

**Bending of light in modified gravity at large distances**

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We discuss the bending of light in a recent model for gravity at large distances containing a Rindler-type acceleration proposed by Grumiller [Phys. Rev. Lett. **105**, 211303 (2010)]. We consider the static, spherically symmetric metric with cosmological constant  $\Lambda$  and Rindler-like term  $2ar$  presented in this model, and we use the procedure by Rindler and Ishak [W. Rindler and M. Ishak, Phys. Rev. D **76**, 043006 (2007).] to obtain the bending angle of light in this metric. Earlier work on light bending in this model by Carloni, Grumiller, and Preis [Phys. Rev. D **83**, 124024 (2011)], using the method normally employed for asymptotically flat space-times, led to a conflicting result (caused by the Rindler-like term in the metric) of a bending angle that increases with the distance of closest approach  $r_0$  of the light ray from the centrally concentrated spherically symmetric matter distribution. However, when using the alternative approach for light bending in nonasymptotically flat space-times, we show that the linear Rindler-like term produces a small correction to the general relativistic result that is inversely proportional to  $r_0$ . This will in turn affect the bounds on Rindler acceleration obtained earlier from light bending and casts doubts on the nature of the linear term  $2ar$  in the metric.

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**I. INTRODUCTION**

In a recent paper, Grumiller [1] has proposed a model for gravity of a central object at large scales, by imposing spherical symmetry, diffeomorphism invariance and additional assumptions, such as power counting renormalizability and analyticity. Starting from the most general spherically symmetric space-time metric [2]

$$ds^2 = g_{AB}(x^i)dx^A dx^B + \Phi^2(x^i)(d\theta^2 + \sin^2\theta d\phi^2), \quad (1)$$

$$A, B = 0, 1, \quad x^i = (t, r),$$

and making the above assumptions they use the process of spherical reduction [3] to reduce the four-dimensional Einstein-Hilbert action to a specific two-dimensional one for the two-dimensional metric  $g_{AB}(x^i)$  and dilation field  $\Phi(x^i)$ . The two-dimensional dilaton gravity model, which depends on two coupling constants  $a$  and  $\Lambda$ , is given by

$$S = - \int d^2x \sqrt{-g} [\Phi^2 R + 2(\partial\Phi)^2 - 6\Lambda\Phi^2 + 8a\Phi + 2]. \quad (2)$$

The solution to the equations of motion derived from this action leads to the line element

$$ds^2 = -K^2 dt^2 + \frac{dr^2}{K^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (3)$$

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$$K^2 = 1 - \frac{2M}{r} - \Lambda r^2 + 2ar, \quad (4)$$

which reduces to the Schwarzschild line element with mass  $M$ , when  $\Lambda = a = 0$ . The additional linear term in the norm  $K$  of the Killing vector  $\partial_t$  for the above metric, which is absent in Einstein gravity, is responsible for Rindler acceleration [4] and provides a constant acceleration  $a$  towards the source provided that  $a$  is positive. The author in [1] claims that this Rindler acceleration may explain the ‘‘Pioneer anomaly,’’ [5,6] i.e. an apparently constant, radial acceleration associated with the trajectories of the Pioneer spacecraft, of order  $a \sim 10^{-11} \text{ ms}^{-2}$ . This was based on the fact that for large distances  $r$  of the order of the Hubble length, both the Rindler term  $2ar$  and the cosmological term  $\Lambda r^2$  become relevant and approach unity for  $a \approx 10^{-10} - 10^{-11} \text{ m/s}^2$ , a value which is of the same order as the Pioneer acceleration and the MOND characteristic acceleration [7]. This effective model for gravity does not exclude the possibility that the Rindler acceleration  $a$  is system dependent, so it does not necessarily spoil the solar-system precision tests. The metric in (3) is also a vacuum solution of conformal Weyl gravity [8] which has also been proposed as a possible alternative to Einstein’s theory of gravity. Unlike the theory proposed by Grumiller, which assumes spherical symmetry in addition to other technical requirements, conformal Weyl gravity uses the principle of local conformal invariance of the space-time manifold (i.e. invariance under conformal transformations  $g_{\mu\nu} \rightarrow \Omega^2(x^\gamma)g_{\mu\nu}$ ;  $\Omega(x^\gamma)$  being a strictly positive function) as the only additional condition that fixes the fourth order

