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# Conformal Weyl gravity and perihelion precession

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We investigate the perihelion shift of planetary motion in conformal Weyl gravity using the metric of the static, spherically symmetric solution discovered by Mannheim and Kazanas [Astrophys. J. **342**, 635 (1989)]. To this end we employ a procedure similar to that used by Weinberg for the Schwarzschild solution, which has also been used recently to study the solar system effects of the cosmological constant $\Lambda$. We show that besides the general relativistic terms obtained earlier from the Schwarzschild–de Sitter solution, the expression for the perihelion shift includes a negative contribution which arises from the linear term $\gamma r$ in the metric. Using data for perihelion shift observations, we obtain constraints on the value of the constant $\gamma$ similar to that obtained earlier using galactic rotational curves.

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## I. INTRODUCTION

One of the possible alternatives to the standard second-order Einstein theory of gravity which has been proposed during the last two decades is conformal Weyl gravity [1–3]. Instead of fixing the gravitational action by demanding that the theory be no higher than second order as in the case of the Einstein-Hilbert action, Weyl gravity employs the principle of local conformal invariance of spacetime as the supplementary condition that fixes the gravitational action. This means that the theory is invariant under local conformal stretching of spacetime geometry of the form

$$g_{\mu\nu}(x) \rightarrow \Omega^2(x) g_{\mu\nu}(x), \tag{1}$$

where $\Omega(x)$ is a smooth, strictly positive function. This restrictive conformal invariance leads to a fourth-order theory with a unique, conformally invariant action

$$\begin{aligned} I_W &= -\alpha \int d^4x(-g)^{1/2} C_{\lambda\mu\nu\kappa} C^{\lambda\mu\nu\kappa} \\ &= -2\alpha \int d^4x(-g)^{1/2}[R_{\mu\kappa}R^{\mu\kappa} - (R^\nu_\nu)^2/3] \\ &\quad + \text{a total derivative}, \end{aligned} \tag{2}$$

where $C_{\lambda\mu\nu\kappa}$ is the conformal Weyl tensor and $\alpha$ is a purely dimensionless coefficient. This action gives rise [1] to the gravitational field equations given by

$$\sqrt{-g} g_{\mu\alpha} g_{\nu\beta} \frac{\delta I_W}{\delta g_{\alpha\beta}} = -2\alpha W_{\mu\nu} = -\frac{1}{2} T_{\mu\nu}, \tag{3}$$

where $T_{\mu\nu}$ is the stress-energy tensor and

$$W_{\mu\nu} = 2C^{\alpha}{}_{\mu\nu}{}^{\beta}{}_{;\alpha\beta} + C^{\alpha}{}_{\mu\nu}{}^{\beta} R_{\alpha\beta}, \tag{4}$$

is the Bach tensor. When $R_{\mu\nu}$ is zero, the tensor $W_{\mu\nu}$, which consists of $R_{\mu\nu}$ and its derivatives, vanishes, too, so that any vacuum solution of Einstein's field equations is also a vacuum solution of Weyl gravity. However, the converse is not necessarily true. Despite the highly nonlinear character of the field equations, a number of exact solutions [4–7] of conformal Weyl gravity has been found in the case of spherical and axial symmetry. Recently, cylindrically symmetric solutions [8,9] in Weyl gravity have also been studied. Moreover, by studying the interior structure of a static, spherically symmetric gravitational source, it was shown [7] that the field inside the source is described exactly by a fourth-order Poisson equation, which under the proper conditions still admits a Newtonian potential $1/r$ term. Therefore, although the second-order Poisson equation in general relativity is sufficient to generate a Newtonian potential, it is not by any means a necessary requirement, so that Newton's law of gravity remains valid in the fourth-order Weyl gravity.

