

# OPTIMAL RECURSIVE DIGITAL FILTERS FOR ACTIVE BENDING STABILIZATION

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In the design of flight control systems for large flexible boosters, it is common practice to utilize active feedback control of the first lateral structural bending mode so as to suppress transients and reduce gust loading. Typically, active stabilization or phase stabilization is achieved by carefully shaping the loop transfer function in the frequency domain via the use of compensating filters combined with the frequency response characteristics of the nozzle/actuator system. In this paper we present a new approach for parametrizing and determining optimal low-order recursive linear digital filters so as to satisfy phase shaping constraints for bending and sloshing dynamics while simultaneously maximizing attenuation in other frequency bands of interest, e.g. near higher frequency parasitic structural modes. By parametrizing the filter directly in the z-plane with certain restrictions, the search space of candidate filter designs that satisfy the constraints is restricted to stable, minimum phase recursive low-pass filters with well-conditioned coefficients. Combined with optimal output feedback blending from multiple rate gyros, the present approach enables rapid and robust parametrization of autopilot bending filters to attain flight control performance objectives. Numerical results are presented that illustrate the application of the present technique to the development of rate gyro filters for an exploration-class multiengine space launch vehicle.

## 1 INTRODUCTION

The design of bending filters that robustly attenuate or active stabilize parasitic structural modes is of paramount importance in the development of a launch vehicle flight control system.<sup>1,2,3,4</sup> The separation of the control input effector, typically a thrust vector control (TVC) system, and the transducer element, a rate gyro, produces non-minimum phase parasitic elastic response with respect to thrust vector inputs. This structural response, in the case of large boosters, is highly coupled with other parasitic modes and appears at relatively low frequencies with respect to the control system bandwidth. Simple low-pass or notch filters commonly employed in aircraft feedback control applications have insufficient performance to ensure robust stabilization of the structural modes, usually due to the penalty in open-loop phase that precludes stable control of the rigid body dynamics. Furthermore, the propellant sloshing modes may be in close proximity to the bending spectrum, further complicating the filter requirements.

While rigorous controller design techniques such as  $H_2$  and  $H_\infty$ <sup>5</sup> yield high-performance compensators, they do so at the expense of high controller order and limited design flexibility for a high-order plant, and do not guarantee classical stability margins.<sup>6</sup> In addition, since large boost vehicles exhibit uncertain nonminimum-phase elastic behavior over a broad spectrum of frequencies, the requisite model reduction of the plant or the resultant compensator can yield poor performance.<sup>5,7</sup>

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Typical integrated vehicle linearized models used in control design are of order 200 or more, and simultaneously consider all adverse control-structure interaction effects.<sup>8</sup> Although  $H_\infty$  techniques have seen limited applications in launch vehicle flight control,<sup>9</sup> when algorithm complexity is considered, experience has shown that classically tuned and highly optimized linear bending filters provide the best performance in this application.<sup>1,10</sup>

Methods of structural stabilization in launch vehicle control system design include active (phase) or passive (gain) stabilization.<sup>3</sup> In the case of active stabilization, a specific phase lag target is achieved by the bending filter in a region near the bending mode frequency. The open-loop peak response of the bending mode thus exceeds unity but does not encircle the  $-1 + j0$  point in the Nyquist plane. In the case of passive stabilization, the bending filter provides sufficient attenuation near the bending mode frequencies such that the open-loop gain of each passively stabilized mode does not exceed unity or some minimum value under all parameter perturbations.

Active stabilization is easier to achieve for modes with high open-loop gain that are near the rigid body control frequency. The phase lag requirement of the filter is combined with the natural parasitic lag of the actuator dynamics, and is substantially less than that required to provide a high degree of stopband attenuation at the same frequency. Conversely, passive stabilization is often employed for high-frequency modes, where the uncertainty levels are substantially higher, especially with respect to open-loop phase characteristics near the inertial coupling resonances or *tail-wag-dog* frequencies of the vectoring engines. Passive stabilization is easier to achieve for high-frequency modes due to the reduced phase penalty incurred by enforcing a stopband attenuation requirement at a frequency well above the rigid-body control frequency.

