Peak Pc Prediction in Conjunction Analysis

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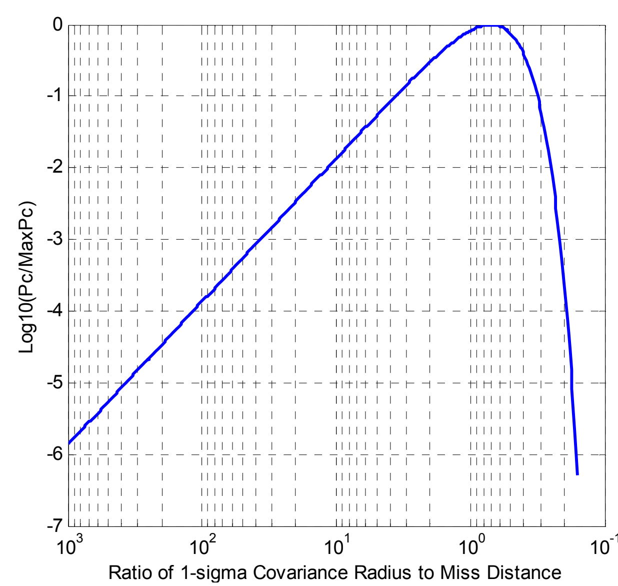
The Problem

- **Satellite conjunction risk typically evaluated through the probability of collision (Pc)**
  - Considers both conjunction geometry and uncertainties in both state estimates
- **Conjunction events initially discovered through JSpOC screenings, usually seven days before Time of Closest Approach (TCA)**
  - However, JSpOC continues to track objects and issue conjunction updates
  - Changes in state estimate and reduced propagation time cause Pc to change as event develops
  - These changes a combination of potentially predictable development and unpredictable changes in state estimate / covariance
- **Operationally useful datum: the peak Pc**
  - If it can reasonably be inferred that the peak Pc value has passed, then risk assessment can be conducted against this peak value
  - If this value below remediation level, then event intensity can be relaxed
- **Can the peak Pc location be reasonably predicted?**

Conjunction Event "Canonical Progression"

- **Conjunction typically first discovered 7 days before TCA**
  - Covariances large, so typically Pc below maximum
- **As event tracked and updated, changes to state estimate are usually relatively small, but covariance shrinks**
  - Because closer to TCA, less uncertainty in projecting positions to TCA
- **Theoretical maximum Pc encountered when 1-sigma covariance size to miss distance ratio is 1/√2**
  - After this, Pc usually decreases rapidly
- **Behavior shown in graph at right**
  - X-axis is covariance / miss distance
  - Y-axis is $\log_{10}(P_c/\max(P_c))$
  - Order of magnitude change in Pc considered significant, thus log-space more appropriate
- **How might this behavior be modeled?**
  - Underlying progression in presence of noise