gravitational action. The fact that in both theories the effects of the linear term in the metric (3) become comparable to those due to the Newtonian potential term  $2M/r$  on length scales comparable to the size of a galaxy, led [1,8] to the fitting of galactic rotation curves without the need to invoke the presence of dark matter as normally done in standard gravitational theory. The classical tests of general relativity in the presence of the Rindler term in the metric (3) with  $\Lambda = 0$  were discussed in [9], and upper bounds for the Rindler acceleration were obtained by comparing results for the perihelion shifts, light bending, and gravitational redshifts with observational data. The perihelion shift data indicated that the Rindler acceleration is system dependent, at least in the case of massive test objects, with the tightest bound of  $|a| < 10^{-14} \text{ m/s}^2$  achieved from the Earth and Mars perihelion shifts. In the case of null geodesics, light bending data for quasars yielded a very loose bound of  $|a| < 6 \times 10^{-2} \text{ m/s}^2$ , while the Cassini data for radar echo time delay lead to  $|a| < 3 \times 10^{-9} \text{ m/s}^2$ , a value which is close to the Pioneer acceleration. In particular, computation of the light deflection by the sun using the metric in (3) showed that besides the positive Einstein bending angle  $4M/r_0$ , where  $r_0$  is the distance of closest approach of the photon's world line to the sun, the expression for the bending angle contains an extra term  $-ar_0/2$ . The angle of deflection was obtained by finding the angle between the asymptotes  $r \rightarrow \infty$  of the photon's trajectory as normally done in the Schwarzschild solution and other asymptotically flat space-times. This leads to the conflicting result that in lensing by a spherically symmetric object, in this case the sun, the larger the light ray's closest approach distance  $r_0$  to the lens, the larger the deflection angle. Moreover, the positive sign required by the Rindler acceleration  $a$ , to explain the Pioneer anomaly or the fitting of galactic rotation curves, leads to an effective repulsion for the light rays thereby producing a defocusing (instead of focusing) of light rays by the lens. In the case of Weyl gravity this latter problem can be solved [10] by utilizing the conformal invariance of null geodesics to find a conformal gauge in which the sign of  $a$  in (3) changes sign so that the theory becomes attractive for null geodesics. In this paper we consider  $\Lambda \neq 0$  and show that when the curvature of the background nonflat geometry in (3) is taken into account, the expression for the angle of deflection lacks the undesirable term  $-ar_0/2$ , and instead contains a term which is inversely proportional to the distance of closest approach  $r_0$ . This approach has been used earlier in [11] to study the bending of light in conformal Weyl gravity and was originally developed by Rindler and Ishak in [12] to show that the bending of light in the Schwarzschild-de Sitter space-time depends on the cosmological constant, even though  $\Lambda$  is not present in the null geodesic equations. The method of Rindler and Ishak has been discussed by several authors (see Refs. [13–19]) and was also used [20] in strong lensing by clusters to obtain observational

constraints on  $\Lambda$ . In Sec. II we present this approach for a general nonasymptotically flat space-time and apply it to the metric in (3) to show that the presence of the Rindler term has only a small effect on the bending angle which diminishes with  $r_0$ . In Sec. III the results are summarized and the nature of the constant  $a$  in the metric (3) is discussed in light of our result for the light bending angle.

