Adaptive Control with Reference Model Modification

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I.   Introduction

The asymptotic behavior of adaptive systems has been a well researched topic during the last couple of decades, and it is well known that asymptotic tracking can be achieved using the Lyapunov redesign method. However, the transient behavior of the input and output signals can be very oscillatory with significant excursions [1]. There has been a great deal of effort to modify the control architecture and the adaptive laws from the perspective of improving the transient behavior of the tracking error. The majority of these efforts led to nonadaptive high gain feedback [2, 3], switching control law [4] or to a parameter dependent persistent excitation condition [5].

On the other hand, the tracking error magnitude can be decreased by increasing the adaptation rate, but this results in unwanted high frequency oscillations and large overshoot in the control signal. This shortcoming is common for the majority of existing adaptive control methods. Recently some results have been appearing in the control community, which explicitly address the input signal transient behavior (see for example [6]).

In this note, instead of modification of the control architecture or the adaptive laws, the reference model is modified by feeding back the tracking error signal. This approach, called here "modified-reference-model MRAC" or M-MRAC in short, is motivated by the fact that the initial large errors in

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the control gains generates large transient excursions both in the system's input and output signals. The idea is to prevent the system's attempt to aggressively maneuver toward the reference model by modifying the reference model by a term proportional to the tracking error. As the tracking error approaches zero, the reference model approaches its original form. Therefore, the system asymptotically tracks not only the modified reference model, but also the original one. In addition, the error feedback term determines the damping in the control signal, which in concert with the adaptation rate makes it possible to regulate the transient of the control signal. A design guideline is provided for the selection of the feedback gain relative to the adaptation rate. The proposed adaptive control method has uniform performance in reference commands and initial conditions without the need for re-tuning.

It is worth to note that the reference model modification is not new. It was used to accommodate for the input saturation in the adaptive system (see for example [7, 8] and references therein).

II. Reference model modification

Consider a multi-input multi-output uncertain linear system

$$\dot{\boldsymbol{x}}(t) \;=\; A\boldsymbol{x}(t) + B\boldsymbol{u}(t), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0 \,, \tag{1}$$

where $\boldsymbol{x} \in \mathbb{R}^n$ and $\boldsymbol{u} \in \mathbb{R}^q$ are the state and input of the system respectively, and $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times q}$ are unknown constant matrices satisfying the following matching conditions.

Assumption II.1 *Given a Hurwitz matrix $A_m \in \mathbb{R}^{n \times n}$ and a matrix $B_m \in \mathbb{R}^{n \times q}$ of full column rank, there exists a matrix $K_1 \in \mathbb{R}^{n \times q}$ and a sign definite matrix $\Lambda \in \mathbb{R}^{q \times q}$ such that the following equations hold*

$$B = B_m \Lambda, \quad A = A_m - B K_1^\top \,.$$

Remark II.1 *The sign definiteness of Λ corresponds to the conventional condition on the high frequency gain matrix of MIMO systems (see for example [9]). Without loss of generality it is assumed that Λ is positive definite. The matching equations (2) complete the conditions for the existence of an adaptive controller.*

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As in the conventional model reference adaptive control (MRAC) framework, the objective is to design a control signal $\boldsymbol{u}(t)$ such that the the system (1) tracks a given reference model

$$\dot{\boldsymbol{x}}_r(t) = A_m \boldsymbol{x}_r(t) + B_m \boldsymbol{r}(t), \quad \boldsymbol{x}_r(0) = x_0,$$

where A_m, B_m are chosen according to performance specifications and satisfy Assumption II.1 and $\boldsymbol{r}(t)$ is a bounded and piecewise uniformly continuous external command. It can be noticed that discontinuous commands such as steps or square waves satisfy these conditions.