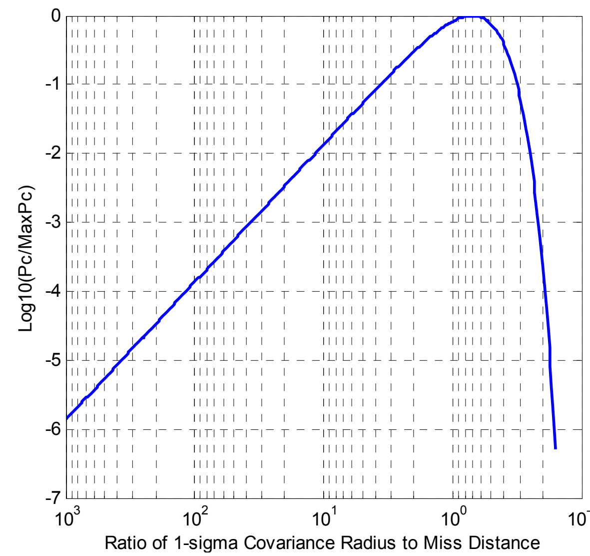


Proposed Choice of Modeling Variables

- **Dependent variable is log10 value of Pc**
  - Need to address problem of very small and 0 values for Pc
  - Majority of Pc values for purposes of operations "essentially 0": < 1E-10
    - Small values of Pc can be "floored" at 1E-10
    - Furthermore, long trains of leading or trailing 1E-10 values can also be eliminated from dataset for model tuning and evaluation; really just a function of when updates happen to occur.
- **Independent variable is time before TCA (usually in fractional days)**
  - Canonical behavior curve uses independent variable as ratio of covariance size to miss distance
  - Problematic independent variable for fitting
    - Not monotonic with time (but it does correlate at least moderately to time)
    - Need temporal independent variable in order to map to operational timelines
  - Thus, use time before TCA as independent variable for model

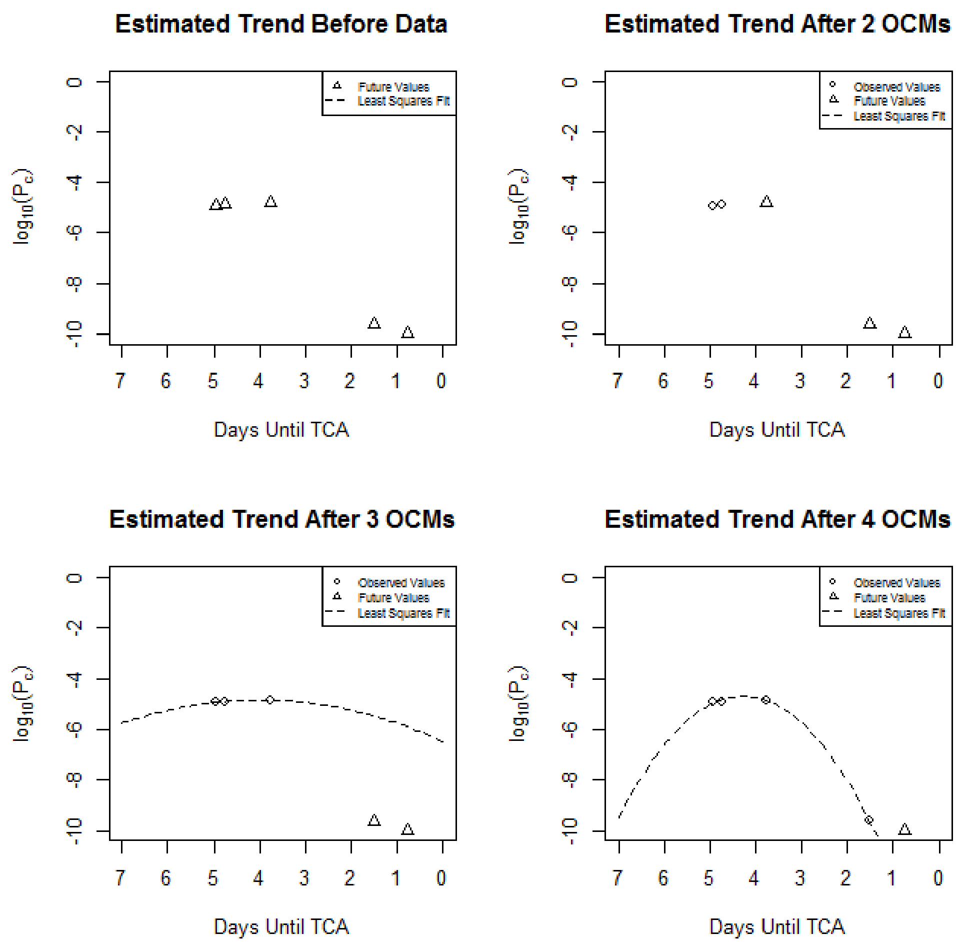
Bayesian Vertex Model

- **Approximate theoretical progression of log(P_c) values using a downward-opening parabola**
  - Equation in vertex form: $Y = a (x - h)^2 + b$
  - Can be recast as: $Y = \beta_0 + \beta_1 x + \beta_2 x^2$
  - Location of peak more important than peak value, so need not match functional form precisely
- **With regression analysis of training dataset, can establish prior distributions of set of β values**
- **Drawing from these priors, can use Bayes' theorem to construct posterior distributions**
  - This allows priors to be combined with unfolding data from current event
  - Can then estimate log(P_c) from mean values from parameter posteriors



