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THE MOTION OF A SATELLITE OF THE MOON

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I. INTRODUCTION

The motion of a satellite of the Moon depends on the potential field due to the Moon as well as the gravitational effects of the Earth and Sun. If one chooses a frame of reference attached to the Moon, it can be shown that the force field resulting from the Sun can be neglected when compared with the perturbing field of the Moon resulting from its oblateness. The effect of the Earth's field on the satellite is of the same order of magnitude as the Moon's perturbing field and must be included in an analysis of the motion of a satellite of the Moon. We will assume that the distance between Earth and Moon remains constant, and we will consider satellite orbits of small eccentricity. It will be shown that a nearly circular polar orbit will digress less than 1 deg from a polar orbit and that the change in eccentricity is less than a factor of e in one year.

II. COMPARISON OF THE EFFECTS OF THE EARTH AND SUN ON THE MOTION OF A SATELLITE OF THE MOON

Let m, M, M, M_0 be the masses of the satellite, Moon, Earth, and Sun, respectively. We assume that a coordinate system centered at the Sun is an inertial frame of reference. From Sketch 1, the motion of the satellite is given by

$$m \frac{d^2 \mathbf{r}}{dt^2} = m \frac{d^2 \mathbf{p}}{dt^2} - m \frac{d^2 \mathbf{s}}{dt^2} \tag{1}$$

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From

$$m\,\frac{d^2\boldsymbol{p}}{dt^2} = \boldsymbol{F}_m(\mathrm{M}_0) + \boldsymbol{F}_m(M) + \boldsymbol{F}_m(\mathrm{M})$$

$$\mathrm{M}\,\frac{d^2\boldsymbol{s}}{dt^2} = \boldsymbol{F}_\mathrm{M}(\mathrm{M}_0) + \boldsymbol{F}_\mathrm{M}(M) \tag{2}$$

with $\boldsymbol{F}_\mathrm{M}(\mathrm{M}_0)$ the force of the Sun on the Moon, etc. (neglecting the force of the satellite on the Moon), Eq. (1) becomes

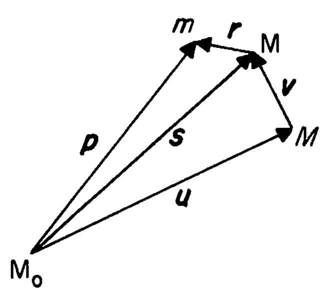
$$m\,\frac{d^2\boldsymbol{r}}{dt^2} = \boldsymbol{F}_m(\mathrm{M}) + \left[\boldsymbol{F}_m(\mathrm{M}_0) - \frac{m}{\mathrm{M}}\boldsymbol{F}_\mathrm{M}(\mathrm{M}_0)\right] + \left[\boldsymbol{F}_m(M) - \frac{m}{\mathrm{M}}\boldsymbol{F}_\mathrm{M}(M)\right]$$

$$= \boldsymbol{F}_m(\mathrm{M}) + \left(-\frac{Gm\mathrm{M}_0}{|\boldsymbol{p}|^3}\boldsymbol{p} + \frac{Gm\mathrm{M}_0}{|\boldsymbol{s}|^3}\boldsymbol{s}\right) + \left[-\frac{Gm\mathrm{M}}{|\boldsymbol{v}+\boldsymbol{r}|^3}(\boldsymbol{v}+\boldsymbol{r}) + \frac{GmM}{|\boldsymbol{v}|^3}\boldsymbol{v}\right] \tag{3}$$

The magnitude of the middle term of Eq. (3) can be approximated by $Gm\mathrm{M}_0|\boldsymbol{r}|/|\boldsymbol{s}|^3$ for $|\boldsymbol{r}| \ll |\boldsymbol{s}|$; the magnitude of the last term of Eq. (3) can be approximated by $GmM|\boldsymbol{r}|/|\boldsymbol{v}|^3$ for $|\boldsymbol{r}| \ll |\boldsymbol{v}|$. The ratio of these terms is

$$\left(\frac{\mathrm{M}_0}{M}\right)\left|\frac{\boldsymbol{v}}{\boldsymbol{s}}\right|^3 \approx (330{,}000)\left(\frac{240{,}000}{93{,}000{,}000}\right)^3 \approx \frac{1}{2}\cdot 10^{-2} \tag{4}$$

Thus, we are justified in omitting the effect of the Sun.



Sketch 1

III. LAGRANGE'S EQUATIONS OF MOTION

Let the Lagrangian, $L = T - V$, of a system of particles be given by $L = L(x^1, x^2, \cdots, x^n, \dot{x}^1, \dot{x}^2, \cdots, \dot{x}^n, t)$. The extremalization of $\int_{t_0}^{t_1} L\, dt$ leads to Lagrange's equations of motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right) - \frac{\partial L}{\partial x^i} = 0 \qquad i = 1, 2, \cdots, n \tag{5}$$