The system (1) can be written in the form

$$\dot{\boldsymbol{x}}(t) = A_m \boldsymbol{x}(t) + B_m \boldsymbol{r}(t) + B_m \Lambda [\boldsymbol{u}(t) - \Theta^\top \boldsymbol{f}(t)], \tag{2}$$

where $\Theta^\top = [K_1^\top \ \Lambda^{-\top}]$, $\boldsymbol{f}(t) = [\boldsymbol{x}^\top(t) \ \boldsymbol{r}^\top(t)]^\top$. Ideally, if Θ were known, one could set the control signal equal to

$$\boldsymbol{u}^0(t) = \Theta^\top \boldsymbol{f}(t), \tag{3}$$

thus reducing the system's dynamics to the given reference model (2). Although the reference model (2) can always be specified from the performance perspectives, the control signal (3), which here is called a reference control signal, can not be implemented. Therefore, the adaptive version of it is implemented in the MRAC framework, that is

$$\boldsymbol{u}(t) = \hat{\Theta}^\top(t) \boldsymbol{f}(t), \tag{4}$$

where $\hat{\Theta}(t)$ is the estimate of the ideal control gain Θ, and is updated online according to the adaptive law

$$\dot{\hat{\Theta}}(t) = -\gamma \boldsymbol{f}(t) \boldsymbol{e}^\top(t) P B_m, \quad \hat{\Theta}(0) = \Theta_0, \tag{5}$$

where $\gamma > 0$ is the adaptation rate, $\boldsymbol{e}(t) = \boldsymbol{x}(t) - \boldsymbol{x}_r(t)$ is the tracking error, and $P = P^\top > 0$ is the solution of the Lyapunov equation

$$A_m^\top P + P A_m = -Q \tag{6}$$

for some $Q = Q^\top > 0$. Introducing the parameter estimation error as $\tilde{\Theta}(t) = \hat{\Theta}(t) - \Theta$, the tracking error dynamics can be written in the form

$$\dot{\boldsymbol{e}}(t) = A_m \boldsymbol{e}(t) + B_m \Lambda \tilde{\Theta}^\top(t) \boldsymbol{f}(t). \tag{7}$$

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It is well known that this control architecture guarantees asymptotic tracking $\boldsymbol{x}(t) \rightarrow \mathbf{x}^0(t)$ as $t \rightarrow \infty$, while ensuring the boundedness of all closed-loop signals. However, the transient behavior of $\boldsymbol{x}(t)$ and $\boldsymbol{u}(t)$ cannot be guaranteed, because high frequency oscillations are generated in the control signal, when the adaptation rate is increased in order to obtain better tracking in transient.

The proposed M-MRAC architecture employs a modified reference model in the form

$$\dot{\boldsymbol{x}}_m(t) = A_m \boldsymbol{x}_m(t) + B_m N \boldsymbol{r}(t) + \lambda \boldsymbol{e}_m(t), \quad \mathbf{x}_m(0) = \boldsymbol{x}_0, \tag{8}$$

where $\lambda > 0$ is a design parameter and $\boldsymbol{e}_m(t) = \boldsymbol{x}(t) - \boldsymbol{x}_m(t)$ is the modified tracking error, and the subscript m is introduced to distinguish between the original reference model (2) and the modified reference model (8), which is used for the actual control design. The original reference model and the reference control signal will be used only for the analysis. The modified tracking error satisfies the dynamic equation

$$\dot{\boldsymbol{e}}_m(t) = (A_m - \lambda \mathbb{I}_n)\boldsymbol{e}_m(t) + B_m \Lambda [\boldsymbol{u}(t) - \Theta^\top \boldsymbol{f}(t)], \tag{9}$$

where $\mathbb{I}_n$ denotes n-dimensional identity matrix. The adaptive control $\boldsymbol{u}(t)$ is defined according to equation (4) as in MRAC design, but the adaptive law now is based on the modified tracking error $\boldsymbol{e}_m(t)$

$$\dot{\hat{\Theta}}(t) = -\gamma \boldsymbol{f}(t)\boldsymbol{e}_m^\top(t)PB_m, \quad \hat{\Theta}(0) = \Theta_0. \tag{10}$$

With the application of the adaptive control $\boldsymbol{u}(t)$ the modified tracking error dynamics reduce to

$$\dot{\boldsymbol{e}}_m(t) = (A_m - \lambda \mathbb{I}_n)\boldsymbol{e}_m(t) + B_m \Lambda \tilde{\Theta}^\top(t)\boldsymbol{f}(t), \tag{11}$$

where the parameter estimation error $\tilde{\Theta}(t)$ is defined similar to MRAC.

