# M-MRAC Backstepping for Systems With Unknown Virtual Control Coefficients

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*Abstract*— **The paper presents an over-parametrization free certainty equivalence state feedback backstepping adaptive control design method for systems of any relative degree with unmatched uncertainties and unknown virtual control coefficients. It uses a fast prediction model to estimate the unknown parameters, which is independent of the control design. It is shown that the system's input and output tracking errors can be systematically decreased by the proper choice of the design parameters. The benefits of the approach is demonstrated in numerical simulations.**

## I. Introduction

Adaptive control problems are challenging for systems with unmatched uncertainties, and backstepping has been excessively used to tackle these problems. However over-parametrization and "explosion of terms" because of repeated differentiations are obstacles in its direct applications. One way of avoiding over-parameterization is the departure from the certainty equivalence principle by modifying the control law to include nonlinear damping terms (see [8] for details). However, this leads to the adaptation rate to enter into the control law and gives rise of high magnitude control signals in the case of fast adaptation, which is desirable from the perspective of the unknown parameter estimations. An alternative way was presented in [2], where certainty equivalence control design avoids over-parametrization for linear and low relative degree (not exceeding two) nonlinear systems in sate feedback settings.

The "explosion of terms" was addressed by combining the standard backstepping with a sliding mode control approach [1], [4], by the Multiple surface sliding control [17], or by the Dynamic scarface control [16]. Alternative approaches include approximation of the virtual control derivatives using sliding mode filters [9], [15], neural networks [11], [12], fuzzy systems [5], first order linear filters [18], and second order linear filters [3]. The authors of [3] used the singular perturbation method and Tikhonov's theorem to prove the closed-loop stability and to obtain the performance bounds.

In [14], we introduced a certainty equivalence state feedback indirect adaptive control approach without over parametrization for nonlinear systems of any relative degree in the parametric strict feedback form. The approach was based on the state prediction model, which is capable of providing fast estimation of unknown parameters independent of the control design. This property is the consequence of feeding back an error term with the gain proportional to the square root of the adaptation rate, like in the modified reference model MRAC (M-MRAC) architecture introduced in [13], where it has been shown that the error feedback gain acts as a damping factor for the adaptive signals, while the adaptation rate determines their frequency.

In this paper we extend the method to the systems with unknown virtual control coefficients. It is shown that the input tracking error (difference between ideal control and command filtered certainty equivalence control signal) and output tracking error can be regulated as desired by the proper choice of design parameters.

The rest of the paper is organized as follows. In Section II, we give the problem statement and the main assumptions. In Section III, we introduce the identification model and give its properties. The control design and performance analysis are presented in Section IV. A simulation example is presented in Section VII, and some concluding remarks are given in Section VIII.

## II. Problem Statement

We consider control design for a system

$$\begin{aligned} \dot{x}_i(t) &= a_i x_{i+1}(t) + \boldsymbol{\vartheta}_i^\top \boldsymbol{f}_i(t), \\ & \quad i = 1, \dots, n-1 \\ \dot{x}_n(t) &= a_n u(t) + \boldsymbol{\vartheta}_n^\top \boldsymbol{f}_n(t)\,, \end{aligned} \tag{1}$$

with some initial conditions $\boldsymbol{x}(0) = \boldsymbol{x}_0$, where $\boldsymbol{x} \in \mathbb{R}^n$ and $u \in \mathbb{R}$ are the state and input of the system, $\boldsymbol{\vartheta} \in \mathbb{R}^p$ is a vector of unknown constant parameters, $\boldsymbol{f}_i : \mathbb{R} \to \mathbb{R}^p$, $i = 1, \dots, n$ are sufficiently smooth known functions, and $a_1, \dots, a_n$ are unknown nonzero virtual control coefficients. We assume that the sign of $a_n$ is known (positive without loss of generality). Here, we use short hand notations $\boldsymbol{f}_i(t) = \boldsymbol{f}_i(x_1(t), \dots, x_i(t)),\ i = 1, \dots, n$.

Our goal is to design a controller such that all closed-loop signals are bounded and the output $y(t) = x_1(t)$ of the system (1) tracks a desired bounded command, which has bounded derivatives up to of order $n-1$. One way of providing such a command is to filter appropriately chosen piecewise continuous signal $y_{com}(t)$ trough a reference model of the same order as the system itself, and track the model's output, which is the approach adopted here. The reference model is given in the controllable canonical form

$$\dot{\boldsymbol{x}}_m(t) = A_m \boldsymbol{x}_m(t) + \boldsymbol{b}_m y_{com}(t), \quad \boldsymbol{x}_m(0) = \boldsymbol{x}_0\,, \tag{2}$$

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where $\boldsymbol{x}_m \in \mathbb{R}^n$ is the state of the reference model, $A_m = A - \boldsymbol{v}_{n,n}\boldsymbol{k}_x^\top$, $A = [0\ \boldsymbol{v}_{n,1}\ \dots\ \boldsymbol{v}_{n,n-1}]$, $\boldsymbol{b}_m = k_r\boldsymbol{v}_{n,n}$, $\boldsymbol{v}_{n,i}$ is the $i$-th coordinate vector in $\mathbb{R}^n$, and the gains $\boldsymbol{k}_x$ and $k_r$ are chosen from the perspective of the performance specifications and to make $A_m$ Hurwitz. The output of the reference model is $y_m(t) = x_{m,1}(t)$.

## III. Identification

We transform (1) into the form

$$\begin{aligned} \dot{x}_i(t) &= x_{i+1}(t) + b_i x_{i+1}(t) + \boldsymbol{\vartheta}_i^\top \boldsymbol{f}_i(t), \\ & \quad i = 1,\dots,n-1 \\ \dot{x}_n(t) &= g_n(t) + a_n[u(t) - b_n g_n(t) + \bar{\boldsymbol{\vartheta}}_n^\top \boldsymbol{f}_n(t)], \end{aligned} \tag{3}$$

where $b_i = a_i - 1,\ i = 1,\dots,n-1$, $b_n = 1/a_n$, $\bar{\boldsymbol{\vartheta}}_n = b_n\boldsymbol{\vartheta}_n$, and $g_n(t)$ is a smooth function to be specified in the control design. This form is better suited for the parameter estimation scheme adopted here and does not require devision by the parameter estimates in the control design.

To estimate the unknown quantities in the first $n-1$ equation in (3) we design an identification model

$$\begin{aligned} \dot{\hat{x}}_i(t) &= \hat{x}_{i+1}(t) + \hat{b}_i(t)x_{i+1}(t) + \hat{\boldsymbol{\vartheta}}_i^\top(t)\boldsymbol{f}_i(t) + \lambda\tilde{x}_i(t) \\ & \quad i = 1,\dots,n-1, \end{aligned} \tag{4}$$

where $\hat{x}_i(t)$ is the prediction of the $i$-th state, $\tilde{x}_i(t) = x_i(t) - \hat{x}_i(t)$ is the prediction error, $\lambda > 0$ is a design parameter, $\hat{b}_i(t),\ \hat{\boldsymbol{\vartheta}}_i(t)$ are the estimates of the unknown constant parameters $b_i,\ \boldsymbol{\vartheta}_i$. These estimates are generated according to adaptive laws

$$\begin{aligned} \dot{\hat{b}}_i(t) &= \gamma\tilde{x}_i(t)x_{i+1}(t) \\ \dot{\hat{\boldsymbol{\vartheta}}}_i(t) &= \gamma\tilde{x}_i(t)\boldsymbol{f}_i(t),\ i = 1,\dots,n-1, \end{aligned} \tag{5}$$

where $\gamma > 0$ is the adaptation rate.

