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BOUNDARY LAYER EFFECT ON THE SURFACE PRESSURE OF AN INFINITE CONE IN SUPERSONIC FLOW

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NOMENCLATURE

C_p = pressure coefficient (dimensionless)

M = Mach number (dimensionless)

p = static pressure (microns Hg in Sections 3.0 and 4.0) } lbs/ft^2 otherwise

p_2 = cone surface pressure (microns Hg in Sections 3.0 and 4.0) } lbs/ft^2 otherwise

p_i = impact pressure (microns Hg in Sections 3.0 and 4.0) } lbs/ft^2 otherwise

r = radius of axisymmetric body (cone plus boundary layer) (feet)

Re = Reynolds number (dimensionless)

S = cross-sectional area of axisymmetric body (cone plus boundary layer) (ft^2)

s = distance along flat plate from leading edge (feet)

T = gas temperature (°Rankine)

U = free stream velocity (ft/sec)

x = axial coordinate (inches in Sections 3.0 and 4.0) (feet in Re_L)

γ = ratio of specific heats = c_p/c_v

δ^* = displacement thickness of boundary layer (in. in Sections 3.0 and 4.0) ft. otherwise

μ = gas viscosity = $lb/sec/ft^2$

ν = kinematic viscosity = μ/ρ = ft^2/sec

ρ = gas density = $slug/ft^3$

θ = cone semi-vertex angle (degrees)

SUBSCRIPTS

Flow properties with no subscript are those of the undisturbed free stream.

Flow properties with subscript L are "local" properties of the potential flow at the seam of the boundary layer, e.g.

$$Re_L = \frac{U_L x}{\nu_L}$$

BOUNDARY LAYER EFFECT ON THE SURFACE PRESSURE OF AN INFINITE CONE IN SUPERSONIC FLOW

1.0 INTRODUCTION

The theory of Taylor and Maccoll (Ref. 1) gives the surface pressure on an infinite cone in supersonic flow as a function of the cone vertex angle and the free stream Mach number and static pressure for a gas of vanishing viscosity. When a slender conical probe is used together with an impact pressure probe to determine the static pressure and Mach number in a low density gas stream, it is desirable to have some theoretical estimate of the effect of viscous boundary layer on the probe readings. Theoretical and experimental results with respect to impact probes have been presented in Refs. 5 and 6. A simple approximation for a conical probe based on linearized supersonic flow and compressible boundary layer theory is presented here.

2.0 ANALYSIS

The cone is oriented parallel to the undisturbed free stream, its axis of symmetry in the x-direction and its vertex at the origin. The semi-vertex angle is θ.

One considers the linearized supersonic flow to take place around a body consisting of the cone plus a displacement thickness of the boundary layer. The radius of this body is (see HYD 2653)

$$r = x \tan\theta + \delta^* \sec\theta \quad \text{- - - - - - - - - - - - - -} \quad (1)$$

Ref. 2 shows that the integration of the laminar boundary layer equations for a cone in supersonic flow can be based on that of a flat plate, and that in particular

$$\delta^* \equiv \delta^* \text{ (cone)} = \frac{\delta^*\text{(plate)}}{\sqrt{3}} \quad \text{- - - - - - - - - - -} \quad (2)$$

where δ^* is calculated from the "local" properties of the potential flow at the cone surface. This calculation, under the additional assumptions that μ is proportional to T, that Prandtl's number = 1.0 and that $\gamma = c_p/c_v = 1.400$, may be obtained from Ref. 3. The result is

$$\delta^*\text{(plate)} = 1.73\ (1 + 0.277\ M_L^2) \left(\frac{\nu_L s}{U_L}\right)^{\frac{1}{2}} \quad \text{- - - -} \quad (3)$$

where the subscript L refers to the aforementioned "local" conditions, and s is the distance along the plate from the leading edge. For the cone s = x secθ, so that the displacement thickness on the cone

$$\delta^* = (1 + 0.277\ M_L^2) \left(\frac{\nu_L\ x \sec\theta}{U_L}\right)^{\frac{1}{2}} \quad \text{- - - - - - - -} \quad (4)$$