We will be interested in transformations of the type

$$x^i = x^i(y^1, \cdots, y^n, y^{n+1}, \cdots, y^{n+m}) \qquad i = 1, 2, \cdots, n \tag{6}$$

such that the Jacobian $|\partial x^i/\partial y^j|$, $i, j = 1, 2, \cdots, n$, does not vanish. It is a simple matter to show that

$$\frac{d}{dt}\left(\frac{\partial \bar{L}}{\partial \dot{y}^i}\right) - \frac{\partial \bar{L}}{\partial y^i} = \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^\alpha}\right) - \frac{\partial L}{\partial x^\alpha}\right]\frac{\partial x^\alpha}{\partial y^i} \qquad i = 1, 2, \cdots, n + m \tag{7}$$

with $\bar{L}(y^1, \cdots, y^{n+m}, \dot{y}^1, \cdots, \dot{y}^{n+m}, t) \equiv L(x^1, \cdots, x^n, \dot{x}^1, \cdots, \dot{x}^n, t)$. The index, α, of Eq. (7) is summed from 1 to n. From Eqs. (5), it follows that

$$\frac{d}{dt}\left(\frac{\partial \bar{L}}{\partial \dot{y}^i}\right) - \frac{\partial \bar{L}}{\partial y^i} = 0 \qquad i = 1, 2, \cdots, n + m \tag{8}$$

Moreover, if $|\partial x^i/\partial y^j| \neq 0$, $i, j = 1, 2, \cdots, n$, one quickly deduces that the vanishing of $d/dt(\partial \bar{L}/\partial \dot{y}^i) - \partial \bar{L}/\partial y^i$ for $i = 1, 2, \cdots, n$, yields the set of Eq. (5). Thus $y^{n+1}, \cdots, y^{n+m}$ can be chosen in any arbitrary manner, and in particular, one can adjoin to Eq. (8) m differential equations involving $y^1, \cdots, y^{n+m}$, provided these new equations are not inconsistent with $d/dt(\partial \bar{L}/\partial \dot{y}^i) - \partial \bar{L}/\partial y^i = 0$, $i = 1, 2, \cdots, n$. We will make use of this result in a subsequent analysis.

IV. THE AVERAGING PROCESS OF KRYLOFF-BOGOLIUBIOFF

Let $x(\psi)$ satisfy the equation

$$\frac{dx}{d\psi} = \epsilon f(x, \sin\psi, \cos\psi), \qquad \epsilon << 1 \tag{9}$$

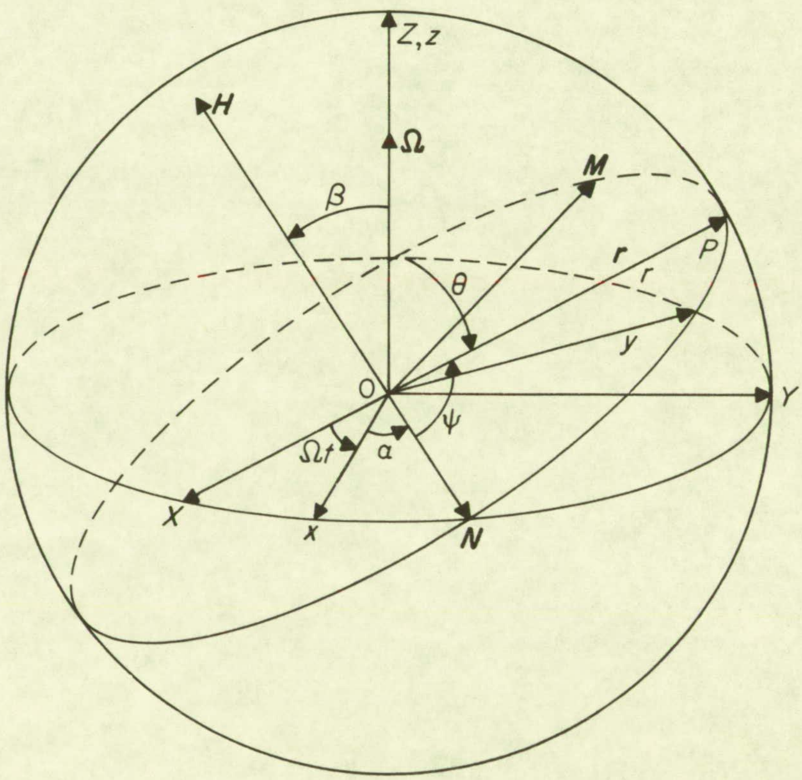
One replaces $f(x, \sin\psi, \cos\psi)$ by its average value, considering x as a parameter, to obtain

$$\frac{dx}{d\psi} = \frac{\epsilon}{2\pi} \int_0^{2\pi} f(x, \sin\theta, \cos\theta)\, d\theta = \epsilon F(x) \tag{10}$$

We are justified in replacing Eq. (9) with Eq. (10) as x is slowly varying and, hence, remains essentially constant as ψ ranges over the interval $(\psi, \psi + 2\pi)$. An integration of Eq. (10) yields $x = x(\psi, \epsilon)$. This method of obtaining an approximate solution of a nonlinear differential equation can be extended to a system of differential equations of the type given by Eq. (9).