The identification model for the estimation of the uncertainties in the last equation in (3) has the form

$$\begin{aligned} \dot{\hat{x}}_n(t) &= g_n(t) + \lambda\tilde{x}_n(t) \\ &+ \hat{a}_n(t)\left[u(t) - \hat{b}_n(t)g_n(t) + \hat{\boldsymbol{\vartheta}}_n^\top(t)\boldsymbol{f}_n(t)\right], \end{aligned} \tag{6}$$

where $\hat{x}_n(t)$ is the prediction of $x_n(t)$, $\tilde{x}_n(t) = x_n(t) - \hat{x}_n(t)$ is the prediction error, $\hat{a}_n(t),\ \hat{b}_n(t),\ \hat{\boldsymbol{\vartheta}}_n(t)$ are the estimates of the unknown constant parameters $a_n,\ b_n,\ \bar{\boldsymbol{\vartheta}}_n$. We notice that if we design the control signal as

$$u(t) = \hat{b}_n(t)g_n(t) - \hat{\boldsymbol{\vartheta}}_n^\top(t)\boldsymbol{f}_n(t), \tag{7}$$

then the prediction model (6) reduces to

$$\dot{\hat{x}}_n(t) = g_n(t) + \lambda\tilde{x}_n(t). \tag{8}$$

Therefore $\hat{a}_n(t)$ is not needed for the control design, and for the generation of the necessary estimates $\hat{b}_n(t)$ and $\hat{\boldsymbol{\vartheta}}_n(t)$, the adaptive laws

$$\begin{aligned} \dot{\hat{b}}_n(t) &= -\gamma\tilde{x}_n(t)g_n(t) \\ \dot{\hat{\boldsymbol{\vartheta}}}_n(t) &= \gamma\tilde{x}_n(t)\boldsymbol{f}_n(t). \end{aligned} \tag{9}$$

can be implemented with $\tilde{x}_n(t)$ generated using the reduced model (8).

The prediction error dynamics are easily derived to be

$$\dot{\tilde{\boldsymbol{x}}}(t) = (A - \lambda\mathbb{I})\tilde{\boldsymbol{x}}(t) + \boldsymbol{\eta}(t), \tag{10}$$

where $\eta_i(t) = \tilde{b}_i(t)x_{i+1}(t) + \tilde{\boldsymbol{\vartheta}}_i^\top(t)\boldsymbol{f}_i(t),\ i = 1,\dots,n-1$, $\eta_n(t) = a_n[-\tilde{b}_n(t)g_n(t) + \tilde{\boldsymbol{\vartheta}}_n^\top(t)\boldsymbol{f}_n(t)]$, and the parameter estimation errors are defined as $\tilde{b}_i(t) = b_i - \hat{b}_i(t),\ i = 1\dots,n,\ \tilde{\boldsymbol{\vartheta}}_i(t) = \boldsymbol{\vartheta}_i - \hat{\boldsymbol{\vartheta}}_i(t),\ i = 1\dots,n-1,\ \tilde{\boldsymbol{\vartheta}}_n(t) = \bar{\boldsymbol{\vartheta}}_n - \hat{\boldsymbol{\vartheta}}_n(t)$.

*Lemma 3.1:* The error signals $\tilde{x}_i(t),\ \tilde{b}_i(t),\ \tilde{\boldsymbol{\vartheta}}_i(t),\ i = 1,\dots,n$ are globally bounded, and $\tilde{x}_i(t) \to 0$ as $t \to \infty$.

*Proof:* Consider a candidate Lyapunov function

$$\begin{aligned} V(t) &= \sum_{i=1}^{n}\tilde{x}_i^2(t) + \frac{1}{\gamma}\sum_{i=1}^{n-1}\left[\tilde{b}_i^2(t) + \tilde{\boldsymbol{\vartheta}}_i^\top(t)\tilde{\boldsymbol{\vartheta}}_i(t)\right] \\ &+ \frac{a_n}{\gamma}\left[\tilde{b}_n^2(t) + \tilde{\boldsymbol{\vartheta}}_n^\top(t)\tilde{\boldsymbol{\vartheta}}_n(t)\right]. \end{aligned}$$

It is straightforward to compute the derivative of $V(t)$ along the trajectories of the prediction error dynamics (10), and the adaptive laws (5) and (9)

$$\dot{V}(t) = -2\lambda\sum_{i=1}^{n}\tilde{x}_i^2(t) + 2\sum_{i=2}^{n}\tilde{x}_{i-1}(t)\tilde{x}_i(t). \tag{11}$$

Since $2\tilde{x}_{i-1}(t)\tilde{x}_i(t) = [x_{i-1}(t)+\tilde{x}_i(t)]^2 - [x_{i-1}^2(t)+\tilde{x}_i^2(t)]$, we conclude that

$$\begin{aligned} \dot{V}(t) &= -(2\lambda-1)\sum_{i=1}^{n}\tilde{x}_i^2(t) - [\tilde{x}_1^2(t) + \tilde{x}_n^2(t)] \\ &\quad -\sum_{i=1}^{n-1}[\tilde{x}_i(t) - \tilde{x}_{i+1}(t)]^2 \le -(2\lambda-1)\sum_{i=1}^{n}\tilde{x}_i^2(t). \end{aligned} \tag{12}$$

If we select $2\lambda > 1$, it follows from the LaSalle-Yoshizawa theorem ([8], p.24) that $\tilde{\boldsymbol{x}}(t),\ \tilde{\boldsymbol{\theta}}_i(t),\ \tilde{b}_i(t),\ i = 1,\dots,n$ are globally uniformly bounded, and $\tilde{\boldsymbol{x}}(t) \to 0$ as $t \to \infty$. ∎

When $\boldsymbol{x}(t)$ and $u(t)$ are bounded (which will be provided by the control design), the following upper bounds on the error signals $\tilde{\boldsymbol{x}}(t)$ and $\eta_i(t),\ i = 1,\dots,n$ can be derived.

*Lemma 3.2:* Let the estimates $\hat{x}_i(t), \hat{b}_i(t)$, and $\hat{\boldsymbol{\vartheta}}_i(t),\ i = 1,\dots,n$ be generated by the identification models (4), (5), (8), and (9). In addition, let $\boldsymbol{x}(t)$ and $u(t)$ be bounded. Then $\eta_i(t)$ and $\tilde{x}_i(t)$ satisfy the following bounds

$$|\eta_i(t)| \le \beta_{i,1}e^{-\nu t} + \frac{\beta_{i,2}}{\sqrt{\gamma}} \tag{13}$$

$$|\tilde{x}_i(t)| \le \beta_{i,3}e^{-\nu t} + \frac{\beta_{i,4}}{\lambda\sqrt{\gamma}}, \tag{14}$$

where the constants $\beta_{i,j} > 0,\ i = 1,\dots,n,\ j = 1,2,3,4$ and $\nu > 0$ are defined in the proof.

*Proof:* Differentiating $\tilde{\eta}_i(t),\ i = 1,\dots,n-1$ and substituting the adaptive laws we obtain

$$\begin{aligned} \dot{\eta}_i(t) &= \dot{\tilde{b}}_i(t)x_{i+1}(t) + \dot{\tilde{\boldsymbol{\vartheta}}}_i^\top(t)\boldsymbol{f}_i(t) + \tilde{b}_i(t)\dot{x}_{i+1}(t) \\ &+ \tilde{\boldsymbol{\vartheta}}_i^\top(t)\dot{\boldsymbol{f}}_i(t) = -\gamma\rho_i(t)\tilde{x}_i(t) + h_i(t), \end{aligned} \tag{15}$$

where $\rho_i(t) = \tilde{x}_{i+1}^2(t) + \boldsymbol{f}_i^\top(t)\boldsymbol{f}_i(t),\ h_i(t) = \tilde{b}_i(t)\dot{x}_{i+1}(t) + \tilde{\boldsymbol{\vartheta}}_i^\top(t)\dot{\boldsymbol{f}}_i(t),\ i = 1,\dots,n-1$. It is straightforward to show that the second derivative of the signals $\eta_i(t),\ i = 1, \ldots, n-1$ satisfy the dynamics

$$\ddot{\eta}_i(t) + \lambda\dot{\eta}_i(t) + \gamma\rho_i(t)\eta_i(t) = \qquad (16)$$
$$-\gamma\rho_i(t)\tilde{x}_{i+1}(t) - \gamma\dot{\rho}_i(t)\tilde{x}_i(t) + \lambda h_i(t) + \dot{h}_i(t)\,.$$

Similarly, for $\eta_n(t)$ we obtain

$$\ddot{\eta}_n(t) + \lambda\dot{\eta}_n(t) + \gamma\rho_n(t)\eta_n(t) = \qquad (17)$$
$$-\gamma\dot{\rho}_n(t)\tilde{x}_n(t) + \lambda h_n(t) + \dot{h}_n(t)\,,$$

where $\rho_n(t) = a_n[g_n^2(t) + \boldsymbol{f}_n^\top(t)\boldsymbol{f}_n(t)]$ and $h_n(t) = a_n[-\tilde{b}_n(t)\dot{g}_n(t) + \tilde{\boldsymbol{\vartheta}}_n^\top(t)\dot{\boldsymbol{f}}_n(t)]$.

