NASA/TM–2012-217589

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July 2012

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Available from:

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Abstract

The term "symmetry" is practically synonomous with the terms "reduction" and "simplification" in the field of structural mechanics. Conditions for the existence of certain symmetries to exist in the buckle patterns of symmetrically laminated composite plates are presented. The plates considered have a general planform with cutouts, variable thickness and stiffnesses, and general support and loading conditions. The symmetry analysis is based on enforcing invariance of the corresponding eigenvalue problem for a group of coordinate transformations associated with buckle patterns commonly exhibited by symmetrically laminated plates. The buckle-pattern symmetries examined include a central point of inversion symmetry, one plane of reflective symmetry, and two planes of reflective symmetry.

Symbols

$\mathcal{B}_1(\ )$, $\mathcal{B}_2(\ )$	differential operators associated with the plate boundary conditions (see equations (8) and (9))
$\mathcal{C}$	set of curves bounding the plate planform
$\mathcal{C}_i$	interior boundary curve of a cutout in the plate planform (see figure 1)
$\mathcal{C}_M$	portion of boundary curve $\mathcal{C}$ with $M_n = 0$ specified
$\mathcal{C}_o$	exterior boundary curve of the plate planform (see figure 1)
$\mathcal{C}_V$	portion of boundary curve $\mathcal{C}$ with $V_n = 0$ specified
$\mathcal{C}_\delta$	portion of boundary curve $\mathcal{C}$ with $\frac{\partial w}{\partial n} = 0$ specified
$\mathcal{C}_\Delta$	portion of boundary curve $\mathcal{C}$ with $w = 0$ specified
$\mathcal{D}$	coordinate domain associated with the plate planform (see figure 1)
D_{11}, D_{12}, D_{16}, D_{22}, D_{26}, D_{66}	plate bending and twisting stiffnesses (see equations (4)), in.-lb
$\mathcal{G}(\ )$	differential operator associated with the equation governing buckling (see equation (6b))
$\mathcal{G}_1(\ )$	differential operator associated with the plate boundary conditions (see equation (8))

$\mathcal{k}$	constant used to specify symmetry or antisymmetry (see equation (15))
$\mathcal{L}(\ )$	differential operator associated with the equation governing buckling (see equation (6a))
M_n, M_{ns}	bending stress resultant defined by equations (3), in-lb/in.
M_x, M_y, M_{xy}	bending stress resultant defined by equations (4), in-lb/in.
$\mathcal{M}_n$, $\mathcal{M}_{ns}$	differential operators defined by equations (7)
n	inplane direction normal to a plate edge, in.
n_x, n_y	components of the unit-magnitude vector field normal to a plate edge (see figure 1)
N_x^0, N_y^0, N_{xy}^0	prebuckling stress resultants (see equation (1)), lb/in.
P_n, P_s	applied inplane edge loads (see equation (1)), lb/in.
$\mathcal{P}_k$	points of boundary curve $\mathcal{C}$ with $M_{ns} = 0$ specified
Q_n	transverse shear stress resultant defined by equation (3e), lb/in.
$\mathcal{Q}_n$	differential operator defined by equation (7f)
s	arc-length parameter used to defined the parametric coordinates of the plate boundary, in.
T[ ]	symmetry transformation operator (see equation (11b))
w(x, y)	buckling-displacement eigenfunction (see equation (1))
V_n	transverse shear stress resultant defined by equation (2a), lb/in.
(x, y)	plate coordinates (see figure 1), in.
$(\bar{x}, \bar{y})$	transformed plate coordinates (see equations (11)), in.
λ	loading parameter and eigenvalue
λ_{cr}	value of the eigenvalue at buckling

Introduction

In the pursuit of high-performance aerospace structures, tailoring of material properties, support conditions, load introduction, and geometry play an important role. For example, previous studies have been conducted to find ways of using curvilinear fiber placement and thickness tailoring to reduce the severity of stress concentrations caused by cutouts, stiffener terminations, and other local stress raisers. Moreover, similar tailoring approaches have been used to find ways to enhance load-carrying, vibration and buckling performance. Specific examples that are representative of these efforts, spanning nearly forty years, are given in references 1-49. These examples, particularly the later ones, illustrate the complexity of the analyses and the computational expense generally involved in optimizing structures with tailored pointwise elastic properties. Thus, ways of reducing computational expense, such as developing relatively simple special-purpose analyses that exploit symmetries are likely to provide more lattitude for optimization efforts conducted in a building-block setting. This scenario is one example that illustrates that even in this era of powerful, general purpose computational tools for structural analysis, there remains a need for special-purpose analysis tools with a limited scope.

The term "symmetry" is practically synonomous with the terms "reduction" and "simplification" in the field of structural mechanics. To illustrate this point, consider a generally laminated, fiber-reinforced rectangular plate with spatially uniform plies. For this general class of plates; extension, shearing, bending, and twisting deformations are fully coupled through the plate constitutive equations (see reference 50, pp. 187-236). As a result of this coupling, the plate behavior, as modelled by classical plate theory, is governed by three fully coupled partial differential equations that generally cannot be solved in closed form. In contrast, for symmetrically laminated plates, the coupling between inplane and bending deformation modes is absent, and the partial differential equations governing the behavior reduce to two equations for the inplane response and one for the out-of-plane bending response. Moreover, when coupling between inplane shear and extension is absent, the two partial differential equations for the inplane response become significantly simpler. Likewise, when coupling between bending and twisting is absent, the corresponding partial differential equation governing the out-of-plane bending behavior reduces to a significantly simpler form. Both of these simplifications correspond to a symmetry exhibited by laminates with orthogonal ply directions that are aligned with the rectangular-plate edges. Thus, the symmetries of the plate lamination correspond to a tremendous simplification of the mathematical analysis that governs laminated-plate behavior. As a result of these symmetry-based simplications, clarity and insight into the physical behavior is gained and the complexity of the corresponding mathematical analyses is reduced. For example, symmetry-based simplications lead to finite element analyses that use only a portion of the original computational domain. Likewise, these simplifications lead to the elimination of unneeded basis functions in classical Rayleigh-Ritz and Bubnov-Galerkin analyses.

Exploiting symmetry in structural mechanics is not a new idea. In particular, using symmetry arguments to simplify the pointwise, local constitutive equations of elastic materials is a cornerstone of the Theory of Elasticity that began in the 19th century.[51] These symmetry-based simplifications ultimately contributed to the development of analytical solutions for a variety of practical boundary-value problems that yielded significant advances in structural mechanics technology. In more recent times, Renton[52] used global, or overall, symmetry arguments to simplify buckling analyses of symmetrical frames. In 1973, Glockner[53] demonstrated how to use symmetry arguments systematically to simplify structural analyses and reduce computational expense. A key ingredient of Glockner's work is his *axiom of symmetry*, which is based on the physical observations and experience base within the structural mechanics discipline up to that time. In essence, it is not unreasonable to expect that the inherent symmetry of a structural system results in a corresponding symmetry of its mechanical response. More specifically, Glockner's axiom is paraphrased as follows. *Given a structure whose geometry, support and loading conditions, elastic stiffness and mass distributions, and thermal expansion properties exhibit a common certain type of symmetry, the structural response will exhibit the same type of symmetry.* Discussions of Glockner's work are found in references 54-58.

In 1973, MacNeal et. al.[59] presented a method for reducing the number of simultaneous equations to be solved in a linear finite element analysis of a general structure composed of two or more identical sub-regions. In this method, the sub-regions are arranged symmetrically with respect to an axis and, as a result, the symmetry is referred to in reference 59 as cyclic symmetry. Later, in 1976, Evensen[60] showed how to exploit symmetry in the analysis of structural dynamics problems. In particular, Evensen showed how to identify the smallest subdomain of a structure, and the appropriate boundary conditions on the subdomain, that will yield all the vibration modes of the overall structure. Also in 1976, and in 1977, Noor and his colleagues[61-64] identified the symmetries exhibited by fiber-reinforced laminated-composite plates and shells, as a function of ply orientation and stacking sequence. In addition, the conditions on the geometry, stiffness and mass coefficients, loading and support conditions, stress resultants, and kinematics quantities are deduced from Glockner's axiom of symmetry. Futhermore, a procedure is presented that exploits a given symmetry to obtain the smallest computational domain that can be used in linear and nonlinear finite element analyses. Everstine[65] presented similar ideas in 1977, and in 1979 Williams[66] presented quantitative structural analysis results that showed substantial benefits in reducing computational costs by exploiting symmetry. In 1992, Li and Reid[67] presented a more detailed examination of the symmetry conditions exhibited by laminated-composite structures, similar to that given by Noor and his colleagues. Several years later, Balaji et. al.[68] presented a computer program for the design of radial impellers that is based on exploiting cyclic symmetry.

The concepts and results presented in reference 59-68 are applications of the mathematical discipline known as Group Theory; as demonstrated by Miller[69] in 1981 and by Zheng et.al.[70] in 1982, and elucidated by Lobry and Broche[71] in 1994. Additional studies that use Group Theory to exploit symmetry in structural analyses are found in references 72-102. Furthermore, reviews of this topic have been given in 2002 and 2009 by Zingoni.[103-104]

For the most part, the previous studies cited herein are focused on simplifying the numerical solution of complicated boundary-value or eigenvalue problems. The studies presented by Noor and his colleagues appear to be the only ones that address the conditions required on the geometry, loading and support conditions, and material properties of plates and shells for certain prescribed symmetries in the response quantities to exist. These conditions are based on Glockner's symmetry axiom previously stated herein and are presented as a direct consequence of the axiom. In contrast, the conditions for certain symmetries to exist in the response quantities of anisotropic and orthotropic plates and shells are derived herein based on enforcing invariance of a given eigenvalue problem; that is, enforcing the primative concept of symmetry. Thus, the objective of the present study is to present the details of a direct procedure for obtaining the necessary and sufficient conditions on the problem parameters for prescribed symmetries to exist in the response quantities, in a simple manner. This procedure is based on the treatment of Schröedinger's equation given by Humi and Miller[105] (pp. 169-170), and on the treatment of functions and operators presented by McWeeny[106] (pp. 166-202). Just as the existence of symmetries were shown to yield reduction of the computational domain in many of the previous studies cited herein that focus on finite element analyses, the existence of symmetries lead to simplification in the basis functions that are used to represent the response quantities in classical Rayleigh-Ritz and Galerkin methods. This simplification is manifested as the elimination of unneeded waveforms, which improves computational efficiency of a special-purpose buckling analysis.

To achieve the objective of the present study, the eigenvalue problem associated with buckling of symmetrically laminated plates is considered first, without becoming entangled in mathematical jargon and notation that are not typically encountered by structural engineers. Specifically, symmetrically laminated plates of general shape with variable stiffness and general support and inplane loading conditions are examined and convenient differential operators are introduced. Then, symmetries associated with plate buckling are discussed and coordinate transformations that are used to characterize the symmetries are presented. After this step, the transformed eigenvalue problem is derived and the conditions that are necessary and sufficient for the plate-buckling symmetries to exist are given. More specifically, these conditions are given for buckle patterns with a central point of inversion symmetry, one plane of reflective symmetry, and two planes of reflective symmetry. Although the analysis presented subsequently is focused on symmetrically laminated plates, its extension to more complicated eigenvalue problems of fully anisotropic shells is straightforward.

Equations Governing Buckling Behavior

The plates considered in the present study are idealized as perfectly flat and have a midplane with the general planform shown in figure 1. Coordinates of points of the midplane are given by (x, y) with respect to the x and y axes and are collectively referred to herein as the domain $\mathcal{D}$. In general, the planform domain $\mathcal{D}$ is bounded by a continuous exterior curve $\mathcal{C}_o$ and by one or more interior curves $\mathcal{C}_i$ that correspond to cutouts. Each of these curves generally consist of a finite number of smooth, connected pieces with coordinates that are specified parametrically by $x = x(s)$ and $y = y(s)$, where s is an arc length parameter. Collectively, the plate boundary is denoted by $\mathcal{C} = \mathcal{C}_o \cup \mathcal{C}_i$. At any given point of a smooth segment of a boundary curve, the unit-magnitude vector field normal to the segment is given by the components $n_x(x, y)$ and $n_y(x, y)$, as shown in figure 1, where $x = x(s)$, $y = y(s)$, $n_x(x, y) = dy/ds$, and $n_y(x, y) = - dx/ds$. Traversal of a boundary curve, associated with increasing values of the arc length parameter s, and orientation of the unit-magnitude vector field are presumed to be consistent with the standard convention of Stokes Integral Theorem; that is, the exterior boundary is traversed counterclockwise and each interior boundary is traversed clockwise, and the unit-magnitude vector field points away from the interior of the plate domain. As depicted in figure 1, the plate is loaded by force-per-unit-length inplane tractions $\lambda P_n(x, y)$ and $\lambda P_s(x, y)$ that are normal and tangent to the boundary curve at a given point, respectively. In general, the tractions are presumed to be distributed over one or more of the segments forming the boundaries of the planform. The symbol λ represents a loading parameter that is increased monotonically from a value of zero to a value λ_{cr} that corresponds to buckling.

