

# Wald Sequential Probability Ratio Test for Analysis of Orbital Conjunction Data

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**We propose a Wald Sequential Probability Ratio Test for analysis of commonly available predictions associated with spacecraft conjunctions. Such predictions generally consist of a relative state and relative state error covariance at the time of closest approach, under the assumption that prediction errors are Gaussian. We show that under these circumstances, the likelihood ratio of the Wald test reduces to an especially simple form, involving the current best estimate of collision probability, and a similar estimate of collision probability that is based on prior assumptions about the likelihood of collision.**

## Nomenclature

$C_k$	Matrix that projects vectors into the conjunction plane, computed from data available at time $t_k$ .
$P_*$	(Unknown) covariance of the distribution of the true relative position vector at time of closest approach.
$\bar{P}_{fa}$	Target false alarm probability.
$\bar{P}_{md}$	Target missed detection probability.
$\hat{P}_{* k}$	Covariance of the error in the current estimate, using all available data up to and including time $t_k$ , of the true relative position vector at time of closest approach (projected into the conjunction plane).
$\hat{P}_{* o}$	Prior covariance of the distribution of the true relative position vector at time of closest approach (projected into the conjunction plane).
$\hat{r}_{* k}$	Current estimate, using all available data up to and including time $t_k$ , of the distribution of the true relative position vector at time of closest approach (projected into the conjunction plane).
$\hat{r}_{* o}$	Prior estimate for the mean of the distribution of the true relative position vector at time of closest approach (projected into the conjunction plane).
$\mathbb{B}$	The set representing the combined hard body volume of the conjuncting spacecraft (projected into the conjunction plane).
$\mathcal{H}_1$	The alternative hypothesis that the conjunction is safe.
$\mathcal{H}_0$	The null hypothesis that the conjunction is unsafe.
$\Lambda_k$	Likelihood ratio, based on all information accumulated up to and including time $t_k$ .
$p(\cdot \cdot)$	Unspecified probability density function of the first argument, conditional on any arguments following the vertical separator.
$\mathbb{Y}_{k:1}$	The reverse-ordered set of all predicted relative positions, $\tilde{r}_{* k}$ .
$\tilde{P}_{* k}$	Covariance of the error in the predicted relative position vector at time of closest approach, based only on information at time $t_k$ (projected into the conjunction plane).
$\tilde{\omega}_{* k}^1$	Predicted angular velocity vector of the orbit of the primary spacecraft at time of closest approach, based only on information at time $t_k$ , relative to the central body.
$\tilde{R}_{* k}^1$	Predicted position vector of the primary spacecraft at time of closest approach, based only on information at time $t_k$ , relative to the central body.

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$\tilde{\mathbf{R}}_{* k}^2$	Predicted position vector of the secondary spacecraft at time of closest approach, based only on information at time $t_k$ , relative to the central body.
$\tilde{\mathbf{r}}_{* k}$	Predicted relative position vector at time of closest approach, based only on information at time $t_k$ (projected into the conjunction plane).
$\tilde{\mathbf{V}}_{* k}^1$	Predicted velocity vector of the primary spacecraft at time of closest approach, based only on information at time $t_k$ , relative to the central body.
$\tilde{\mathbf{V}}_{* k}^2$	Predicted velocity vector of the secondary spacecraft at time of closest approach, based only on information at time $t_k$ , relative to the central body.
$\tilde{\mathbf{v}}_{* k}$	Predicted relative velocity vector at time of closest approach, based only on information at time $t_k$ .
$\boldsymbol{\mu}_*$	(Unknown) mean of the distribution of the true relative position vector at time of closest approach.
$A$	Threshold for rejecting the null hypothesis (dismissal threshold).
$B$	Threshold for accepting the null hypothesis (alarm threshold).
$P_{c k}$	Probability of collision, based on all information accumulated up to and including time $t_k$ .
$P_{c o}$	Probability of collision, based on <i>a priori</i> information.
$P_{fa}$	False alarm probability.
$P_{md}$	Missed detection probability.

## I. Introduction

WHEN a maneuverable spacecraft confronts a potentially unsafe conjunction with another space object, its operators must decide whether to maneuver to mitigate the risk of a collision. Such decisions may not be straightforward, since the operators must balance their confidence in the predictions which detected the conjunction, the actual likelihood of a collision, any risk inherent in performing the maneuver, interruptions to the mission's ongoing operations, and long-term consequences such as depletion of consumable propellant. In principle, operators could quantify their tolerance for performing a maneuver which was not required in terms of an acceptable rate of *false alarms*, and their tolerance for failing to maneuver when a collision was going to occur as an acceptable rate of *missed detections*. Whether such tolerances are explicitly defined, or only implicitly considered, operators may seek to inform their maneuver decisions using empirical thresholds on various metrics associated with the conjunction. An alternative to such *ad hoc* methods is to use a Wald Sequential Probability Ratio Test<sup>1</sup> (WSPRT) to inform the collision avoidance decision process. The WSPRT guides decisions based explicitly on false alarm and missed detection criteria.