Substituting Eq.4 into Eq.1,

$$r = ax + bx^{\frac{1}{2}} \qquad (5)$$

where

$$a = \tan\theta$$

$$b = \sec^{3/2}\theta \; (1 + 0.277\, M_L^2) \left(\frac{\nu_L}{U_L}\right)^{\frac{1}{2}}$$

Thus the cross-sectional area of the "effective body" is

$$S = \pi\,(ax + bx^{\frac{1}{2}})^2 \qquad (6)$$

Let p_2 be the pressure at the surface of the cone and define the surface pressure coefficient C_p as

$$C_p = \frac{p_2 - p}{\frac{1}{2}\rho U^2} \qquad (7)$$

where quantities without subscript are those of the undisturbed free stream. Ref. 4 relates this pressure coefficient to the body geometry and the free stream Mach number M, by a first-order solution of the linearized equations of supersonic flow about uniformly continuous bodies of revolution. The result is

$$C_p = \frac{1}{\pi}\left[\frac{S'}{x} - \left(1 + \ln\frac{r\sqrt{M^2-1}}{2x}\right) S'' - \frac{x}{2} S''' + \frac{x^2}{12} S^{(4)} - \frac{x^3}{72} S^{(5)} + \frac{x^4}{480} S^{(6)} + \ldots\right]$$
$$- (r')^2 + 0(r^3) \qquad (8)$$

where primes and superscripts in parentheses indicate differentiation with respect to x. It is pointed out further in Ref.4 that for the case of an infinite cone, the rigorous first order linearized solution is

$$C_p \approx -\frac{S''}{\pi} \ln\frac{r}{x}\,,\; M > \sqrt{2} \qquad (9)$$

Since all applications in the wind tunnel facilities of this project are for $M > \sqrt{2}$, this expression for C_p will be used. Substituting Eqs.5 and 6 into Eq.9

$$C_p \approx -2\left[a^2 + \frac{3ab}{4} x^{-\frac{1}{2}}\right] \ln\,(a + bx^{-\frac{1}{2}}) \qquad (10)$$

For small-angled cones we may make the approximations $\tan\theta \approx \theta$, $\sec\theta \approx 1$, whereby

$$a \approx \theta\,,\; b \approx (1 + 0.277\, M_L^2) \left(\frac{\nu_L}{U_L}\right)^{\frac{1}{2}} \qquad (11)$$

Then

$$c_p \approx -2\left[\theta^2 + \frac{3\theta}{4}\,\frac{(1 + 0.277\,M_L^2)}{\sqrt{Re_L}}\right]\ln\left(\theta + \frac{1 + 0.277\,M_L^2}{\sqrt{Re_L}}\right) \quad - - - \quad (12)$$

where

$$Re_L = \frac{U_L x}{\nu_L}$$

This expression may be simplified by expansion of the logarithm, and the dropping of all terms containing powers $> \frac{1}{2}$ in $1/Re_L$, since terms of these higher orders are neglected in all boundary layer theory. One obtains

$$c_p \approx 2\theta^2 \ln\left(\frac{1}{\theta}\right) + \left(\frac{3}{2}\theta \ln\left(\frac{1}{\theta}\right) - 2\theta\right)\frac{(1 + 0.277\,M_L^2)}{\sqrt{Re_L}} \quad - - - - \quad (13)$$

To a consistent degree of approximation in the linearized non-viscous flow, M_L is related to M, θ, by

$$M_L^2 \approx M^2\left[1 + \theta^2(1 + 2\ln\theta)\right] \quad - - - - - - - - - - - - - - - - - - \quad (14)$$

Note that for vanishing viscous action, $(Re_L \rightarrow \infty)$,

$$c_{p_{ideal}} \approx 2\theta^2 \ln\left(\frac{1}{\theta}\right) \quad - - - - - - - - - - - - - - - - - - - \quad (15)$$

3.0 RESULTS

For wind tunnel applications the ratio $p_2/p_{2_{ideal}}$ is of interest. From the perfect gas law,

$$\rho U^2 = \gamma M^2 p$$

and we obtain from the pressure coefficients

$$\frac{p_2}{p_{2_{ideal}}} = \frac{1 + \frac{\gamma}{2} M^2 c_p}{1 + \frac{\gamma}{2} M^2 c_{p_{ideal}}} \quad - \quad (16)$$