Since $\boldsymbol{x}(t)$ and $u(t)$ are bounded, and $g(t),\ \boldsymbol{f}_i(t),\ i = 1, \ldots, n$ are smooth functions, there exist positive constants $\delta_{i,1},\ \delta_{i,2},\ \delta_{i,3}$ such that $\|\rho_i(t)\|_{\mathcal{L}_\infty} \le \delta_{i,1},\ \|\dot{\rho}_i(t)\|_{\mathcal{L}_\infty} \le \delta_{i,2}$ and $\|h_i(t)\|_{\mathcal{L}_\infty} \le \delta_{i,3}$. Then, it follows from the results of [13] that choosing $\lambda \ge 2\sqrt{\delta_0\gamma}$, where $\delta_0 = \max[\delta_{1,1}, \ldots, \delta_{n,1}]$, damps the oscillations in $\eta_i(t)$ and guarantees the bound

$$|\eta_i(t)| \le \beta_i e^{-\nu_1 t} + \delta_{i,2}\|\tilde{\boldsymbol{x}}(t)\| + \frac{\delta_{i,4}}{\sqrt{\gamma}}|h_i(t)|\,, \qquad (18)$$

where $\nu_1$ is proportional to $\sqrt{\gamma}$, and the positive constants $\beta_i$ and $\delta_{i,4}$ are independent of $\gamma$ (see details in [13]).

It follows from Lemma 3.1 that there exists $\delta_1 > 0$ such that $\sum_{i=1}^n[\|\tilde{\boldsymbol{\theta}}_i(t)\|^2 + b_i^2(t)] \le \delta_1^2$. Therefore, the inequality (12) can be written as

$$\dot{V}(t) \le -(2\lambda - 1)\left[V(t) - \frac{1}{\gamma}\delta_1^2\right]\,. \qquad (19)$$

Integrating (19) we conclude that

$$V(t) \le \left[V(0) - \frac{\delta_1^2}{\gamma}\right]e^{-(2\lambda-1)t} + \frac{\delta_1^2}{\gamma}\,. \qquad (20)$$

Since $\|\tilde{\boldsymbol{x}}(t)\|^2 = \sum_{i=1}^n\ \tilde{x}_i^2(t) \le V(t)$, we obtain

$$\|\tilde{\boldsymbol{x}}(t)\| \le \sqrt{\left[2V(0) - \frac{\delta_1^2}{\gamma}\right]e^{-(2\lambda-1)t} + \frac{\delta_1^2}{\gamma}}\,, \qquad (21)$$

Recalling that $\sqrt{a+b} \le \sqrt{a} + \sqrt{b}$ for any $a \ge 0,\ b \ge 0$, we arrive at

$$\|\tilde{\boldsymbol{x}}(t)\| \le \delta_2 e^{-(\lambda-1/2)t} + \frac{\delta_1}{\sqrt{\gamma}}\,, \qquad (22)$$

where $\delta_2 = \sqrt{|2V(0) - \frac{\delta_1^2}{\gamma}|}$.

Substituting (22) into (18) we obtain (13) with $\nu = \min(\nu_1,\ \lambda - 1/2)$, $\beta_{i,1} = \beta_i + \delta_{i,2}\delta_2$ and $\beta_{i,2} = \delta_{i,2}\delta_1 + \delta_{i,3}\delta_{i,4}$.

The bound (14) is obtained by direct integration of (10), the last equation of which has the form

$$\dot{\tilde{x}}_n(t) = -\lambda\tilde{x}_n(t) + \eta_n(t)\,. \qquad (23)$$

Therefore

$$|\tilde{x}_n(t)| \le |\tilde{x}_n(t)|e^{-\lambda t} + \frac{\beta_{n,1}}{\lambda - \nu}\left[e^{-\nu t} - e^{-\lambda t}\right] \qquad (24)$$
$$+ \frac{\beta_{n,2}}{\lambda\sqrt{\gamma}}\left[1 - e^{-\lambda t}\right] \le \beta_{n,3}e^{-\nu t} + \frac{\beta_{n,4}}{\lambda\sqrt{\gamma}}\,,$$

where $\beta_{n,3}$ and $\beta_{n,4}$ are readily computed. Working backward and taking into account the bounds

$$|\tilde{x}_{i+1}(t) + \eta_i(t)| \le (\beta_{i,1} + \beta_{i+1,3})\,e^{-\nu t} + \frac{\lambda\beta_{i,2} + \beta_{i+1,4}}{\lambda\sqrt{\gamma}}\,,$$

we obtain in the similar way the inequalities

$$|\tilde{x}_i(t)| \le \beta_{i,3}e^{-\nu t} + \frac{\beta_{i,4}}{\lambda\sqrt{\gamma}} \qquad (25)$$

for each $i = n-1, \ldots, 1$. The proof is complete. ■

## IV. Control Design for Known Systems

In this section we design two controllers, which are solely used for the performance analysis.

### A. Conventional Backstepping

We first design a controller assuming that $\boldsymbol{b},\ \boldsymbol{\vartheta}_1, \ldots, \boldsymbol{\vartheta}_n$ are known (ideal control) and formally applying the conventional backstepping procedure to the system (3). The error variables are defined as

$$e_1^0(t) = x_1^0(t) - x_{m,1}(t) \qquad (26)$$
$$e_i^0(t) = x_i^0(t) - x_{m,i}(t) - g_{i-1}^0(t), \quad i = 2, \ldots, n\,,$$

where the stabilizing functions have the form

$$g_1^0(t) = -b_1x_2^0(t) - \boldsymbol{\vartheta}_1^\top\boldsymbol{f}_1^0(t)$$
$$g_i^0(t) = -b_ix_{i+1}^0(t) - \boldsymbol{\vartheta}_i^\top\boldsymbol{f}_i^0(t) + \dot{g}_{i-1}^0(t), \quad (27)$$
$$i = 2, \ldots, n-1\,.$$

To distinguish between the designs we furnish all variables with a superscript 0, which indicates that the loop is closed by the backstepping controlled designed for (3) in the ideal case of known dynamics.

The error variables satisfy the equations

$$\dot{e}_i^0(t) = e_{i+1}^0(t), \quad i = 1, \ldots, n-1 \qquad (28)$$
$$\dot{e}_n^0(t) = g_n^0(t) + [u^0(t) - b_ng_n^0(t) + \bar{\boldsymbol{\vartheta}}_n^\top\boldsymbol{f}_n^0(t)]$$
$$- \dot{x}_{m,n}(t) - \dot{g}_{n-1}(t)\,.$$

When the control signal is designed as

$$u^0(t) = b_ng_n^0(t) - \bar{\boldsymbol{\vartheta}}_n^\top\boldsymbol{f}_n^0(t)\,, \qquad (29)$$

the error dynamics reduce to an exponentially stable system

$$\dot{\boldsymbol{e}}^0(t) = A_m\boldsymbol{e}^0(t)\,, \qquad (30)$$

if we select

$$g_n^0(t) = -\boldsymbol{k}_x^\top\boldsymbol{e}^0(t) + \dot{x}_{m,n}(t) + \dot{g}_{n-1}(t)\,. \qquad (31)$$

Although the stabilizing functions depend on the corresponding virtual controls, it can be easily shown that the algebraic loops are solvable and the controller could be implemented if the uncertainties were known.

We notice that since $\boldsymbol{e}^0(t)$ is exponentially stable, it follows that $y^0(t)$ exponentially converges to $y_m(t)$. In addition, $y_{com}(t) \in \mathcal{L}_\infty$ implies that $\boldsymbol{x}_m(t) \in \mathcal{L}_\infty$, which along with $\boldsymbol{e}^0(t) \in \mathcal{L}_\infty$ can be used to recursively show that $g_i^0(t),\ \dot{g}_i^0(t) \in \mathcal{L}_\infty,\ i = 2, \ldots, n$. Then, $u^0(t) \in \mathcal{L}_\infty$, and the following lemma has been proven.

*Lemma 4.1:* The controller defined by (26), (27) and (29) guarantees the boundedness of all closed loop signals in the system (3) and exponential tracking of the reference model's output.

