_4 = 0.0095$ ,  $a_5 = 0.0215$ ,  $b = 1$ . The disturbance  $d(t)$  represents unknown atmospheric effects and is a square wave of amplitude 0.15 and of frequency 0.5 rad/sec. Only the sign of the control effectiveness  $b$  is assumed to be known (positive). The parameters of the reference model are chosen

as follows  $A_m = [0 \ 1; -1 \ -1.6]$ ,  $b_m = [0; 1]$ . The external input to follow is a step command of magnitude  $-15$  degrees at  $t = 15$ . We run two simulations respectively from the small initial conditions ( $6\text{deg.}$ ,  $3\text{deg/sec}$ ) and large initial conditions ( $30\text{deg.}$ ,  $10\text{deg/sec}$ ). The adaptation rate is set to  $\gamma = 10000$  with  $\lambda$  defined according to equation (30). The simulation results are displayed on Figures 1 and 2 respectively. It can be seen that a good tracking is achieved for both output and control signals, and the later does not exhibit any high frequency oscillations even for the selected high adaptation rate. The disturbance effect is completely attenuated.

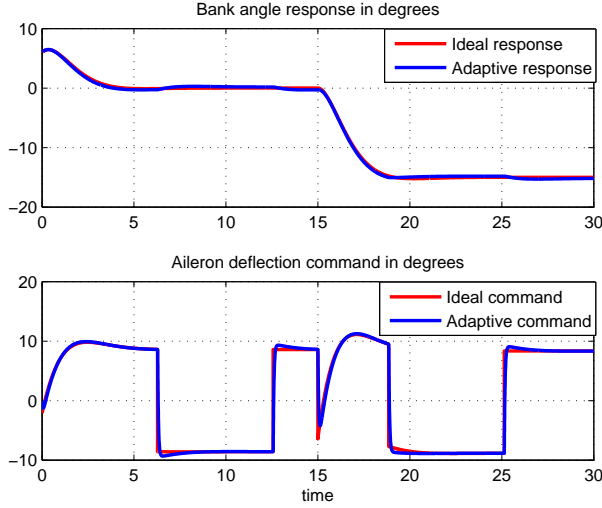


Fig. 1. Tracking a step command from the small initial conditions.

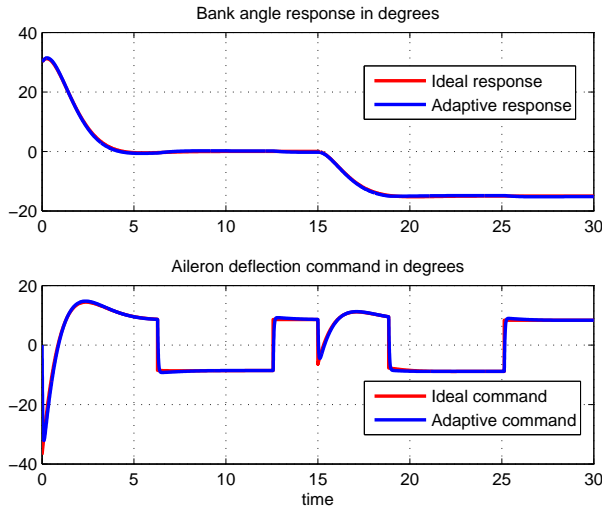


Fig. 2. Tracking a step command from the large initial conditions.

## VII. CONCLUSIONS

We have presented design and performance analysis of M-MRAC architecture for a class of uncertain systems subject

to bounded disturbances. It is shown that the systems' input and output tracking errors can be decreased as desired by increasing the adaptation rate, when the error feedback gain is selected according to derived rule. This design method prevents high frequency oscillations in the control signal, which are unavoidable in conventional MRAC systems. The performance of M-MRAC is demonstrated on a benchmark problem of controlling wing rock motion of slender delta wings in a turbulent atmosphere.