Substituting from Eqs. 13, 14, 15

$$\frac{p_2}{p_{2_{ideal}}} = 1 + \frac{f(M, \theta)}{\sqrt{Re_L}} \quad - \quad (17)$$

where

$$f(M,\theta) = \frac{\gamma}{4}\ \frac{M^2\left(3\theta \ln\left(\frac{1}{\theta}\right) - 4\theta\right)\left\{1 + 0.277\, M^2\left[1 - \theta^2\left(2\ln\left(\frac{1}{\theta}\right) - 1\right)\right]\right\}}{1 + \gamma M^2\theta^2 \ln\left(\frac{1}{\theta}\right)} \quad (18)$$

$f(M,\theta)$ is plotted versus M in HYD 2654, for $\theta = 5°, 7.5°, 10°$.

A somewhat less general result is more convenient for use in work with the No.3 Wind Tunnel (Ref.7). Assuming adiabatic flow from a stagnation reservoir temperature of 540° Rankine, and employing the Sutherland formula for the air viscosity, one may write

$$\frac{Re_L}{p_2 x} = g(M_L) \quad \text{- - - - - - - - - - - - - - -} \quad (19)$$

This function is plotted for p_2 in microns Hg and x in inches (HYD 2655). To maintain the degree of approximation expressed in Eq.17, p_2 may be replaced by $p_{2_{ideal}}$ in Eq.19. Then, since $p_{2_{ideal}}/p$ and M_L are both functions of M and θ, we modify Eq.19 to read

$$\frac{Re_L}{p\ x} = h(M,\theta) \quad \text{- - - - - - - - - - - - - -} \quad (20)$$

or, substituting Eq.20 in Eq.17,

$$\frac{p_2}{p_{2_{ideal}}} = 1 + \frac{\varphi(M,\theta)}{\sqrt{p\ x}}, \quad \text{- - - - - - - - -} \quad (21)$$

where $\varphi(M,\theta)$ has the dimensions necessary to render the equation dimensionless. $\varphi(M,\theta)$ in (microns Hg - in.)$^{\frac{1}{2}}$ is shown in HYD 2656.

4.0 NUMERICAL EXAMPLES - NO.3 WIND TUNNEL

Two examples characteristic of the smallest and largest boundary layer effects on conical probe pressure measurements in the No.3 Wind Tunnel may be calculated from Eq.21. Both examples refer to the No.15 probe, $\theta = 5°$, $x = 0.700$ in.

(1) No.4 Nozzle, 21 lbs/hr, M = 2.75, p = 120 microns (nominal values of M and p, ignoring viscous effect). From HYD 2656, $\varphi(M,\theta) = 0.7$. From Eq.21,

$$\frac{p_2}{p_{2_{ideal}}} = 1 + \frac{0.7}{\sqrt{(120)(0.7)}} = 1.08$$

The effect of this ratio on M and p may be computed roughly if the Rayleigh (ideal fluid) impact pressure, $p_{i_{ideal}}$, is known. As explained in Ref.8, M may be found as a function of the ratio $(p_{i_{ideal}}/p_{2_{ideal}})$.

Our nominal $M = 2.75$ was computed from the ratio $(p_{i_{ideal}}/p_2)$, which had the value 8.84, whereas the true free stream Mach number corresponds to the ratio,

$$\frac{p_{i_{ideal}}}{p_{2_{ideal}}} = \frac{p_{i_{ideal}}}{p_2} \times \frac{p_2}{p_{2_{ideal}}} = (8.84)\ (1.08) = 9.55$$

and has the value $M = 2.88$. The true static pressure p follows from the relationship between M and $p_{2_{ideal}}/p$. The measured value of p_2 for this example was 139 microns. Then,

$$p_{2_{ideal}} = p_2 \times \frac{p_{2_{ideal}}}{p_2} = \frac{139}{1.08} = 129 \text{ microns}$$

For $M = 2.88$, $p_{2_{ideal}}/p = 1.167$, and $p = 129/1.167 = 111$ microns is the true static pressure. According, then, to this highly approximate calculation, the use of the No.15 probe without regard to the effect of viscous boundary layer causes an underestimate of M by 5 percent, and an overestimate of p by 8 percent, in this case. The effect on the free stream Reynolds number may be read from HYD 2655, and is an underestimate by 5 percent.