In Reference 2, Carpenter and Markley proposed the use of such a WSPRT for conjunction avoidance. Some limitations of the method proposed in Reference 2 include assumptions that the observations are statistically independent, and its reliance on a set of assumptions that reduce the complexity of the encounter. These limitations were overcome in Reference 3 which reformulated the WSPRT using innovations from a filter bank consisting of two norm-inequality-constrained epoch-state extended Kalman filters. In that approach one filter models a null hypothesis that the miss distance between two conjuncting spacecraft is inside their combined hard body radius at the predicted time of closest approach, and one is constrained by an alternative complementary hypothesis. The epoch-state filter developed for that method explicitly accounts for any process noise present in the system. Because of its epoch-state formulation however, that method still required potentially inaccurate approximations to mapping probability density forward through time. In Reference 4, Carpenter and Markley constructed a WSPRT that does not require prediction of probability densities. Instead, one uses solutions to a set of Lambert problems after each measurement update in a sigma-point transformation, to approximate the boundary of the set of current-state velocities that will result in a collision. We used this boundary to define two inequality-constrained *current*-state filters whose innovations formed the likelihood ratio for the Wald test. Although this work relaxes many assumptions of our prior work, it still requires accurate knowledge of the time of closest approach. Also, References 3 and 4 require filtering of the measurement data, so they may not be suitable in circumstances when such data are not available.

In the present work, we return to the scenario considered in Reference 2, which was previously considered in many works, including Foster and Estes,<sup>5</sup> Akella and Alfriend,<sup>6</sup> Patera,<sup>7</sup> Chan,<sup>8</sup> and Alfano.<sup>9</sup> Alfano<sup>10</sup> provides a summary and comparison of many of these methods. Each of these works assumes that we know, or can accurately approximate, the probability density of the relative position at the time of closest approach. In such cases, we can accurately compute a collision probability, and computing this probability accurately and efficiently has been the focus of many such works. By re-examining some of the assumptions

of Reference 2, the present work significantly simplifies and improves the WSPRT. Furthermore, the WSPRT proposed herein is cast in a form that may allow it to leverage currently ongoing work to improve collision probability computations.<sup>11,12,13</sup>

## II. Problem Statement

The true position and velocity vectors of two space objects at their point of closest approach are given by  $\mathbf{R}_*^i$ , and  $\mathbf{V}_*^i$ ,  $i = 1, 2$ , respectively. The origin of these vectors is the center of mass of the central body the spacecraft jointly orbit. Within the context of this paper, predictions of the relative position and velocity of the objects are viewed as random variables, with known probability density functions. We denote predictions of the relative position and velocity of the objects, based only on information available at time  $t_k < t_*$ , as follows,

$$\tilde{\mathbf{r}}_{*|k} = \tilde{\mathbf{R}}_{*|k}^2 - \tilde{\mathbf{R}}_{*|k}^1 \quad (1)$$

$$\tilde{\mathbf{v}}_{*|k} = \tilde{\mathbf{V}}_{*|k}^2 - \tilde{\mathbf{V}}_{*|k}^1 - \tilde{\boldsymbol{\omega}}_{*|k}^1 \times \Delta \tilde{\mathbf{R}}_{*|k} \quad (2)$$

We seek to avoid the condition that at  $t_*$  the two spacecraft reside within a common region  $\mathbb{B}$  that bounds their combined hard body volumes. If we can determine that such a condition is likely, we can maneuver one or both spacecraft so as to avoid it. At the same time, we wish to avoid unnecessary maneuvers. We quantify our tolerance for failing to maneuver when a maneuver was needed as a probability of missed detection. We quantify our tolerance for maneuvering when it was unnecessary as a false alarm probability. We therefore seek a decision procedure that will guide our maneuver decisions, in the context of meeting the specified missed detection and false alarm rates.