The equations governing the buckling behavior of plates are found in references 107 and 108. From the equations in these references, and reference 50, the general form of the partial differential equation governing bifurcation buckling of perfectly flat plates with flexural-twist anisotropy, in terms of the Cartesian coordinates (x, y), is found to be given by

$$\begin{aligned}&\frac{\partial^2}{\partial x^2}\left(D_{11}\frac{\partial^2 w}{\partial x^2} + D_{12}\frac{\partial^2 w}{\partial y^2} + 2D_{16}\frac{\partial^2 w}{\partial x \partial y}\right) + \frac{\partial^2}{\partial y^2}\left(D_{12}\frac{\partial^2 w}{\partial x^2} + D_{22}\frac{\partial^2 w}{\partial y^2} + 2D_{26}\frac{\partial^2 w}{\partial x \partial y}\right)\\ &+ 2\frac{\partial^2}{\partial x \partial y}\left(D_{16}\frac{\partial^2 w}{\partial x^2} + D_{26}\frac{\partial^2 w}{\partial y^2} + 2D_{66}\frac{\partial^2 w}{\partial x \partial y}\right) = \lambda\left(N_x^0\frac{\partial^2 w}{\partial x^2} + N_y^0\frac{\partial^2 w}{\partial y^2} + 2N_{xy}^0\frac{\partial^2 w}{\partial x \partial y}\right)\end{aligned} \tag{1}$$

for the set of points that form the domain $\mathcal{D}$. The function $w(x, y)$ is the out-of-plane buckling deflection and the functions $\lambda N_x^0(x, y)$, $\lambda N_y^0(x, y)$, and $\lambda N_{xy}^0(x, y)$ denote the prebuckling, inplane stress resultants produced by the edge loads $\lambda P_n(x, y)$ and $\lambda P_s(x, y)$. Positive values of these stress resultants are shown in figure 1. The terms D_{11}, D_{12}, D_{22}, and D_{66} are the plate bending stiffnesses that account for flexural orthotropy, and the terms D_{16} and D_{26} are the plate bending stiffnesses that account for flexural-twist anisotropy. For the general case of plates with variable thickness and nonuniform fiber placement, these stiffnesses are functions of the coordinates (x, y). Herein, these functions are presumed to be differentiable. Formulas for computing these bending stiffnesses for laminated-composite plates are found in reference 50. It is important to note that the determination of the prebuckling stress resultants generally requires the solution of a separate boundary-value problem. However, for several practical cases, they are statically determinant.

The general form of the boundary conditions that correspond to equation (1) are given by the conjugate pairs

$$w = 0 \quad \text{or} \quad V_n \equiv Q_n + \frac{\partial M_{ns}}{\partial s} + \lambda\left(P_n\frac{\partial w}{\partial n} + P_s\frac{\partial w}{\partial s}\right) = 0 \tag{2a}$$

$$\frac{\partial w}{\partial n} = 0 \quad \text{or} \quad M_n = 0 \tag{2b}$$

evaluated on one or more segments of the boundary curves and

$$M_{ns} = 0 \tag{2c}$$

evaluated at points of the boundary curves with sharp corners (discontinuous derivatives). In these equations,

$$\frac{\partial w}{\partial n} = n_x \frac{\partial w}{\partial x} + n_y \frac{\partial w}{\partial y} \tag{3a}$$

$$\frac{\partial w}{\partial s} = -\, n_y \frac{\partial w}{\partial x} + n_x \frac{\partial w}{\partial y} \tag{3b}$$

$$M_n = M_x n_x^2 + 2M_{xy} n_x n_y + M_y n_y^2 = 0 \tag{3c}$$

$$M_{ns} = \left(M_y - M_x\right) n_x n_y + \left(n_x^2 - n_y^2\right) M_{xy} \tag{3d}$$

$$Q_n = \left(\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y}\right) n_x + \left(\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y}\right) n_y \tag{3e}$$

where

$$M_x = -\left[D_{11}\frac{\partial^2 w}{\partial x^2} + D_{12}\frac{\partial^2 w}{\partial y^2} + 2D_{16}\frac{\partial^2 w}{\partial x \partial y}\right] \tag{4a}$$

$$M_y = -\left[D_{12}\frac{\partial^2 w}{\partial x^2} + D_{22}\frac{\partial^2 w}{\partial y^2} + 2D_{26}\frac{\partial^2 w}{\partial x \partial y}\right] \tag{4b}$$

$$M_{xy} = -\left[D_{16}\frac{\partial^2 w}{\partial x^2} + D_{26}\frac{\partial^2 w}{\partial y^2} + 2D_{66}\frac{\partial^2 w}{\partial x \partial y}\right] \tag{4c}$$

are the bending stress resultants. Together, the homogeneous partial differential equation and the homogeneous boundary conditions, given by equations (1) and (2), constitute an eigenvalue problem in which λ is the eigenvalue and $w(x, y)$ is the corresponding eigenfunction. An eigenfunction is referred to herein as a buckle pattern.

To facilitate further analysis presented herein, it is convenient to express equation (1) in operator form as

$$\mathcal{L}\big(w(x, y)\big) = \lambda\, \mathcal{G}\big(w(x, y)\big) \tag{5}$$

where the linear differential operator $\mathcal{L}(\ )$ is defined by

$$\mathcal{L}(\ ) = \frac{\partial^2}{\partial x^2}\left(D_{11}(x, y)\, \frac{\partial^2(\ )}{\partial x^2} + D_{12}(x, y)\, \frac{\partial^2(\ )}{\partial y^2} + 2D_{16}(x, y)\, \frac{\partial^2(\ )}{\partial x \partial y} \right)$$

$$+ \frac{\partial^2}{\partial y^2}\left(D_{12}(x, y)\, \frac{\partial^2(\ )}{\partial x^2} + D_{22}(x, y)\, \frac{\partial^2(\ )}{\partial y^2} + 2D_{26}(x, y)\, \frac{\partial^2(\ )}{\partial x \partial y} \right) \tag{6a}$$

$$+ 2\frac{\partial^2}{\partial x \partial y}\left(D_{16}(x, y)\, \frac{\partial^2(\ )}{\partial x^2} + D_{26}(x, y)\, \frac{\partial^2(\ )}{\partial y^2} + 2D_{66}(x, y)\, \frac{\partial^2(\ )}{\partial x \partial y} \right)$$

and the linear differential operator $\mathcal{G}(\ )$ is defined by

$$\mathcal{G}(\ ) = N_x^0(x, y)\, \frac{\partial^2(\ )}{\partial x^2} + N_y^0(x, y)\, \frac{\partial^2(\ )}{\partial y^2} + 2N_{xy}^0(x, y)\, \frac{\partial^2(\ )}{\partial x \partial y} \tag{6b}$$

Similarly, the boundary conditions are expressed in operator from as follows. First, the boundary stress resultants are expressed in operator form as

$$M_n = -\mathcal{M}_n\big(w(x, y)\big) \tag{7a}$$

$$M_{ns} = -\mathcal{M}_{ns}\big(w(x, y)\big) \tag{7b}$$

$$Q_n = -\mathcal{Q}_n\big(w(x, y)\big) \tag{7c}$$

where

$$\mathcal{M}_n(\ ) = \left[n_x^2(x, y) D_{11}(x, y) + n_y^2(x, y) D_{12}(x, y) + 2n_x(x, y)\, n_y(x, y) D_{16}(x, y) \right] \frac{\partial^2(\ )}{\partial x^2}$$

$$+ \left[n_x^2(x, y) D_{12}(x, y) + n_y^2(x, y) D_{22}(x, y) + 2n_x(x, y)\, n_y(x, y) D_{26}(x, y) \right] \frac{\partial^2(\ )}{\partial y^2} \tag{7d}$$

$$+ 2\left[n_x^2(x, y) D_{16}(x, y) + n_y^2(x, y) D_{26}(x, y) + 2n_x(x, y)\, n_y(x, y) D_{66}(x, y) \right] \frac{\partial^2(\ )}{\partial x \partial y}$$

$$\mathcal{M}_{ns}(\ ) = \left[n_x(x,y) n_y(x,y) \Big(D_{12}(x,y) - D_{11}(x,y) \Big) + \Big(n_x^2(x,y) - n_y^2(x,y) \Big) D_{16}(x,y) \right] \frac{\partial^2(\ )}{\partial x^2}$$
$$+ \left[n_x(x,y) n_y(x,y) \Big(D_{22}(x,y) - D_{12}(x,y) \Big) + \Big(n_x^2(x,y) - n_y^2(x,y) \Big) D_{26}(x,y) \right] \frac{\partial^2(\ )}{\partial y^2} \tag{7e}$$
$$+ 2 \left[n_x(x,y) n_y(x,y) \Big(D_{26}(x,y) - D_{16}(x,y) \Big) + D_{66}(x,y) \Big(n_x^2(x,y) - n_y^2(x,y) \Big) \right] \frac{\partial^2(\ )}{\partial x \partial y}$$

$$\mathcal{Q}_n(\ ) = n_x(x,y) \frac{\partial}{\partial x} \left[D_{11}(x,y) \frac{\partial^2(\ )}{\partial x^2} + D_{12}(x,y) \frac{\partial^2(\ )}{\partial y^2} + 2D_{16}(x,y) \frac{\partial^2(\ )}{\partial x \partial y} \right]$$
$$+ n_y(x,y) \frac{\partial}{\partial y} \left[D_{12}(x,y) \frac{\partial^2(\ )}{\partial x^2} + D_{22}(x,y) \frac{\partial^2(\ )}{\partial y^2} + 2D_{26}(x,y) \frac{\partial^2(\ )}{\partial x \partial y} \right] \tag{7f}$$
$$+ \left(n_x(x,y) \frac{\partial(\ )}{\partial y} + n_y(x,y) \frac{\partial(\ )}{\partial x} \right) \left[D_{16}(x,y) \frac{\partial^2(\ )}{\partial x^2} + D_{26}(x,y) \frac{\partial^2(\ )}{\partial y^2} + 2D_{66}(x,y) \frac{\partial^2(\ )}{\partial x \partial y} \right]$$

Then, equation (2a) is expressed in operator form as

$$\mathcal{B}_1\Big(w(x,y)\Big) = \lambda\, \mathcal{G}_1\Big(w(x,y)\Big) \tag{8a}$$

where for the displacement boundary condition $w = 0$,

$$\mathcal{B}_1(\ ) = (\ ) \quad \text{and} \quad \mathcal{G}_1(\ ) = 0 \tag{8b}$$

and for the conjugate force boundary condition $V_n = 0$,

$$\mathcal{B}_1(\ ) = \mathcal{Q}_n(\ ) + \left(n_x(x,y) \frac{\partial(\ )}{\partial y} - n_y(x,y) \frac{\partial(\ )}{\partial x} \right) \mathcal{M}_{ns}(\ ) \tag{8c}$$

and

$$\mathcal{G}_1(\ ) = \Big(n_x(x,y) P_n(x,y) - n_y(x,y) P_s(x,y) \Big) \frac{\partial(\ )}{\partial x} + \Big(n_x(x,y) P_s(x,y) + n_y(x,y) P_n(x,y) \Big) \frac{\partial(\ )}{\partial y} \tag{8d}$$

Likewise, equation (2b) is expressed in operator form as

$$\mathcal{B}_2\Big(w(x,y)\Big) = 0 \tag{9a}$$

where for the slope boundary condition $\frac{\partial w}{\partial n} = 0$,

$$\mathcal{B}_2\left(\; \right) = n_x(x, y)\,\frac{\partial\left(\; \right)}{\partial x} + n_y(x, y)\,\frac{\partial\left(\; \right)}{\partial y} \tag{9b}$$

and for the conjugate moment boundary condition $M_n = 0$,

$$\mathcal{B}_2\left(\; \right) = \mathcal{M}_n\left(\; \right) \tag{9c}$$

Furthermore, equation (2c) is expressed as

$$\mathcal{M}_{ns}\left(w(x, y)\right) = 0 \tag{10}$$

for points of the boundary curves with discontinuous derivatives.