(2) Similar computations may be performed for the example $M = 2.78$, $p = 38$ microns (3 lbs/hr, No.3 nozzle). Here the neglect of boundary layer in the No.15 probe causes an indicated underestimate of M by 8 percent, an overestimate of p by 17 percent, and an underestimate of Re by 7 percent.

5.0 DISCUSSION - NATURE AND LIMITATIONS OF RESULTS

Eqs.17 or 21 show that the viscous boundary layer on the surface of an infinite cone at zero angle of attack in a uniform supersonic air stream causes an increase of the surface pressure on the cone, and makes this pressure dependent on the distance from the vertex of the cone.

Roughly speaking, the real surface pressure will differ appreciably (say by 1 percent) from the ideal pressure when the local Reynolds number of the flow at the seam of the boundary layer is less than 10^5, the difference increasing in proportion to $1/\sqrt{Re_L}$ as Re_L decreases. For free stream Mach numbers greater than 2.5, the theoretically calculated ratio $p_2/p_{2_{ideal}}$ is sensitive to variation in M and in θ, the cone semi-vertex angle.

The limitations of this theoretical analysis are severe and should be recognized in some detail. By the nature of its basic assumptions boundary layer theory is applicable only when viscous layers are reasonably thin, (i.e. when $1/\sqrt{Re_L}$ is fairly small). Consequently the results of the present analysis should be numerically significant only in the range of Reynolds numbers $10^4 < Re_L < 10^5$ (say) and serve only to indicate a trend of events for lower Re_L. Similarly, linearized supersonic flow theory is best restricted to very slender bodies and moderately low Mach numbers, although the form of theory employed here does not become progressively worse for increasing M.

Such physical approximations as Prandtl number = 1 and $\mu \propto T$ are not serious as respects theoretical trends, but will have considerable effect on numerical results.

6.0 CONCLUSION

A simple approximate analysis based on compressible boundary layer theory and first order linearized supersonic flow theory indicates that the pressure on the surface of an infinite cone in a viscous fluid is greater than that in an ideal fluid. The difference between real and ideal pressures is proportional to the inverse square root of the local Reynolds number for given M and θ. For given Re_L and θ, this difference increases with increasing M, for Re and M fixed, it decreases with increasing θ. For $5° < \theta < 10°$, $M > 1.5$, the difference is not negligible for $Re_L < 10^5$.

An experimental program is being developed for the investigation of this matter in the No.3 Low Density Wind Tunnel.

7.0 REFERENCES

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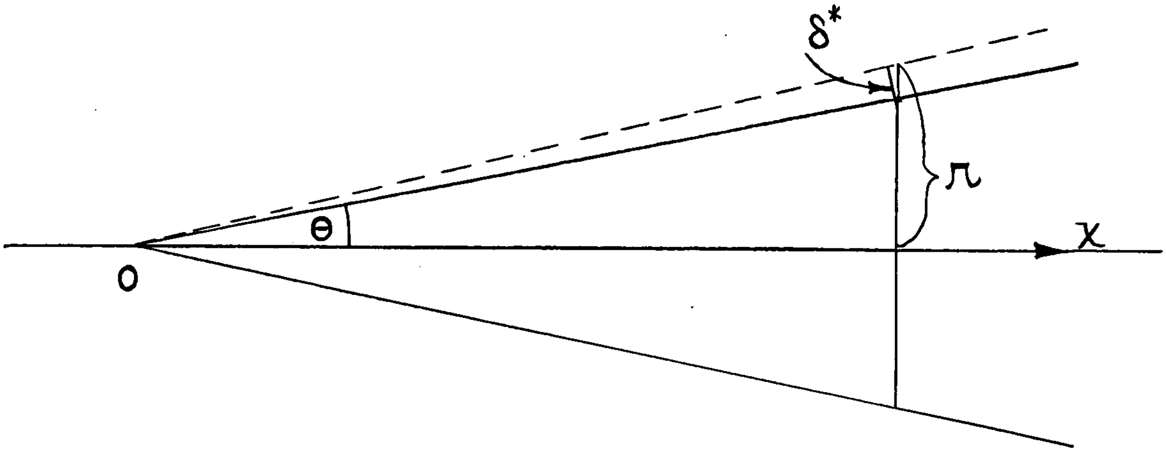