### B. Command Filtering

Following [3], we introduce the command filtered version of the design (26), (27) and (29). In this case the error variables are introduced as

$$\begin{aligned} e_1^f(t) &= x_1^f(t) - x_{m,1}(t) \\ e_i^f(t) &= x_i^f(t) - x_{m,i}(t) - \sigma_{i-1,1}^0(t), \quad i = 2,\ldots,n\,, \end{aligned} \tag{32}$$

where $\sigma_{i,1}^0(t),\ i = 1,\ldots,n-1$ are the outputs of the command filters

$$\begin{aligned} \dot{\sigma}_{i,1}^0(t) &= \omega\sigma_{i,2}^0(t) \\ \dot{\sigma}_{i,2}^0(t) &= -2\zeta\omega\sigma_{i,2}^0(t) - \omega[\sigma_{i,1}^0(t) - g_i^f(t)] \end{aligned} \tag{33}$$

with initial conditions $\sigma_{i,1}^0(0) = g_i^f(0)$ and $\sigma_{i,2}^0(0) = 0$. The stabilizing functions are defined similar to the ideal case as

$$\begin{aligned} g_1^f(t) &= -b_1 x_2^f(t) - \boldsymbol{\vartheta}_1^\top \boldsymbol{f}_1^f(t) \\ g_i^f(t) &= -b_i x_{i+1}^f(t) - \boldsymbol{\vartheta}_i^\top \boldsymbol{f}_i^f(t) + \omega\sigma_{i-1,2}^0(t), \\ & \quad i = 2,\ldots,n-1\,. \end{aligned} \tag{34}$$

where the superscript $f$ indicates that the corresponding quantities are computed with the command filtered control in the loop. The system (3) is written in error variable as

$$\begin{aligned} \dot{e}_i^f(t) &= e_{i+1}^f(t) - g_i^f(t) + \sigma_{i,1}^0(t),\ i = 1,\ldots,n-1 \\ \dot{e}_n^f(t) &= g_n^f(t) + a_n\left[u(t) - b_n g_n^f(t) + \bar{\boldsymbol{\vartheta}}_i^\top \boldsymbol{f}_i^f(t)\right] \\ &- \dot{x}_{m,n}(t) - \omega\sigma_{n-1,2}^0(t)\,. \end{aligned} \tag{35}$$

The command filtered control signal is defined according to equations

$$\begin{aligned} u^f(t) &= b_n g_n^f(t) - \bar{\boldsymbol{\vartheta}}_i^\top \boldsymbol{f}_i^f(t) \\ g_n^f(t) &= -\boldsymbol{k}_x^\top \boldsymbol{e}^f(t) + \dot{x}_{m,n}(t) + \omega\sigma_{n-1,2}^0(t)\,, \end{aligned} \tag{36}$$

which translates the error dynamics (35) into

$$\dot{\boldsymbol{e}}^f(t) = A_m\boldsymbol{e}^f(t) + \boldsymbol{\alpha}(t)\,, \tag{37}$$

where

$$\boldsymbol{\alpha}(t) = \begin{bmatrix} \sigma_{1,1}^0(t) - g_1^f(t) \\ \vdots \\ \sigma_{n-1,1}^0(t) - g_{n-1}^f(t) \\ 0 \end{bmatrix}.$$

For the stability analysis we introduce the compensated error signal as $\boldsymbol{e}_c^f(t) = \boldsymbol{e}^f(t) - \boldsymbol{\xi}(t)$, where $\boldsymbol{\xi}(t)$ satisfies the dynamics

$$\dot{\boldsymbol{\xi}}(t) = A_m\boldsymbol{\xi}_i(t) + \boldsymbol{\alpha}(t) \tag{38}$$

with the initial condition $\boldsymbol{\xi}(0) = 0$ and $\xi_n(t) \equiv 0$ for all $t \geq 0$. It is straightforward to verify that

$$\dot{\boldsymbol{e}}_c^f(t) = A_m\boldsymbol{e}_c^f(t)\,. \tag{39}$$

Obviously, the compensated error dynamics are exponentially stable.

*Lemma 4.2:* The command filtered controller defined by (32), (33), (34), and (36) guarantees the following relationships

$$\begin{aligned} \boldsymbol{e}^f(t) - \boldsymbol{e}^0(t) &= \mathcal{O}(\varepsilon)\,, \quad \boldsymbol{\xi}(t) = \mathcal{O}(\varepsilon) \\ \sigma_{i,1}^0(t) - g_i^0(t) &= \mathcal{O}(\varepsilon)\,, \quad i = 1,\ldots,n-1 \\ \omega\sigma_{i,2}^0(t) - \dot{g}_i^0(t) &= \mathcal{O}(\varepsilon)\,, \quad i = 1,\ldots,n-1\,, \end{aligned} \tag{40}$$

where $\varepsilon = 1/\omega$ (the proper choice of $\zeta$ and $\omega$ is discussed in [3]), and the notation $\mathcal{O}(\varepsilon)$ is adopted from [6] (p. 383).

The proof of the lemma follows the steps of Theorem 2 from [3].

## V. Adaptive Control Design

### A. Certainty Equivalent Control

Here, we design a controller for the identification models (4) and (6). In essence, the form of the control signal is given by (7) and we need to only specify $g(t)$. To distinguish from the ideal design we use "hat" notation for the corresponding variables. Following the conventional backstepping design steps we define the stabilizing functions as

$$\begin{aligned} \hat{g}_1(t) &= -\hat{b}_1(t)x_2(t) - \hat{\boldsymbol{\vartheta}}_1^\top(t)\boldsymbol{f}_1(t) \\ \hat{g}_i(t) &= -\hat{b}_i(t)x_{i+1}(t) - \hat{\boldsymbol{\vartheta}}_i^\top(t)\boldsymbol{f}_i(t) + \dot{\hat{g}}_{i-1}(t)\,, \\ & \quad i = 2,\ldots,n-1\,. \end{aligned} \tag{41}$$

and the error variables as

$$\begin{aligned} \hat{e}_1(t) &= \hat{x}_1(t) - x_{m,1}(t) \\ \hat{e}_i(t) &= \hat{x}_i(t) - x_{m,i}(t) - \hat{g}_{i-1}(t), \quad i = 2,\ldots,n\,. \end{aligned} \tag{42}$$

The identification model (4) in the error variables takes the form

$$\dot{\hat{e}}_i(t) = \hat{e}_{i+1}(t) + \lambda\tilde{x}_i(t),\ i = 2,\ldots,n-1\,.$$

The dynamics of $\hat{e}_n(t)$ is derived from the equation (6) as follows

$$\begin{aligned} \dot{\hat{e}}_n(t) &= \hat{g}_n(t) + a_n\left[u(t) - \hat{b}_n(t)\hat{g}_n(t) + \hat{\boldsymbol{\vartheta}}_n^\top(t)\boldsymbol{f}_n(t)\right] \\ &- \dot{x}_{m,n}(t) - \dot{\hat{g}}_{n-1}(t) + \lambda\tilde{x}_n(t)\,. \end{aligned} \tag{43}$$

We notice that the error system is translated into

$$\dot{\hat{e}}_n(t) = -\boldsymbol{k}_x^\top\hat{\boldsymbol{e}}(t) + \lambda\tilde{x}_i(t)\,, \tag{44}$$

which is asymptotically stable, if we set $\hat{g}_n(t) = -\boldsymbol{k}_x^\top\hat{\boldsymbol{e}}(t) + \dot{x}_{m,n}(t) + \dot{\hat{g}}_{n-1}(t)$ and the certainty equivalence control law

$$\hat{u}(t) = \hat{b}_n(t)\hat{g}_n(t) - \hat{\boldsymbol{\vartheta}}_n^\top(t)\boldsymbol{f}_n(t)\,. \tag{45}$$

It is easy to see that the error dynamics can be written as

$$\dot{\hat{\boldsymbol{e}}}(t) = A_m(t)\hat{\boldsymbol{e}}(t) + \lambda\tilde{\boldsymbol{x}}(t)\,. \tag{46}$$

*Lemma 5.1:* The controller defined by (41), (42) and (45) guarantees boundedness of all closed-loop signals and the inequality

$$|\hat{e}_i(t)| \leq \beta_{i,5}e^{-\nu_2 t} + \frac{\beta_{i,6}}{\nu_2\lambda\sqrt{\gamma}}\,, \tag{47}$$

where $\beta_{i,j} > 0,\ i = 1,\dots,n,\ j = 5,6$ and $\nu_2 > 0$ are defined in the proof.