Symmetries Associated With Plate Buckling

A buckle pattern $w(x, y)$ that is a legitimate solution to the eigenvalue problem for plate buckling defined herein may exhibit certain types of symmetries, depending on the circumstances of a given problem. These circumstances are determined by the geometry of the midplane domain $\mathcal{D}$, the distributions of the edge loading and supports, and the stiffnesses associated with the plate material composition and construction. For example, the buckle pattern may exhibit one plane of reflective symmetry, as shown by the contour plots in figures 2 and 3 for a rectangular plate domain, or one plane of reflective antisymmetry, as shown in figures 4 and 5. Inspection of figure 2 reveals that a buckle pattern with one plane of reflective symmetry, given by $x = 0$, satisfies the condition $w(-x, y) = w(x, y)$. Similarly, in figure 3 the plane of reflective symmetry given by $y = 0$ satisfies the condition $w(x, -y) = w(x, y)$. In contrast, the contour plots in figures 4 and 5 reveal that a buckle pattern with one plane of reflective antisymmetry satisfies the conditions $w(-x, y) = - w(x, y)$ or $w(x, -y) = -w(x, y)$, respectively. In many cases, a buckle pattern may exhibit two planes of reflective symmetry, as shown in figure 6, for which $w(-x, y) = w(x, y)$ and $w(x, -y) = w(x, y)$. Similarly, a buckle pattern may exhibit one plane of reflective symmetry and one plane of reflective antisymmetry where $w(-x, y) = -w(x, y)$ and $w(x, -y) = w(x, y)$ are satisfied (figure 7) or where $w(-x, y) = w(x, y)$ and $w(x, -y) = -w(x, y)$ are satisifed (figure 8). Furthemore, a buckle pattern may exhibit two planes of reflective antisymmetry where $w(-x, y) = -w(x, y)$ and $w(x, -y) = -w(x, y)$ are satisifed, as shown in figure 9. Two other types of symmetries often encountered are depicted in figures 10 and 11. In figure 10, the buckle pattern exhibits a central point of inversion symmetry for which $w(-x, -y) = w(x, y)$, and in figure 11 the buckle pattern exhibits a central point of inversion antisymmetry for which $w(-x, -y) = -w(x, y)$.

It is significant to note that buckle patterns simlilar to those with the reflective symmetries described above have been observed in experiments conducted on actual plates that are not perfectly symmetric, but may be reasonbly close for engineering purposes. Examples of real buckle patterns with reflective quasi-symmetries are found in references 109 and 110. An example of a real plate buckle pattern similar to the idealized central point of inversion symmetry shown in figure 10, is shown in figure 12. This plate is a highly anisotropic $[+60_6/-60_6]_s$ laminated-composite plate made of a typical high-strength graphite-epoxy material and loaded in compression. This plate was tested at the NASA Langley Research Center by the author circa 1990.

Although the symmetries associated with plate buckling have been illustrated for a rectangular plate domain, they can exist for certain types of nonrectangular domains. For example, an elliptical domain with its principal axes coincident with the x and y axes can exhibit the symmetries described in this section, depending on the loading and support conditions and on the material properties and corresponding stiffness distributions.

Necessary and Sufficient Conditions for a Prescribed Symmetry

Each of the symmetries depicted in figures 2-11 are characterized by a relationship between the values of the eigenfunction w at the points $(x, y) \in \mathcal{D}$ and at a corresponding set of related points $(\bar{x}, \bar{y}) \in \mathcal{D}$ that are reckoned with respect to the same coordinate frame. This relationship between the sets of points is quantified by a general unique orthogonal coordinate transformation of the domain $\mathcal{D}$ onto itself. Herein, this transformation is denoted symbolically by

$$\bar{x} = \bar{x}(x, y) \quad \text{and} \quad \bar{y} = \bar{y}(x, y) \tag{11a}$$

or by

$$(\bar{x}, \bar{y}) = \mathbf{T}\big[(x, y)\big] \tag{11b}$$

where $\mathbf{T}[\ \ ]$ denotes the transformed image of the quantity within the brackets. In this context, equations (11) are viewed as a mapping of points in which the coordinate axes remain fixed. In addition, because the transformations are orthogonal, the lengths of lines in $\mathcal{D}$, and the angles between them remain unaltered; that is, invariant. Moreover, it follows that since $\mathcal{D}$ is bounded by the curves forming the boundary $\mathcal{C}$, that for every $(x, y) \in \mathcal{C}$, the corresponding point $(\bar{x}, \bar{y})$ is also an element of $\mathcal{C}$ for this class of transformations. Thus, the actual form of the functional relationships given by equations (11) places restrictions on the shape of the domain and its boundary; that is, $\mathbf{T}[\,\mathcal{D}\,] = \mathcal{D}$ and $\mathbf{T}[\,\mathcal{C}\,] = \mathcal{C}$.

Three coordinate transformations are used in the present study to characterize symmetries commonly exhibited by the buckle patterns of idealized plates described herein previously. The first transformation corresponds to a reflection of the points forming the region $\mathcal{D} \cup \mathcal{C}$ about the plane x = 0, and is given by

$$\bar{x}(x, y) = -x \quad \text{and} \quad \bar{y}(x, y) = y \tag{12}$$

For this transformation, the domain $\mathcal{D}$ and boundary $\mathcal{C}$ are required to be symmetric about the plane x = 0, as shown in figure 13. The second transformation corresponds to a reflection of the points comprizing $\mathcal{D} \cup \mathcal{C}$ about the plane y = 0, and is given by

$$\bar{x}(x, y) = x \quad \text{and} \quad \bar{y}(x, y) = -y \tag{13}$$

For this transformation, the domain $\mathcal{D}$ and boundary $\mathcal{C}$ are required to be symmetric about the plane y = 0. The third transformation corresponds to a reflection of the points comprizing $\mathcal{D} \cup \mathcal{C}$ about the plane normal to the line passing through the point (x, y) and the origin. This transformation is given by

$$\bar{x}(x, y) = -x \quad \text{and} \quad \bar{y}(x, y) = -y \tag{14}$$

and is depicted in figure 14. It is important to observe that the partial derivatives of $\bar{x}$ and $\bar{y}$ with respect to x and y are constants for these three very special transformations. Moreover, the derivatives $\frac{\partial y}{\partial \bar{x}} = 0$, $\frac{\partial x}{\partial \bar{y}} = 0$, and the values of $\frac{\partial x}{\partial \bar{x}}$ and $\frac{\partial y}{\partial \bar{y}}$ are equal to either 1 or -1.

Furthermore, the inverses are given by $\frac{\partial x}{\partial \bar{x}} = \frac{\partial \bar{x}}{\partial x}$ and $\frac{\partial y}{\partial \bar{y}} = \frac{\partial \bar{y}}{\partial y}$. These simplifications are used subsequently to simplify the analysis.

Necessary Conditions

Each of the three transformations given by equations (12)-(14) corresponds to a reflective symmetry about a given plane that may be either symmetric or antisymmetric about the plane. If a buckle pattern exists that is symmetric about the symmetry plane, with respect to a given coordinate transformation, then the buckle pattern defined in both sets of coordinates satisfies the condition $w(\bar{x}, \bar{y}) = w(x, y)$. In contrast, if a buckle pattern exists that is antisymmetric about the symmetry plane, then the buckle pattern defined in both coordinate systems satisfies the condition $w(\bar{x}, \bar{y}) = -w(x, y)$. For convenience, both of these cases are represented herein by

$$w(\bar{x}, \bar{y}) = \mathscr{k}\, w(x, y) \tag{15}$$

where $\mathscr{k} = 1$ and $\mathscr{k} = -1$ for buckle patterns that are symmetric and antisymmetric about the plane of reflection symmetry, respectively. The conditions that are necessary for the existence of equation (15) are found by examining the transformed eigenvalue problem as follows.

First, equations (5)-(10) are expressed in terms of the $(\bar{x}, \bar{y})$ coordinates by simply replacing x with $\bar{x}$ and y with $\bar{y}$ to obtain expressions for the transformed eigenvalue problem. Then, the coordinate transformation given by equations (11) is introduced and the chain rule of differentiation is applied to get the transformed eigenvalue problem.

$$\mathbf{T}\left[\mathcal{L}\left(w(x, y)\right)\right] = \lambda\, \mathbf{T}\left[\mathcal{G}\left(w(x, y)\right)\right] \tag{16a}$$

for every $(x, y) \in \mathcal{D}$ and by

$$\mathbf{T}\left[\mathcal{B}_1\left(w(x, y)\right)\right] = \lambda\, \mathbf{T}\left[\mathcal{G}_1\left(w(x, y)\right)\right] \tag{16b}$$

and

$$\mathbf{T}\left[\mathcal{B}_2\left(w(x, y)\right)\right] = 0 \tag{16c}$$

for every $(x(s), y(s)) \in \mathcal{C}$. Likewise,

$$\mathbf{T}\left[\mathcal{M}_{ns}\left(w(x, y)\right)\right] = 0 \tag{16d}$$

for points of $\mathcal{C}$ where the derivatives are discontinuous. In these transformed equations,

$$\begin{aligned}\mathbf{T}\left[\mathcal{L}(\;)\right] = {} & \frac{\partial^2}{\partial x^2}\left(D_{11}(\bar{x}, \bar{y})\frac{\partial^2(\;)}{\partial x^2} + D_{12}(\bar{x}, \bar{y})\frac{\partial^2(\;)}{\partial y^2} + 2D_{16}(\bar{x}, \bar{y})\left(\frac{\partial x}{\partial \bar{x}}\right)^3 \frac{\partial y}{\partial \bar{y}}\,\frac{\partial^2(\;)}{\partial x \partial y}\right) \\ & + 2\frac{\partial^2}{\partial x \partial y}\left(D_{16}(\bar{x}, \bar{y})\left(\frac{\partial x}{\partial \bar{x}}\right)^3 \frac{\partial y}{\partial \bar{y}}\,\frac{\partial^2(\;)}{\partial x^2} + D_{26}(\bar{x}, \bar{y})\frac{\partial x}{\partial \bar{x}}\left(\frac{\partial y}{\partial \bar{y}}\right)^3 \frac{\partial^2(\;)}{\partial y^2} + 2D_{66}(\bar{x}, \bar{y})\frac{\partial^2(\;)}{\partial x \partial y}\right) \\ & + \frac{\partial^2}{\partial y^2}\left(D_{12}(\bar{x}, \bar{y})\frac{\partial^2(\;)}{\partial x^2} + D_{22}(\bar{x}, \bar{y})\frac{\partial^2(\;)}{\partial y^2} + 2D_{26}(\bar{x}, \bar{y})\frac{\partial x}{\partial \bar{x}}\left(\frac{\partial y}{\partial \bar{y}}\right)^3 \frac{\partial^2(\;)}{\partial x \partial y}\right)\end{aligned} \tag{17a}$$

$$\mathbf{T}\left[\mathcal{G}(\;)\right] = N_x^0(\bar{x}, \bar{y})\frac{\partial^2(\;)}{\partial x^2} + N_y^0(\bar{x}, \bar{y})\frac{\partial^2(\;)}{\partial y^2} + 2N_{xy}^0(\bar{x}, \bar{y})\frac{\partial x}{\partial \bar{x}}\frac{\partial y}{\partial \bar{y}}\,\frac{\partial^2(\;)}{\partial x \partial y} \tag{17b}$$

where the arguments of the stiffnesses and the prebuckling stress resultants are given in terms of (x, y) by equations (11). In obtaining these equations, it is noted that even powers of the derivatives $\frac{\partial x}{\partial \bar{x}}$ and $\frac{\partial y}{\partial \bar{y}}$ are equal to unity for the particular class of transformations considered in the present study. Likewise, the operators in equations (16b)-(16c) are transformed in two steps as follows. First, the partial derivatives in these operators are transformed to get

$$\begin{aligned}\mathbf{T}\left[\mathscr{M}_{n}(\ )\right] &= \left[n_x^2(\bar{x},\bar{y})D_{11}(\bar{x},\bar{y}) + n_y^2(\bar{x},\bar{y})D_{12}(\bar{x},\bar{y}) + 2n_x(\bar{x},\bar{y})\,n_y(\bar{x},\bar{y})D_{16}(\bar{x},\bar{y})\right]\frac{\partial^2(\ )}{\partial x^2}\\ &+\left[n_x^2(\bar{x},\bar{y})D_{12}(\bar{x},\bar{y}) + n_y^2(\bar{x},\bar{y})D_{22}(\bar{x},\bar{y}) + 2n_x(\bar{x},\bar{y})\,n_y(\bar{x},\bar{y})D_{26}(\bar{x},\bar{y})\right]\frac{\partial^2(\ )}{\partial y^2}\\ &+2\left[n_x^2(\bar{x},\bar{y})D_{16}(\bar{x},\bar{y}) + n_y^2(\bar{x},\bar{y})D_{26}(\bar{x},\bar{y}) + 2n_x(\bar{x},\bar{y})\,n_y(\bar{x},\bar{y})D_{66}(\bar{x},\bar{y})\right]\frac{\partial x}{\partial \bar{x}}\frac{\partial y}{\partial \bar{y}}\frac{\partial^2(\ )}{\partial x\partial y}\end{aligned} \tag{18}$$