## III. Problem Solution

Our solution to this problem employs the WSPRT. In the present case, our WSPRT uses a ratio involving the set of  $k$  predictions of the relative positions,

$$\mathbb{Y}_{k:1} = \{\tilde{\mathbf{r}}_{*|k}, \tilde{\mathbf{r}}_{*|k-1}, \dots, \tilde{\mathbf{r}}_{*|1}\} \quad (3)$$

The WSPRT ratio divides the likelihood of these predictions having occurred under the alternative hypothesis,  $\mathcal{H}_1$ , that the conjunction is safe, by the likelihood that they have occurred under the null hypothesis,  $\mathcal{H}_0$ , that the conjunction is unsafe:

$$\Lambda_k = \frac{\text{p}(\mathbb{Y}_{k:1}|\mathcal{H}_1)}{\text{p}(\mathbb{Y}_{k:1}|\mathcal{H}_0)} = \frac{\text{p}(\mathbb{Y}_{k:1}|\mathbf{r}_* \notin \mathbb{B})}{\text{p}(\mathbb{Y}_{k:1}|\mathbf{r}_* \in \mathbb{B})} \quad (4)$$

In a Wald test, one compares  $\Lambda_k$  to decision limits  $A$  and  $B$  such that whenever  $B < \Lambda_k < A$  one should, if possible, seek another observation. If  $\Lambda_k \leq B$ , then one should accept the null hypothesis, and in the present case, we would recommend a collision avoidance maneuver. If  $\Lambda_k \geq A$ , then one should accept the alternative hypothesis, and hence we would dismiss the conjunction alarm<sup>a</sup>. Wald's explanations for the thresholds  $A$  and  $B$  are that we will accept the alternative hypothesis if it is  $A$  times more likely than the null, and accept the null hypothesis if it is  $1/B$  times more likely than the alternative. Wald shows that such a procedure will terminate with probability one, and that the resulting false alarm probability,  $P_{fa}$ , and missed detection probability,  $P_{md}$ , satisfy the inequalities

$$\frac{1 - P_{fa}}{P_{md}} \geq A \quad \text{and} \quad \frac{P_{fa}}{1 - P_{md}} \leq B \quad (5)$$

We follow Wald's suggestion to define the decision limits  $A$  and  $B$  in terms of a target false alarm probability,  $\bar{P}_{fa}$ , and missed detection probability,  $\bar{P}_{md}$ , as

$$A = \frac{1 - \bar{P}_{fa}}{\bar{P}_{md}} \quad \text{and} \quad B = \frac{\bar{P}_{fa}}{1 - \bar{P}_{md}} \quad (6)$$

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<sup>a</sup>In the present case, there may be minimal penalty in waiting until all possible measurements have been collected. If the test is still indeterminate at that time, it may be prudent to maneuver, although this would imply an increased false alarm rate.

It must be noted that Eqs. (5) and (6) do *not* guarantee that  $P_{fa} \leq \bar{P}_{fa}$  and  $P_{md} \leq \bar{P}_{md}$ . They do provide the weaker inequalities  $P_{fa} + P_{md} \leq \bar{P}_{fa} + \bar{P}_{md}$ ,  $P_{fa} \leq \bar{P}_{fa}/(1 - \bar{P}_{md})$ , and  $P_{md} \leq \bar{P}_{md}/(1 - \bar{P}_{fa})$ , however. These inequalities guarantee that at most one of the probabilities  $P_{fa}$  or  $P_{md}$  can be greater than its target value<sup>b</sup>, and it cannot be much greater in the usual case that both target values are much less than unity<sup>b</sup>.

In order to evaluate the conditional densities involving compound hypotheses in (4), we will need to make use of the following theorem.

**Theorem 1** (Bayes' Rule for Compound Conditionals). *The probability density of a random vector,  $\mathbf{z}$ , on the condition that a vector of its parameters,  $\boldsymbol{\alpha}$ , is restricted to lie within a region  $\mathbb{S}$ , is the ratio of the integral over  $\mathbb{S}$  of the joint density of  $\mathbf{z}$  and  $\boldsymbol{\alpha}$ , divided by the marginal probability that  $\boldsymbol{\alpha}$  has taken on any value in  $\mathbb{S}$ ; that is,*

$$p(\mathbf{z}|\boldsymbol{\alpha} \in \mathbb{S}) = \frac{\int_{\mathbb{S}} p(\mathbf{z}, \boldsymbol{\alpha}) d\boldsymbol{\alpha}}{\int_{\mathbb{S}} p(\boldsymbol{\alpha}) d\boldsymbol{\alpha}} \quad (7)$$