In both cases, a numerical optimization technique is utilized to produce highly optimized infinite impulse response (IIR) filter designs that provide maximum robustness to structural mode uncertainty while minimizing phase lag at critical control system and sloshing mode frequencies. Various optimization and design methods have been employed for a variety of IIR filter applications, including spacecraft and launch vehicles.<sup>11,12,13,14</sup> In the following sections, the structure of a straightforward numerical optimization technique as well as its objectives and constraints will be detailed. This technique relies on direct pole-zero optimization in the complex ( $z$ ) plane and is able to simultaneously satisfy phase constraints and attenuation objectives by employing the well-known sequential quadratic programming approach.<sup>15</sup>

## 2 FILTER OPTIMIZATION

It is known that direct numerical optimization of transfer function coefficients is a very ill-conditioned optimization problem due to singularities in the gradient of the objective function. Given that the target filter is in discrete form, details of its structure can be presumed *a priori* so as to reduce the search space to a set of already-known feasible solutions. For example, we desire a stable, nonminimum phase SISO discrete transfer function. As such, all poles and zeros of the discretized system must lie strictly within the unit circle. Similarly to the technique presented in (Reference 16), we choose to parametrize each filter in zero-pole-gain form as

$$H(z) = \frac{k_0 \prod (z^2 - \beta_i z + \alpha_i)}{\prod (z^2 - \beta_j z + \alpha_j)} \quad (1)$$

with  $i, j = 1 \dots k$  where  $k = n/2$  and  $n$  is the filter order. Furthermore,  $\beta_i^2 - 4\alpha_i < 0$  and  $\beta_j^2 - 4\alpha_j < 0$ , so the poles and zeros of the discrete filter always appear in complex pairs. In this

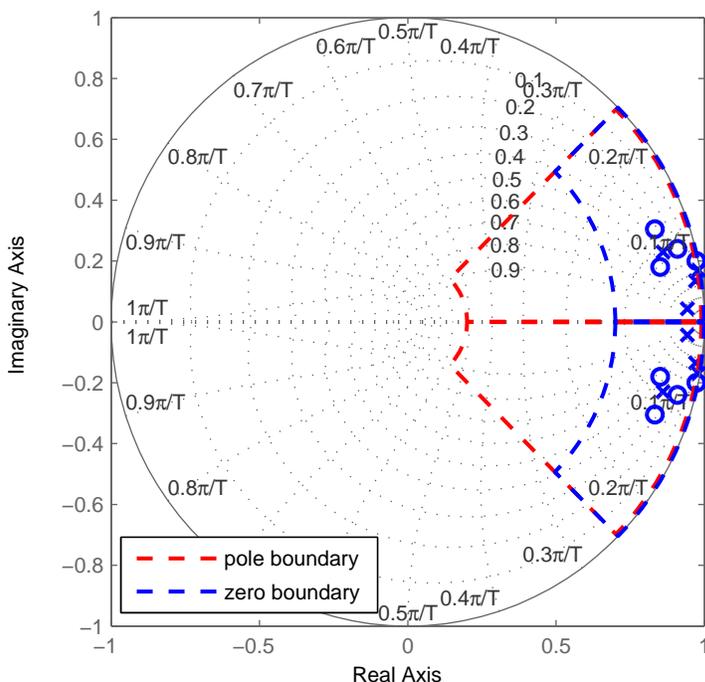
form, we may introduce a transformation to polar coordinates in the  $z$ -plane. Let  $\lambda_i = r_{zi}e^{\pm j\theta_{zi}}$  ( $\lambda_j = r_{pj}e^{\pm j\theta_{pj}}$ ) be the roots of the  $i^{\text{th}}$  ( $j^{\text{th}}$ ) zero (pole) polynomial in Eq. (1). We may then construct a parameter vector

$$x = [ r_{z1} \ \theta_{z1} \ r_{p1} \ \theta_{p1} \ \dots \ r_{zk} \ \theta_{zk} \ r_{pk} \ \theta_{pk} ]^T \quad (2)$$