$$\begin{aligned}\mathbf{T}\left[\mathscr{M}_{ns}(\ )\right] &= \left[n_x(\bar{x},\bar{y})n_y(\bar{x},\bar{y})\Big(D_{12}(\bar{x},\bar{y}) - D_{11}(\bar{x},\bar{y})\Big) + \Big(n_x^2(\bar{x},\bar{y}) - n_y^2(\bar{x},\bar{y})\Big)D_{16}(\bar{x},\bar{y})\right]\frac{\partial^2(\ )}{\partial x^2}\\ &+\left[n_x(\bar{x},\bar{y})n_y(\bar{x},\bar{y})\Big(D_{22}(\bar{x},\bar{y}) - D_{12}(\bar{x},\bar{y})\Big) + \Big(n_x^2(\bar{x},\bar{y}) - n_y^2(\bar{x},\bar{y})\Big)D_{26}(\bar{x},\bar{y})\right]\frac{\partial^2(\ )}{\partial y^2}\\ &+2\left[n_x(\bar{x},\bar{y})n_y(\bar{x},\bar{y})\Big(D_{26}(\bar{x},\bar{y}) - D_{16}(\bar{x},\bar{y})\Big) + D_{66}(\bar{x},\bar{y})\Big(n_x^2(\bar{x},\bar{y}) - n_y^2(\bar{x},\bar{y})\Big)\right]\frac{\partial x}{\partial \bar{x}}\frac{\partial y}{\partial \bar{y}}\frac{\partial^2(\ )}{\partial x\partial y}\end{aligned} \tag{19}$$

$$\begin{aligned}\mathbf{T}\left[\mathscr{Q}_{n}(\ )\right] &= n_x(\bar{x},\bar{y})\frac{\partial x}{\partial \bar{x}}\frac{\partial}{\partial x}\left[D_{11}(\bar{x},\bar{y})\frac{\partial^2(\ )}{\partial x^2} + D_{12}(\bar{x},\bar{y})\frac{\partial^2(\ )}{\partial y^2} + 2D_{16}(\bar{x},\bar{y})\frac{\partial x}{\partial \bar{x}}\frac{\partial y}{\partial \bar{y}}\frac{\partial^2(\ )}{\partial x\partial y}\right]\\ &+ n_y(\bar{x},\bar{y})\frac{\partial y}{\partial \bar{y}}\frac{\partial}{\partial y}\left[D_{12}(\bar{x},\bar{y})\frac{\partial^2(\ )}{\partial x^2} + D_{22}(\bar{x},\bar{y})\frac{\partial^2(\ )}{\partial y^2} + 2D_{26}(\bar{x},\bar{y})\frac{\partial x}{\partial \bar{x}}\frac{\partial y}{\partial \bar{y}}\frac{\partial^2(\ )}{\partial x\partial y}\right]\\ &+\left(n_x(\bar{x},\bar{y})\frac{\partial y}{\partial \bar{y}}\frac{\partial(\ )}{\partial y} + n_y(\bar{x},\bar{y})\frac{\partial x}{\partial \bar{x}}\frac{\partial(\ )}{\partial x}\right)\left[D_{16}(\bar{x},\bar{y})\frac{\partial^2(\ )}{\partial x^2} + D_{26}(\bar{x},\bar{y})\frac{\partial^2(\ )}{\partial y^2} + 2D_{66}(\bar{x},\bar{y})\frac{\partial x}{\partial \bar{x}}\frac{\partial y}{\partial \bar{y}}\frac{\partial^2(\ )}{\partial x\partial y}\right]\end{aligned} \tag{20}$$

For the displacement boundary condition $w = 0$,

$$\mathbf{T}\left[\mathscr{B}_1(\ )\right] = (\ ) \quad \text{and} \quad \mathbf{T}\left[\mathscr{G}_1(\ )\right] = 0 \tag{21a}$$

For the conjugate force boundary condition $V_n = 0$,

$$\mathbf{T}\left[\mathcal{B}_1(\;)\right] = \mathbf{T}\left[\mathcal{Q}_n(\;)\right] + \left(n_x(\bar{x}, \bar{y})\frac{\partial y}{\partial \bar{y}}\frac{\partial(\;)}{\partial y} - n_y(\bar{x}, \bar{y})\frac{\partial x}{\partial \bar{x}}\frac{\partial(\;)}{\partial x} \right)\mathbf{T}\left[\mathcal{M}_{ns}(\;)\right] \tag{21b}$$

and

$$\begin{aligned}\mathbf{T}\left[\mathcal{G}_1(\;)\right] = &\left(n_x(\bar{x}, \bar{y})P_n(\bar{x}, \bar{y}) - n_y(\bar{x}, \bar{y})P_s(\bar{x}, \bar{y}) \right)\frac{\partial x}{\partial \bar{x}}\frac{\partial(\;)}{\partial x} \\ &+ \left(n_x(\bar{x}, \bar{y})P_s(\bar{x}, \bar{y}) + n_y(\bar{x}, \bar{y})P_n(\bar{x}, \bar{y}) \right)\frac{\partial y}{\partial \bar{y}}\frac{\partial(\;)}{\partial y}\end{aligned} \tag{21c}$$

For the slope boundary condition $\frac{\partial w}{\partial n} = 0$,

$$\mathbf{T}\left[\mathcal{B}_2(\;)\right] = n_x(\bar{x}, \bar{y})\frac{\partial x}{\partial \bar{x}}\frac{\partial(\;)}{\partial x} + n_y(\bar{x}, \bar{y})\frac{\partial y}{\partial \bar{y}}\frac{\partial(\;)}{\partial y} \tag{22a}$$

and for the conjugate moment boundary condition $M_n = 0$,

$$\mathbf{T}\left[\mathcal{B}_2(\;)\right] = \mathbf{T}\left[\mathcal{M}_n(\;)\right] \tag{22b}$$

In these equations for the transformed boundary conditions, $n_x(\bar{x}, \bar{y})$ and $n_y(\bar{x}, \bar{y})$ are the components of the unit-magnitude vector field normal at points of plate boundary $\mathcal{C}$ given by $\bar{x} = \bar{x}(x(s), y(s)) = \bar{x}(s)$ and $\bar{y} = \bar{y}(x(s), y(s)) = \bar{y}(s)$. The second step in the transformation process deals with transforming these components. Specifically, the relationships between the components of the unit-magnitude normal-vector fields are obtained by applying the chain rule of differentiation to the definitions $n_x(x, y) = dy/ds$, and $n_y(x, y) = - dx/ds$. This process gives

$$n_x(x, y) = n_x(\bar{x}, \bar{y})\frac{\partial y}{\partial \bar{y}} \quad \text{or} \quad n_x(\bar{x}, \bar{y}) = n_x(x, y)\frac{\partial \bar{y}}{\partial y} \tag{23a}$$

$$n_y(x, y) = n_y(\bar{x}, \bar{y})\frac{\partial x}{\partial \bar{x}} \quad \text{or} \quad n_y(\bar{x}, \bar{y}) = n_y(x, y)\frac{\partial \bar{x}}{\partial x} \tag{23b}$$

for the special class of coordinate transformations considered herein. Applying equations (23) to equations (18)-(20) yields

$$\mathbf{T}\left[\mathcal{M}_n(\ )\right] = \left[n_x^2(x,y)D_{11}(\bar{x},\bar{y}) + n_y^2(x,y)D_{12}(\bar{x},\bar{y}) + 2n_x(x,y)n_y(x,y)\frac{\partial\bar{x}}{\partial x}\frac{\partial\bar{y}}{\partial y}D_{16}(\bar{x},\bar{y})\right]\frac{\partial^2(\ )}{\partial x^2}$$
$$+ \left[n_x^2(x,y)D_{12}(\bar{x},\bar{y}) + n_y^2(x,y)D_{22}(\bar{x},\bar{y}) + 2n_x(x,y)n_y(x,y)\frac{\partial\bar{x}}{\partial x}\frac{\partial\bar{y}}{\partial y}D_{26}(\bar{x},\bar{y})\right]\frac{\partial^2(\ )}{\partial y^2} \tag{24}$$
$$+ 2\left[n_x^2(x,y)\frac{\partial x}{\partial\bar{x}}\frac{\partial y}{\partial\bar{y}}D_{16}(\bar{x},\bar{y}) + n_y^2(x,y)\frac{\partial x}{\partial\bar{x}}\frac{\partial y}{\partial\bar{y}}D_{26}(\bar{x},\bar{y}) + 2n_x(x,y)n_y(x,y)D_{66}(\bar{x},\bar{y})\right]\frac{\partial^2(\ )}{\partial x\partial y}$$

$$\mathbf{T}\left[\mathcal{M}_{ns}(\ )\right] = \left[n_x(x,y)n_y(x,y)\frac{\partial\bar{x}}{\partial x}\frac{\partial\bar{y}}{\partial y}\Big(D_{12}(\bar{x},\bar{y}) - D_{11}(\bar{x},\bar{y})\Big) + \Big(n_x^2(x,y) - n_y^2(x,y)\Big)D_{16}(\bar{x},\bar{y})\right]\frac{\partial^2(\ )}{\partial x^2}$$
$$+ \left[n_x(x,y)n_y(x,y)\frac{\partial\bar{x}}{\partial x}\frac{\partial\bar{y}}{\partial y}\Big(D_{22}(\bar{x},\bar{y}) - D_{12}(\bar{x},\bar{y})\Big) + \Big(n_x^2(x,y) - n_y^2(x,y)\Big)D_{26}(\bar{x},\bar{y})\right]\frac{\partial^2(\ )}{\partial y^2} \tag{25}$$
$$+ 2\left[n_x(x,y)n_y(x,y)\Big(D_{26}(\bar{x},\bar{y}) - D_{16}(\bar{x},\bar{y})\Big) + \frac{\partial x}{\partial\bar{x}}\frac{\partial y}{\partial\bar{y}}D_{66}(\bar{x},\bar{y})\Big(n_x^2(x,y) - n_y^2(x,y)\Big)\right]\frac{\partial^2(\ )}{\partial x\partial y}$$

$$\mathbf{T}\left[\mathcal{Q}_n(\ )\right] = n_x(x,y)\frac{\partial x}{\partial\bar{x}}\frac{\partial\bar{y}}{\partial y}\frac{\partial}{\partial x}\left[D_{11}(\bar{x},\bar{y})\frac{\partial^2(\ )}{\partial x^2} + D_{12}(\bar{x},\bar{y})\frac{\partial^2(\ )}{\partial y^2} + 2D_{16}(\bar{x},\bar{y})\frac{\partial x}{\partial\bar{x}}\frac{\partial y}{\partial\bar{y}}\frac{\partial^2(\ )}{\partial x\partial y}\right]$$
$$+ n_y(x,y)\frac{\partial\bar{x}}{\partial x}\frac{\partial y}{\partial\bar{y}}\frac{\partial}{\partial y}\left[D_{12}(\bar{x},\bar{y})\frac{\partial^2(\ )}{\partial x^2} + D_{22}(\bar{x},\bar{y})\frac{\partial^2(\ )}{\partial y^2} + 2D_{26}(\bar{x},\bar{y})\frac{\partial x}{\partial\bar{x}}\frac{\partial y}{\partial\bar{y}}\frac{\partial^2(\ )}{\partial x\partial y}\right]$$
$$+ \frac{\partial x}{\partial\bar{x}}\frac{\partial y}{\partial\bar{y}}\left(n_x(x,y)\frac{\partial(\ )}{\partial y} + n_y(x,y)\frac{\partial(\ )}{\partial x}\right) \times \tag{26}$$
$$\left[\frac{\partial\bar{x}}{\partial x}\frac{\partial\bar{y}}{\partial y}\left(D_{16}(\bar{x},\bar{y})\frac{\partial^2(\ )}{\partial x^2} + D_{26}(\bar{x},\bar{y})\frac{\partial^2(\ )}{\partial y^2}\right) + 2D_{66}(\bar{x},\bar{y})\frac{\partial^2(\ )}{\partial x\partial y}\right]$$