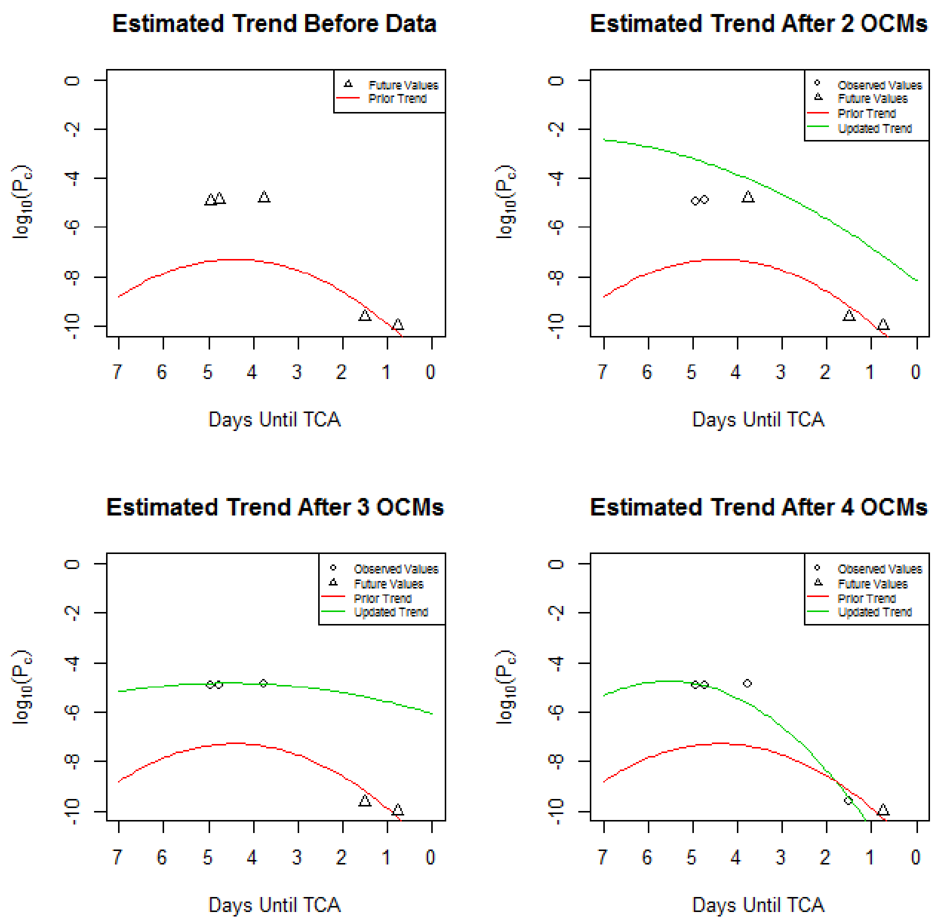
Using Frequentist Methods

- **If we refit the line each time we receive a new OCM using frequentist methods (*i.e.* least squares), we would see something like this**



Using Bayesian Methods

- **If we use Bayesian methods, it is possible to incorporate prior information into the estimates**



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Comparisons Between the Bayesian and Frequentist Models

- **Using the Bayesian methods, we can make predictions using only two OCMs (though generally these are not particularly informative)**
  - This is not possible with the frequentist model
- **The frequentist model fits the points as closely as possible, whereas the Bayesian model incorporates prior information, compromising between the current and previous data**
- **The fits are generally similar, but the Bayesian fit is generally more conservative**
  - The Bayesian model takes into account the uncertainty of the estimates, thus it is less likely to fit the data "too well"
  - As a result, the Bayesian model generally has wider error bounds, which are usually more realistic
  - The frequentist approach tends to chase the action, whereas the Bayesian approach is more realistically predictive (because it considers prior information)

Methodology Details (1 of 3)