The exact static and spherically symmetric vacuum solution for conformal gravity is given, up to a conformal factor, by the metric [1]

$$ds^2 = -B(r)dt^2 + \frac{dr^2}{B(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{5}$$

where

$$B(r) = 1 - \frac{\beta(2 - 3\beta\gamma)}{r} - 3\beta\gamma + \gamma r - kr^2, \tag{6}$$

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and $\beta$, $\gamma$, and $k$ are integration constants. This solution includes as special cases the Schwarzschild solution ($\gamma = k = 0$) and the Schwarzschild–de Sitter ($\gamma = 0$) solution, the latter requiring the presence of a cosmological constant in Einstein gravity. The constant $\gamma$ has dimensions of acceleration and so the solution provides a characteristic, constant acceleration without having introduced one at the Lagrangian (such as in the relativistic implementation of MOND with TeVeS [10]). It should be noted that a metric with a similar linear term was also introduced recently [11] in a model for gravity at large distances based, however, on very different considerations.

Although the magnitude and the nature of the integration constant $\gamma$ in (6) remain uncertain, it has been associated [1] with the inverse Hubble length, i.e., $\gamma \simeq 1/R_H$. In this case the effects of the acceleration are comparable to those due to the Newtonian potential term $2\beta/r \equiv r_s/r$ ($r_s$ is the Schwarzschild radius) on length scales given by

$$r_s/r^2 \simeq \gamma \simeq 1/R_H \quad \text{or} \quad r \simeq (r_s R_H)^{1/2}. \tag{7}$$

As noted in Ref. [1], for a galaxy of mass $M \simeq 10^{11}\ M_\odot$ with $r_s \simeq 10^{16}$ cm and $R_H \simeq 10^{28}$ cm, this scale is $r \sim 10^{22}$ cm, i.e., roughly the size of the galaxy, a fact that prompted Mannheim and Kazanas to produce fits to the galactic rotation curves using the metric of Eq. (6) above. One should point out that Eq. (7) describes not a particular length scale but a continuum of sizes at which the contribution from the linear term becomes significant. Objects along this sequence encompass not only galaxies but also, at larger scales, galaxy clusters and, at lower scales, globular clusters, only recently found to require the presence of dark matter in order to account for the observed dynamics [12,13]. As we know, to account for the dynamics of objects along this sequence within the standard gravitational theory, one would need to invoke the presence of dark matter.

The classical tests in Einstein's gravity for the metric (6) with vanishing $\gamma$ have been well documented since the discovery of the theory and, therefore, it would be natural to study any effects of the linear term in the metric on such tests. The issue of the propagation of null geodesics, particularly the computation of the bending of light by a spherically symmetric object using the metric in (6), has been studied in detail. The first results obtained in Refs. [14–17] showed that besides the positive Einstein deflection of $4\beta/b$, the expression for the deflection of light in Weyl gravity contained an extra term $-\gamma b$, where $b$ is the impact parameter. This led to the paradoxical situation where the bending angle in lensing increased with the light ray's impact parameter with respect to the lens. This problem was later solved in Ref. [18], where it was shown that when the curvature of the background, asymptotically nonflat geometry of (6) is taken into account, the total bending angle is given by

$$\Delta\psi = \frac{4\beta}{b} - \frac{2\beta^2\gamma}{b} - \frac{kb^3}{2\beta}, \tag{8}$$

such that the contribution from the linear term in the metric is inversely proportional to the impact parameter. Moreover, its ratio to that of the standard $1/r$ component is of order $\beta\gamma$, which, given the associations and magnitudes of these constants for a galaxy, we get $\beta\gamma \simeq 10^{-12}$, i.e., insignificant for all practical purposes.

In this paper we study timelike geodesics in Weyl gravity, particularly the effect of the linear term in the metric on the perihelion shift. Then using the available data for planetary perihelion shifts, we get constraints on the magnitude of $\gamma$ similar to that obtained earlier [1] from the fitting of galactic rotational curves. Earlier studies [19–22] of the geodesic structure in the Schwarzschild–de Sitter solution showed that the cosmological constant $\Lambda$ increases the perihelion shift in the Schwarzschild solution by $\pi\Lambda a^3(1-\epsilon^2)^3/m$, where $a$, $\epsilon$ are the length of the semimajor axis and eccentricity, respectively, and $m = GM/c^2$, with $M$ being the mass of the source. Some authors [23,24] have used this correction to the perihelion shift of Mercury to obtain upper bounds for the cosmological constant, while others [25,26] showed that the effect of the cosmological constant is only significant at large radii and since it is unmeasurably small for Mercury's orbit, it cannot be used to limit the cosmological constant.