*Proof.* Using Bayes' Rule, we can write the probability that  $\mathbf{z}$  occurs within some region  $\mathbf{z}' - \frac{\epsilon}{2} < \mathbf{z} < \mathbf{z}' + \frac{\epsilon}{2}$ , under the condition that  $\boldsymbol{\alpha} \in \mathbb{S}$ , as

$$\Pr(\mathbf{z}' - \epsilon/2 < \mathbf{z} < \mathbf{z}' + \epsilon/2 | \boldsymbol{\alpha} \in \mathbb{S}) = \frac{\Pr(\mathbf{z}' - \frac{\epsilon}{2} < \mathbf{z} < \mathbf{z}' + \frac{\epsilon}{2}, \boldsymbol{\alpha} \in \mathbb{S})}{\Pr(\boldsymbol{\alpha} \in \mathbb{S})} \quad (8)$$

$$= \frac{\int_{\mathbf{z}' - \epsilon/2}^{\mathbf{z}' + \epsilon/2} \int_{\mathbb{S}} p(\mathbf{z}, \boldsymbol{\alpha}) d\boldsymbol{\alpha} d\mathbf{z}}{\int_{\mathbb{S}} p(\boldsymbol{\alpha}) d\boldsymbol{\alpha}} \quad (9)$$

Letting  $\epsilon \rightarrow 0$ , the conditional probability on the left-hand side becomes a conditional probability density in terms of  $\mathbf{z}$ :

$$\lim_{\epsilon \rightarrow 0} \Pr(\mathbf{z}' - \epsilon/2 < \mathbf{z} < \mathbf{z}' + \epsilon/2 | \boldsymbol{\alpha} \in \mathbb{S}) = p(\mathbf{z}|\boldsymbol{\alpha} \in \mathbb{S}) \quad (10)$$

and dropping the prime notation, the integral over  $\mathbf{z}$  on the right-hand side drops away, so that we arrive at the result we sought to prove. □

**Corollary 1.1.** *The joint probability density of a set of  $N$  random vectors,  $\{\mathbf{z}_i\}, i = 1, \dots, N$ , on the condition that a vector of parameters,  $\boldsymbol{\alpha}$ , associated with the marginal densities of each  $\mathbf{z}_i$  is restricted to lie within a region  $\mathbb{S}$ , is the ratio of the integral over  $\mathbb{S}$  of the joint density of the  $\mathbf{z}_i$  and  $\boldsymbol{\alpha}$ , divided by the marginal probability that  $\boldsymbol{\alpha}$  has taken on a value in  $\mathbb{S}$ ; that is,*

$$p(\mathbf{z}_1, \dots, \mathbf{z}_N | \boldsymbol{\alpha} \in \mathbb{S}) = \frac{\int_{\mathbb{S}} p(\mathbf{z}_1, \dots, \mathbf{z}_N, \boldsymbol{\alpha}) d\boldsymbol{\alpha}}{\int_{\mathbb{S}} p(\boldsymbol{\alpha}) d\boldsymbol{\alpha}} \quad (11)$$

*Proof.* Stack the  $\mathbf{z}_i$  into a single vector  $\mathbf{z} = [\mathbf{z}_1^T, \dots, \mathbf{z}_N^T]^T$ . Then  $p(\mathbf{z}_1, \dots, \mathbf{z}_N | \boldsymbol{\alpha} \in \mathbb{S}) = p(\mathbf{z}|\boldsymbol{\alpha} \in \mathbb{S})$  and Theorem 1 gives the result we sought. □

*Remark.* If the  $\{\mathbf{z}_i\}$  are independent, then

$$p(\mathbf{z}_1, \dots, \mathbf{z}_N, \boldsymbol{\alpha}) = \prod_{i=1}^N p(\mathbf{z}_i|\boldsymbol{\alpha}) p(\boldsymbol{\alpha}) \quad (12)$$

which implies that

$$p(\mathbf{z}_1, \dots, \mathbf{z}_N | \boldsymbol{\alpha} \in \mathbb{S}) = \frac{\int_{\mathbb{S}} \prod_{i=1}^N p(\mathbf{z}_i|\boldsymbol{\alpha}) p(\boldsymbol{\alpha}) d\boldsymbol{\alpha}}{\int_{\mathbb{S}} p(\boldsymbol{\alpha}) d\boldsymbol{\alpha}} \quad (13)$$

$$\neq \frac{\prod_{i=1}^N \int_{\mathbb{S}} p(\mathbf{z}_i|\boldsymbol{\alpha}) p(\boldsymbol{\alpha}) d\boldsymbol{\alpha}}{\int_{\mathbb{S}} p(\boldsymbol{\alpha}) d\boldsymbol{\alpha}} \quad (14)$$

Equation (33) in a prior work by the first two authors<sup>2</sup> erroneously used Eq. (14), rather than using Eq. (13).