V. QUASI-EULERIAN COORDINATES

A description of the quasi-Eulerian coordinates associated with a moving point P is given in Sketch 2. The x-axis points towards the Earth, the y-axis points in the direction of the orbit of the Moon, and the z-axis is the polar axis (axis of rotation of the Moon). The X-Y-Z frame is an inertial frame of reference coinciding with the x-y-z frame at $t = 0$. The unit vector $\mathbf{N}$ lies in the x-y plane, and $\mathbf{M}$ is a unit vector normal to $\mathbf{N}$ in the plane formed by $\mathbf{N}$ and the position vector $\mathbf{r}$. We define $\mathbf{H}$ such that $\mathbf{H} = \mathbf{N} \times \mathbf{M}$.



Sketch 2

The coordinate transformation between the spherical coordinates (θ, ϕ) and the quasi-Eulerian coordinates (α, β, ψ) can be obtained as follows:

$$
\begin{aligned}
\boldsymbol{N} &= \cos\alpha\,\boldsymbol{i} + \sin\alpha\,\boldsymbol{j} \\
\boldsymbol{H} &= \sin\beta\sin\alpha\,\boldsymbol{i} - \sin\beta\cos\alpha\,\boldsymbol{j} + \cos\beta\,\boldsymbol{k} \\
\boldsymbol{M} = \boldsymbol{H}\times\boldsymbol{N} &= -\sin\alpha\cos\beta\,\boldsymbol{i} + \cos\alpha\cos\beta\,\boldsymbol{j} + \sin\beta\,\boldsymbol{k} \\
\boldsymbol{u}_r &= \cos\psi\,\boldsymbol{N} + \sin\psi\,\boldsymbol{M} \\
&= (\cos\psi\cos\alpha - \sin\psi\sin\alpha\cos\beta)\,\boldsymbol{i} + (\cos\psi\sin\alpha + \sin\psi\cos\alpha\cos\beta)\,\boldsymbol{j} + \sin\psi\sin\beta\,\boldsymbol{k} \\
&= \sin\theta\cos\phi\,\boldsymbol{i} + \sin\theta\sin\phi\,\boldsymbol{j} + \cos\theta\,\boldsymbol{k}
\end{aligned}
\tag{11}
$$

so that

$$
\begin{aligned}
\sin\theta\cos\phi &= \cos\psi\cos\alpha - \sin\psi\sin\alpha\cos\beta \\
\sin\theta\sin\phi &= \cos\psi\sin\alpha + \sin\psi\cos\alpha\cos\beta \\
\cos\theta &= \sin\psi\sin\beta
\end{aligned}
\tag{12}
$$

yielding $r = r$, $\theta = \theta(\psi, \beta)$, $\phi = \phi(\alpha, \beta, \psi)$. This is the type of transformation of coordinates discussed in Sec. III.

The angular velocity of the $\boldsymbol{H}, \boldsymbol{N}, \boldsymbol{M}$ frame of reference is given by

$$
\begin{aligned}
\boldsymbol{\omega} &= \frac{d\beta}{dt}\boldsymbol{N} + \left(\frac{d\alpha}{dt} + \Omega\right)\boldsymbol{k} \\
&= \cos\beta(\dot{\alpha} + \Omega)\,\boldsymbol{H} + \dot{\beta}\boldsymbol{N} + \sin\beta(\dot{\alpha} + \Omega)\,\boldsymbol{M}
\end{aligned}
\tag{13}
$$

The velocity of the point P is

$$
\boldsymbol{v} = \frac{d\boldsymbol{r}}{dt} = \frac{D\boldsymbol{r}}{dt} + \boldsymbol{\omega}\times\boldsymbol{r} \tag{14}
$$

with $\mathbf{r} = r \cos\psi\, \mathbf{N} + r \sin\psi\, \mathbf{M}$. Thus

$$\mathbf{v} = r\,[\sin\psi\,\dot{\beta} - \cos\psi \sin\beta\,(\dot{\alpha} + \Omega)]\,\mathbf{H} + [\dot{r}\cos\psi - r\sin\psi\,\dot{\psi} - r\sin\psi\cos\beta\,(\dot{\alpha}+\Omega)]\,\mathbf{N}$$

$$+ [\dot{r}\sin\psi + r\cos\psi\,\dot{\psi} + r\cos\psi\cos\beta\,(\dot{\alpha}+\Omega)]\,\mathbf{M} \tag{15}$$

which yields the kinetic energy per unit mass given by

$$T = \frac{1}{2}\,[\dot{r}^2 + r^2\dot{\psi}^2 + r^2\sin^2\psi\,\dot{\beta}^2 + r^2(1 - \sin^2\psi\sin^2\beta)(\dot{\alpha}+\Omega)^2$$

$$- 2r^2\sin\psi\cos\psi\sin\beta\,\dot{\beta}(\dot{\alpha}+\Omega) + 2r^2\cos\beta\,\dot{\psi}(\dot{\alpha}+\Omega)] \tag{16}$$

After Lagrange's equations of motion are set up, we will impose the constraint that $\mathbf{v}$ be perpendicular to $\mathbf{H}$, yielding

$$\sin\psi\,\dot{\beta} - \cos\psi\sin\beta\,(\dot{\alpha}+\Omega) = 0 \tag{17}$$