- **We can calculate what is known as the posterior density of the parameters given the data**
  - $p(\beta|y) \propto p(y|\beta) * p(\beta)$
  - Thus, we specify a prior distribution for the beta parameters $p(\beta)$, update it with the data that we have seen $p(y|\beta)$, and get an updated probability distribution of the beta parameters given the data $p(\beta|y)$
- **Now, we can force the parabola to open downwards by choosing priors the allow only this shape**
  - Consider the model $Y = \beta_0 + \beta_1 x + \beta_2 x^2 + \varepsilon$, where ε is the noise in the measurement
  - If we force β_0 and β_2 to be negative, this will ensure a downward opening parabola will be fit each time and ensure that the vertex be realizable (*e.g.,* not have a y-value greater than 1, which is not possible for a Pc value)
  - This presents one potential hazard with the model: what if the observed data actually had the shape of an upward opening parabola? It would be fit with a horizontal line, which is not the correct shape

Methodology Details (2 of 3)

- **The resulting constraints are**

$$\beta_0 < 0$$
$$(\beta_1)^2 < 4\,\beta_0\,\beta_2$$
$$\beta_2 < 0$$

- **In order to specify these priors, we use truncated Normal distributions, so that**

$$\beta_0 \sim \mathrm{Normal}(\mu_0,\sigma_0^2)I(-\infty,0)$$
$$\beta_1 \sim \mathrm{Normal}(\mu_0,\sigma_0^2)I(-2\sqrt{(\beta_0\,\beta_2)},\ 2\sqrt{(\beta_0\,\beta_2)})$$
$$\beta_2 \sim \mathrm{Normal}(\mu_2,\sigma_2^2)I(0,\infty)$$

- **While other prior distributions are possible, we find that the truncated normal have the best convergence properties**
  – Gamma distributions were also attempted but exhibited high levels of autocorrelation and overall slow convergence

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Methodology Details (3 of 3)

- **Assume that ε ~ Normal(0, σ^2)**
- **Assume a Gamma prior on the inverse of the variance $1/\sigma^2$**
  - Common practice.
- **Choosing the parameters of these prior distributions**
  - Use restricted maximum likelihood to estimate downward opening parabolas on a set of test data (we examined over 1000 events)
  - Collect all of the betas from the fits
  - Find parameters of a truncated normal distribution that is close to the observed distribution of each parameter by matching quantiles

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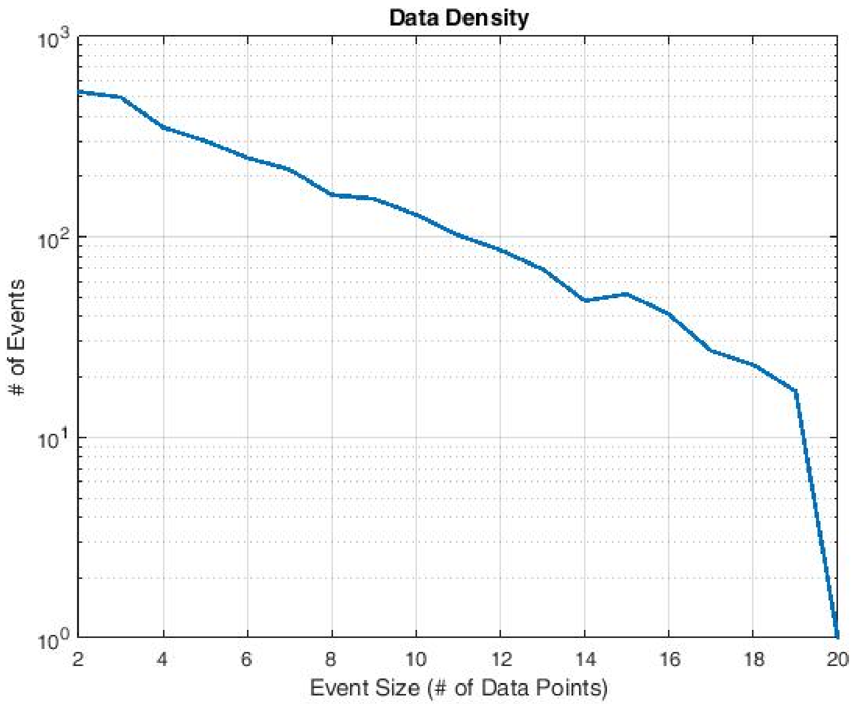
Bayesian Vertex Model: Model Performance Investigation