*Proof:* According to Lemma 3.2, $\tilde{\boldsymbol{x}}(t) \in \mathcal{L}_\infty$. Since $A_m$ is Hurwitz, it follows from (46) that $\hat{\boldsymbol{e}}(t) \in \mathcal{L}_\infty$. In addition, from $y_{com}(t) \in \mathcal{L}_\infty$ it follows that $\boldsymbol{x}_m(t) \in \mathcal{L}_\infty$. Therefore $\hat{x}_1(t) = \hat{e}_1(t) + x_{m1}(t) \in \mathcal{L}_\infty$ and $x_1(t) = \hat{x}_1(t) + \tilde{x}_1(t) \in \mathcal{L}_\infty$. Then $\hat{g}_1(t) \in \mathcal{L}_\infty$, since $f_1(x_1(t))$ and $\boldsymbol{h}_1(x_1(t))$ are continuous and $\hat{b}_1(t),\ \hat{\boldsymbol{\theta}}_1(t) \in \mathcal{L}_\infty$ according to Lemma 3.1. This implies that $\hat{x}_2(t) = \hat{e}_2(t) + \hat{g}_1(t) \in \mathcal{L}_\infty$ and $x_2(t) = \hat{x}_2(t) + \tilde{x}_2(t) \in \mathcal{L}_\infty$. Continuing this recursion we conclude that $\hat{\boldsymbol{x}}(t) \in \mathcal{L}_\infty$, $\boldsymbol{x}(t) \in \mathcal{L}_\infty$, $\hat{g}_i(t) \in \mathcal{L}_\infty,\ i = 1,\dots,n$, $\dot{\hat{g}}_i(t) \in \mathcal{L}_\infty,\ i = 1,\dots,n-1$, and $\hat{u}(t) \in \mathcal{L}_\infty$. Thus all closed-loop signals are bounded. To derive the bound (47) we notice that $e^{A_m t} \le \kappa e^{-\nu_2 t}$ for some $\kappa > 0$, where $\nu_2 > 0$ is the decay rate of the reference model. Therefore integrating (46) and taking into account (14) we obtain

$$\begin{aligned}|\hat{e}_i(t)| &\le \frac{\kappa\beta_{i,3}}{\nu-\nu_2}\left[e^{-\nu_2 t} - e^{-\nu t}\right] + \frac{\kappa\beta_{i,4}}{\nu_2\lambda\sqrt{\gamma}}\left[1 - e^{-\nu_2 t}\right] \\ &\le \beta_{i,5}e^{-\nu_2 t} + \frac{\beta_{i,6}}{\nu_2\lambda\sqrt{\gamma}}\,. \end{aligned} \tag{48}$$

This control design is used for the performance analysis. ■

### B. Actual Control Design

Now we design the actual control applied to the system as the command filtered version of the certainty equivalent control from the previous subsection. The uncompensated error variables are introduced as

$$\begin{aligned}\hat{e}_1^f(t) &= \hat{x}_1^f(t) - x_{m,1}(t) \\ \hat{e}_i^f(t) &= \hat{x}_i(t) - x_{m,i}(t) - \hat{\sigma}_{i-1,1}(t), \quad i = 2,\dots,n.\end{aligned} \tag{49}$$

where $\hat{\sigma}_{i-1,1}(t)$ is the first state of the command filter

$$\begin{aligned}\dot{\hat{\sigma}}_{i,1}(t) &= \omega\hat{\sigma}_{i,2}(t) \\ \dot{\hat{\sigma}}_{i,2}(t) &= -2\zeta\omega\hat{\sigma}_{i,2}(t) - \omega[\hat{\sigma}_{i,1}(t) - \hat{g}_i^f(t)] \\ & i = 0,\dots,n-1\,,\end{aligned} \tag{50}$$

with the initial conditions $\hat{\sigma}_{i,1}(0) = \hat{g}_i^f(0)$ and $\hat{\sigma}_{i,2}(0) = 0$. The stabilizing functions have the form

$$\begin{aligned}\hat{g}_1^f(t) &= -\hat{b}_1(t)x_2^f(t) - \hat{\boldsymbol{\vartheta}}_1^\top(t)\boldsymbol{f}_1^f(t) \\ \hat{g}_i^f(t) &= -\hat{b}_i(t)x_{i+1}^f(t) - \hat{\boldsymbol{\vartheta}}_i^\top(t)\boldsymbol{f}_i^f(t) + \omega\sigma_{i-1,2}(t)\,, \\ & i = 2,\dots,n-1\,.\end{aligned} \tag{51}$$

where the superscript $f$ indicates that the corresponding quantities are computed with the command filtered control in the loop. The identification model in error variables takes the form

$$\begin{aligned}\dot{\hat{e}}_i^f(t) &= \hat{e}_{i+1}(t) - \hat{g}_i^f(t) + \hat{\sigma}_{i,1}(t) + \lambda\tilde{x}_i^f(t), \\ & i = 1,\dots,n-1 \\ \dot{\hat{e}}_n^f(t) &= \hat{g}_n^f(t) + [u(t) - \hat{b}_n(t)\hat{g}_n^f(t) + \hat{\boldsymbol{\vartheta}}_n^\top(t)\boldsymbol{f}_n^f(t)] \\ & \quad -\dot{x}_{m,n}(t) + \omega\hat{\sigma}_{n-1,2}(t) + \lambda\tilde{x}_n^f(t)\,.\end{aligned} \tag{52}$$

where $\tilde{\boldsymbol{x}}^f(t) = \boldsymbol{x}^f(t) - \hat{\boldsymbol{x}}^f(t)$. Designing the controller as

$$\begin{aligned}\hat{u}^f(t) &= \hat{b}_n(t)\hat{g}_n^f(t) - \hat{\boldsymbol{\vartheta}}_n^\top(t)\boldsymbol{f}_n^f(t) \\ \hat{g}_n^f(t) &= -\boldsymbol{k}_x^\top\hat{\boldsymbol{e}}^f(t) + \dot{x}_{m,n}(t) + \omega\hat{\sigma}_{n-1,2}(t)\,.\end{aligned} \tag{53}$$

reduces the error dynamic to

$$\dot{\hat{\boldsymbol{e}}}^f(t) = A_m\hat{\boldsymbol{e}}^f(t) + \hat{\boldsymbol{\alpha}}(t) + \lambda\tilde{\boldsymbol{x}}^f(t)\,, \tag{54}$$

where $\hat{\boldsymbol{\alpha}}(t)$ is obtained from $\boldsymbol{\alpha}(t)$ by replacing $\sigma_{i,1}^0(t)$ with $\sigma_{i,1}(t)$ and $g_i^f(t)$ with $\hat{g}_i^f(t)$.

The compensated error signal is $\hat{\boldsymbol{e}}_c^f(t) = \hat{\boldsymbol{e}}^f(t) - \hat{\boldsymbol{\xi}}(t)$, where $\hat{\boldsymbol{\xi}}(t)$ now satisfies the dynamics

$$\dot{\hat{\boldsymbol{\xi}}}(t) = A_m\hat{\boldsymbol{\xi}}_i(t) + \hat{\boldsymbol{\alpha}}(t) \tag{55}$$

with the initial condition $\boldsymbol{\xi}(0) = 0$. It is straightforward to verify that

$$\dot{\hat{\boldsymbol{e}}}_c^f(t) = A_m\hat{\boldsymbol{e}}_c^f(t) + \lambda\tilde{\boldsymbol{x}}^f(t)\,. \tag{56}$$

*Lemma 5.2:* The controller defined by (49), (50), (51), and (53) guarantees the following relationships

$$\begin{aligned}&\hat{\boldsymbol{e}}^f(t) - \hat{\boldsymbol{e}}(t) = \mathcal{O}(\varepsilon)\,, \quad \hat{\boldsymbol{\xi}}(t) = \mathcal{O}(\varepsilon) \\ &\hat{\sigma}_{i,1}(t) - \hat{g}_i^f(t) = \mathcal{O}(\varepsilon)\,, \quad i = 1,\dots,n-1 \\ &\omega\hat{\sigma}_{i,2}(t) - \dot{\hat{g}}_i^f(t) = \mathcal{O}(\varepsilon)\,, \quad i = 1,\dots,n-1\,.\end{aligned} \tag{57}$$

*Proof:* Since the exponential convergence of $\tilde{\boldsymbol{x}}^f(t)$ is not guaranteed, Tikhonov's theorem ([6], Theorem 11.2) cannot be directly applied to the system comprised of (52), (55) and (56). However, since $\tilde{\boldsymbol{x}}^f(t)$ does not depend on $\varepsilon$, the state transformation $\boldsymbol{s}^f(t) = \hat{\boldsymbol{e}}^f(t) - \boldsymbol{\mu}(t)$, where $\boldsymbol{\mu}(t)$ is dynamically defined as $\dot{\boldsymbol{\mu}}(t) = A_m\boldsymbol{\mu}(t) + \lambda\tilde{\boldsymbol{x}}^f(t)$, results in the system

$$\dot{\boldsymbol{s}}^f(t) = A_m\boldsymbol{s}^f(t) + \hat{\boldsymbol{\alpha}}(t)\,. \tag{58}$$