Neglecting higher order terms, it can be shown that the Moon's potential is given by (see Ref. 1)

$$V_1 = -\frac{GM}{R}\left\{\frac{R}{r} + \frac{R^3 J}{r^3}\left(\frac{1}{3} - \sin^2\psi\sin^2\beta\right)\right.$$

$$\left. + \frac{R^3 K}{r^3}\,[\cos 2\alpha\,(\cos^2\psi - \sin^2\psi\cos^2\beta) - \sin 2\alpha\sin 2\psi\cos\beta]\right\} \tag{18}$$

The principal moments of inertia of the Moon about the x-y-z axes, respectively, are A, B, C. M is the mass of the Moon, and R is the radius of the Moon in the direction of the orbit of the Moon (y – axis). The numerical values of J and K are obtained from Ref. 2., although slightly more accurate values can be obtained from Ref. 1. The values of J and K are

$$J = \frac{3}{2MR^2}\left(C - \frac{1}{2}A - \frac{1}{2}B\right) \approx 0.00034$$

$$K = \frac{3}{4MR^2}(B - A) \approx 0.00035 \pm 0.000015 \tag{19}$$

One must also obtain the potential due to the Earth. The x-axis points toward the Earth, thus, the third term of Eq. (3) can be written (for a unit mass) as

$$\mathbf{F} = GM\left\{-\frac{(x-D)\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{\left[(x-D)^2 + y^2 + z^2\right]^{\frac{3}{2}}} - \frac{\mathbf{i}}{D^2}\right\} \tag{20}$$

For $x^2 + y^2 + z^2 \ll D^2$, with D the fixed distance between Earth and Moon, Eq. (20) can be replaced by

$$\mathbf{F} \approx \frac{GM}{D^3}\left(2x\,\mathbf{i} - y\,\mathbf{j} - z\,\mathbf{k}\right) = -\nabla\left[\frac{GM}{2D^3}(-3x^2 + r^2)\right] \tag{21}$$

Thus the potential due to the Earth is

$$V_2 = \frac{GM}{2D^3}(r^2 - 3x^2) = \frac{GM}{2D^3}r^2(1 - 3\sin^2\theta\cos^2\phi)$$

$$= \frac{GM}{2D^3}r^2\left[1 - 3(\cos\psi\cos\alpha - \sin\psi\sin\alpha\cos\beta)^2\right] \tag{22}$$

from Eq. (12). The total potential is $V = V_1 + V_2$, as given by Eqs. (18) and (22). Comparing R^3K/r^3 with $(GM/2D^3)r^2$ for a satellite near the Moon yields

$$\frac{R^3K}{R^3}\,\frac{2D^3}{GMR^2} = \frac{2KD^3}{GMR^2} = \frac{2K\,D^3}{\dfrac{GM}{R^2}\,\dfrac{M}{\mathrm{M}}\,R^4} \approx \frac{2\cdot 3\cdot 10^{-5}\,(240{,}000)^3}{\dfrac{32}{6\cdot 5280}\,(80)\,(1100)^4} \approx 10 \tag{23}$$

Thus, the K-term of V_1 may be slightly more important than the V_2 term in determining the motion of the satellite.

VI. EQUATIONS OF MOTION

For $J = K = 0$, $\Omega = 0$, $D = \infty$, one has Keplerian motion in the $\boldsymbol{M}$, $\boldsymbol{N}$ plane, provided that $\boldsymbol{H}$ is chosen normal to $\mathbf{v}$. For this case, $\beta =$ constant, $\alpha =$ constant, $u = 1/r = GM/h^2 + A \sin\psi + B \cos\psi$, with A, B, h constants of the motion. Thus, for J, $K \ll 1$, $\Omega/\dot{\psi} \ll 1$, $R/D \ll 1$, one assumes that β, α, h, A, B will be slowly varying quantities. Thus, we will neglect the terms $(\dot{\alpha} + \Omega)^2$, $\dot{\beta}^2$, $(\dot{\alpha} + \Omega)\dot{\beta}$, Ω^2, $\dot{h}\,\dot{\alpha}$, etc., in Lagrange's equations of motion given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{u}}\right) = \frac{\partial L}{\partial u}, \qquad u = \beta, \psi, r \tag{24}$$

The equation for $u = \alpha$ is a consequence of the Eqs. (24), and will be replaced by Eq. (17).

The equations of motion are, respectively,

$$\begin{aligned}\frac{d}{dt}\left[r^2 \sin^2\psi\,\dot{\beta} - r^2 \sin\psi \cos\psi \sin\beta(\dot{\alpha}+\Omega)\right] &= -r^2 \sin\beta\dot{\psi}(\dot{\alpha}+\Omega) - \frac{2GMR^2J}{r^3}\sin^2\psi\sin\beta\cos\beta\\ &+ \frac{GMR^2K}{r^3}\left[2\cos 2\alpha\sin^2\psi\sin\beta\cos\beta + \sin 2\alpha\sin 2\psi\sin\beta\right]\\ &+ \frac{3GMr^2}{D^3}(\cos\psi\cos\alpha - \sin\psi\sin\alpha\cos\beta)\sin\psi\sin\alpha\sin\beta\end{aligned} \tag{25}$$