- **Conjunction data archive assembled for 2013-14 for well-populated orbit regime**
  - Perigee height between 500 and 750 km and eccentricity < 0.25
  - Thousands of events per year
- **Use part of 2013 data to "train" model—set prior distribution coefficients**
- **Use 2014 data as validation dataset**
- **Segregate performance results**
  - First, by total number of data points (CDMs) in the event
    - Data-poor events may perform worse than data-rich ones
  - Second, by data point number
    - How does model perform after point 3 versus point 6 or 10?

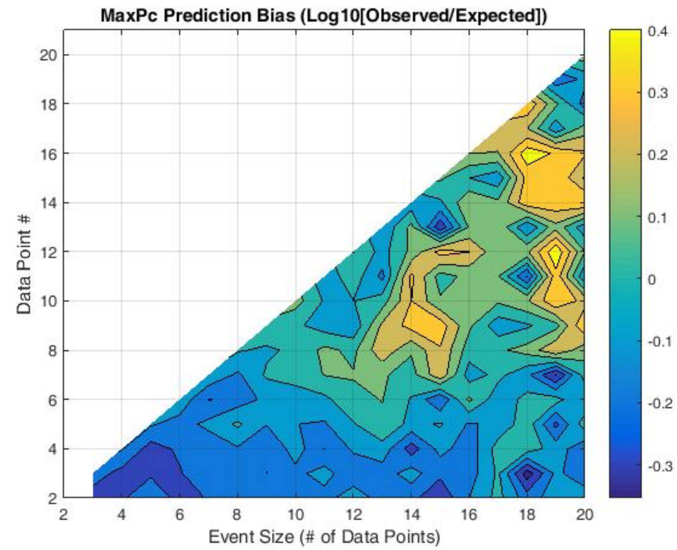
Bayesian Vertex Model: LEO2: Data Density

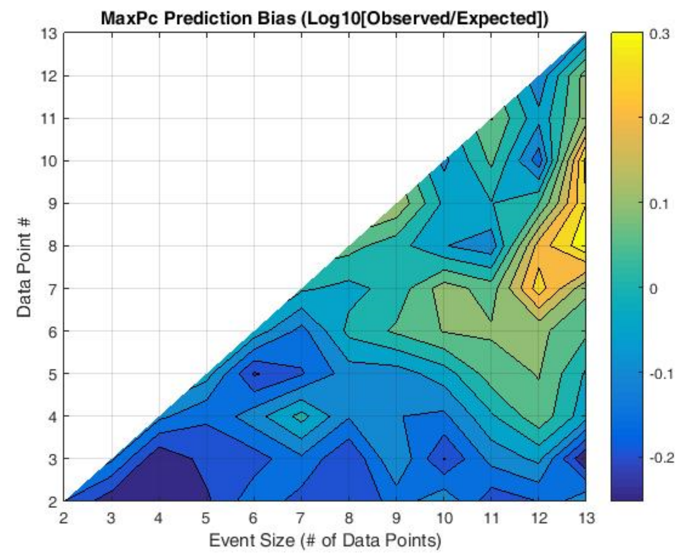
- **Probably want at least 50 events surveyed to feel confident about model performance conclusions**
- **This achieved only for event sizes smaller than 14 data points**
- **Should focus on performance results for these shorter events—sampling more plentiful**