and impose hard limits on the parameters  $x$  that restrict the admissible solutions to yield reasonable filter designs having a low-pass characteristic. The gain parameter  $k_0$  is always explicitly calculated to normalize the DC gain of the filter,  $H(z)_{z=1} = 1$ , where

$$k_0 = H(z) / |H(z)|_{z=1}. \quad (3)$$

In the most general sense, limit constraints on  $x_i$  impose sector constraints in the  $z$ -plane. For example, in order to yield stable, nonminimum phase filters, we require  $r_{zi}, r_{pj} < 1$ ; in order to enhance robustness of the filter to truncation errors, the location of complex poles and zeros can be confined to a sector that is separated from the unit circle by some nonzero  $\varepsilon$ .<sup>16,17</sup> An example of sector constraints are shown below in Figure 1 along with a typical filter pole-zero pattern. In this case, the zeros are confined to a sector closer to the boundary of the unit circle so as to ensure that all zeros are effective in shaping the response of the filter.



**Figure 1. Pole-zero sector boundary constraints**

The use of a transfer function for optimization of a general SISO linear system has the advantage of being a minimal realization.<sup>18</sup> If we consider the response of an  $n^{\text{th}}$  order discrete SISO linear

system to be a nonlinear process as a function of its parameters  $x$ ,

$$H = H_n(x, z), \quad (4)$$

and if the parameters  $x$  are the transfer function coefficients, the  $2n$  parameters in  $x$  are a minimal realization of  $H_n$  and completely span the objective space. Clearly, other discrete linear time-invariant realizations, such as the discrete-time state equations, are not minimal realizations, but linear transformations can render the realization sparse and eliminate the redundant parameters. Examples of such transformations include conversion to phase variable canonical form or diagonalization.

In the present application, it is neither necessary nor desired to span the entire objective space. By assuming that the class of discrete-time filters consisting of only complex pole and zero pairs will adequately satisfy the optimization objectives, one half of the search space can be eliminated via symmetry in the complex plane. It is also assumed that the number of zeros and the number of poles are equal (the resultant transfer function is biproper). The addition of the hard limits on the parameters confines the search space to specific sectors in the complex plane, and the result is that only stable, minimum-phase filters with a prescribed characteristic are considered in the space of admissible solutions. The dimensionality of the search space remains the same ( $2n$ ) but singularities in the parametrization are eliminated.

An additional nonlinear constraint is valuable for optimization algorithm convergence and filter implementation stability. When performing finite differencing to compute gradients of the objectives, superposition of filter poles and zeros may yield undesirable “masking” of pole-zero pairs. In addition, it is known that sensitivity to coefficient truncation is approximately proportional to the proximity of the poles to the unit circle and to the system zeros.<sup>17</sup> In order to avoid these phenomena, a minimum norm distance constraint is enforced in the complex plane. Given  $n$  zeros  $\lambda_i$  and  $n$  poles  $\lambda_j$ , taken pairwise, there are  $p = n!/(2(n-2)!)$  unique pole-zero pairs. For relatively low-order filters as are typical for bending filters, this is a modest number ( $p = \{6, 28, 66\}$  for  $n = \{4, 8, 12\}$ , respectively). Let  $P$  be the set of all unique pole-zero indices. A supplementary constraint of the form

$$c(x) = \min_{i,j \in P} \|\lambda_i(x) - \lambda_j(x)\| - \delta \quad (5)$$

ensures that any adjacent pole-zero pair has the minimum separation distance  $\delta$ , where  $\delta$  is a scalar parameter, and the constraint  $c(x) \leq 0$ . Due to symmetry, the constraint is applied only to the roots having positive complex parts. Since the index set  $P$  can be computed offline, determining  $c(x)$  is not a substantial computational burden even for orders above  $n = 12$ .