<sup>b</sup>In cases for which  $\bar{P}_{fa} + \bar{P}_{md} \geq 1$ , the WSPRT fails to provide a sensible decision procedure.

According to Corollary 1.1, we can express the conditional densities required for the likelihood ratio as ratios of the joint density integrated over the region associated with each hypothesis, divided by the marginal probability that each hypothesis is true, independent of any predictions:

$$p(\tilde{\mathbf{r}}_{*|k}, \tilde{\mathbf{r}}_{*|k-1}, \dots, \tilde{\mathbf{r}}_{*|1} | \mathbf{r}_* \in \mathbb{B}) = \frac{\int_{\mathbb{B}} p(\tilde{\mathbf{r}}_{*|k}, \tilde{\mathbf{r}}_{*|k-1}, \dots, \tilde{\mathbf{r}}_{*|1}, \mathbf{r}_*) d\mathbf{r}_*}{\int_{\mathbb{B}} p(\mathbf{r}_*) d\mathbf{r}_*} \quad (15)$$

$$p(\tilde{\mathbf{r}}_{*|k}, \tilde{\mathbf{r}}_{*|k-1}, \dots, \tilde{\mathbf{r}}_{*|1} | \mathbf{r}_* \notin \mathbb{B}) = \frac{\int_{\mathbb{B}} p(\tilde{\mathbf{r}}_{*|k}, \tilde{\mathbf{r}}_{*|k-1}, \dots, \tilde{\mathbf{r}}_{*|1}, \mathbf{r}_*) d\mathbf{r}_*}{\int_{\mathbb{B}} p(\mathbf{r}_*) d\mathbf{r}_*} \quad (16)$$

We assume the true relative position has a Gaussian distribution, with some fixed but unknown mean and covariance:

$$p(\mathbf{r}_*) = N(\mathbf{r}_* | \boldsymbol{\mu}_*, \mathbf{P}_*) = \frac{1}{(2\pi)^{\frac{n}{2}} \sqrt{|\mathbf{P}_*|}} \exp\left(-\frac{1}{2}(\mathbf{r}_* - \boldsymbol{\mu}_*)^\top \mathbf{P}_*^{-1}(\mathbf{r}_* - \boldsymbol{\mu}_*)\right) \quad (17)$$

where  $n$  is the dimension of the relative position. We assume the predicted relative positions have Gaussian distributions, centered on the true relative position, so the joint density, after  $k$  observations, becomes

$$\begin{aligned} & p(\tilde{\mathbf{r}}_{*|k}, \tilde{\mathbf{r}}_{*|k-1}, \dots, \tilde{\mathbf{r}}_{*|1}, \mathbf{r}_*) \\ &= \prod_{i=1}^k \left( \frac{1}{(2\pi)^{\frac{n}{2}} \sqrt{|\tilde{\mathbf{P}}_{*|i}|}} \right) \frac{1}{(2\pi)^{\frac{n}{2}} \sqrt{|\mathbf{P}_*|}} \exp\left(-\frac{1}{2} \left[ \sum_{i=1}^k (\tilde{\mathbf{r}}_{*|i} - \mathbf{r}_*)^\top \tilde{\mathbf{P}}_{*|i}^{-1}(\tilde{\mathbf{r}}_{*|i} - \mathbf{r}_*) + (\mathbf{r}_* - \boldsymbol{\mu}_*)^\top \mathbf{P}_*^{-1}(\mathbf{r}_* - \boldsymbol{\mu}_*) \right]\right) \end{aligned} \quad (18)$$

Since  $\boldsymbol{\mu}_*$  and  $\mathbf{P}_*$  are unknown, we will use *a priori* estimates  $\hat{\mathbf{r}}_{*|o}$  and  $\hat{\mathbf{P}}_{*|o}$  instead. Collecting terms as follows will allow us to simplify the result considerably.

$$\hat{\mathbf{P}}_{*|k} = (\hat{\mathbf{P}}_{*|o}^{-1} + \sum_{i=1}^k \tilde{\mathbf{P}}_{*|i}^{-1})^{-1} \quad (19)$$

$$\hat{\mathbf{r}}_{*|k} = \hat{\mathbf{P}}_{*|k} \left( \hat{\mathbf{P}}_{*|o}^{-1} \hat{\mathbf{r}}_{*|o} + \sum_{i=1}^k \tilde{\mathbf{P}}_{*|i}^{-1} \tilde{\mathbf{r}}_{*|i} \right) \quad (20)$$

$$\alpha = \hat{\mathbf{r}}_{*|o}^\top \hat{\mathbf{P}}_{*|o}^{-1} \hat{\mathbf{r}}_{*|o} + \sum_{i=1}^k \tilde{\mathbf{r}}_{*|i}^\top \tilde{\mathbf{P}}_{*|i}^{-1} \tilde{\mathbf{r}}_{*|i} - \hat{\mathbf{r}}_{*|k}^\top \hat{\mathbf{P}}_{*|k}^{-1} \hat{\mathbf{r}}_{*|k} \quad (21)$$