$$\begin{aligned}\frac{d}{dt}\left[r^2\dot{\psi} + r^2\cos\beta(\dot{\alpha}+\Omega)\right] &= -\frac{2GMR^2J}{r^3}\sin\psi\cos\psi\sin^2\beta - \frac{2GMR^2K}{r^3}\\ &\left[\cos 2\alpha(1+\cos^2\beta)\sin\psi\cos\psi + \sin 2\alpha\cos\beta\cos 2\psi\right]\\ &- \frac{3GMr^2}{D^3}(\cos\psi\cos\alpha - \sin\psi\sin\alpha\cos\beta)(\sin\psi\cos\alpha + \cos\psi\sin\alpha\\ &\cos\beta)\end{aligned} \tag{26}$$

$$\frac{d^2r}{dt^2} = r\dot{\psi}^2 + 2r\cos\beta\dot{\psi}(\dot{\alpha}+\Omega) - \frac{GM}{r^2} - \frac{3GMR^2J}{r^4}\left(\frac{1}{3} - \sin^2\psi\sin^2\beta\right)$$

$$- \frac{3GMR^2K}{r^4}\left[\cos 2\alpha\left(\cos^2\psi - \sin^2\psi\cos^2\beta\right) - \sin 2\alpha\sin 2\psi\cos\beta\right] \tag{27}$$

$$- \frac{GMr}{D^3}\left[1 - 3\left(\cos\psi\cos\alpha - \sin\psi\sin\alpha\cos\beta\right)^2\right]$$

We note that, if $\beta = 0$ at $t = 0$, then $\beta \equiv 0$ satisfies Eq. (25) and the initial condition, so that an equatorial orbit remains equatorial. For $\beta \neq 0, \pi$, making use of Eq. (17) along with $r^2\dot{\psi} = h \neq$ constant, Eq. (25) becomes

$$\frac{d\alpha}{d\psi} = -\frac{\Omega r^2}{h} - \frac{2GMR^2J}{h^2r}\sin^2\psi\cos\beta + \frac{GMR^2K}{h^2r}\left(2\cos 2\alpha\sin^2\psi\cos\beta + \sin 2\alpha\sin 2\psi\right)$$

$$+ \frac{3GMr^4}{h^2D^3}\left(\cos\psi\cos\alpha - \sin\psi\sin\alpha\cos\beta\right)\sin\psi\sin\alpha \tag{28}$$

Applying the method of Sec. IV with

$$u = \frac{1}{r} = \frac{GM}{h^2} + A\sin\psi + B\cos\psi = \frac{GM}{h^2}\left(1 + \frac{Ah^2}{GM}\sin\psi + \frac{Bh^2}{GM}\cos\psi\right)$$

$$r \approx \frac{h^2}{GM}\left(1 - \frac{Ah^2}{GM}\sin\psi - \frac{Bh^2}{GM}\cos\psi\right) \text{ for small eccentricity, } \quad \frac{Ah^2}{GM} \ll 1, \quad \frac{Bh^2}{GM} \ll 1,$$

one obtains

$$\left\langle \frac{d\alpha}{d\psi} \right\rangle = -\frac{\Omega h^3}{C^2 M^2} - \frac{G^2 M^2 R^2 J}{h^4} \cos\beta + \frac{G^2 M^2 R^2 K}{h^4} \cos 2\alpha \cos\beta - \frac{3}{2} \frac{Mh^6}{G^3 M^4 D^3} \cos\beta \sin^2\alpha$$

$$= -\frac{\Omega h^3}{C^2 M^2} - \frac{C^2 M^2 R^2}{h^4} \cos\beta \left(J - K \cos 2\alpha + \frac{3Mh^{10}}{2G^5 M^6 R^2 D^3} \sin^2\alpha \right) \tag{29}$$

From Eqs. (17) and (28), one obtains

$$\frac{d\beta}{d\psi} = \frac{\cos\psi}{\sin\psi} \sin\beta \left(\frac{d\alpha}{d\psi} + \frac{\Omega r^2}{h} \right) = -\frac{2GMR^2 J}{h^2 r} \sin\psi \cos\psi \cos\beta \sin\beta$$

$$+ \frac{2GMR^2 K}{h^2 r} (\cos 2\alpha \sin\psi \cos\psi \sin\beta \cos\beta + \sin 2\alpha \sin\beta \cos^2\psi) \tag{30}$$

$$+ \frac{3GMr^4}{h^2 D^3} (\cos\psi \cos\alpha - \sin\psi \sin\alpha \cos\beta) \sin\alpha \sin\beta \cos\psi$$

Applying the averaging process of Sec. IV yields

$$\left\langle \frac{d\beta}{d\psi} \right\rangle = \frac{G^2 M^2 R^2}{h^4} \left(K + \frac{3Mh^{10}}{4G^5 M^6 R^2 D^3} \right) \sin 2\alpha \sin\beta \tag{31}$$