Next, define $\hat{\boldsymbol{s}}(t) = \hat{\boldsymbol{e}}(t) - \boldsymbol{\mu}(t)$, which satisfies the exponentially stable dynamics

$$\dot{\hat{\boldsymbol{s}}}(t) = A_m\hat{\boldsymbol{s}}(t)\,. \tag{59}$$

Following the steps from [3], it can be verified that Thikhonov's theorem's conditions are satisfied for the systems (58) and (59). Therefore the last three relationships in (57) can be concluded along with $\boldsymbol{s}^f(t) - \hat{\boldsymbol{s}}(t) = \mathcal{O}(\varepsilon)$. Then, it follows that $\hat{\boldsymbol{e}}^f(t) - \hat{\boldsymbol{e}}(t) = \boldsymbol{s}^f(t) - \hat{\boldsymbol{s}}(t) = \mathcal{O}(\varepsilon)$, which completes the proof. ■

## VI. Performance Analysis

*Lemma 6.1:* Let the command filtered controller for system (3) be defined by (32), (33), (34), and (36). Then all closed-loop signals are bounded and

$$\begin{aligned}\boldsymbol{x}^f(t) - \boldsymbol{x}^0(t) &= \mathcal{O}(\varepsilon) \\ u^f(t) - u^0(t) &= \mathcal{O}(\varepsilon)\,.\end{aligned} \tag{60}$$

*Proof:* From Lemma 4.1 we have $\boldsymbol{e}^0(t) \in \mathcal{L}_\infty$, $g_i^0(t),\ \dot{g}_i^0(t) \in \mathcal{L}_\infty$ for $i = 1,\dots,n-1$. Therefore, (40) implies that $\boldsymbol{e}^f(t) \in \mathcal{L}_\infty$, $\sigma_{i,1}^0(t) \in \mathcal{L}_\infty$ and $\sigma_{i,2}^0(t) \in \mathcal{L}_\infty$ for $i = 1,\dots,n-1$. Next, (32) implies that $\boldsymbol{x}^f(t) \in \mathcal{L}_\infty$. Therefore $\boldsymbol{f}_i^f(t) \in \mathcal{L}_\infty$, and equations (34) imply that $g_i^f(t) \in \mathcal{L}_\infty$ for all $i = 1,\dots,n$. It follows from the definition (36) that $u^f(t) \in \mathcal{L}_\infty$. Further, we notice that

$$x_i^f(t) - x_i^0(t) = e_i^f(t) - e_i^0(t) + \sigma_{i,1}^0(t) - g_i^0(t)\,.$$

Since $e_i^f(t) - e_i^0(t) = \mathcal{O}(\varepsilon)$ and $\sigma_{i,1}^0(t) - g_i^0(t) = \mathcal{O}(\varepsilon)$, it follows that $x_i^f(t) - x_i^0(t) = \mathcal{O}(\varepsilon)$ for all $i = 1, \ldots, n$. From Lemma 4.2 we also have $\omega\sigma_{n-1,2}(t) - \dot{g}_{n-1}^0(t) = \mathcal{O}(\varepsilon)$, therefore

$$g_n^f(t) - g_n^0(t) = \mathcal{O}(\varepsilon) \ .$$

It follows from the smoothness of $\boldsymbol{f}_n$ that $\boldsymbol{f}_n(\boldsymbol{x}^f(t)) - \boldsymbol{f}_n(\boldsymbol{x}^0(t)) = \mathcal{O}(\varepsilon)$. Then

$$u^f(t) - u^0(t) = \mathcal{O}(\varepsilon) \ .$$

The proof is complete. ■

*Lemma 6.2:* Let the controller for system (3) and the identification models (4) and (8) be defined by (49), (50), (51), and (53). Then all signals are bounded and

$$\hat{\boldsymbol{x}}^f(t) - \hat{\boldsymbol{x}}(t) = \mathcal{O}(\varepsilon) \qquad (61)$$
$$\hat{u}^f(t) - \hat{u}(t) = \mathcal{O}(\varepsilon) \ .$$

*Proof:* From Lemma 5.1 we have $\hat{\boldsymbol{e}}(t) \in \mathcal{L}_\infty$, $\hat{g}_i(t),\ \dot{\hat{g}}_i(t) \in \mathcal{L}_\infty$ for $i = 1, \ldots, n-1$. Therefore, Lemma 5.2 implies that $\boldsymbol{e}^f(t) \in \mathcal{L}_\infty$, $\sigma_{i,1}(t) \in \mathcal{L}_\infty$ and $\sigma_{i,2}(t) \in \mathcal{L}_\infty$ for $i = 1, \ldots, n-1$. It follows from (49) that $\hat{\boldsymbol{x}}^f(t) \in \mathcal{L}_\infty$. Since $\tilde{\boldsymbol{x}}^f(t) \in \mathcal{L}_\infty$, it follows that $\boldsymbol{x}^f(t) \in \mathcal{L}_\infty$ as well. Therefore $\boldsymbol{f}_i^f(t) \in \mathcal{L}_\infty$, and equations (51) imply that $\hat{g}_i^f(t) \in \mathcal{L}_\infty$ for all $i = 1, \ldots, n$. The definition (53) implies that $\hat{u}^f(t) \in \mathcal{L}_\infty$ as well. Next, we observe that

$$\hat{x}_i^f(t) - \hat{x}_i(t) = \hat{e}_i^f(t) - \hat{e}_i(t) + \hat{\sigma}_{i,1}(t) - \hat{g}_i(t) \, .$$

Since $\hat{e}_i^f(t) - \hat{e}_i(t) = \mathcal{O}(\varepsilon)$ and $\hat{\sigma}_{i,1}(t) - \hat{g}_i(t) = \mathcal{O}(\varepsilon)$, it follows that $\hat{x}_i^f(t) - \hat{x}_i(t) = \mathcal{O}(\varepsilon)$ for all $i = 1, \ldots, n$. On the other hand, $\omega\hat{\sigma}_{n-1,2}(t) - \dot{\hat{g}}_{n-1}(t) = \mathcal{O}(\varepsilon)$ is true according to Lemma 5.2, therefore

$$\hat{g}_n^f(t) - \hat{g}_n(t) = \mathcal{O}(\varepsilon) \ .$$

Next, $\boldsymbol{f}_n(\boldsymbol{x}^f(t)) - \boldsymbol{f}_n(\boldsymbol{x}(t)) = \mathcal{O}(\varepsilon)$ holds because $\boldsymbol{f}_n$ is smooth. Since $\hat{b}_n(t)$ and $\hat{\boldsymbol{\vartheta}}_n(t)$ are bounded, it follows that $\hat{u}^f(t) - \hat{u}(t) = \mathcal{O}(\varepsilon)$. The proof is complete. ■

Now we are ready to prove the main result.

*Theorem 6.1:* Let the controller for the system (1) be defined according to command filtered scheme given by (4), (5), (8), (9), (49), (50), (51), and (53). Then the input and output tracking errors satisfy the following upper bounds

$$|y(t) - y_m(t)| \leq \beta_3 e^{-\nu_2 t} + \frac{\beta_4}{\lambda\sqrt{\gamma}} + \mathcal{O}(\varepsilon) \qquad (62)$$

$$|\hat{u}^f(t) - u^0(t)| \leq \beta_5 e^{-\nu_2 t} + \frac{\beta_6}{\sqrt{\gamma}} + \mathcal{O}(\varepsilon) \ , \qquad (63)$$

where $\beta_3$, $\beta_4$, $\beta_5$, $\beta_6$, and $\nu_2$ are positive constants defined in the proof.