For the force boundary condition $V_n = 0$, equations (21b) and (21c) become

$$\mathbf{T}\left[\mathcal{B}_1(\ )\right] = \mathbf{T}\left[\mathcal{Q}_n(\ )\right] + \left(n_x(x,y)\frac{\partial(\ )}{\partial y} - n_y(x,y)\frac{\partial(\ )}{\partial x}\right)\mathbf{T}\left[\mathcal{M}_{ns}(\ )\right] \tag{27a}$$

and

$$\mathbf{T}\left[\mathcal{G}_1(\ )\right] = \left(n_x(x, y)\frac{\partial \bar{y}}{\partial y}\frac{\partial x}{\partial \bar{x}}\, P_n(\bar{x}, \bar{y}) - n_y(x, y)\, P_s(\bar{x}, \bar{y})\right)\frac{\partial(\ )}{\partial x} + \left(n_x(x, y)\, P_s(\bar{x}, \bar{y}) + n_y(x, y)\frac{\partial \bar{x}}{\partial x}\frac{\partial y}{\partial \bar{y}}\, P_n(\bar{x}, \bar{y})\right)\frac{\partial(\ )}{\partial y} \tag{27b}$$

For the slope boundary condition $\frac{\partial w}{\partial n} = 0,$ equation (22a) becomes

$$\mathbf{T}\left[\mathcal{B}_2(\ )\right] = n_x(x, y)\frac{\partial \bar{y}}{\partial y}\frac{\partial x}{\partial \bar{x}}\frac{\partial(\ )}{\partial x} + n_y(x, y)\frac{\partial \bar{x}}{\partial x}\frac{\partial y}{\partial \bar{y}}\frac{\partial(\ )}{\partial y} \tag{28}$$

The *necessary conditions* that correspond to the existence of a buckle pattern that is either symmetric or antisymmetric about a symmetry plane are obtained by enforcing invariance of the eigenvalue problem under the transformations given by equations (11). In other words, if equation (15) is valid, as presumed, the two corresponding eigenvalue problems must be identical. Specifically, invariance of the differential equation is given by

$$\mathbf{T}\left[\mathcal{L}\big(w(x, y)\big) - \lambda\mathcal{G}\big(w(x, y)\big)\right] = \mathcal{L}\big(w(x, y)\big) - \lambda\,\mathcal{G}\big(w(x, y)\big) \tag{29}$$

and yields the *necessary conditions*

$$D_{11}(\bar{x}, \bar{y}) = D_{11}(x, y) \tag{30a}$$

$$D_{12}(\bar{x}, \bar{y}) = D_{12}(x, y) \tag{30b}$$

$$D_{16}(\bar{x}, \bar{y})\left(\frac{\partial x}{\partial \bar{x}}\right)^3\left(\frac{\partial y}{\partial \bar{y}}\right) = D_{16}(x, y) \tag{30c}$$

$$D_{22}(\bar{x}, \bar{y}) = D_{22}(x, y) \tag{30d}$$

$$D_{26}(\bar{x}, \bar{y})\frac{\partial x}{\partial \bar{x}}\left(\frac{\partial y}{\partial \bar{y}}\right)^3 = D_{26}(x, y) \tag{30e}$$

$$D_{66}(\bar{x}, \bar{y}) = D_{66}(x, y) \tag{30f}$$

on the stiffnesses and

$$N_x^0(\bar{x}, \bar{y}) = N_x^0(x, y) \tag{31a}$$

$$N_y^0(\bar{x}, \bar{y}) = N_y^0(x, y) \tag{31b}$$

$$N_{xy}^0(\bar{x}, \bar{y})\left(\frac{\partial x}{\partial \bar{x}}\right)\left(\frac{\partial y}{\partial \bar{y}}\right) = N_{xy}^0(x, y) \tag{31c}$$

on the prebuckling stress resultants, for all values of (x, y) in the domain $\mathcal{D}$.

As for invariance of the boundary conditions, consider the general case in which $w(x, y) = 0$ for an arbitrary subset of the plate boundary denoted by $\mathcal{C}_\Delta \subset \mathcal{C}$, as depicted in figures 13 and 14. In general, $\mathcal{C}_\Delta$ consists of a finite number of smooth disjoint boundary segments. Applying the coordinate transformation given by equations (11) to this displacement constraint, and enforcing equation (15), yields the *necessary condition* that $w(x, y) = 0$ on points of the plate boundary given by $\mathbf{T}[\mathcal{C}_\Delta]$. Similarly, if $\frac{\partial w(x, y)}{\partial n} = 0$ is specified on a given subset of the plate boundary $\mathcal{C}_\delta \subset \mathcal{C}$, then the transformation of $\frac{\partial w(\bar{x}, \bar{y})}{\partial n} = 0$ that is obtained from equations (16c) and (27), and enforcement of equation (15), yields the *necessary condition* that $\frac{\partial w(x, y)}{\partial n} = 0$ on $\mathbf{T}[\mathcal{C}_\delta]$, for the class of transformations considered herein.

Next, consider invariance of the boundary condition $M_n(x, y) = 0$, given by equations (7c) and (9c), for an arbitrary subset of the plate boundary denoted by $\mathcal{C}_M \subset \mathcal{C}$. Invariance of the corresponding transformed boundary condition, given by equations (16c), (22b) and (24), yields the *necessary conditions* given by equations (28a), (28b), (28d), (28f), and

$$D_{16}(\bar{x}, \bar{y})\frac{\partial \bar{x}}{\partial x}\frac{\partial \bar{y}}{\partial y} = D_{16}(x, y) \tag{32a}$$

$$D_{26}(\bar{x}, \bar{y})\frac{\partial \bar{x}}{\partial x}\frac{\partial \bar{y}}{\partial y} = D_{26}(x, y) \tag{32b}$$

Equations (32a) and (32b) are effectively the same as equations (30c) and (30e), respectively, for the class of transformations considered herein. Applying these necessary conditions to equation (24) gives

$$\mathbf{T}\left[\mathcal{M}_n(\ )\right] = \mathcal{M}_n(\ ) \tag{33a}$$

Thus, the transformed boundary condition yields the additional necessary condition that

$M_n(x, y) = 0$ be specified on $\mathbf{T}[\mathscr{C}_M]$.

Now, consider the invariance of the operator $\mathscr{M}_{ns}(\ )$ that is given by equations (7b). Substituting the necessary conditions given by equations (28a), (28b), (28d), (28f), and (30) into the corresponding transformed quantity, given by equation (25), yields

$$\mathbf{T}\left[\mathscr{M}_{ns}(\ )\right] = \frac{\partial \bar{x}}{\partial x}\frac{\partial \bar{y}}{\partial y}\mathscr{M}_{ns}(\ ) \tag{33b}$$

Likewise, equation (26) becomes

$$\mathbf{T}\left[\mathscr{Q}_n(\ )\right] = \frac{\partial \bar{x}}{\partial x}\frac{\partial \bar{y}}{\partial y}\mathscr{Q}_n(\ ) \tag{34}$$

Next, substituting equations (33b) and (34) into equation (27a) yields

$$\mathbf{T}\left[\mathscr{B}_1(\ )\right] = \frac{\partial \bar{x}}{\partial x}\frac{\partial \bar{y}}{\partial y}\mathscr{B}_1(\ ) \tag{35}$$

Moreover, comparing equations (8d) and (27b) indicates that invariance of the operator $\mathscr{G}_1(\ )$ yields the following *necessary conditions* on the applied loads

$$P_n(x, y) = P_n(\bar{x}, \bar{y}) \tag{36a}$$

and

$$P_s(x, y) = \frac{\partial \bar{x}}{\partial x}\frac{\partial \bar{y}}{\partial y}P_s(\bar{x}, \bar{y}) \tag{36b}$$

Enforcing these necessary conditions in equation (27b) yields

$$\mathbf{T}\left[\mathscr{G}_1(\ )\right] = \frac{\partial \bar{x}}{\partial x}\frac{\partial \bar{y}}{\partial y}\mathscr{G}_1(\ ) \tag{37}$$

Now, consider the boundary condition $V_n(x, y) = 0$, given by equations (8a), (8c), and (8d), for an arbitrary subset of the plate boundary denoted by $\mathscr{C}_V \subset \mathscr{C}$. Invariance requirements for this boundary condition are obtained by substituting equation (15) into equation (16b) and making use of equations (35) and (37). This process yields the additional *necessary condition* that $V_n(x, y) = 0$ on $\mathbf{T}[\mathscr{C}_V]$.

Lastly, consider invariance of the boundary condition $M_{ns}(x, y) = 0$, given by equations (7b) and (7e), for N points of the boundary with sharp corners, denoted by $\mathcal{P}_\kappa \in \mathcal{C}$, for k = 1, 2, ..., N. From equation (33), it is seen that invariance of this boundary condition produces the additional *necessary condition* that $M_{ns}(x, y) = 0$ on $\mathbf{T}[\mathcal{P}_\kappa]$, for k = 1, 2, ..., N.

Sufficient Conditions

In the previous section, a prescribed symmetry was presumed to exist and the conditions that are necessary for the eigenvalue problem to be invariant under a corresponding coordinate transformation were detemined. In this section, the following question is addressed: "*If the eigenvalue problem is invariant under a specified coordinate transformation, does the invariance imply that the eigenfunctions exhibit the symmetry inherent to the coordinate transformation?*" If the answer to this question is yes, then the necessary conditions for invariance of the eigenvalue problem are also sufficient conditions for the existence of the corresponding symmetry.

To obtain the sufficient conditions for a buckle pattern to possess one of the symmetries described herein, it is presumed that the eigenvalue problem is invariant under the action of a coordinate transformation of the general form given by equations (11). Let the buckle pattern $w(x, y)$ be a solution to the original eigenvalue problem for a given eigenvalue, and let $w(\bar{x}, \bar{y})$ be the solution to the corresponding transformed eigenvalue problem, for the same eigenvalue. Because of the presumed invariance, both eigenfunctions are solutions to the original eigenvalue problem, for a given eigenvalue. Moreover, both eigenvalue problems possess the same spectrum of eigenfunctions and, as a result, there exists a pair of eigenfunctions from the two identical eigenvalue problems, for a given eigenvalue, that possesses a unique correspondance. Additionally, since the amplitude of an eigenfunction is indeterminate, it follows that every corresponding pair of eigenfunctions $w(x, y)$ and $w(\bar{x}, \bar{y})$ can differ only in sign and magnitude. These differences are represented by a multiplicative real constant; that is,

$$w(\bar{x}, \bar{y}) = \mathcal{k}\, w(x, y) \tag{38}$$

To have complete equivalence of the eigenfunctions appearing in this equation, $\mathcal{k} w(x, y)$ and $w(\bar{x}, \bar{y})$ are scaled to have the same maximum amplitude. Thus, the constant $\mathcal{k}$ has admissible values of +1 and -1 and, as a result, equation (38) corresponds to equation (15) that was used to determine the necessary conditions. This correspondance between equations (15) and (38), implies that the necessary conditions given previously for the existence of symmetries are also *sufficient conditions*.

Up to this point in the analysis, the functions that define the bending stiffnesses have been presumed to be differentiable on the domain $\mathcal{D}$. However, for laminated plates that are tailored in a patchwork fashion, the bending-stiffness functions are sectionally differentiable. For these cases, the necessary and sufficient conditions for a buckle pattern to possess one of the symmetries described herein are found by partitioning the domain $\mathcal{D}$ into a finite number of subdomains and then enforcing the symmetry requirements for each subdomain and its boundary.

This process places additional conditions on the distribution of the patchwork bending stiffnesses that must be satisfied.

The Three Specific Cases

The general necessary and sufficient conditions for the existence of a buckle pattern that is either symmetric or antisymmetric about a plane of reflective symmetry were derived in the previous section of the present study. The corresponding results for the three specific cases defined by equations (12)-(14) are presented subsequently.