In order to determine the change in the angular momentum, $h = r^2 \dot{\psi}$, we return to Eq. (26). Thus,

$$\frac{dh}{d\psi} = -h\cos\beta \frac{d}{d\psi}\left(\frac{d\alpha}{d\psi} + \frac{\Omega r^2}{h}\right) - \frac{2GMR^2J}{hr}\sin\psi\cos\psi\sin^2\beta$$

$$-\frac{2GMR^2K}{hr}\left[\cos 2\alpha\,(1+\cos^2\beta)\sin\psi\cos\psi + \sin 2\alpha\cos\beta\cos 2\psi\right] \tag{32}$$

$$-\frac{3GMr^4}{hD^3}(\cos\psi\cos\alpha - \sin\psi\sin\alpha\cos\beta)(\sin\psi\cos\alpha + \cos\psi\sin\alpha\cos\beta)$$

neglecting the terms involving $\dot{\beta}(\dot{\alpha}+\Omega)$, $\dot{h}(\dot{\alpha}+\Omega)$. Making use of Eq. (28) to eliminate $d/d\psi\,(d\alpha/d\psi + \Omega r^2/h)$ yields

$$\frac{dh}{d\psi} = \frac{4GMR^2Ju}{h}\cos 2\beta\sin\psi\cos\psi + \frac{2GMR^2J}{h}\frac{du}{d\psi}\sin^2\psi\cos^2\beta - \frac{2GMR^2Ku}{h}\cos\beta(\cos 2\alpha\sin 2\psi\cos\beta + \sin 2\alpha\cos 2\psi)$$

$$-\frac{GMR^2K}{h}\frac{du}{d\psi}(2\cos 2\alpha\cos\beta\sin^2\psi + \sin 2\alpha\sin 2\psi) - \frac{3GM}{hD^3u^4}\left[\sin\alpha\cos\alpha(\cos^2\psi - \sin^2\psi) - \sin^2\alpha\cos\beta\sin 2\psi\right]\cos\beta$$

$$+\frac{12GM}{hD^3u^5}\frac{du}{d\psi}\cos\beta(\cos\psi\cos\alpha - \sin\psi\sin\alpha\cos\beta)\sin\psi\sin\alpha - \frac{2GMR^2J}{h}u\sin\psi\cos\psi\sin^2\beta \tag{33}$$

$$-\frac{2GMR^2K}{h}u\left[\cos 2\alpha\,(1+\cos^2\beta)\sin\psi\cos\psi + \sin 2\alpha\cos\beta\cos 2\psi\right]$$

$$-\frac{3GM}{hD^3u^4}(\cos\psi\cos\alpha - \sin\psi\sin\alpha\cos\beta)(\sin\psi\cos\alpha + \cos\psi\sin\alpha\cos\beta)$$

neglecting the terms involving $J\ (dh/d\psi)$, $K\ (dh/d\psi)$. Now $u = GM/h^2 + A \sin\psi + B \cos\psi$, with h, A, B constants of the motion for $J = K = 0$, $\Omega = 0$, $D = \infty$, so that $v = du/d\psi = A \cos\psi - B \sin\psi$. We define $v = du/d\psi \equiv A \cos\psi - B \sin\psi$ in a subsequent analysis. Applying the averaging process of Sec. IV yields

$$\left\langle \frac{dh}{d\psi} \right\rangle = 0 \tag{34}$$

Thus, for nearly circular orbits, the angular momentum $h = r^2 \dot{\psi}$ remains essentially constant. We are now in a position to find a relationship between α and β from Eqs. (29) and (31). Removing the < > signs one obtains

$$\left(K + \frac{3Mh^{10}}{4G^5M^6R^2D^3} \right) \sin 2\alpha \sin\beta \, d\alpha + \left[\frac{\Omega h^7}{G^4M^4R^2} + \cos\beta \left(J - K \cos 2\alpha + \frac{3Mh^{10}}{2G^5M^6R^2D^3} \sin^2\alpha \right) \right] d\beta = 0 \tag{35}$$

with $h \equiv$ constant from Eq. (34). Equation (35) becomes exact upon multiplication by the integrating factor $\sin\beta$. An integration yields

$$-\frac{2\Omega h^7}{G^4M^4R^2} \cos\beta + \left(J - K \cos 2\alpha + \frac{3Mh^{10}}{2G^5M^6R^2D^3} \sin^2\alpha \right) \sin^2\beta \equiv \text{constant} \tag{36}$$

Accurate observations of a nearly circular satellite of the moon would yield values of α, β, and h, from which one could obtain values of J and K from Eq. (36).