In Sec. II we start with the differential equation representing timelike geodesics and use a method similar to that used by Weinberg for the Schwarzschild solution to obtain an expression for the perihelion shift of a test particle in the exterior geometry described by the metric in (6). Then in Sec. III, the results are summarized and discussed.

## II. TIMELIKE GEODESICS, PERIHELION PRECESSION

The timelike orbits for the metric

$$ds^2 = -B(r)dt^2 + A(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{9}$$

are given by [27]

$$\frac{A(r)}{r^4}\left(\frac{dr}{d\phi}\right)^2 + \frac{1}{r^2} - \frac{1}{J^2 B(r)} = -\frac{E}{J^2}, \tag{10}$$

where $J$ and $E$ are constants of the motion. Since $\beta\gamma \ll 1$, for simplicity we take

$$B(r) = A^{-1}(r) = 1 - \frac{2\beta}{r} + \gamma r - kr^2. \tag{11}$$

Then the angular distance between the perihelion $r_-$ and aphelion $r_+$ is given by

$$\phi(r_+) - \phi(r_-) = \int_{r_-}^{r_+} A^{1/2}(r)\left[\frac{1}{J^2 B(r)} - \frac{E}{J^2} - \frac{1}{r^2}\right]^{-1/2}\frac{dr}{r^2}. \tag{12}$$

Using the fact that $dr/d\phi$ vanishes at $r_-$ and $r_+$, one can derive the following values for the constants of the motion

$$E = \frac{\frac{r_+^2}{B(r_+)} - \frac{r_-^2}{B(r_-)}}{r_+^2 - r_-^2}, \tag{13}$$

and

$$J^2 = \frac{\frac{1}{B(r_+)} - \frac{1}{B(r_-)}}{\frac{1}{r_+^2} - \frac{1}{r_-^2}}, \tag{14}$$

such that the angular distance in (12) becomes

$$\phi(r_+) - \phi(r_-) = \int_{r_-}^{r_+} A(r)^{1/2}\left[\frac{r_-^2(B^{-1}(r) - B^{-1}(r_-)) - r_+^2(B^{-1}(r) - B^{-1}(r_+))}{r_+^2 r_-^2(B^{-1}(r_+) - B^{-1}(r_-))} - \frac{1}{r^2}\right]^{-1/2}\frac{dr}{r^2}. \tag{15}$$

The perihelion precession per orbit is given by

$$\Delta\phi = 2|\phi(r_+) - \phi(r_-)| - 2\pi. \tag{16}$$

The expression in the square brackets in (15) vanishes at $r_+$ and $r_-$ and, hence, for slightly eccentric orbits we can write

$$\left[\frac{r_-^2(B^{-1}(r) - B^{-1}(r_-)) - r_+^2(B^{-1}(r) - B^{-1}(r_+))}{r_+^2 r_-^2(B^{-1}(r_+) - B^{-1}(r_-))} - \frac{1}{r^2}\right] \approx C\left(\frac{1}{r_-} - \frac{1}{r}\right)\left(\frac{1}{r} - \frac{1}{r_+}\right), \tag{17}$$

where $C$ is a constant. Then letting $u = 1/r$ and differentiating twice with respect to $u$ gives

$$C \approx 1 - \frac{(u_+ - u_-)(u_+ + u_-)A''(u)}{2(A(u_+) - A(u_-))}\bigg|_{u=L^{-1}}, \tag{18}$$

where $L = 2/(u_+ + u_-) = a(1 - \epsilon^2)$ is the *semilatus rectum* of the elliptic orbit. Again, for slightly elliptic orbits we can write

$$A(u_+) - A(u_-) \approx (u_+ - u_-)A'\left(\frac{u_+ + u_-}{2}\right), \tag{19}$$

so that

$$C \approx 1 - \frac{uA''(u)}{A'(u)}\bigg|_{u=L^{-1}}. \tag{20}$$

Now

$$\begin{aligned} A(u) &= B^{-1}(u) \\ &= 1 + 2\beta u + 4\beta^2 u^2 + \frac{4\beta k}{u} - \frac{\gamma}{u} + \frac{k}{u^2} - 4\beta\gamma + \cdots. \end{aligned} \tag{21}$$