CONE GEOMETRY

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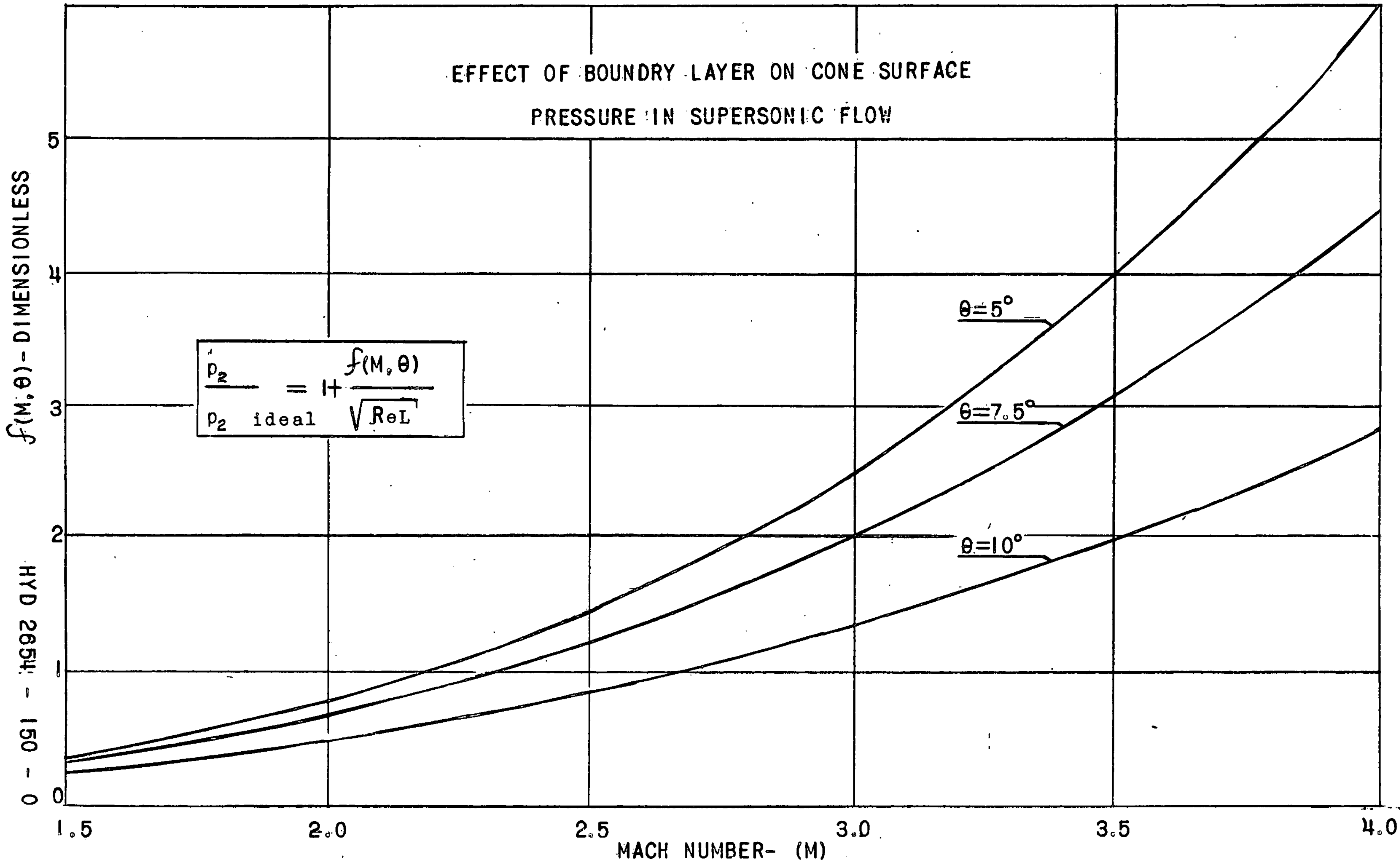