## APPENDIX A

### AN UPPER BOUND FOR A SECOND ORDER LTV SYSTEM

Consider a second order time variant linear system

$$\ddot{x}(t) + 2a\dot{x}(t) + \gamma k(t)x(t) = b_1 f(t) + b_2 \dot{f}(t) \quad (36)$$

with  $x(0) = x_0$ ,  $\dot{x}(0) = \dot{x}_0$ , where  $\gamma > 0$  is a constant parameter,  $k(t)$  is continuous with  $k^* \geq k(t) \geq k_* > 0$  and has a bounded derivative. The function  $f(t)$  is assumed to be piecewise continuous and bounded. The equation (36) can be written in the matrix form as

$$\dot{z}(t) = Az(t) + Bf(t) \quad (37)$$

where

$$z(t) = \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 \\ -\gamma k(t) & -2a \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix},$$

We are interested in minimizing the upper bound on  $x(t)$  by the choice of the parameter  $a$ . To this end we introduce notations  $\omega^2 = \gamma k_0$ ,  $k_0 = \frac{k^* + k_*}{2}$ ,  $a = \zeta\omega$  and represent the system (37) in the following equivalent form

$$\dot{z}(t) = Dz(t) + Bf(t) + C[\omega^2 - \gamma k(t)]x(t) \quad (38)$$

where

$$D = \begin{bmatrix} 0 & 1 \\ -\omega^2 & -2\zeta\omega \end{bmatrix}, \quad C = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

For the convenience of derivations we decompose  $z(t)$  into initial response  $z^i(t)$  of the homogeneous system

$$\dot{z}^i(t) = Az^i(t) \quad (39)$$

with the initial condition  $z_0 = [x_0 \ \dot{x}_0]^T$ , and force response  $z^f(t)$  of the system (38) with zero initial conditions, which can be represented in the equivalent integral form

$$z^f(t) = \int_0^t G(t-\tau)Bf(\tau)d\tau + \int_0^t G(t-\tau)C[\omega^2 - \gamma k(t)]x^f(\tau)d\tau. \quad (40)$$

Here,  $G(t) = e^{Dt}$  is the state transition matrix, which can be computed by direct integration (see for example [5]).

In order to minimize the bound on  $z_1^f(t) = x^f(t)$  we compute the  $\mathcal{L}_1$  norm of the elements in the first row of matrix  $G(t)$ . For  $g_{12}(t)$  we obtain

$$\|g_{12}(t)\|_{\mathcal{L}_1} = \begin{cases} \frac{1}{\omega^2} \frac{e^{\frac{\zeta\omega\pi}{\delta}} + 1}{e^{\frac{\zeta\omega\pi}{\delta}} - 1}, & \zeta < 1 \\ \frac{1}{\omega^2}, & \zeta \geq 1 \end{cases}, \quad (41)$$

where  $\delta = \omega\sqrt{1-\zeta^2}$  and  $0 < \varphi = \tan^{-1}(\frac{\delta}{\zeta\omega}) < \frac{\pi}{2}$ . Obviously,  $\|g_{12}(t)\|_{\mathcal{L}_1}$  reaches its minimum of  $\frac{1}{\omega^2}$  for all  $\zeta \geq 1$ . On the other hand, the  $\mathcal{L}_1$  norm of  $g_{11}(t)$  is computed to be

$$\|g_{11}(t)\|_{\mathcal{L}_1} = \begin{cases} \frac{2}{\omega} \left( \zeta + \frac{e^{\frac{\zeta\omega\varphi}{\zeta\omega\pi-1}}}{e^{\frac{\zeta\omega\varphi}{\zeta\omega\pi-1}} - 1} \right), & \zeta < 1 \\ \frac{2\zeta}{\omega}, & \zeta \geq 1 \end{cases}, \quad (42)$$

It can be shown that the minimum of  $\|g_{11}(t)\|_{\mathcal{L}_1}$  is reached at some  $\zeta^* < 1$ . The analytical computations are somewhat involved, but numerical computations results in  $\zeta^* = 0.66$  with the minimum value of  $\frac{2c}{\omega}$ , where  $c = 0.8026$ .

Since there is no common minimum point for the entries of  $G(t)$ , one can use a "suboptimal" value  $\zeta = 1$ , which is good enough for our purposes. In this case  $\|g_{11}(t)\|_{\mathcal{L}_1} = \frac{2}{\omega}$  and  $\|g_{12}(t)\|_{\mathcal{L}_1} = \frac{1}{\omega^2}$ . We notice that selecting a larger value of  $\zeta$  while leaving  $\|g_{12}(t)\|_{\mathcal{L}_1}$  intact, increases  $\|g_{11}(t)\|_{\mathcal{L}_1}$  proportional to  $\frac{1}{\omega}$ . Therefore, we can select any  $\zeta \geq 1$  with  $\|g_{11}(t)\|_{\mathcal{L}_1} = \frac{c_1}{\omega}$ , where  $c_1 \geq 2$  is determined by the selected  $\zeta$  and is independent of  $\omega$ .