With the present parametrization, the filter can be optimized as a general nonlinear goal attainment problem of the form

$$\min \gamma \quad \text{subject to} \quad \begin{cases} F(x) - \gamma w & \leq 0 \\ c(x) & \leq 0 \\ x_{min} & \leq x \leq x_{max} \end{cases} \quad (6)$$

where  $F(x)$  is a vector objective function,  $w$  is a goal weight vector,  $\gamma$  is a scalar slack variable, and  $x_{min}$ ,  $x_{max}$  are the parameter limits. The nonlinear objective  $F(x)$  encodes filter design criteria, such as stopband attenuation, maximum passband ripple, phasing targets at particular frequencies, and so on, and the aforementioned constraint  $c(x)$  provides supplementary limits on the pole-zero separation. If all objectives cannot be simultaneously satisfied for a given filter order, it is often

pertinent to let those  $w_i$  associated with the phase targets be zero (converting them to constraints), and weight the attenuation constraints according to their relative importance. Note that the number of objectives can be less than, equal to, or greater than the number of parameters, so the Jacobian  $\nabla_x F$  is not square in general. This type of goal attainment problem can be readily solved using finite differencing and general nonlinear multiobjective optimization methods, such as sequential quadratic programming (SQP).<sup>19,15</sup>

### 3 DESIGN CRITERIA

Based on conservative heritage design guidelines for boost vehicles,<sup>3</sup> filters are designed to attenuate structural flexibility, including all uncertainties, by a minimum of 6 dB below the critical gain in the case that bending modes are passively stabilized. In the case of active stabilization, bending filters are designed to provide a minimum of 30 degrees of phase margin for phase stabilized bending modes when all uncertainties are accounted for. The assumed viscoelastic damping in the structural model strongly influences the modal attenuation requirements imposed on the filter. All bending modes are assumed to have a viscous damping ratio not exceeding one half percent since launch vehicle structural analysis and test has not demonstrated conclusive results supporting damping ratios above this level with flight-like boundary conditions.

Many launch vehicle autopilots are of the simple proportional-derivative type, where the attitude errors are a small-angle linearization of an error quaternion and the rate errors are sensed based on a platform-mounted rate gyro. The open-loop frequency response of the controlled vehicle dynamics can therefore be expressed in the form

$$G_{OL}(j\omega) = K_p H_{att}(j\omega) G_{att}(j\omega) + K_d H_{rat}(j\omega) G_{rat}(j\omega) \quad (7)$$

where  $K_p$ ,  $K_d$  are the proportional and derivative gains, respectively, and  $G_{att}$  and  $G_{rat}$  are the open-loop system frequency response with respect to a thrust vector command input in terms of the attitude and rate errors. The frequency responses of the attitude and attitude-rate filter are given by  $H_{att}$  and  $H_{rat}$ .

Suppose a critical bending mode appears in the open loop at the frequency  $\omega = \omega_b$ . If it is desired to phase-stabilize this mode, the total open-loop phase  $\angle G_{OL}(j\omega)$  at the critical frequency  $\omega_b$  should be near the critical phase  $\phi_b = -3\pi/2$ . Let the open-loop magnitude response at the critical frequency with  $H_{rat}(j\omega) = 1$  be given by  $r = |G'_{OL}(j\omega_b)|$ . To achieve a phase perturbation of the open-loop response without a substantial change in the open-loop magnitude at that frequency, the equality

$$r e^{j\phi_b} = G_{OL}(j\omega_b) \quad (8)$$

must be satisfied. For the purposes of determining the rate filter phase lag requirement, it will be assumed that the attitude filter is fixed. Equation (7) can be solved as

$$H_{rat}(j\omega_b) = (K_d G_{rat}(j\omega_b))^{-1} \left( r e^{j\phi_b} - K_p H_{att}(j\omega_b) G_{att}(j\omega_b) \right) \quad (9)$$

and the phase requirement for the filter at this frequency is simply the argument of the result.