One Plane of Reflective Symmetry at x = 0

Consider the case in which the plane $x = 0$ is a proposed plane of reflective symmetry, as depicted in figure 2. The coordinate transformation used herein to characterize this type of symmetry is given by equation (12). For this transformations, equations (30) give

$$D_{11}(-x, y) = D_{11}(x, y) \tag{39a}$$

$$D_{12}(-x, y) = D_{12}(x, y) \tag{39b}$$

$$D_{16}(-x, y) = -D_{16}(x, y) \tag{39c}$$

$$D_{22}(-x, y) = D_{22}(x, y) \tag{39d}$$

$$D_{26}(-x, y) = -D_{26}(x, y) \tag{39e}$$

$$D_{66}(-x, y) = D_{66}(x, y) \tag{39f}$$

on the stiffnesses and

$$N_x^0(-x, y) = N_x^0(x, y) \tag{40a}$$

$$N_y^0(-x, y) = N_y^0(x, y) \tag{40b}$$

$$N_{xy}^0(-x, y) = -N_{xy}^0(x, y) \tag{40c}$$

on the prebuckling stress resultants. Likewise, the inplane loads given by equations (36) must obey

$$P_n(-x, y) = P_n(x, y) \tag{41a}$$

$$P_s(-x, y) = -P_s(x, y) \tag{41b}$$

In addition, the transformation requires that for all $(x(s), y(s)) \in \mathcal{C}$, the condition $(-x(s), y(s)) \in \mathcal{C}$, must hold, as illustrated in figure 13. As shown in this figure, a cutout placed on the y-axis must be symmetric about this axis, and a cutout located off the y-axis must have a correspond cutout on the other side of the axis with the appropriate symmetry. Moreover, for any boundary condition specified over all or parts of $\mathcal{C}$, with coordinates $(x(s), y(s))$, it must also be specified over the corresponding parts of $\mathcal{C}$, with coordinates $(-x(s), y(s))$. It is also important to note than when the bending stiffnesses are constants, $D_{16} = D_{26} = 0$ must hold. Each of these necessary and sufficient conditions are completely consistent with Glockner's symmetry axiom as applied in references 61-64.

Two Planes of Reflective Symmetry

Consider the case in which $y = 0$ is a reflective-symmetry plane, in addition to the plane $x = 0$. Because the plane $x = 0$ is taken as a pre-existing symmetry plane, the corresponding necessary and sufficient conditions given in the previous section must be enforced prior to applying the transformation used to characterize the plane $y = 0$ as a reflective-symmetry plane. The coordinate transformation used herein to characterize symmetries with respect to the plane $y = 0$ is given by equation (13). For this transformations, and the previous set of necessary and sufficient conditions, equations (30) give

$$D_{11}(-x, y) = D_{11}(x, y) \quad \text{and} \quad D_{11}(x, -y) = D_{11}(x, y) \tag{42a}$$

$$D_{12}(-x, y) = D_{12}(x, y) \quad \text{and} \quad D_{12}(x, -y) = D_{12}(x, y) \tag{42b}$$

$$D_{16}(-x, y) = -D_{16}(x, y) \quad \text{and} \quad D_{16}(x, -y) = -D_{16}(x, y) \tag{42c}$$

$$D_{22}(-x, y) = D_{22}(x, y) \quad \text{and} \quad D_{22}(x, -y) = D_{22}(x, y) \tag{42d}$$

$$D_{26}(-x, y) = -D_{26}(x, y) \quad \text{and} \quad D_{26}(x, -y) = -D_{26}(x, y) \tag{42e}$$

$$D_{66}(-x, y) = D_{66}(x, y) \quad \text{and} \quad D_{66}(x, -y) = D_{66}(x, y) \tag{42f}$$

on the stiffnesses and

$$N_x^0(-x, y) = N_x^0(x, y) \quad \text{and} \quad N_x^0(x, -y) = N_x^0(x, y) \tag{43a}$$

$$N_y^0(-x, y) = N_y^0(x, y) \quad \text{and} \quad N_y^0(x, -y) = N_y^0(x, y) \tag{43b}$$

$$N_{xy}^0(-x, y) = -N_{xy}^0(x, y) \quad \text{and} \quad N_{xy}^0(x, -y) = -N_{xy}^0(x, y) \tag{43c}$$

on the prebuckling stress resultants. Likewise, the inplane loads given by equations (36) must obey

$$P_n(-x, y) = P_n(x, y) \quad \text{and} \quad P_n(x, -y) = P_n(x, y) \tag{44a}$$

$$P_s(-x, y) = -P_s(x, y) \quad \text{and} \quad P_s(x, -y) = -P_s(x, y) \tag{44b}$$

In addition, the transformation requires that for all $(x(s), y(s)) \in \mathcal{C}$, the conditions $(-x(s), y(s)) \in \mathcal{C}$ and $(x(s), -y(s)) \in \mathcal{C}$ must hold. Thus, a cutout placed on the x-axis must be symmetric about this axis, and a cutout located off the x-axis must have a correspond cutout on the other side of the axis with the appropriate symmetry. Moreover, for any boundary condition specified over all or parts of $\mathcal{C}$, with coordinates $(x(s), y(s))$, it must also be specified over the corresponding parts of $\mathcal{C}$, with coordinates $(-x(s), y(s))$ and $(x(s), -y(s))$. Each of these necessary and sufficient conditions are also completely consistent with Glockner's symmetry axiom as applied in references 61-64.

A Central Point of Inversion Symmetry or Antisymmetry

Consider the case in which the origin of the coordiinates is a proposed point of inversion symmetry or antisymmetry, as depicted in figures 10 and 11. The coordinate transformation used herein to characterize this type of symmetry is given by equation (14). For this transformation, equations (30) give

$$D_{11}(-x, -y) = D_{11}(x, y) \tag{45a}$$

$$D_{12}(-x, -y) = D_{12}(x, y) \tag{45b}$$

$$D_{16}(-x, -y) = D_{16}(x, y) \tag{45c}$$

$$D_{22}(-x, -y) = D_{22}(x, y) \tag{45d}$$

$$D_{26}(-x, -y) = D_{26}(x, y) \tag{45e}$$

$$D_{66}(-x, -y) = D_{66}(x, y) \tag{45f}$$

on the stiffnesses and

$$N_x^0(-x, -y) = N_x^0(x, y) \tag{46a}$$

$$N_y^0(-x, -y) = N_y^0(x, y) \tag{46b}$$

$$N_{xy}^0(-x, -y) = N_{xy}^0(x, y) \tag{46c}$$

on the prebuckling stress resultants. Likewise, the inplane loads given by equations (36) must obey

$$P_n(-x, -y) = P_n(x, y) \tag{47a}$$

$$P_s(-x, -y) = P_s(x, y) \tag{47b}$$

In addition, the transformation requires that for all $(x(s), y(s)) \in \mathcal{C}$, the condition $(-x(s), -y(s)) \in \mathcal{C}$ must hold, as illustrated in figure 14. As shown in this figure, a cutout placed at the origin must have a central point of inversion symmetry, and a cutout located away from the origin must have a correspond cutout that produce a domain with the appropriate inversion symmetry. Moreover, for any boundary condition specified over all or parts of $\mathcal{C}$, with coordinates $(x(s), y(s))$, it must also be specified over the corresponding parts of $\mathcal{C}$, with coordinates $(-x(s), -y(s))$. Each of these necessary and sufficient conditions are also completely consistent with Glockner's symmetry axiom as applied in references 61-64.

Concluding Remarks

Necessary and sufficient conditions for the existence of certain symmetries to exist in the buckle patterns of symmetrically laminated composite plates have been derived. The plates considered have a general planform with cutouts, variable thickness and stiffnesses, and general support and loading conditions. The specific form of the symmetry conditions are based on the invariance properties of the corresponding eigenvalue problem for a group of coordinate transformations associated with buckle patterns commonly exhibited by symmetrically laminated plates. The buckle-pattern symmetries examined include a central point of inversion symmetry, one plane of reflective symmetry, and two planes of reflective symmetry. The necessary and sufficient conditions presented herein are consistent with Glockner's axiom of symmetry. Although only symmetrically laminated plates have been considered in the present study, the analysis presented herein is applicable to generally laminates plates and shells.

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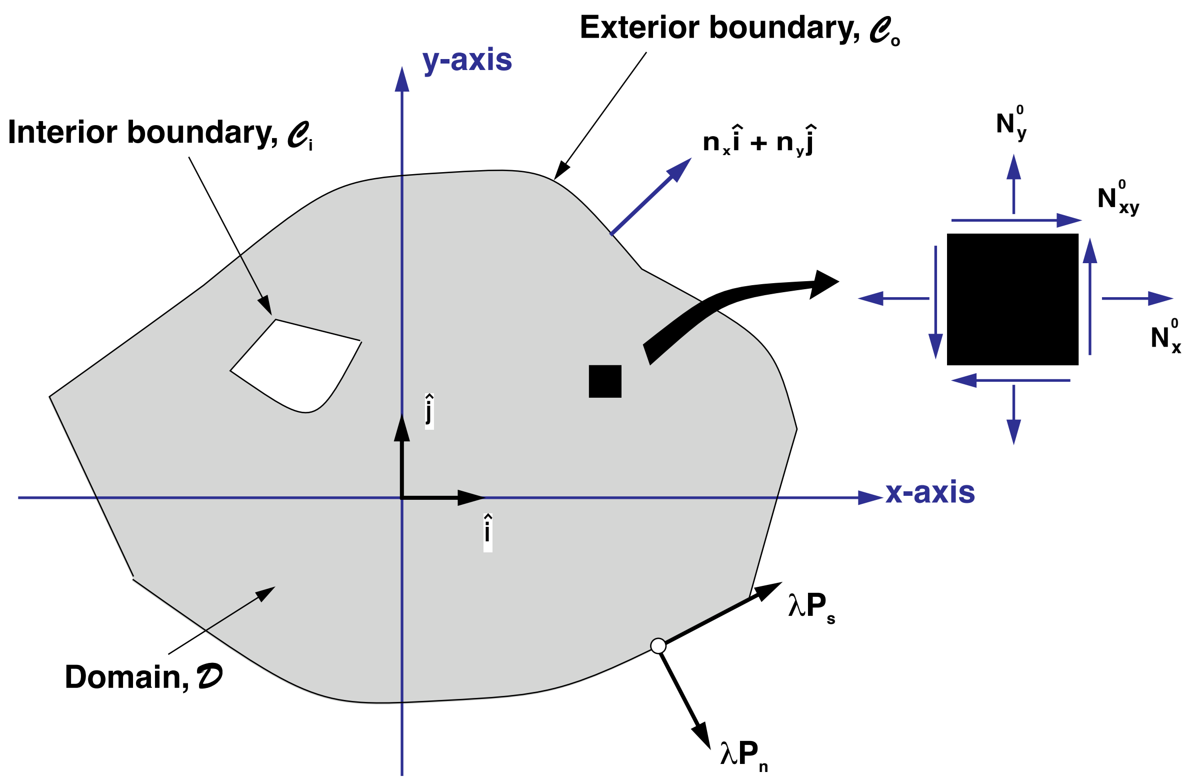

Figure 1. Plate middle surface, coordinate system, prebuckling stress resultants, and edge loads.

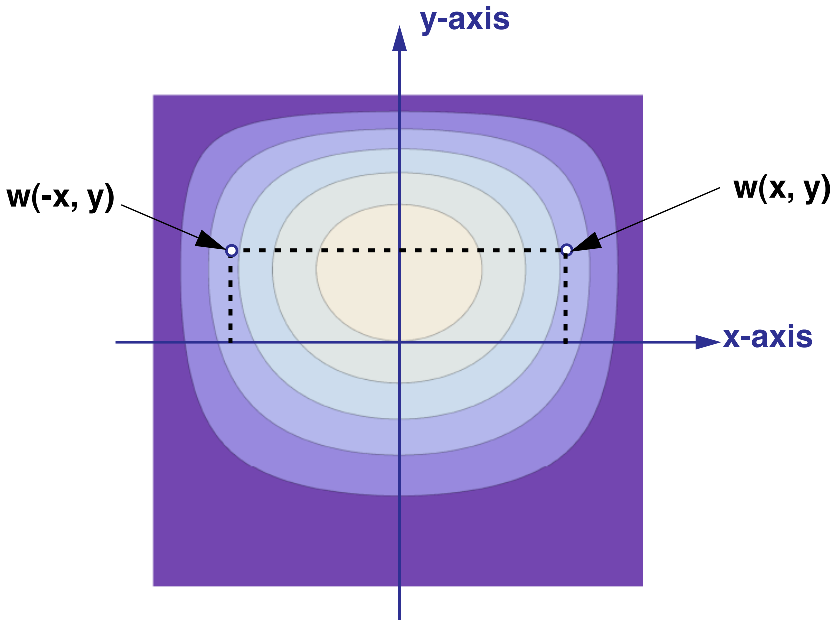

Figure 2. Buckle pattern with one plane of reflective symmetry given by w(-x, y) = w(x, y).