Next we compute  $\mathcal{L}_\infty$  bound on  $x^f(t)$

$$x^f(t) = \int_0^t [b_1 g_{11}(t-\tau) + b_2 g_{12}(t-\tau)] f(\tau) d\tau + \int_0^t g_{12}(t-\tau) [\omega^2 - \gamma k(t)] x^f(\tau) d\tau. \quad (43)$$

Since  $\|\omega^2 - \gamma k(t)\|_{\mathcal{L}_\infty} = \omega^2 - \gamma k_* = \gamma \frac{k^* - k_*}{2}$ , we obtain (see [7], p. 199 for details)

$$\|x^f(t)\|_{\mathcal{L}_\infty} \leq \|f(t)\|_{\mathcal{L}_\infty} [\|b_1\| \|g_{11}(t)\|_{\mathcal{L}_1} + \|b_2\| \|g_{12}(t)\|_{\mathcal{L}_1}] + \gamma \frac{k^* - k_*}{2} \|x^f(t)\|_{\mathcal{L}_\infty} \|g_{12}(t)\|_{\mathcal{L}_1}. \quad (44)$$

Substituting the  $\mathcal{L}_1$  norm values and solving the resulting inequality for  $\|x^f(t)\|_{\mathcal{L}_\infty}$  we obtain

$$\left(1 - \frac{k^* - k_*}{2k_0}\right) \|x^f(t)\|_{\mathcal{L}_\infty} \leq \left[ \frac{c_1 \|b_1\|}{\omega} + \frac{\|b_2\|}{\omega^2} \right] \|f(t)\|_{\mathcal{L}_\infty}, \quad (45)$$

which results in

$$\|x^f(t)\|_{\mathcal{L}_\infty} \leq \left[ \frac{c_1 \|b_1\| \sqrt{k_0}}{k_* \gamma} + \frac{\|b_2\|}{k_* \gamma} \right] \|f(t)\|_{\mathcal{L}_\infty}. \quad (46)$$

To obtain a bound for  $z^i(t)$ , we recall that according to Theorem 8.7 [13] the origin of the system (39) is uniformly exponentially stable, since  $\|A(t)\|$  is bounded,  $\|\dot{A}(t)\|$  is essentially bounded, and the point wise eigenvalues of matrix  $A(t)$  have negative right hand sides. Therefore

$$\|z^i(t)\| \leq c_3 \|z(0)\| e^{-\nu t} \quad (47)$$

for some positive constants  $c_3$  and  $\nu$ ,

Since (49) is true for each component of  $z^i(t)$ , adding the corresponding inequalities we finally arrive at

$$\|x(t)\| \leq c_3 \|z(0)\| e^{-\nu t} + \left[ \frac{c_1 \|b_1\| \sqrt{k_0}}{k_* \gamma} + \frac{\|b_2\|}{k_* \gamma} \right] \|f(t)\|_{\mathcal{L}_\infty} \quad (48)$$

for all  $a \geq \sqrt{\gamma k_0}$ .

We notice that when  $x$  and  $f$  are  $q$ -dimensional vectors in the equation (36), then  $z = [x_1 \ \dot{x}_1 \ \dots \ x_q \ \dot{x}_q]^\top$  and the matrices  $A$ ,  $B$ ,  $C$ ,  $D$  and  $G$  have repeated block structures.

Therefore the equation (44) and the upper bound (46) hold for each component  $x_i^f(t)$  of vector  $x^f(t)$  with  $f(t)$  replaced with  $f_i(t)$ . On the other hand, the inequality (47) is true for the  $2q$ -vector  $z(t)$ , hence it is true for the vectors  $x^f(t)$  and  $\dot{x}^f(t)$ . That is

$$\|x^i(t)\| \leq c_3 e^{-\nu t} \sqrt{\|x_0\|^2 + \|\dot{x}_0\|^2} \equiv c_4 e^{-\nu t} \quad (49)$$

It follows that the inequality

$$\|x(t)\| \leq c_4 e^{-\nu t} + \left[ \frac{c_1 \|b_1\| \sqrt{k_0}}{k_* \gamma} + \frac{\|b_2\|}{k_* \gamma} \right] \|f(t)\|_{\mathcal{L}_\infty} \quad (50)$$

holds in the vector case as well, when  $a \geq \sqrt{\gamma k_0}$ .

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