Bayesian Vertex Model: Mean Peak Estimation Error

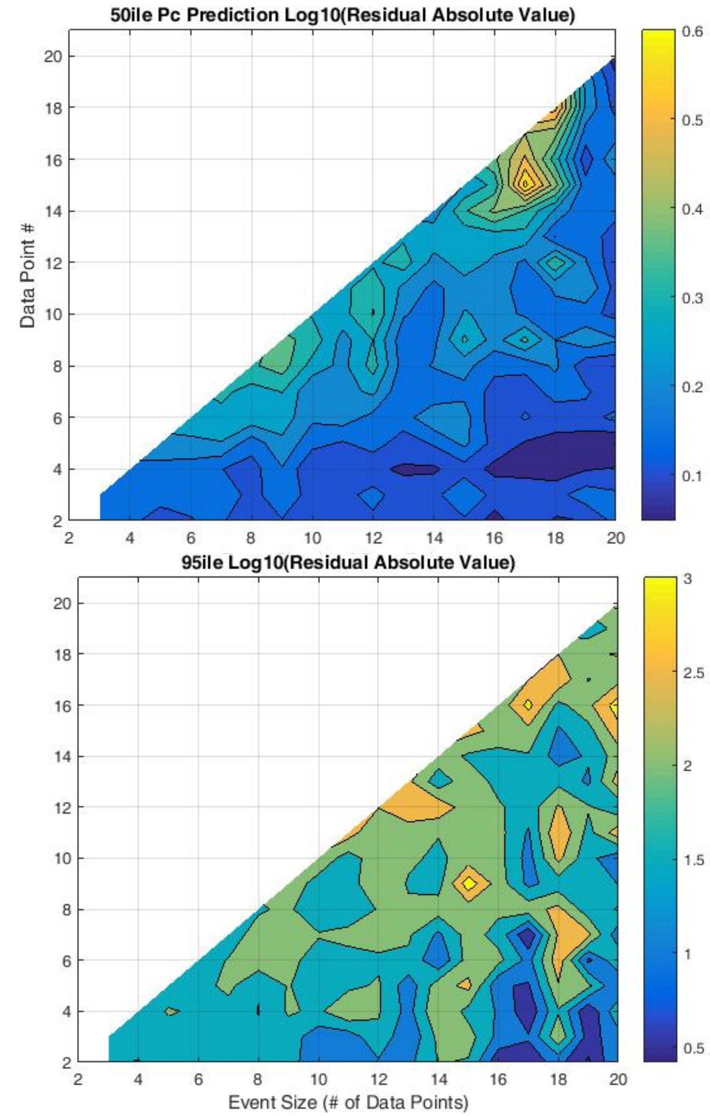
- **mean(Y – Yhat) for all the events of each size**
- **Value becomes unstable beginning at event sizes of about 13 observations**
- **Stable region shows mean values ranging from around 0 +- half an order of magnitude**
- **Model is biased but biased in a favorable direction**
  - Overpredicting leads to conservative safety-of-flight decisions—better than the reverse





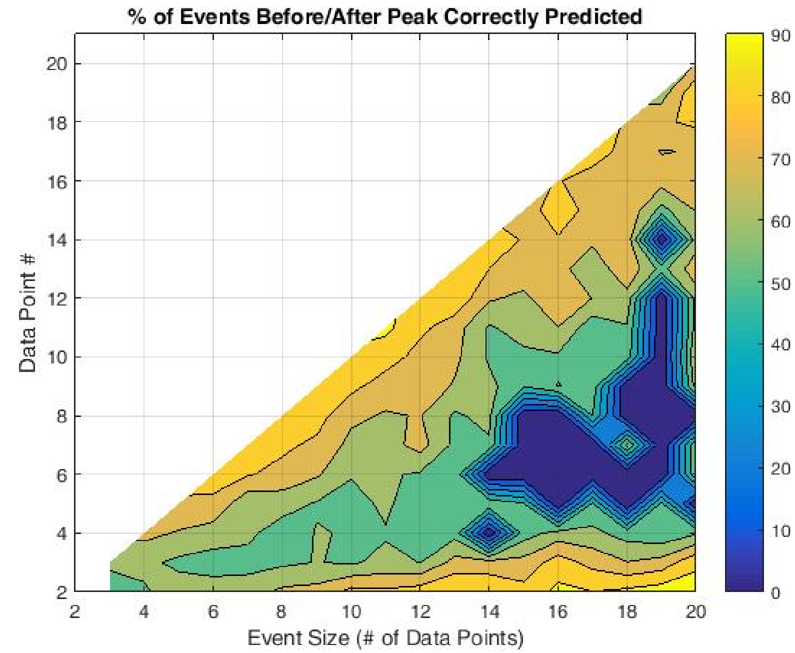
Bayesian Vertex Model: 50^{th} and 95^{th} percentile Peak Absolute Residual Errors