So the conditional densities associated with the two hypotheses become

$$p(\mathbb{Y}_{k:1} | \mathbf{r}_* \in \mathbb{B}) = \frac{\prod_{i=1}^k \left( \frac{1}{(2\pi)^{\frac{n}{2}} \sqrt{|\tilde{\mathbf{P}}_{*|i}|}} \right) \sqrt{\frac{|\tilde{\mathbf{P}}_{*|k}|}{|\tilde{\mathbf{P}}_{*|o}|}} e^{-\frac{1}{2}\alpha} P_{c|k}}{P_{c|o}} \quad (22)$$

$$p(\mathbb{Y}_{k:1} | \mathbf{r}_* \notin \mathbb{B}) = \frac{\prod_{i=1}^k \left( \frac{1}{(2\pi)^{\frac{n}{2}} \sqrt{|\tilde{\mathbf{P}}_{*|i}|}} \right) \sqrt{\frac{|\tilde{\mathbf{P}}_{*|k}|}{|\tilde{\mathbf{P}}_{*|o}|}} e^{-\frac{1}{2}\alpha} (1 - P_{c|k})}{1 - P_{c|o}} \quad (23)$$

where

$$P_{c|k} = \frac{1}{(2\pi)^{\frac{n}{2}} \sqrt{|\hat{\mathbf{P}}_{*|k}|}} \int_{\mathbb{B}} \exp\left(-\frac{1}{2}(\hat{\mathbf{r}}_{*|k} - \mathbf{r}_*)^\top \hat{\mathbf{P}}_{*|k}^{-1}(\hat{\mathbf{r}}_{*|k} - \mathbf{r}_*)\right) d\mathbf{r}_* \quad (24)$$

which we can recognize as the collision probability, and

$$P_{c|o} = \frac{1}{(2\pi)^{\frac{n}{2}} \sqrt{|\hat{\mathbf{P}}_{*|o}|}} \int_{\mathbb{B}} \exp\left(-\frac{1}{2}(\mathbf{r}_* - \hat{\mathbf{r}}_{*|o})^\top \hat{\mathbf{P}}_{*|o}^{-1}(\mathbf{r}_* - \hat{\mathbf{r}}_{*|o})\right) d\mathbf{r}_* \quad (25)$$

which we can view as a prior estimate of collision, determined from considerations independent of the data that produced the predictions presently available. Hence, the likelihood ratio becomes

$$\Lambda_k = \frac{1 - P_{c|k}}{P_{c|k}} \frac{P_{c|o}}{1 - P_{c|o}} \quad (26)$$

*Remark.* It is apparent that

$$\Lambda_k = 1 \Leftrightarrow P_{c|k} = P_{c|o} \quad (27)$$

The left-hand equality means that the predictions,  $\mathbb{Y}_{k:1}$ , are equally likely under either hypothesis  $\mathcal{H}_0$  or  $\mathcal{H}_1$ . The right-hand equality means that the predictions do not change our prior estimate of the probability of collision.

Although we reached Eq. (26) via assuming independent, Gaussian-distributed conjunction predictions, the form of Eq. (26) suggests the following conjecture:

**Conjecture 1.** *The likelihood ratio for the WSPRT may be computed using collision probabilities computed by any means.*

We can simplify the decision procedure further by eliminating the likelihood ratio as follows. Solving Eq. (26) for  $P_{c|k}$  gives

$$P_{c|k} = \frac{P_{c|o}}{\Lambda_k + (1 - \Lambda_k)P_{c|o}} \quad (28)$$

Since we will recommend a maneuver if  $\Lambda_k \leq B$  and recommend a dismissal if  $\Lambda_k > A$ , we can therefore write the decision thresholds as thresholds on  $P_{c|k}$  as follows:

$$P_{c|k} \geq \frac{P_{c|o}}{B + (1 - B)P_{c|o}} \Rightarrow \text{Maneuver} \quad (29)$$

$$P_{c|k} < \frac{P_{c|o}}{A + (1 - A)P_{c|o}} \Rightarrow \text{Dismiss} \quad (30)$$