In the following analysis, the control error signal is needed, which is defined as $\tilde{\boldsymbol{u}}(t) = \boldsymbol{u}(t) - \boldsymbol{u}^0(t)$. From the above constructions it follows that

$$\tilde{\boldsymbol{u}}(t) = \tilde{\Theta}^\top(t)\boldsymbol{f}(t). \tag{12}$$

Since the reference control signal is the best achievable signal, it is in our interest to minimize the control error $\tilde{\boldsymbol{u}}(t)$, as well as the state errors $\boldsymbol{e}(t)$ and $\boldsymbol{e}_m(t)$, both in transient and steady state by selecting proper values for the adaptation rate γ and feedback parameter λ. This is the main objective of the analysis in the following sections.

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Remark II.2 *It is worth to point out that the modified reference model is used for the control design purposes. The asymptotic and thransient analysis of the adaptive system is conducted with respect to the original reference model, which implies that the desired dynamic characteristics built in the specified reference model are preserved with the proposed control method.*

III. Analysis of M-MRAC Design

A. Asymptotic properties

The following theorem summarizes the asymptotic properties of the proposed M-MRAC architecture.

Theorem III.1 *Let the system (1) satisfy Assumption II.1, and the reference model be given by (2). Let the system be controlled by the M-MRAC scheme given by (4), (8) and (10). Then*

1) all closed-loop signals are bounded

2) $e_m(t) \to 0$, $e(t) \to 0$ *and* $\tilde{u}(t) \to 0$ *as* $t \to \infty$.

The theorem can be proved using the standard Lyapunov tools and the extended Barabalat's lemma from [10].

Theorem III.1 shows that M-MRAC architecture guarantees asymptotic tracking of not only the modified reference model (8), but also the original reference model of the conventional MRAC architecture (2). That is the asymptotic performances of both conventional MRAC and proposed M-MRAC designs are equivalent.

B. State transient properties

Using the Lyapunov analysis it can be shown

$$\|e_m(t)\|_{\mathcal{L}_\infty} \le \frac{\sigma}{\sqrt{\lambda_{\min}(P)}} \frac{1}{\sqrt{\gamma}}, \tag{13}$$

where $\sigma = \sqrt{\operatorname{trace}\left(\Lambda \tilde{\Theta}^\top(0)\tilde{\Theta}(0)\right)}$ and

$$\|e_m(t)\|_{\mathcal{L}_2} \le \frac{\sigma}{\sqrt{\lambda_{\min}(Q) + 2\lambda\lambda_{\min}(P)}} \frac{1}{\sqrt{\gamma}}. \tag{14}$$

The bounds on the tracking error signal $e(t)$ are obtained using the equation

$$\frac{d}{dt}[e(t) - e_m(t)] = A_m[e(t) - e_m(t)] + \lambda e_m(t) \tag{15}$$

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and setting $\lambda = c_0\sqrt{\gamma}$, where $c_0 > 0$ is selected form the control error perspective in the next subsection. These bounds have form

$$\|e(t)\|_{\mathcal{L}_\infty} \frac{\sigma}{\sqrt{\lambda_{\min}(P)}}\left(k_m c_0 + \frac{1}{\sqrt{\gamma}}\right)$$

$$\|e(t)\|_{\mathcal{L}_2} \le \frac{\sigma}{\sqrt{\lambda_{\min}(Q) + 2\lambda\lambda_{\min}(P)}}\left(k_m c_0 + \frac{1}{\sqrt{\gamma}}\right),\tag{16}$$

where k_m is the $\mathcal{L}_1$ norm of the state transition matrix $e^{A_m t}$. It follows from the above inequalities that the oscillations ($\mathcal{L}_2$ norm) in the error $e(t)$ are reduced by simultaneously increasing λ and γ. The overshoot ($\mathcal{L}_\infty$ norm) is regulated by the choice of γ and matrix P (or Q in the Lyapunov equation (6)).