Hence, substituting this in (20) we get

$$C \approx \frac{2\beta + 3\gamma u^{-2} - 12k\beta u^{-2} - 8ku^{-3}}{2\beta + 8\beta^2 u - 4k\beta u^{-2} + \gamma u^{-2} - 2ku^{-3}}\bigg|_{u=L^{-1}}, \tag{22}$$

or

$$C \approx 1 - 4\beta u - \frac{2\gamma}{u} + \frac{\gamma}{\beta u^2} + \frac{4k}{u^2} - \frac{3k}{\beta u^3}\bigg|_{u=L^{-1}}. \tag{23}$$

The expression for the angular distance in (15) can be written as

$$\phi(r_+) - \phi(r_-) = -\int_{u_-}^{u_+}\frac{A(u)^{1/2}du}{[C(u_- - u)(u - u_+)]^{1/2}}. \tag{24}$$

Using the substitution

$$u = \frac{1}{2}(u_+ + u_-) + \frac{1}{2}(u_+ - u_-)\sin\psi, \tag{25}$$

simplifies the integral to

$$\phi(r_+) - \phi(r_-) = \frac{1}{C^{1/2}}\int_{-\pi/2}^{\pi/2} A(\psi)^{1/2}d\psi. \tag{26}$$

This leads to

$$\begin{aligned} \phi(r_+) - \phi(r_-) \approx \pi\Big(1 &+ \frac{3\beta}{a(1-\epsilon^2)} + \frac{17}{2}ka^2(1-\epsilon^2)^2 \\ &+ \frac{3k}{2\beta}a^3(1-\epsilon^2)^3 - \frac{\gamma}{2\beta}a^2(1-\epsilon^2)^2 \\ &- 2\gamma a(1-\epsilon^2)\Big), \end{aligned} \tag{27}$$

so that the perihelion shift per orbit is given by

$$\Delta\phi \approx \frac{6\pi\beta}{a(1-\epsilon^2)} + \frac{3\pi}{\beta}ka^3(1-\epsilon^2)^3 - \frac{\pi}{\beta}\gamma a^2(1-\epsilon^2)^2. \tag{28}$$

## III. DISCUSSION AND CONCLUSION

The expression for the perihelion precession in conformal Weyl gravity obtained in (28) constitutes the main result of this paper. Besides the conventional term $6\pi\beta/a(1-\epsilon^2)$ representing the precession in the Schwarzschild geometry, the expression includes two other terms that originate due to the large-scale structure of the embedding spacetime. The $k$-dependent term is clearly associated with the $kr^2$ de Sitter term of the metric of Weyl gravity and is the same contribution arising from the cosmological constant obtained earlier in Refs. [19–22] for the Schwarzschild–de Sitter solution. This contribution was expected considering the fact that the metric in (6) reduces to the Schwarzschild–de Sitter solution when $\gamma$ approaches zero. The additional term $\pi\gamma a^2(1-\epsilon^2)^2/\beta$ in (28) originates from the linear $\gamma r$ term in the metric, and due to its negative sign, it reduces the amount of precession per orbit. This has also been observed before in Ref. [18] [see Eq. (8) in this paper], where it was shown that both the linear and de Sitter terms in the metric diminish the light-bending angle. In this case, however, the effects on the precession from these two terms are opposite.