In order to investigate the stability of a nearly circular polar orbit, let $\beta = \pi/2 - \sigma$, $\sigma << 1$, $\sin\beta \approx 1$, $\cos\beta \approx \alpha$; thus, from Eqs. (29) and (31)

$$-\frac{d\sigma}{d\psi} \approx \frac{G^2M^2R^2}{h^4} \left(K + \frac{3Mh^{10}}{4G^5M^6R^2D^3} \right) \sin 2\alpha$$

$$\frac{d\alpha}{d\psi} \approx -\frac{\Omega h^3}{G^2M^2} \tag{37}$$

An integration, with $\sigma = 0$ for $\psi = \psi_0$, yields

$$\sigma = \frac{G^4 M^4 R^2}{2\Omega h^7}\left(K + \frac{3Mh^{10}}{4G^5 M^6 R^2 D^3}\right)\left\{-\cos\left[2\left(\alpha_0 - \frac{\Omega h^3}{G^2 M^2}\psi\right)\right] + \cos\left[2\left(\alpha_0 - \frac{\Omega h^3}{G^2 M^2}\psi_0\right)\right]\right\} \tag{38}$$

so that

$$|\sigma| \leqq \frac{G^4 M^4 R^2}{\Omega h^7}\left(K + \frac{3Mh^{10}}{4G^5 M^6 R^2 D^3}\right) \tag{39}$$

For a satellite near the Moon's surface, $h^2 \approx GMR \approx R^4 \dot{\psi}^2$, $<\dot{\psi}> \approx 2\pi/100$ rad/min, $\Omega \approx 2\pi/43000$ rad/min

$$|\sigma| \leqq \frac{\langle\dot{\psi}\rangle}{\Omega}\left[K + \frac{3}{4}\left(\frac{M}{\mathrm{M}}\right)\left(\frac{R}{D}\right)^3\right] \approx (430)\ [(3.5)\,10^{-5} + 6 \cdot 10^{-6}] \approx (1.5)\,10^{-2}\ \text{rad} < 1\ \text{deg} \tag{40}$$

thus, an almost circular polar orbit near the Moon's surface will deviate at most 1 deg from the polar plane. For a near circular polar orbit with $r \approx 2R$,

$$|\sigma| \leqq \frac{430}{\sqrt{2^7}}\ [(3.5)\,10^{-5} + 2^5 \cdot 6 \cdot 10^{6}] \approx 8 \cdot 10^{-3}\ \text{rad} < 0.5\ \text{deg} \tag{41}$$

Since the preceding analysis requires an orbit of small eccentricity, it is necessary to study the radial motion of the satellite in order to determine the rate of change of the eccentricity, $\epsilon = h^2/GM\sqrt{A^2 + B^2}$. We return now to Eq. (27). From $r^2\dot{\psi} = h$, it follows that

$$\frac{dr}{dt} = \frac{h}{r^2}\frac{dr}{d\psi} = -h\frac{du}{d\psi}, \qquad \frac{d^2r}{dt^2} = -h^2u^2\frac{d^2u}{d\psi^2} - hu^2\frac{dh}{d\psi}\frac{du}{d\psi} \tag{42}$$

Thus, Eq. (27) becomes

$$\frac{du}{d\psi} = v$$

$$\frac{dv}{d\psi} = -u + \frac{GM}{h^2} - \frac{1}{h}\frac{dh}{d\psi}v - \frac{2\cos\beta}{hu}\left(hu^2\frac{d\alpha}{d\psi} + \Omega\right) + \frac{3GMR^2Ju^2}{h^2}\left(\frac{1}{3} - \sin^2\psi\sin^2\beta\right)$$

$$+ \frac{3GMR^2Ku^2}{h^2}\left[\cos 2\alpha\left(\cos^2\psi - \sin^2\psi\cos^2\beta\right)\sin 2\alpha\sin 2\psi\cos\beta\right] \tag{43}$$

$$+ \frac{GM}{h^2D^3u^3}\left[1 - 3(\cos\psi\cos\alpha - \sin\psi\sin\alpha\cos\beta)^2\right]$$

The solution of Eq. (43) for $J = K = 0$, $\Omega = 0$, $D = \infty$, is

$$u = \frac{GM}{h^2} + A\sin\psi + B\cos\psi$$

$$v = A\cos\psi - B\sin\psi \tag{44}$$

The parameters A and B are now varied, with the result that

$$\frac{dA}{d\psi} = \frac{2GM}{h^3}\frac{dh}{d\psi}\sin\psi - \frac{1}{h}\frac{dh}{d\psi}v\cos\psi - \frac{2\cos\beta}{hu}\left(hu^2\frac{d\alpha}{d\psi} + \Omega\right)\cos\psi + \frac{3GMR^2J}{h^2}u^2\left(\frac{1}{3} - \sin^2\psi\sin^2\beta\right)\cos\psi$$

$$+ \frac{3GMR^2K}{h^2}u^2\left[\cos 2\alpha(\cos^2\psi - \sin^2\psi\cos^2\beta) - \sin 2\alpha\sin 2\psi\cos\beta\right]\cos\psi$$

$$+ \frac{GM}{h^2D^3u^3}\left[1 - 3(\cos\psi\cos\alpha - \sin\psi\sin\alpha\cos\beta)^2\right]\cos\psi$$

(45)

$$\frac{dB}{d\psi} = \frac{2GM}{h^3}\frac{dh}{d\psi}\cos\psi + \frac{1}{h}\frac{dh}{d\psi}v\sin\psi + \frac{2\cos\beta}{hu}\left(hu^2\frac{d\alpha}{d\psi} + \Omega\right)\sin\psi - \frac{3GMR^2J}{h^2}u^2\left(\frac{1}{3} - \sin^2\psi\sin^2\beta\right)\sin\psi$$

$$- \frac{3GMR^2K}{h^2}u^2\left[\cos 2\alpha(\cos^2\psi - \sin^2\psi\cos^2\beta) - \sin 2\alpha\sin 2\psi\cos\beta\right]\sin\psi$$

$$- \frac{GM}{h^2D^3u^3}\left[1 - 3(\cos\psi\cos\alpha - \sin\psi\sin\alpha\cos\beta)^2\right]\sin\psi$$