## II. LIGHT BENDING IN CURVED BACKGROUND GEOMETRIES

The null geodesic equation for the static and spherically symmetric line element in Eq. (3) is given by

$$\frac{d^2 u}{d\phi^2} + u = 3Mu^2 - a, \quad (5)$$

where  $u = 1/r$ . Note that the effects of the cosmological  $\Lambda r^2$  term in the metric are not present in the null geodesic equation. Usually the null orbit is obtained as a perturbation of the undeflected straight line in flat space-time, i.e.,

$$r = \frac{R}{\sin\phi}, \quad (6)$$

which is the solution of Eq. (5) without the right hand side, and  $R$  is the impact parameter, i.e., the distance of closest approach of the light ray from the concentrated spherically symmetric matter distribution. This first approximation is then substituted in the right hand side of Eq. (5), and the resulting differential equation for  $u$  solved in the usual way. This has the solution

$$\frac{1}{r} = u = \frac{\sin\phi}{R} + \frac{M}{2R^2}(3 + \cos 2\phi) - a. \quad (7)$$

Note that in our case the parameter  $R$  is related to the distance  $r_0$  of closest approach (at which  $\phi = \pi/2$ ) by

$$\frac{1}{r_0} = \frac{1}{R} + \frac{M}{R^2} - a. \quad (8)$$

We now follow [12] and consider the subspace  $\theta = \pi/2$ ,  $t = \text{const}$  in (3) and let  $\psi$  be the angle between the two directions  $\mathbf{d} = d^i$  and  $\delta = \delta^j$  in the plane graph of the orbit Eq. (7), as shown in Fig. 1.

The angle  $\psi$  represents the angle that the photon orbit makes with the coordinate plane  $\phi = \text{const}$  and is given by the invariant formula

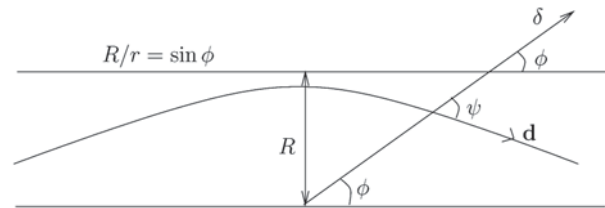


FIG. 1. The plane graph corresponding to the orbit in (7) with the one-sided deflection angle given by  $\epsilon = \psi - \phi$  (adapted from [12]).

$$\cos\psi = \frac{g_{ij}d^i\delta^j}{\sqrt{g_{ij}d^id^j}\sqrt{g_{ij}\delta^i\delta^j}}, \quad (9)$$

where  $d^i = (dr, d\phi)$ ,  $\delta^j = (\delta r, 0)$ , and  $g_{ij}$  denotes the metric on the  $\theta = \pi/2$ ,  $t = \text{const}$  surface. Differentiating (7) gives

$$A(r, \phi) \equiv \frac{dr}{d\phi} = \frac{Mr^2}{R^2} \sin 2\phi - \frac{r^2}{R} \cos\phi. \quad (10)$$

Substituting this in (9), we get

$$\cos\psi = \frac{|A|}{(A^2 + g^{11}r^2)^{1/2}}, \quad (11)$$

or

$$\tan\psi = \frac{(g^{11})^{1/2}r}{|A|}. \quad (12)$$

As can be seen from Fig. 1, the one-sided bending angle at a general point  $(r, \phi)$  along the orbit is given by  $\epsilon = \psi - \phi$ . Hence the total one-sided bending angle corresponds to  $\phi = 0$  or  $\epsilon = \psi_0$ , which corresponds to

$$r = \frac{R^2}{2M - aR^2}, \quad |A| = \frac{R^3}{(2M - aR^2)^2}. \quad (13)$$

Substituting these expressions in (12) and using the fact that  $\psi_0$  is a small angle and including only linear terms in  $a$ , we get

$$\psi_0 \approx \frac{2M}{R} + \frac{4M^2a}{R} - \frac{\Lambda R^3}{4M}. \quad (14)$$

Thus the total bending angle for the photon orbit is given by

$$2\psi_0 \approx \frac{4M}{R} + \frac{8M^2a}{R} - \frac{\Lambda R^3}{2M}. \quad (15)$$