We can use Eq. (6) to write the decision procedure solely in terms of the four probabilities  $P_{c|k}$ ,  $P_{c|o}$ ,  $\bar{P}_{md}$ , and  $\bar{P}_{fa}$ :

$$P_{c|k} \geq P_{c|k}^A = \frac{(1 - \bar{P}_{md})P_{c|o}}{\bar{P}_{fa} + (1 - \bar{P}_{md} - \bar{P}_{fa})P_{c|o}} \Rightarrow \text{Maneuver} \quad (31)$$

$$P_{c|k} < P_{c|k}^D = \frac{\bar{P}_{md}P_{c|o}}{1 - [\bar{P}_{fa} + (1 - \bar{P}_{md} - \bar{P}_{fa})P_{c|o}]} \Rightarrow \text{Dismiss} \quad (32)$$

## IV. Test Procedure

According to the literature, for example References 5, 6, 8, it is often beneficial to project the geometry of the conjunction into a plane normal to the relative velocity at the time of closest approach; one of several advantages of this approach is that it removes some of the sensitivity to uncertainty in the time of closest approach. This projection must be used with care if the objects approach one another slowly, and their relative motion is not approximately rectilinear during a brief interval when conjunction risk is high. 14, 15, 16 While Conjecture 1 suggests the method we have described above may be generalized to cases in which such projections are inappropriate, we believe it to be most applicable, in current practice, to cases where the projection method is suitable. Define the projection matrix as<sup>5, 6</sup>

$$\mathbf{C}_k = \begin{bmatrix} (\tilde{\mathbf{V}}_{*|k}^2 \times \tilde{\mathbf{V}}_{*|k}^1)^\top / \|\tilde{\mathbf{V}}_{*|k}^2 \times \tilde{\mathbf{V}}_{*|k}^1\| \\ \tilde{\mathbf{v}}_{*|k}^\top \times (\tilde{\mathbf{V}}_{*|k}^2 \times \tilde{\mathbf{V}}_{*|k}^1)^\top / \|\tilde{\mathbf{v}}_{*|k} \times (\tilde{\mathbf{V}}_{*|k}^2 \times \tilde{\mathbf{V}}_{*|k}^1)\| \end{bmatrix} \quad (33)$$

Then, the predictions will be projected down into the conjunction plane, and we will re-use our previous notation with the following replacements:

$$\tilde{\mathbf{r}}_{*|k} \leftarrow \mathbf{C}_k \cdot \tilde{\mathbf{r}}_{*|k} \quad (34)$$

$$\tilde{\mathbf{P}}_{*|k} \leftarrow \mathbf{C}_k \tilde{\mathbf{P}}_{*|k} \mathbf{C}_k^\top \quad (35)$$

and similarly for the other symbols.

Table 1. Parameter values used in testing procedure.

Dimensions of combined hard body region	120 m $\times$ 120 m
Standard deviation of prior distribution	Uniformly sampled from (0, 1000 m]
Correlation coefficient of prior distribution	Uniformly sampled from $[-.8, 0.8]$
Standard deviation of predictions	Uniformly sampled from (0, 100 m]
Correlation coefficient of predictions	Uniformly sampled from $[-.8, 0.8]$

To test our method, we performed Monte Carlo trials using Gaussian draws to simulate sequences of orbital conjunction data projected into the conjunction plane, as defined above. For each Monte Carlo trial  $i$ , our simulation draws a true separation vector,  $\mathbf{r}_i$ , from a normal distribution with mean  $\hat{\mathbf{r}}_{*|o} = 0$ , and covariance  $\hat{\mathbf{P}}_{*|o}$ . The simulation independently varies each of the standard deviations and the correlation coefficient of  $\hat{\mathbf{P}}_{*|o}$  for each trial, by drawing from a uniform distribution with the parameters Table 1 lists. Within each trial, the simulation draws up to 30 predictions from a normal distribution with mean equal to the true separation for that trial,  $\mathbf{r}_i$ , and covariance  $\tilde{\mathbf{P}}_{*|k}$ . The simulation varies each of the standard deviations and the correlation coefficient of  $\tilde{\mathbf{P}}_{*|k}$  for each prediction, using the parameters Table 1 also lists. Figure 1 shows three simulated predictions for a typical Monte Carlo trial.