Remark III.1 *The derived bounds still hold, when the reference model cannot be initialized at the system's initial conditions. The difference is an additional term in σ, which is now given by the equation*

$$\sigma = \sqrt{e_m{}^\top(0)Pe_m(0) + \operatorname{trace}\left(\Lambda\tilde{\Theta}^\top(0)\tilde{\Theta}(0)\right)}.$$

C. Input transient properties

It can be noticed that $\tilde{u}(t)$ does not explicitly depend on design parameters λ and γ. Instead, $\dot{\tilde{u}}(t)$ depend on γ through the adaptive laws, and $\ddot{\tilde{u}}(t)$ depend on λ through the tracking error dynamics. Therefore we conduct dynamic analysis of this signal. It satisfies the differential equation

$$\ddot{\tilde{u}}(t) + \lambda\dot{\tilde{u}}(t) + \gamma\rho(t)G\tilde{u}(t) = \gamma\eta(t)B_m^\top Pe_m(t) + (s+\lambda)h(t),\tag{17}$$

where $\rho(t) = f^\top(t)f(t)$, $G = B_m^\top PB_m\Lambda$, $\eta(t) = -\rho(t)B_m^\top PA_me_m(t) - \dot{\rho}(t)B_m^\top Pe_m(t)$ and $h(t) = \tilde{\Theta}^\top(t)\dot{f}(t)$. Since all closed-loop signals are bounded, it follows that $\|\rho(t)\|_{\mathcal{L}_\infty} \le \alpha_1$, $\|\dot{\rho}(t)\|_{\mathcal{L}_\infty} \le \alpha_2$ and $\|h(t)\|_{\mathcal{L}_\infty} \le \alpha_3$ for some positive α_1, α_2 and α_3. Therefore, the equation (17) can be considered as a second order linear equation with time varying coefficients in $\tilde{u}(t)$. It can be inferred from the equation (17) that the adaptation rate γ determines the frequency of $\tilde{u}(t)$ and hence the frequency of the control signal $u(t)$, since the reference control $u^0(t)$ is in the low frequency range. Therefore, increasing γ increases the oscillations in $u(t)$ as it is the case for the conventional MRAC design. On the other hand, λ determines the damping ratio. Therefore increasing λ suppresses the oscillations in $\tilde{u}(t)$ and hence in the control signal $u(t)$.

A proper value for the parameter λ is selected from the perspective of minimizing the norm bound on $\tilde{\boldsymbol{u}}(t)$. Following the steps from [11], it can be shown that selection of $\lambda = \sqrt{2\alpha_1\gamma\lambda_{\max}(B_m^\top P B_m)}$ results in the bound

$$\|\tilde{\boldsymbol{u}}(t)\| \leq \beta_3 e^{-\nu t} + \beta_4 \gamma^{-\frac{1}{6}}, \tag{18}$$

where the positive constants β_3 and β_4 do not depend on γ and ν is proportional to $\sqrt{\gamma}$.

Using the inequality (18), a tighter bound can be obtained for tracking error form the error dynamics (7). This bound has a form of

$$\|\boldsymbol{e}_m(t)\| \leq \beta_5 e^{-\nu^* t} + \beta_6 \gamma^{-\frac{2}{3}}, \tag{19}$$

where $\nu^* = \min[\nu, \nu_m]$ and ν_m is the rate of decay of $e^{A_m t}$.

It follows from the inequalities (18) and (??) that the input and output transient errors can be decreased as desired by increasing the adaptation rate.

IV.   Application to a Generic Transport Model

In this section, the advantages of the M-MRAC architecture are demonstrated in simulations for a dynamic model that represents the lateral-directional motion of a generic transport aircraft (GTM) [12]. The nominal model is the linearized lateral-directional dynamics of GTM at the altitude of $9144\ m$ and speed of $0.8M$ and is given by the equation

$$\dot{\boldsymbol{x}}(t) = A_n \boldsymbol{x}(t) + B_n \boldsymbol{u}(t), \tag{20}$$

where $\boldsymbol{x} = [\beta \quad r \quad p \quad \phi]^\top$ is the lateral-directional state vector, in which β is the sideslip angle, r is the yaw rate, p is the roll rate, ϕ is the bank angle, and $\boldsymbol{u} = [\delta_a \quad \delta_r]^\top$ is the control signal that includes the aileron deflection δ_a and the rudder deflection δ_r. The controlled output is selected to be the sideslip angle and the bank angle of GTM. The open loop system has a slow Roll mode and negligible damping in the Dutch-Roll mode (see Table 1 for details).

The reference model is selected from the perspective of improving the performance characteristics of the nominal dynamics and is given by the matrices $A_m = A_n - B_n K_0$ and $B_m = B_n N$, where the feedback and feedforward matrices are selected to speeds up the Roll mode and increases the damping ratio of the Dutch-Roll mode.