- **Focus on more stable region (event sizes of 13 or fewer points)**
- **At the 50^{th} percentile all of that area is less than 0.5 of an order of magnitude**
  - An acceptable result
- **At the 95^{th} percentile, that area varies between 0.5 and 3 orders of magnitude**
  - Probably not an acceptable result
- **Model probably not useful for peak prediction**
- **However, could still be useful for predicting whether peak has occurred**



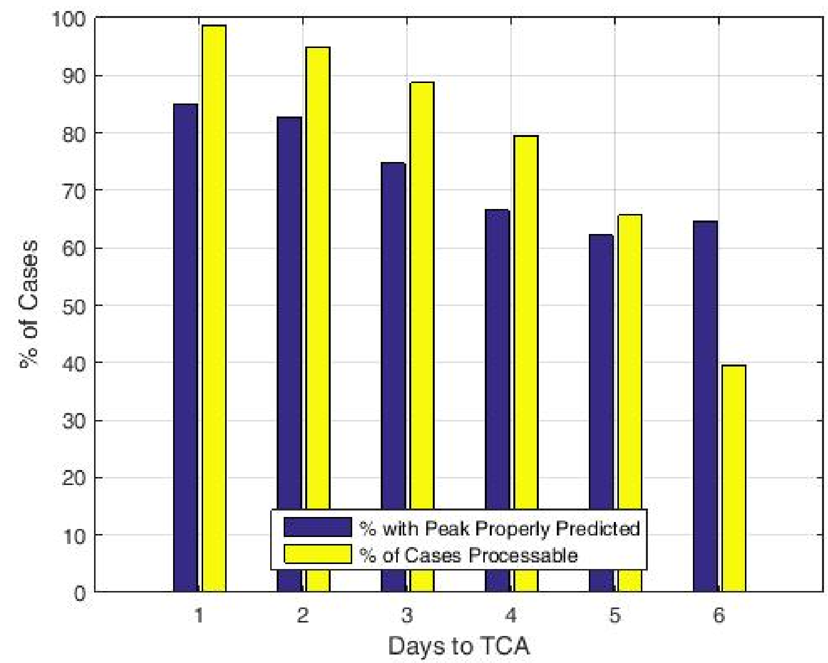
Bayesian Vertex Model: Peak Prediction Performance

- **Operational question: has the event reached its peak Pc value?**
- **Plot at right shows, for all events of a certain size after a certain data point, the percent correct peak predictions**
  - % of the time the model indicates the peak has already passed, and in fact it has
- **In region of interest (< 14 data points), performance always better than 50% once half the event points received**
  - Performance moves to 80-100% as number of points reaches total event size
- **However, difficult to use result, since # of points not known in advance**
  - Examine predictive force at "times to TCA" of operational interest



Bayesian Vertex Model: Peak Prediction Performance (cont'd)

- **Examine situation at typical maneuver planning and commit times**
  - 4, 3, 2, and 1 days before TCA
- **Blue bars show percentage of correct before/after peak predictions at these time points**
- **Yellow bars show number of events for which prediction was possible**
  - At least two points needed
  - MCMC fails to converge occasionally
- **Not stunning performance, but could be an operational tool of some utility**



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Conclusion/Future Work

- **A simple statistical model shows operational promise in determining whether the peak Pc value has occurred**
- **Additional areas requiring exploration**
  - Event Pc histories need categorization
    - May be that algorithm performs well only in “obvious” cases; may not be helpful more ambiguous situations where greater operational need
  - Different overall functional forms may yield better results
    - For instance, the log probabilities of collision are effectively bounded between -10 and 0, suggesting a different distribution (Beta) may be more appropriate
  - Other modeling paradigms
    - Other ways of borrowing information, *e.g.* mixed models
  - Longitudinal data analysis, because the observations are repeated measurements on different events