The simulation computes the likelihood ratio for the WSPRT as follows. At the beginning of each trial, the simulation evaluates Eq.(25) using a multivariate Gaussian cumulative distribution function, with that trial's values for  $\hat{\mathbf{r}}_{*|o}$  and  $\hat{\mathbf{P}}_{*|o}$ . Then, as the simulation obtains each new prediction, it updates the estimated covariance and state for that trial using Eqs. (19) and (20), and then uses these values in the multivariate Gaussian cumulative distribution function to compute  $P_{c|k}$  using Eq.(24). The simulation then computes the likelihood ratio using Eq. (26), and compares it to the outer bounds for the decision limits  $A$  and  $B$ , given by Eq. (6). If the likelihood ratio exceeds either limit, the trial terminates; otherwise, the simulation obtains another prediction for that trial, until it reaches the maximum number of predictions we allowed. Since we do not permit the WSPRT to continue beyond a fixed number of predictions for each trial, we do not expect it to always terminate with a decision.

## V. Results

We performed 1,200,000 Monte Carlo trials for each of three variations of the WSPRT thresholds. Table 2 describes these variations, and summarizes the results. One may notice several trends from these results. First, in all cases, the achieved false alarm and missed detection rates were better than the target thresholds; the missed detection rates, in particular, were almost two orders of magnitude lower than those sought. In all cases, since we limited the number of observations available, there were many cases in which the WSPRT failed to reach a decision, but even in the most stringent case we tested, the no-decision rate was less than 1%. Moving from left to right in the table, as we made it easier to reach a decision by relaxing the decision limits, the no-decision rate went down, as did the average number of observations required to reach a decision.

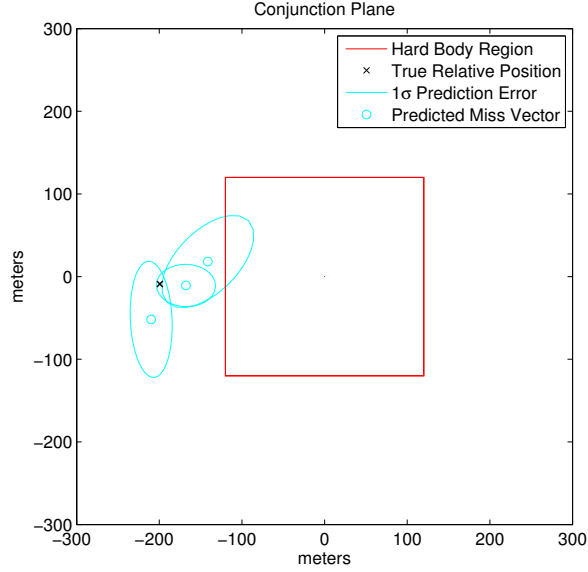


Figure 1. Three simulated predictions for a typical trial.



**Table 2. Comparison of 1,200,000 trials using various decision limits.**

Acceptable False Alarm Rate	5%	10%	33.3%
Acceptable Missed Detection Rate	0.1%	1%	10%
Dismissal Limit, $A$	950	90	6.667
Alarm Limit, $B$	0.05005	0.10101	0.3704
Number of Misses	1,068,232	1,124,677	1,103,506
Number of Hits	131,786	75,323	96,494
Number of False Alarms	18,989	32,545	74,090
Number of Missed Detections	7	34	468
Number of No Decisions	6,923	1,329	116
Achieved False Alarm Rate	1.8%	2.9%	6.7%
Achieved Missed Detection Rate	0.0053%	0.045%	0.49%
No Decision Rate	0.58%	0.11%	0.0097%
Avg. Number of Observations	7.8	3.0	1.5

When contemplating these results, one should appreciate that our test procedure drew the true relative positions from the same distribution that we used to compute the prior probability of collision. Had we drawn the truth values from a different distribution, the results could have been worse, depending on how the distributions differed. Since the true distribution of the relative position predictions will never be known, the margin in performance that Table 2 depicts is reassuring.

## VI. Conclusions

We may conclude that the WSPRT presented in this work represents a significant improvement over Reference 2, our previous WSPRT aimed at using the type of conjunction data most commonly available in the current state of the practice. We consider the improvements with respect to Reference 2 to be the following. Firstly, Reference 2 failed to exceed the acceptable false alarm and missed detection rates, even though that work attempted to derive tighter tolerances for the decision limits  $A$  and  $B$  than Wald's outer bounds. Secondly, the final result for the likelihood ratio in the present work, Eq. (26), is significantly simpler, both in form and in ease of computation, than either the Frequentist or Bayesian methods that Reference 2 describes. Lastly, although this work derives and tests the new result under assumptions of independence and normality, the form of Eq. (26) suggests our conjecture that a WSPRT based on Eq. (26) would be equally valid with other means of computing collision probability, which relax or otherwise improve upon the assumptions we made in this work.

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