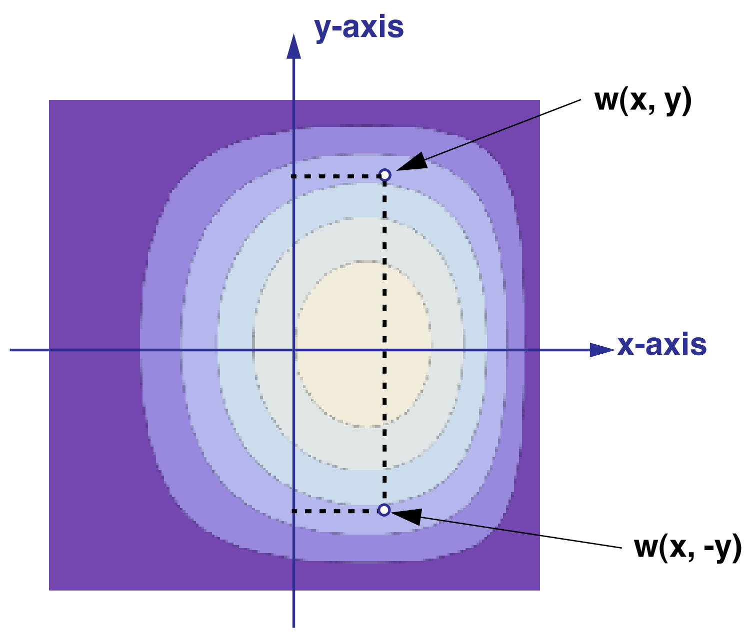

Figure 3. Buckle pattern with one plane of reflective symmetry given by w(x, -y) = w(x, y).

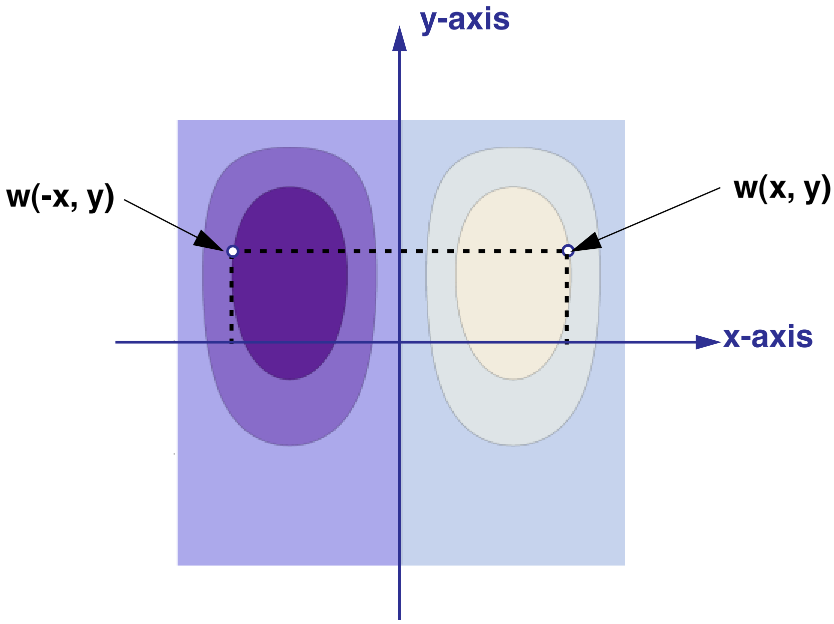

Figure 4. Buckle pattern with one plane of reflective antisymmetry given by w(-x, y) = -w(x, y).

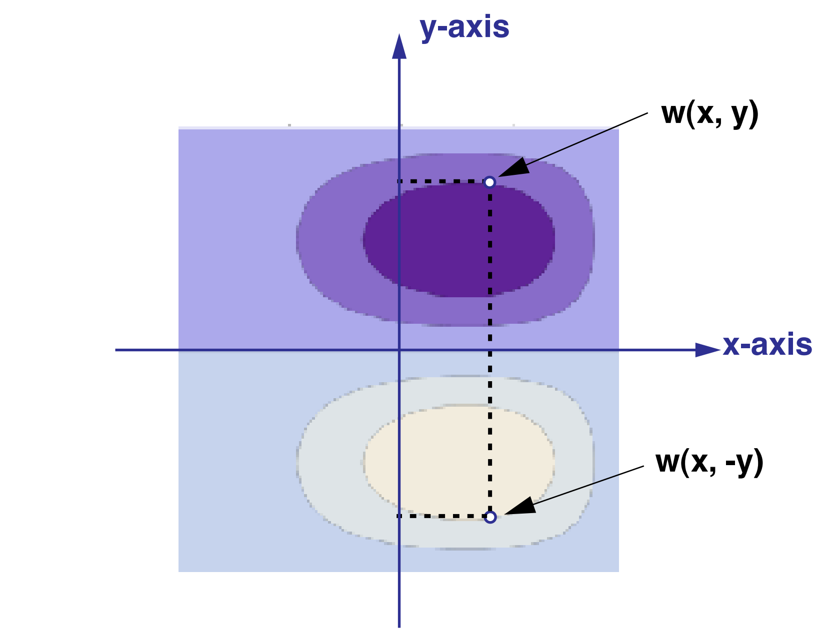

Figure 5. Buckle pattern with one plane of reflective antisymmetry given by w(x, -y) = -w(x, y).

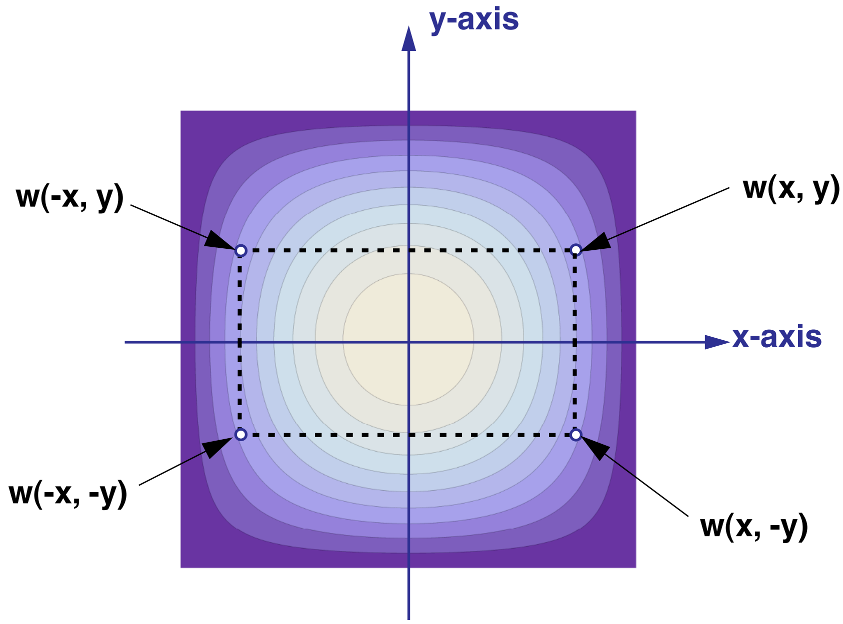

Figure 6. Buckle pattern with two planes of reflective symmetry given by w(-x, y) = w(x, y) and w(x,-y) = w(x, y).

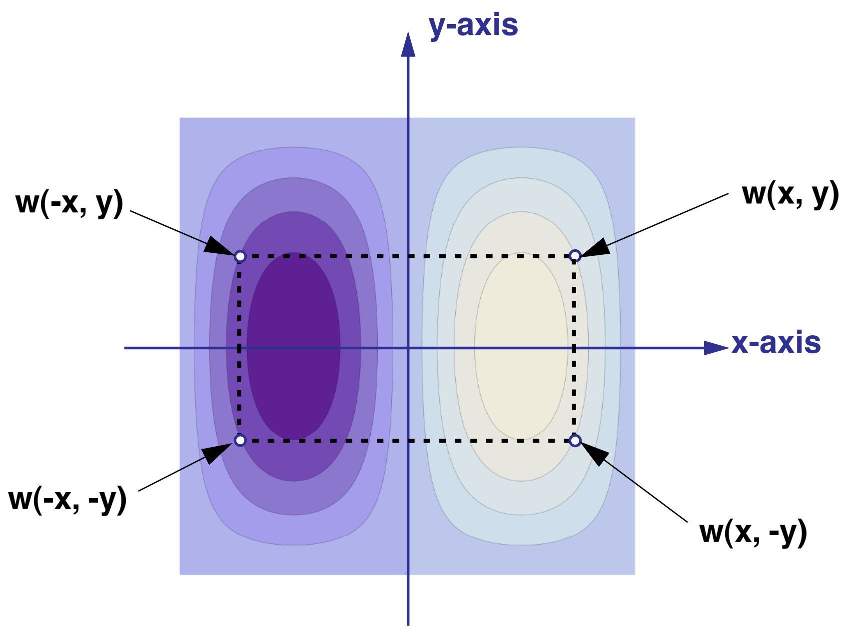

Figure 7. Buckle pattern with one plane of reflective symmetry given by w(x,-y) = w(x,y) and one plane of reflective antisymmetry given by w(-x, y) = -w(x, y).

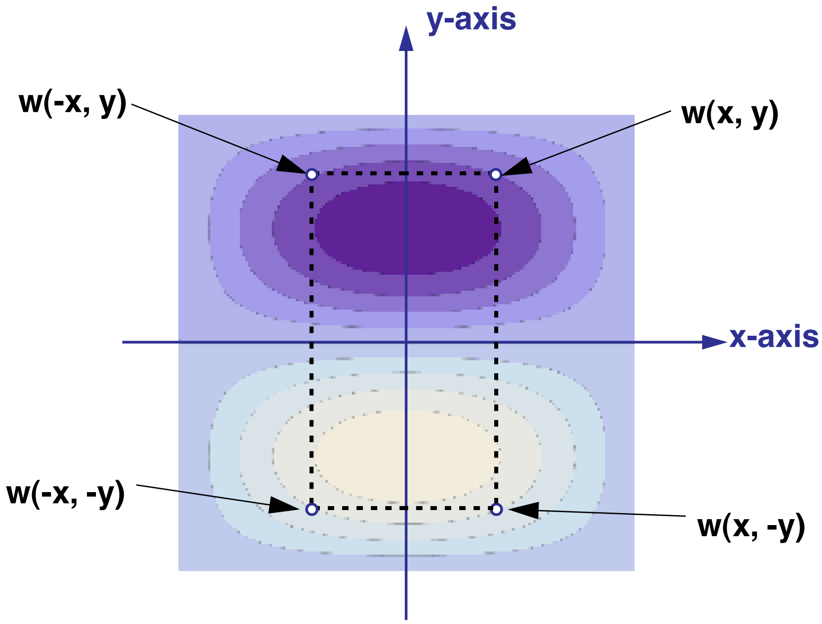

Figure 8. Buckle pattern with one plane of reflective symmetry given by w(-x, y) = w(x, y) and one plane of reflective antisymmetry given by w(x, -y) = -w(x, y).

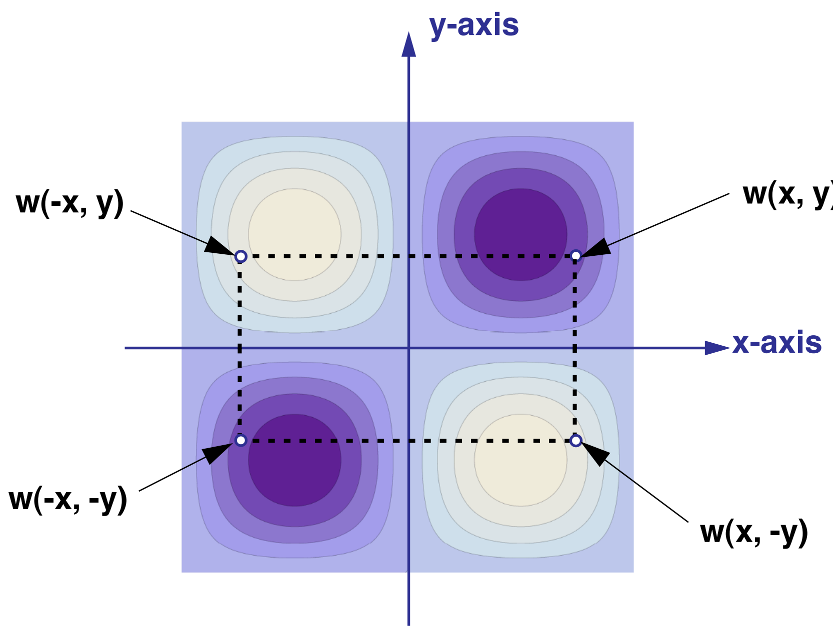

Figure 9. Buckle pattern with two planes of reflective antisymmetry given by w(-x, y) = -w(x, y) and w(x, -y) = -w(x, y).