One now substitutes $dh/d\psi$, as given by Eq. (33), and $[hu^2(d\alpha/d\psi) + \Omega]$, as given by Eq. (28), into Eqs. (45), with $u = GM/h^2 + A\sin\psi + B\cos\psi$, $du/d\psi = v = A\cos\psi - B\sin\psi$. Neglecting higher powers of Ah^2/GM, Bh^2/GM, and applying the averaging process of Sec. IV, one obtains

$$\left\langle\frac{dA}{d\psi}\right\rangle = \frac{G^2M^2R^2}{h^4}\left\{J\left(2-\frac{5}{2}\sin^2\beta\right)B + K\left[-3\cos\beta\sin 2\alpha A + \frac{1}{2}(3-5\cos^2\beta)\cos 2\alpha B\right]\right\}$$

$$+\frac{Mh^6}{D^3G^3M^4}\left[-\frac{15}{4}A\sin 2\alpha\cos\beta + B\left(6\cos^2\alpha - \frac{3}{2}\right)\right]$$

(46)[1]

$$\left\langle\frac{dB}{d\psi}\right\rangle = -\frac{G^2M^2R^2}{h^4}\left\{J\left(2-\frac{5}{2}\sin^2\beta\right)A + K\left[3\cos\beta\sin 2\alpha B + \frac{1}{2}(3-5\cos^2\beta)\cos 2\alpha A\right]\right\}$$

$$+\frac{Mh^6}{D^3G^3M^4}\left[\frac{15}{4}B\sin 2\alpha\cos\beta + \frac{3A}{2}(1+\cos^2\alpha - 5\sin^2\alpha\cos^2\beta)\right]$$

The terms in $< dA/d\psi >$, $<dB/d\psi>$ involving J and K arose without the assumption of small eccentricity. The presence of the Earth's field on the satellite accounts for the nonsymmetric terms in Eqs. (46); the assumption of small eccentricity was introduced to derive these terms. R. E. Roberson (Ref. 3) first obtained the critical angle given by $\sin^2\beta = 4/5$ (see Eq. 46).

Removing the < > signs, Eqs. (46) yield

$$A\frac{dA}{d\psi} + B\frac{dB}{d\psi} = -\frac{3G^2M^2R^2}{h^4}K\cos\beta\sin 2\alpha(A^2+B^2)$$

$$+\frac{Mh^6}{D^3G^3M^4}\left[-\frac{15}{4}\sin 2\alpha\cos\beta(A^2-B^2) + \frac{15}{2}(\cos^2\alpha - \sin^2\alpha\cos^2\beta)AB\right] \tag{47}$$

For an orbit which remains nearly polar, $\beta \approx \pi/2$, $\cos\beta \approx 0$, and

[1] F. Yagi, JPL, assisted in checking the results of this equation.

$$\frac{A\dfrac{dA}{d\psi}+B\dfrac{dB}{d\psi}}{A^2+B^2} \approx \frac{15Mh^6}{2D^3G^3M^4}\cos^2\alpha\,\frac{AB}{A^2+B^2} \leqq \frac{15}{4}\,\frac{Mh^6}{D^3G^3M^4}$$

$$\frac{d}{d\psi}\ln\left(A^2+B^2\right) \leqq \frac{15}{2}\,\frac{Mh^6}{D^3G^3M^4} \tag{48}$$

$$\frac{\left(A^2+B^2\right)^{1/2}}{\left(A_0^2+B_0^2\right)^{1/2}} \leqq \exp\left(\frac{15}{4}\,\frac{Mh^6\psi}{D^3G^3M^4}\right)$$

with $A = A_0$, $B = B_0$ for $\psi = 0$.

Now the eccentricity of the orbit is given by $\epsilon = h^2/GM\sqrt{A^2+B^2}$; thus,

$$\epsilon \leqq \epsilon_0 \exp\left(\frac{15}{4}\,\frac{Mh^6}{D^3G^3M^4}\,\psi\right) \tag{49}$$

with ϵ_0 the initial eccentricity. The eccentricity will increase at most, by a factor of e when $15/4\ Mh^6/D^3G^3M^4\,\psi = 1$, or $\psi = 4/15\ D^3G^3M^4/Mh^6$ rad. For an orbit near the Moon's surface, $h^2 \approx GMR$, and $\psi = 4/15\,(M/M)(D/R)^3$ rad $\approx 1/3 \cdot 10^5$ rad. The time, τ, associated with this increase of eccentricity is

$$\tau = \frac{10^5}{6\pi}\,\frac{100}{60\cdot 24\cdot 365}\ \text{yr} \approx 1\ \text{year} \tag{50}$$

We have shown that a satellite of the Moon will tend to remain in an almost circular polar orbit for a considerable length of time.

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