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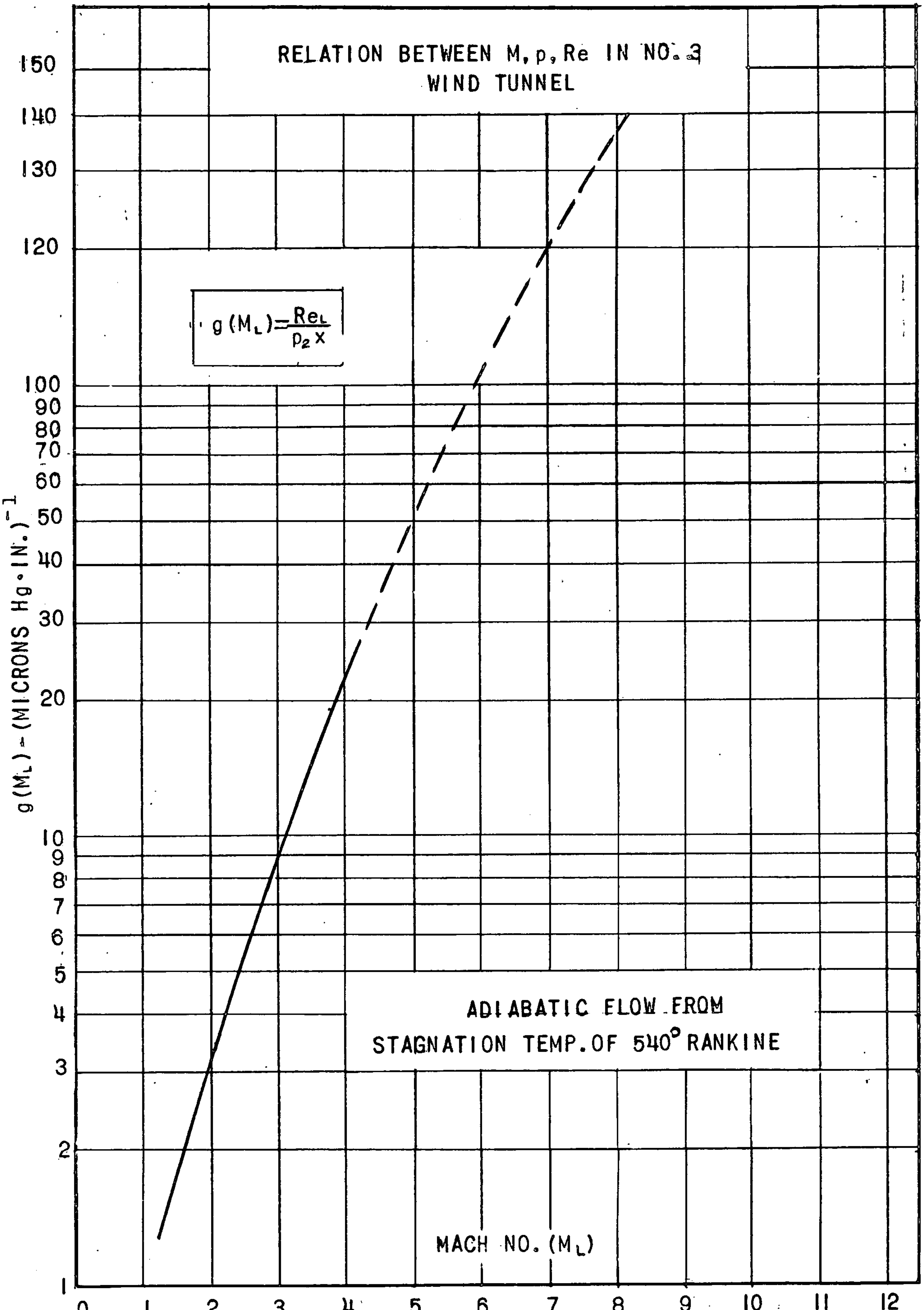