*Proof:* First of all, we notice that $y(t) - y_m(t) = x_1^f(t) - x_{m1}(t) = \tilde{x}_1^f(t) + \hat{x}_1^f(t) - x_{m1}(t) = \tilde{x}_1^f(t) + \hat{e}_1^f(t)$. Since $\hat{e}_1^f(t) = \hat{e}_1(t) + \mathcal{O}(\varepsilon)$, it follows that $y(t) - y_m(t) = \tilde{x}_1^f(t) + \hat{e}_1(t) + O(\varepsilon)$. Using (14) and (47) one can obtain

$$|y(t) - y_m(t)| \leq \beta_{1,3} e^{-\nu t} + \frac{\beta_{1,4}}{\lambda\sqrt{\gamma}} + \beta_{1,5} e^{-\nu_2 t}$$
$$+ \frac{\beta_{1,6}}{\nu_2\lambda\sqrt{\gamma}} + \mathcal{O}(\varepsilon) \leq \beta_3 e^{-\nu_2 t} + \frac{\beta_4}{\lambda\sqrt{\gamma}} + \mathcal{O}(\varepsilon) \ ,$$

since $\nu > \nu_2$ for fast adaptation, and $\beta_3$ and $\beta_4$ are readily computed.

To compute the bound for the control signal, we consider two closed-loop systems. Namely, the system (1) with the controller defined by (32), (33), (34) and (36), and the system (1) with the actual controller. For the clarity, we denote $y(u^f)$, when $y(t)$ is generated by the controller $u^f(t)$, and $y(\hat{u}^f)$ when $y(t)$ is generated by the actual control $\hat{u}^f(t)$. The error between this variables is $\tilde{y}^f(t) = y(u^f) - y(\hat{u}^f)$.

First, we recursively compute $\tilde{x}_i^f(t) = x_i(u^f) - x_i(\hat{u}^f)$, $\tilde{\boldsymbol{f}}_i^f(t) = \boldsymbol{f}_i(u^f) - \boldsymbol{f}_i(\hat{u}^f)$, $\tilde{g}_i^f(t) = g_i(u^f) - g_i(\hat{u}^f)$, $i = 1, \ldots, n$. For $i = 1$ we have $\tilde{x}_1^f(t) = x_1(u^f) - x_{m,1}(t) + x_{m,1}(t) - \hat{x}_1(\hat{u}^f) + \hat{x}_1(\hat{u}^f) - x_1(\hat{u}^f) = e_1^f(t) - \hat{e}_1^f(t) - \tilde{x}_1(t) = e_1^0(t) + \mathcal{O}(\varepsilon) - \tilde{x}_1(t) - \hat{e}_1^f(t)$. Since $|e^0(t)| \leq |e^0(0)|\kappa e^{-\nu_2 t}$, it follows from (14) and (47) that

$$|\tilde{x}_1^f(t)| \leq \|e^0(0)\|\kappa e^{-\nu_2 t} + \mathcal{O}(\varepsilon) + \beta_{1,3} e^{-\nu t} + \tfrac{\beta_{1,4}}{\lambda\sqrt{\gamma}}$$
$$+ \beta_{1,5} e^{-\nu_2 t} + \tfrac{\beta_{1,6}}{\nu_2\lambda\sqrt{\gamma}} \leq \beta_{x,1} e^{-\nu_2 t} + \tfrac{\delta_{x,2}}{\sqrt{\gamma}} + O(\varepsilon) \ , (64)$$

where $\beta_{x,1} = \|e^0(0)\|\kappa + \beta_{1,3} + \beta_{1,5}$ and $\delta_{x,2} = \frac{\beta_{1,4}}{\lambda} + \frac{\beta_{1,6}}{\nu_2\lambda}$. Since $\boldsymbol{x}(t)$ is bounded and $\boldsymbol{f}_1(t)$ is smooth, it follows from (64) that

$$\|\tilde{\boldsymbol{f}}_1^f(t)\| \leq \beta_{f,1} e^{-\nu_2 t} + \frac{\delta_{f,1}}{\sqrt{\gamma}} + O(\varepsilon) \ . \qquad (65)$$

where $\beta_{f,1} = L_1\beta_{x,1}$, $\delta_{f,1} = L_1\delta_{x,2}$, and $L_1$ is the Lipschitz constant for $\boldsymbol{f}_1$. Using (34) and (51) it is straightforward to compute that

$$\tilde{g}_1^f(t) = -b_1\tilde{x}_2^f(t) - \boldsymbol{\vartheta}_1^\top \tilde{\boldsymbol{f}}_1^f(t) - \eta_1(t) \, . \qquad (66)$$

On the other hand, it follows from (32) and (49) that

$$\tilde{x}_2^f(t) = e_2^f(t) + \sigma_{1,1}(u^f) - \tilde{x}_2(t) - \hat{e}_2^f(t) - \hat{\sigma}_{1,1}(\hat{u}^f) \, . \ (67)$$

From Lemmas 4.2 and 5.2 it follows that $\sigma_{1,1}(u^f) = g_1(u^f) + O(\varepsilon)$ and $\hat{\sigma}_{1,1}(\hat{u}^f) = g_1(\hat{u}^f) + O(\varepsilon)$, therefore

$$\tilde{x}_2^f(t) = e_2^f(t) - \tilde{x}_2(t) - \hat{e}_2^f(t) + O(\varepsilon)$$
$$-b_1\tilde{x}_2^f(t) - \boldsymbol{\vartheta}_1^\top \tilde{\boldsymbol{f}}_1^f(t) - \eta_1(t) \, . \qquad (68)$$

Solving (68) for $\tilde{x}_2^f(t)$ and recalling that $1 + b_1 = a_1 0$, we conclude

$$\tilde{x}_2^f(t) = \frac{e_2^f(t) - \tilde{x}_2(t) - \hat{e}_2^f(t) + O(\varepsilon) - \boldsymbol{\vartheta}_1^\top \tilde{\boldsymbol{f}}_1^f(t) - \eta_1(t)}{a_1} \ ,$$

from which we obtain

$$|\tilde{x}_2^f(t)| \leq \tfrac{1}{|a_1|}\left[\|e^0(0)\|\kappa e^{-\nu_2 t} + \beta_{2,3} e^{-\nu t} + \tfrac{\beta_{2,4}}{\lambda\sqrt{\gamma}}\right.$$
$$+ \beta_{2,5} e^{-\nu_2 t} + \tfrac{\beta_{2,6}}{\nu_2\lambda\sqrt{\gamma}} + \|\boldsymbol{\vartheta}_1\| \left[\beta_{f,1} e^{-\nu_2 t} + \tfrac{\delta_{f,1}}{\sqrt{\gamma}}\right] \qquad (69)$$
$$\left. + \beta_{1,1} e^{-\nu t} + \tfrac{\beta_{1,2}}{\sqrt{\gamma}}\right] + \mathcal{O}(\varepsilon) \leq \beta_{x,2} e^{-\nu_2 t} + \tfrac{\delta_{x,2}}{\sqrt{\gamma}} + O(\varepsilon) \ ,$$

where $\beta_{x,2}$ and $\delta_{x,2}$ are readily computed. Taking into account (69), we conclude from (66) that

$$|\tilde{g}_1^f(t)| \leq |b_1| \left[\beta_{x,2} e^{-\nu_2 t} + \tfrac{\delta_{x,2}}{\sqrt{\gamma}}\right] + \|\boldsymbol{\vartheta}_1\| \left[\beta_{f,1} e^{-\nu_2 t} + \tfrac{\delta_{f,1}}{\sqrt{\gamma}}\right]$$
$$+ \beta_{1,1} e^{-\nu t} + \tfrac{\beta_{1,2}}{\sqrt{\gamma}} + O(\varepsilon) \leq \beta_{g,1} e^{-\nu_2 t} + \tfrac{\delta_{g,1}}{\sqrt{\gamma}} + O(\varepsilon) \ .$$

To prepare the next step, we observe that the signals $\tilde{\sigma}_{i,1}(t) = \sigma^0_{i,1}(t) - \sigma_{i,1}(t)$ and $\tilde{\sigma}_{i,2}(t) = \sigma^0_{i,2}(t) - \sigma_{i,2}(t)$ satisfy the operator equations

$$\begin{aligned} \tilde{\sigma}_{i,1}(s) &= G_1(s)\tilde{g}^f_i(s), \quad G_1(s) = \tfrac{\omega^2}{s^2+2\zeta\omega s+\omega^2} \\ \tilde{\sigma}_{i,2}(s) &= G_2(s)\tilde{g}^f_i(s), \quad G_2(s) = \tfrac{\omega s}{s^2+2\zeta\omega s+\omega^2}\,. \end{aligned} \tag{70}$$