### 4 NUMERICAL EXAMPLE

An example of the results of the present optimization method are shown below. The 8<sup>th</sup> order digital filter design is implemented with a sampling period of 20 ms (50 Hz) and provides robust

Specification	Rationale	Frequency	Value
Phase	Rigid-body phase margin	0.20 Hz	$\phi > -15$ deg
Phase	First bending mode phase stability	0.80 Hz	$-95 < \phi < -85$ deg
Phase	First bending mode phase stability	0.95 Hz	$-120 < \phi < -110$ deg
Phase	Transition band phase shaping	1.20 Hz	$\phi < -150$ deg
Attenuation	Passband ripple	0.10-0.60 Hz	$-1 <  H  < 1$ dB
Attenuation	Transition band peak	0.60-2.00 Hz	$ H  < 1$ dB
Attenuation	High-frequency robust passive stabilization	1.90-15.0 Hz	$ H  < -26$ dB
Attenuation	Notch high gain structural mode	1.90-2.28 Hz	$ H  < -32$ dB

**Table 1. Example filter specifications**

phase stabilization of the first lateral bending mode of a large, multiengineed exploration-class launch vehicle having a first lateral bending frequency near 0.9 Hz and a control system angular frequency of 0.2 Hz. The remaining structural attenuation is passive, and is achieved with strict attenuation requirements below the control system Nyquist frequency of 25 Hz. In addition, a particularly high-amplitude structural response near 2 Hz is suppressed by the addition of a supplementary attenuation constraint. The filter specifications and associated rationale are shown in Table 1. For implementation, the 8<sup>th</sup> order optimal filter is decomposed into two 4<sup>th</sup> order sections to avoid loss of precision on the target hardware.

The response of the optimized filter is shown in Figure 2. The 8<sup>th</sup> order filter is able to over-achieve the attenuation objectives while satisfying the phase shaping constraints that ensure rigid-body, slosh, and first bending mode stability. The resultant filter has additional robustness to uncertainty in the bending dynamics by increasing the stopband attenuation in the high-frequency portion of the spectrum.

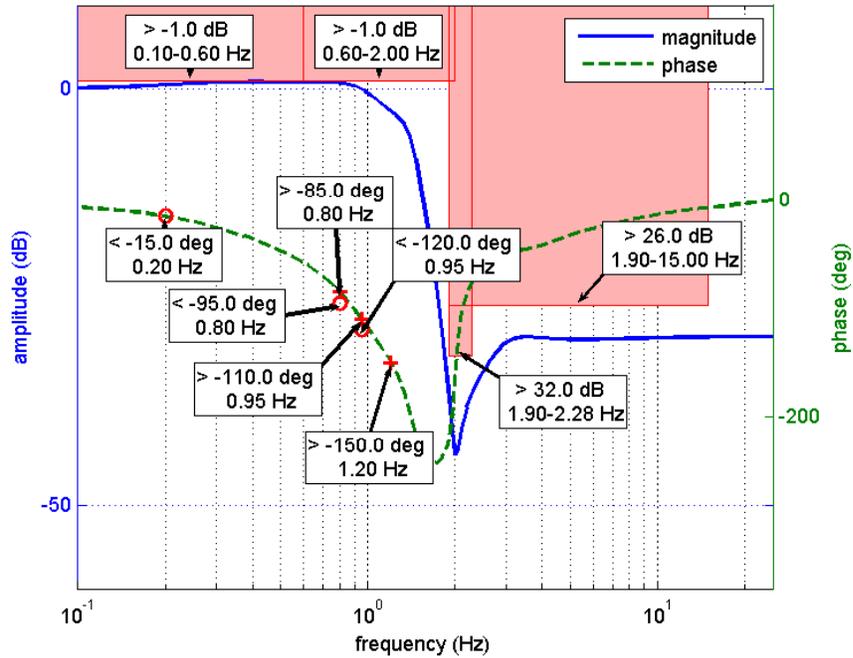


Figure 2. Example filter response

## 5 DISCUSSION

The present parametrization technique offers a flexible design method which is compatible with standard software packages for nonlinear multiobjective optimization without overburdening the underlying numerical process. While the launch vehicle application drove specific requirements to shape phase and gain characteristics well outside of the capabilities of any existing analytic filter design methodology, this framework has seen successful applications in other aspects of aerospace vehicle signal processing. In addition to autopilot bending filter design, the software toolbox implementing this optimization method has been used to design downsampling filters to attenuate the resonant modes of sensor mechanical isolators, and to create linear approximations of noise processes based on their spectral descriptions.

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