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Table 1 Dynamics characteristics of the nominal model.

Open-loop Eigenvalues	-0.0177, -1.6339, $-0.3212 \pm 1.7404i$
Closed-loop Eigenvalues	-1.1623, -3.5583, $-1.2106 \pm 1.2008i$

The uncertain model of GTM corresponds to about 28% loss of left wing tip and 55% loss of rudder surface. The control objective is to track a reference command $r(t) = [\beta_{com}(t) \quad \phi_{com}(t)]^\top$ form the zero initial conditions. The reference command is chosen to be a series of coordinated turn maneuvers. That is, $\beta_{com}(t) = 0$ and the bank angle command is chosen to be a square wave of the amplitude of 15 degrees and of the frequency $\frac{\pi}{10}$ rad/sec. In order to make $\phi_{com}(t)$ differentiable, it is filtered through a first order stable filter $\frac{10}{s+10}$. For each γ, parameter λ is selected with $\alpha_1 = \frac{\pi^2}{12^2}$.

First, a simulation is run with $\gamma = 100$ and $Q = \mathbb{I}_4$. Figure 1 displays the output tracking of the adaptive system with M-MRAC and MRAC versus the output of the original reference model. Clearly M-MRAC has a superior performance even for the chosen low adaptation rate. The corresponding adaptive control signals and the reference control signals are displayed in Figure 2. It can be observed that the M-MRAC control signals tracks the reference control signal quite well without any oscillations unlike the MRAC control signal.
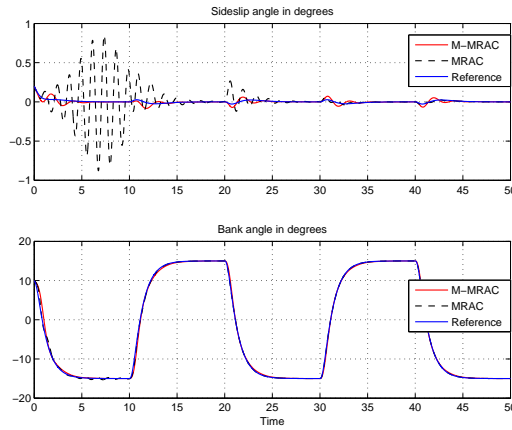
Fig. 1 Output performance of the proposed M-MRAC with $\gamma = 100$.

Second, it is demonstrated that both the output and input tracking errors can be decreased by increasing γ, without generating the oscillations in the control signal. For this purpose, simulations are run with $\gamma = 40000$ and $\gamma = 200000$. The results are presented in Table 2. As it can be seen
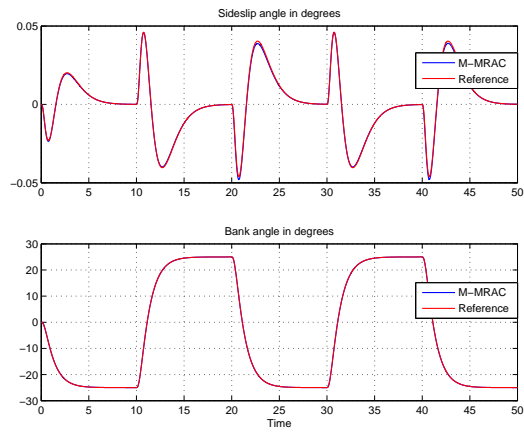
Fig. 2 Input performance of the proposed M-MRAC with $\gamma = 100$.

from the table, the control errors are systematically decreased when γ is increased in the case of M-MRAC design. In the MRAC design case these errors are large for large values of γ. The tracking errors are decreased when γ is increased for both designs. In general, the tracking error may be smaller in the case of MRAC, since the term $\frac{k_m c_0 \sigma}{\sqrt{\lambda_{\min}(P)}}$ on the right hand side of the first inequality in (16) is absent in the case of MRAC design ($c_0 = 0$).