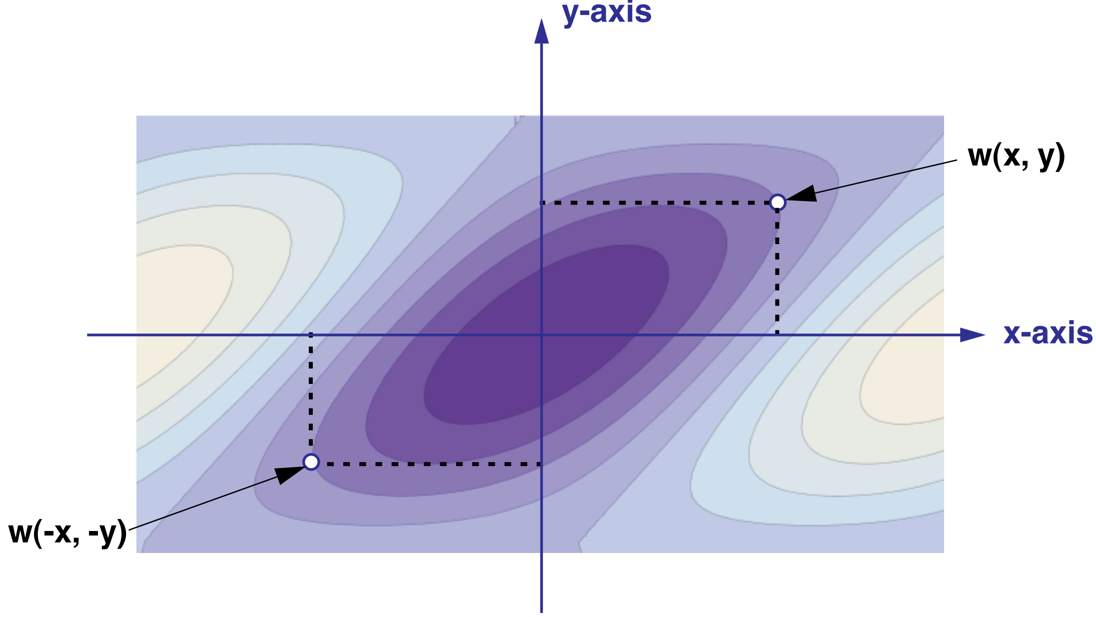

Figure 10. Buckle pattern with a central point of inversion symmetry given by $w(-x, -y) = w(x, y)$.

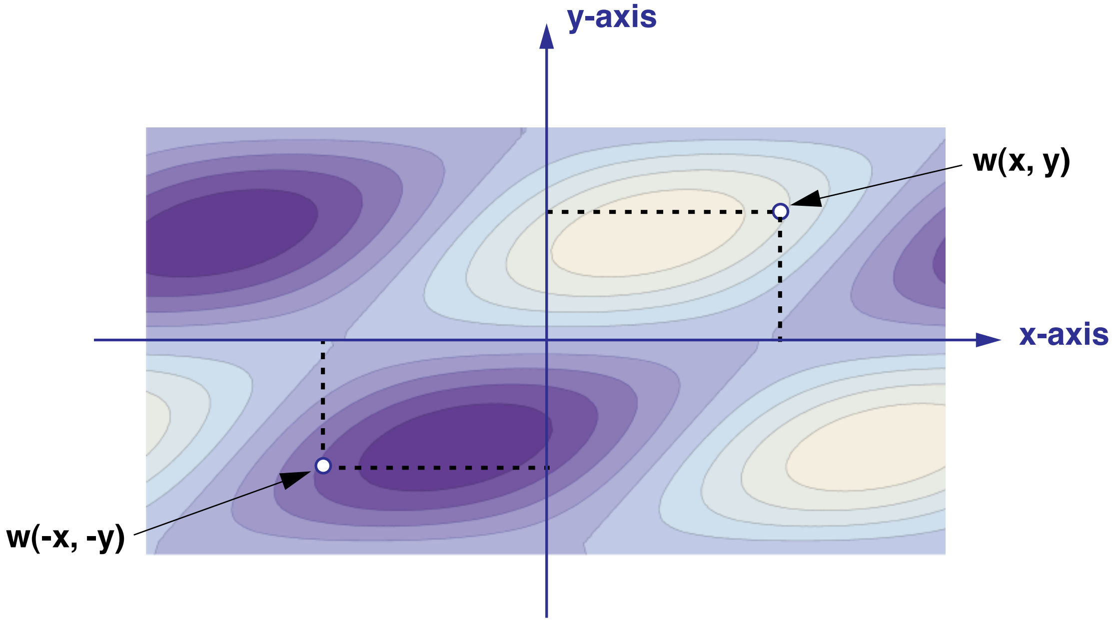

Figure 11. Buckle pattern with a central point of inversion antisymmetry given by $w(-x, -y) = -w(x, y)$.

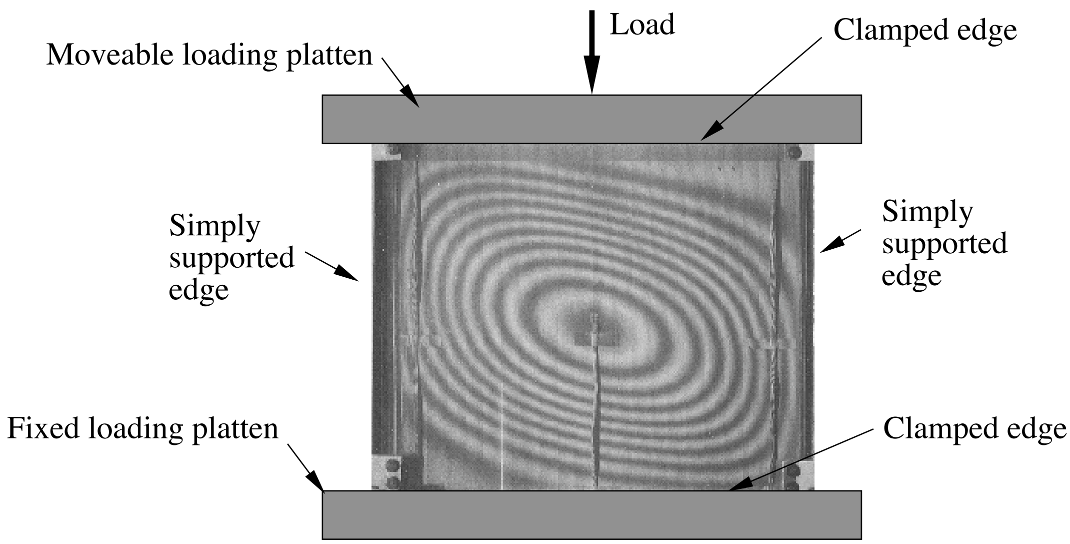

Figure 12. Buckle pattern of a compression-loaded $[+60_6/-60_6]_s$ laminated-composite plate made of a typical high-strength graphite-epoxy material.

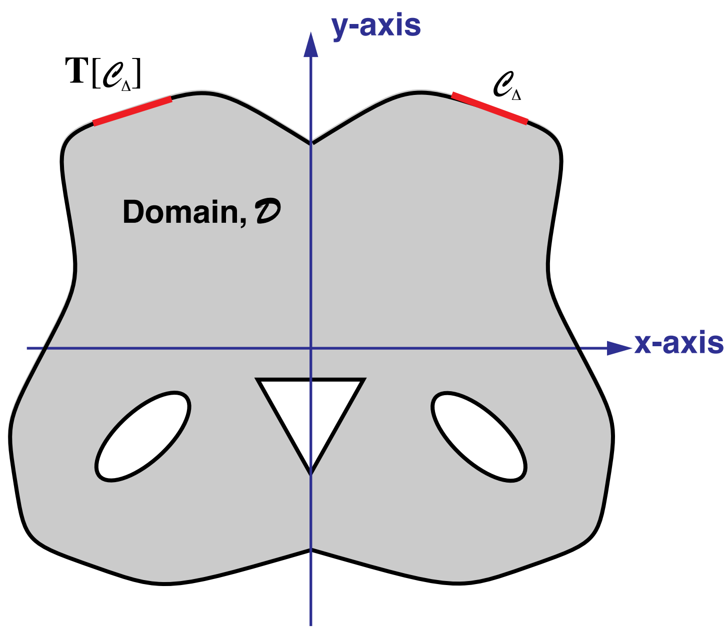

Figure 13. Geometry of domain $\mathcal{D}$ and boundary $\mathcal{C}$ imposed by the coordinate transformation $(\bar{x}, \bar{y}) = (-x, y)$.

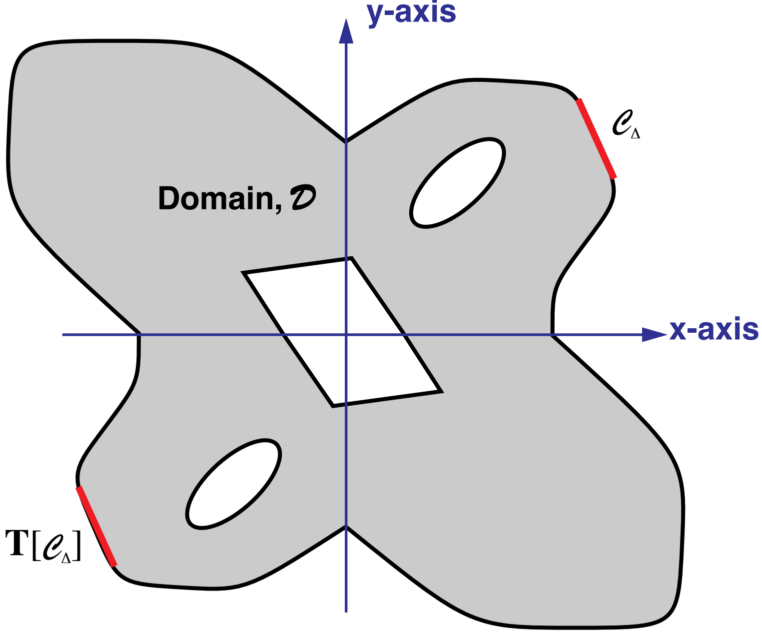

Figure 14. Geometry of domain $\mathcal{D}$ and boundary $\mathscr{C}$ imposed by the coordinate transformation $(\bar{x}, \bar{y}) = (-x, -y)$.

REPORT DOCUMENTATION PAGE

Form Approved
OMB No. 0704-0188

1. REPORT DATE *(DD-MM-YYYY)*
01- 07 - 2012

2. REPORT TYPE
Technical Memorandum

3. DATES COVERED *(From - To)*

4. TITLE AND SUBTITLE
Conditions for Symmetries in the Buckle Patterns of Laminated-Composite Plates

5a. CONTRACT NUMBER

5b. GRANT NUMBER

5c. PROGRAM ELEMENT NUMBER

6. AUTHOR(S)
Nemeth, Michael P.

5d. PROJECT NUMBER

5e. TASK NUMBER

5f. WORK UNIT NUMBER
869021.04.07.01.13

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)
NASA Langley Research Center
Hampton, VA 23681-2199

8. PERFORMING ORGANIZATION REPORT NUMBER
L-20159

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)
National Aeronautics and Space Administration
Washington, DC 20546-0001

10. SPONSOR/MONITOR'S ACRONYM(S)
NASA

11. SPONSOR/MONITOR'S REPORT NUMBER(S)
NASA/TM-2012-217589

12. DISTRIBUTION/AVAILABILITY STATEMENT
Unclassified - Unlimited
Subject Category 39
Availability: NASA CASI (443) 757-5802

13. SUPPLEMENTARY NOTES

14. ABSTRACT

Conditions for the existence of certain symmetries to exist in the buckle patterns of symmetrically laminated composite plates are presented. The plates considered have a general planform with cutouts, variable thickness and stiffnesses, and general support and loading conditions. The symmetry analysis is based on enforcing invariance of the corresponding eigenvalue problem for a group of coordinate transformations associated with buckle patterns commonly exhibited by symmetrically laminated plates. The buckle-pattern symmetries examined include a central point of inversion symmetry, one plane of reflective symmetry, and two planes of reflective symmetry.

15. SUBJECT TERMS
Buckling; Laminated composites; Plates; Symmetry

16. SECURITY CLASSIFICATION OF: a. REPORT	b. ABSTRACT	c. THIS PAGE	17. LIMITATION OF ABSTRACT	18. NUMBER OF PAGES	19a. NAME OF RESPONSIBLE PERSON
					STI Help Desk (email: help@sti.nasa.gov)
					19b. TELEPHONE NUMBER ***(Include area code)***
U	U	U	UU	45	(443) 757-5802