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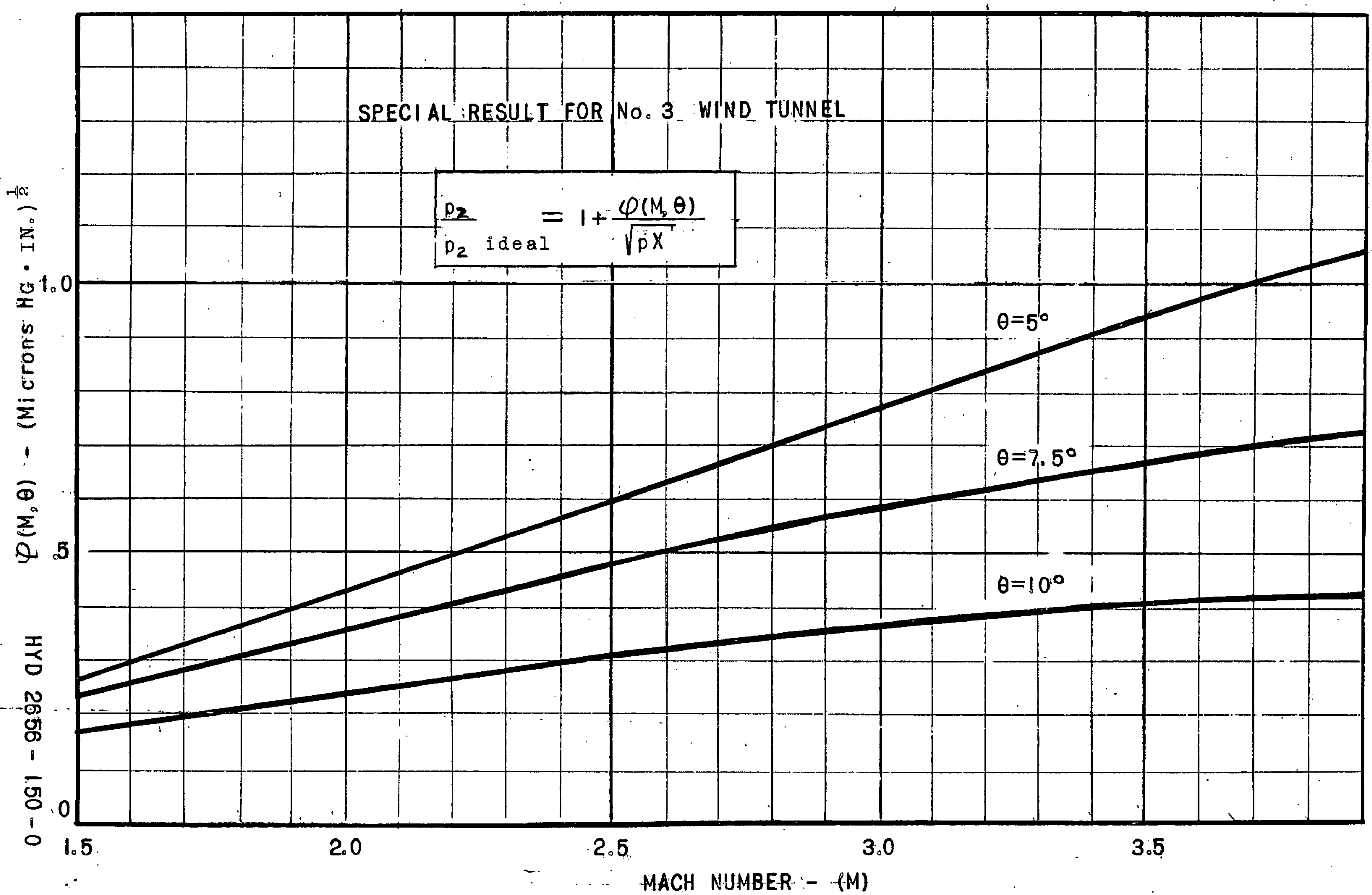

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