Table 2 Output and input tracking error norms in degrees

γ	Design	$\|e_\beta(t)\|_{\mathcal{L}_\infty}$	$\|e_\phi(t)\|_{\mathcal{L}_\infty}$	$\left\|\tilde{\delta}_a(t)\right\|_{\mathcal{L}_\infty}$	$\left\|\tilde{\delta}_r(t)\right\|_{\mathcal{L}_\infty}$
400	MRAC	0.2826	0.7513	15.0577	23.0739
	M-MRAC	0.0255	1.3035	11.7093	0.6349
40000	MRAC	0.0056	0.0172	27.2261	47.1725
	M-MRAC	0.0036	0.1958	4.6656	0.2307
200000	MRAC	0.0025	0.0051	27.3205	47.1719
	M-MRAC	0.0016	0.1162	3.7371	0.1922

Next, to demonstrate the scaling properties of M-MRAC, a simulation is run with 2-times scaled $\phi_{com}(t)$. In this case, the output performance does not change, as can be seen from Figure 3, and the input is scaled 2-times as in the case of a linear time invariant system.

The final simulation is run with nonzero initial conditions to show that M-MRAC does not degrade with the change of initial conditions, unlike conventional MRAC. The output and input performance of M-MRAC with $\gamma = 200000$ and $x_0 = [0.2\ 0.5\ -3.0\ 10]^\top \frac{\pi}{180}$ is displayed in Figures

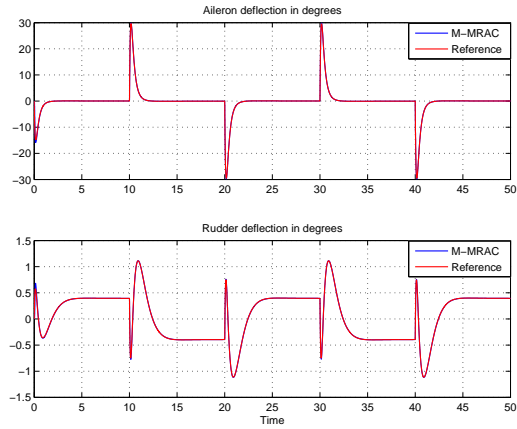**Fig. 3 Output performance of the proposed M-MRAC for scaled command.**

Fig. 4 Input performance of the proposed M-MRAC for scaled command.

5 and 6.

It can be observed that a good tracking of the output and input signals is achieved both in transient and in steady state.

V.   Conclusions

A simple yet powerful modification of the conventional MRAC design is presented, which has a guaranteed transient and asymptotic performances in both the input and output signals of the system. The proposed M-MRAC design enables one to achieve close tracking of reference output and input signals by increasing the adaptation rate without generating high frequency oscillations in the control signal. The performance of M-MRAC is uniform in initial conditions and in reference
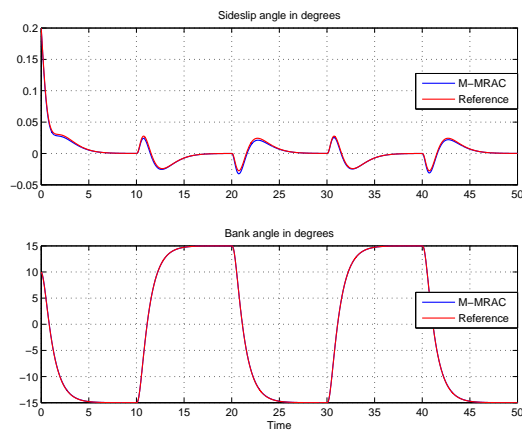
Fig. 5 Output performance of the proposed M-MRAC for nonzero initial conditions.
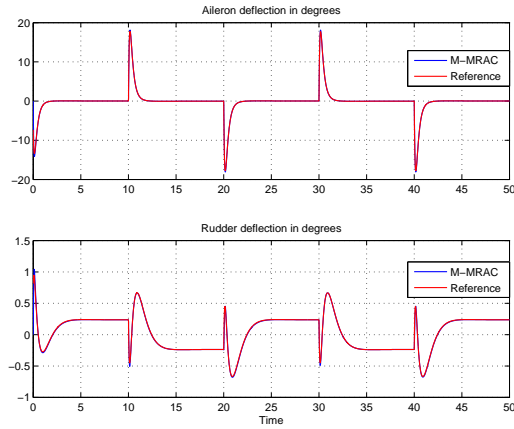

Fig. 6 Input performance of the proposed M-MRAC for nonzero initial conditions.

command.

Since only the reference model is modified leaving the adaptive laws intact, this method can be applied along with any known adaptive laws, such as dead-zone modification, e-modification, σ-modification, etc.

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