### III. DISCUSSION AND CONCLUSION

In an asymptotically flat space-time such as the Schwarzschild solution, the angle of deflection of the null trajectory is obtained by letting  $r \rightarrow \infty$  or  $u \rightarrow 0$  in (7). However, in the case of a nonasymptotically flat space-time such as (3) the value of  $r$  is limited by the presence of a cosmological event horizon and so in general it would not be possible to take infinite values for the radial coordinate as was done by Carloni *et al.* in [9]. In fact, when  $M/r \ll 1$  such that the  $M$  dependent terms in (3) can be ignored, the metric can be rewritten under the coordinate transformation

$$\rho = \frac{2r}{(1 + 2ar - \Lambda r^2)^{1/2} + 1 + ar}, \quad \text{and} \quad \tau = \int R(t)dt, \quad (16)$$

in the form

$$ds^2 = \frac{[1 - (a^2 + \Lambda)\rho^2/4]^2}{R^2(\tau)[(1 - a\rho/2)^2 + \Lambda\rho^2/4]^2} \times \left[ -d\tau^2 + \frac{R^2(\tau)}{[1 - (a^2 + \Lambda)\rho^2/4]^2} \times (d\rho^2 + \rho^2(d\theta^2 + \sin^2\theta d\phi^2)) \right]. \quad (17)$$

The metric is therefore asymptotically conformal to a Friedmann-Robertson-Walker metric with arbitrary scale factor  $R(\tau)$  and spatial curvature  $\kappa = -(\Lambda + a^2)$ . So instead of taking an infinite value for  $r$ , one takes the value of  $r$  at  $\phi = 0$  which corresponds to that region of space-time between the Schwarzschild space-time and cosmological background space-time in (17), where no more significant bending occurs. It is in this region far from the spherically symmetric matter distribution that the source and observer are assumed to be located.

We note that besides the conventional Einstein bending angle of  $4M/R$ , the expression in Eq. (15) contains contributions from the cosmological and Rindler terms in the metric (3). As expected a positive  $\Lambda$  diminishes the bending angle, while a positive Rindler acceleration increases its value. When  $a = 0$  the expression for the bending angle reduces to that obtained by Rindler and Ishak in [12] for the Schwarzschild-de Sitter space-time. Unlike the previous result of [9], where the contribution to the bending angle from the Rindler term was found to be proportional to the distance of closest approach  $r_0$  and hence in direct contradiction with observations, the present treatment indicates that it is inversely proportional to  $R$  or  $r_0$ , just like the conventional term. Moreover, its magnitude is very small; its ratio to that of the standard  $1/r$  component is of the order  $Ma \approx 10^{-6}$ , when using the bound  $|a| < 3 \text{ nm/s}^2$  derived by Carloni *et al.* in [9], and therefore insignificant for all practical purposes. The bound on the Rindler acceleration  $|a| < 6 \times 10^{-2} \text{ m/s}^2$  mentioned earlier, obtained by Carloni *et al.* in [9] using light bending, becomes even less tight when the contribution from the Rindler term in (15) is used. This and the very different bounds that were obtained in [9] from the other classical tests add to the mystery about the true nature and magnitude of the constant  $a$  in the metric. As seen in Eq. (17) for large values of  $r$ , the metric in (3) becomes conformal to Friedmann-Robertson-Walker, so one can say that it describes a spherically symmetric object embedded in a conformally flat background space. The fact the curvature of this background space depends on  $a$  and  $\Lambda$  points towards a cosmological origin of  $a$ . On the other hand, the different bounds obtained from the perihelion shift data for different planets as discussed in the Introduction, implies that  $a$  is system dependent. Hence one can suggest that the Rindler term  $2ar$  in the metric provides the necessary changes in the space-time geometry to allow the embedding of a spherically symmetric matter distribution in a cosmological background.

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