The nature and magnitude of the constant $\gamma$ are still unknown. When $\beta = 0$ or when $r$ is sufficiently large so that all $\beta-$ dependent terms in (6) can be ignored, the metric can be rewritten under the coordinate transformation

$$\rho = \frac{4r}{2(1+\gamma r - kr^2)^{1/2} + 2 + \gamma r}, \quad \text{and} \qquad \tau = \int R(t)dt, \tag{29}$$

in the form

$$ds^2 = \frac{[1-\rho^2(\gamma^2/16 + k/4)]^2}{R^2(\tau)[(1-\gamma\rho/4)^2 + k\rho^2/4]^2} \times \left[-d\tau^2 + \frac{R^2(\tau)}{[1-\rho^2(\gamma^2/16+k/4)]^2} \times (d\rho^2 + \rho^2(d\theta^2 + \sin^2\theta d\phi^2))\right]. \tag{30}$$

The metric is, therefore, asymptotically conformal to a FLRW metric with arbitrary scale factor $R(\tau)$ and spatial curvature $\kappa = -k - \gamma^2/4$, i.e., it describes a spherically symmetric object embedded in a conformally flat background space. The fact that the curvature of this background space depends on $\gamma$ and $k$ points toward a cosmological origin of $\gamma$. This led to its association with the inverse Hubble length in Ref. [1], so that $\gamma \sim 10^{-28}\ \mathrm{cm}^{-1}$, and this explained quite successfully the flat rotation curves in galaxies and galaxy clusters.

On the other hand, there is nothing in the theory that forbids $\gamma$ from being also system dependent (i.e., like the parameter $\beta$, which is associated with the mass of the gravitating body), and in this case one can suggest that the linear term $\gamma r$ in the metric provides the necessary changes in the spacetime geometry to allow the embedding of a spherically symmetric matter distribution in a cosmological background. Assuming a spatially flat matter-dominated universe with $k = 0$, one can use (28) to get an upper bound for the magnitude of $\gamma$ from the difference between the observed and the general relativistic values of the precession of perihelia. So, for example, in the case of Mercury [28],

$$\Delta\phi_{\mathrm{obs}} - \Delta\phi_{\mathrm{gr}} = -0.0036 \pm 0.0050\ \mathrm{arcsec/century}, \tag{31}$$

where $\Delta\phi_{\mathrm{obs}}$ refers to the observed value of the perihelion precession once corrected for the general precession of the equinoxes and for the perturbations due to other planets. The contribution to the precession of the perihelion from the linear term in the metric can be written in the form

$$\Delta\phi_\gamma = -\frac{2\pi\gamma L^2}{r_s}, \tag{32}$$

where $r_s = 2\beta = \frac{2GM}{c^2}$. For Mercury $r_s L^{-1} = 5.3\times 10^{-8}$ so that

$$\Delta\phi_\gamma = 1.2\times 10^8 \gamma L\ \mathrm{rad/orbit}. \tag{33}$$

Therefore from (31) we get $\gamma < 1.5\times 10^{-31}\ \mathrm{cm}^{-1}$, which is about three orders of magnitude smaller than the large-scale estimate obtained from the fitting of galactic rotational curves. A similar calculation for other planets such as the Earth leads to tighter upper bounds for $\gamma$. However, when comparing bounds on $\gamma$ derived from planetary data with those obtained from the large-scale geometry of the universe, one should keep in mind that the calculation of the perihelion precession assumes that planets are treated as test particles devoid of self-gravity, and so this weakens the validity of such bounds. In fact, in the case of the asteroid Icarus with a diameter of just 1 km, (so that the effect of self-gravity in this case is insignificant), the difference between the observed and general relativistic perihelion precession is given by [27]

$$\Delta\phi_{\mathrm{obs}} - \Delta\phi_{\mathrm{gr}} = -0.5 \pm 0.8\ \mathrm{arcsec/century}, \tag{34}$$

which gives, using (32), the value $\gamma < 1.3\times 10^{-28}\ \mathrm{cm}^{-1}$. This fits nicely with the cosmological estimate of $10^{-28}\ \mathrm{cm}^{-1}$ obtained earlier from galactic rotational curves.

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