Since $\|G_1(s)\|_{\mathcal{H}_\infty} = 1$ for $\zeta \geq \sqrt{2}/2$ and $\|G_2(s)\|_{\mathcal{H}_\infty} = (2\zeta)^{-1}$, it follows from (70) that

$$|\tilde{\sigma}_{i,2}(t)| \leq \beta_{\sigma,1}e^{-\nu_2 t} + \frac{\delta_{\sigma,1}}{\sqrt{\gamma}} + O(\varepsilon)\ , \tag{71}$$

where $\beta_{\sigma,1} = (2\zeta)^{-1}\beta_{g,1}$ and $\delta_{\sigma,1} = (2\zeta)^{-1}\delta_{g,1}$. Using again (34) and (51) we compute

$$\tilde{g}^f_2(t) = -b_2\tilde{x}^f_3(t) - \boldsymbol{\vartheta}_2^\top\tilde{\boldsymbol{f}}^f_2(t) - \eta_2(t) + \omega\tilde{\sigma}_{1,2}(t)\,.$$

Now, we use (32) and (49) to write

$$\tilde{x}^f_3(t) = e^f_3(t) + \sigma_{2,1}(u^f) - \tilde{x}_3(t) - \hat{e}^f_3(t) - \hat{\sigma}_{2,1}(\hat{u}^f)\,, \tag{72}$$

where $\sigma_{2,1}(u^f) - \sigma_{2,1}(\hat{u}^f) = \tilde{g}^f_2(t) + O(\varepsilon)$ according to Lemmas 4.2 and 5.2. Therefore

$$\begin{aligned} \tilde{x}^f_3(t) = e^f_3(t) - \tilde{x}_3(t) - \hat{e}^f_3(t) - b_2\tilde{x}^f_3(t) - \boldsymbol{\vartheta}_2^\top\tilde{\boldsymbol{f}}^f_2(t) \\ -\eta_2(t) + \omega\tilde{\sigma}_{1,2}(t) + O(\varepsilon)\ , \end{aligned} \tag{73}$$

which after solving for $\tilde{x}^f_3(t)$ reduces to

$$\begin{aligned} \tilde{x}^f_3(t) = \tfrac{1}{a_2}\big[e^f_3(t) - \tilde{x}_3(t) - \hat{e}^f_3(t) - \boldsymbol{\vartheta}_2^\top\tilde{\boldsymbol{f}}^f_2(t) - \eta_2(t) \\ +\omega\tilde{\sigma}_{1,2}(t) + O(\varepsilon)\big]\,, \end{aligned} \tag{74}$$

where $\tilde{\boldsymbol{f}}^f_2(t)$ satisfies the bound

$$\|\tilde{\boldsymbol{f}}^f_2(t)\| \leq \beta_{f,2}e^{-\nu_2 t} + \frac{\delta_{f,2}}{\sqrt{\gamma}} + O(\varepsilon)\ , \tag{75}$$

which follows from the smoothness of $\boldsymbol{f}_2(\cdot)$ with respect to $x_1$ and $x_2$ and from the bounds (64) and (69). Substituting the corresponding bounds in (74) we obtain

$$|\tilde{x}^f_3(t)| \leq \beta_{x,3}e^{-\nu_2 t} + \frac{\delta_{x,3}}{\sqrt{\gamma}} + O(\varepsilon)\ , \tag{76}$$

which enables us to compute the following bound for $\tilde{g}^f_2(t)$

$$\begin{aligned} |\tilde{g}^f_2(t)| &\leq |b_2|\left[\beta_{x,3}e^{-\nu_2 t} + \tfrac{\delta_{x,3}}{\sqrt{\gamma}}\right] + \beta_{2,1}e^{-\nu t} + \tfrac{\beta_{2,2}}{\sqrt{\gamma}} \\ &+\|\boldsymbol{\vartheta}_2\|\left[\beta_{f,2}e^{-\nu_2 t} + \tfrac{\delta_{f,2}}{\sqrt{\gamma}}\right] + \tfrac{\omega}{2\zeta}\left[\beta_{\sigma,1}e^{-\nu_2 t} + \tfrac{\delta_{\sigma,1}}{\sqrt{\gamma}}\right] \\ &+O(\varepsilon) \leq \beta_{g,2}e^{-\nu_2 t} + \tfrac{\delta_{g,2}}{\sqrt{\gamma}} + O(\varepsilon)\ . \end{aligned} \tag{77}$$

Continuing in the same manner, we can derive the following bounds

$$\begin{aligned} \|\tilde{\boldsymbol{f}}^f_i(t)\| &\leq \beta_{h,i}e^{-\nu_2 t} + \frac{\delta_{h,i}}{\sqrt{\gamma}} + O(\varepsilon) \\ |\tilde{x}^f_i(t)| &\leq \beta_{x,i}e^{-\nu_2 t} + \frac{\delta_{x,i}}{\sqrt{\gamma}} + O(\varepsilon) \\ |\tilde{g}^f_i(t)| &\leq \beta_{g,i}e^{-\nu_2 t} + \frac{\delta_{g,i}}{\sqrt{\gamma}} + O(\varepsilon) \end{aligned} \tag{78}$$

for all $i = 1, \ldots, n$.

Now, we are ready to derive a bound for the control signal. To this end, we recall that $u^f(t) - u^0(t) = \mathcal{O}(\varepsilon)$ according to Lemma 6.1. Therefore

$$u^0(t) - \hat{u}^f(t) = \mathcal{O}(\varepsilon) + u^f(t) - \hat{u}^f(t)\,. \tag{79}$$

Substituting the expressions of $u^f(t)$ from (36) and $\hat{u}^f(t)$ from (53) into (79) we obtain

$$\begin{aligned} u^0(t) - \hat{u}^f(t) &= \mathcal{O}(\varepsilon) - b_n g_n(u^f) - \boldsymbol{\vartheta}_n^\top\boldsymbol{f}_n(u^f) \\ &\quad+\hat{b}_n(t)g_n(\hat{u}^f) + \hat{\boldsymbol{\vartheta}}_n^\top(t)\boldsymbol{f}_n(\hat{u}^f) \\ &= -\eta_n(t) - b_n\tilde{g}_n(t) - \boldsymbol{\vartheta}_n^\top\tilde{\boldsymbol{f}}^f_n(t) + \mathcal{O}(\varepsilon)\,, \end{aligned} \tag{80}$$

which upon substitution of the corresponding bounds results in

$$\begin{aligned} |u^0(t) - \hat{u}^f(t)| &\leq |b_n|\left[\beta_{g,n}e^{-\nu_2 t} + \tfrac{\delta_{g,n}}{\sqrt{\gamma}}\right] + \beta_{2,1}e^{-\nu t} \\ &+\tfrac{\beta_{2,2}}{\sqrt{\gamma}} + \|\boldsymbol{\vartheta}_n\|\left[\beta_{f,n}e^{-\nu_2 t} + \tfrac{\delta_{f,n}}{\sqrt{\gamma}}\right] + \mathcal{O}(\varepsilon) \\ &= \beta_5 e^{-\nu_2 t} + \tfrac{\beta_6}{\sqrt{\gamma}} + \mathcal{O}(\varepsilon)\ , \end{aligned}$$

which completes the proof. ■

*Remark 6.1:* It follows form Theorem 6.1 that the bounds on the input and output tracking errors can be systematically decreased by choosing large values for $\omega$ and $\gamma$. It should be notice that $\omega$ appears in the bounds of the control signal not only in $\mathcal{O}(\varepsilon)$ term as $\varepsilon = \frac{1}{\omega}$, but also in the coefficient $\beta_6$ in the form of a factor $\omega^n$. Therefore in order to make the term $\frac{\beta_6}{\sqrt{\gamma}}$ smaller the adaptation must be much faster than the command filter's response.

## VII. Numerical Simulations

As a simulation example we consider a third order system with $\boldsymbol{f}_1(x_1) = 2.2x_1^2$, $\boldsymbol{f}_2(x_1, x_2) = 1.3x_1x_2$ and $\boldsymbol{f}_3(\boldsymbol{x}) = 0$. For the numerical simulations the uncertain parameters are set to $\boldsymbol{\vartheta}_1 = 2.2$, $\boldsymbol{\vartheta}_2 = 1.3$, $a_1 = 1.2$, $a_2 = 2.4$ and $a_3 = 2.5$. The reference model is implemented with $\boldsymbol{k}_x = [8\ 10.4\ 5.2]^\top$ and $k_r = 16$. The frequency and damping ratio of the command filters are respectively set to $\omega = 500$ and $\zeta = 0.8$. The adaptation rate is set to $\gamma = 50000$ and the error feedback gain is $\lambda = 2\sqrt{\gamma}$.

Figure 1 displays the controller performance for the external step command of magnitude 2. Clearly asymptotic tracking is achieved for both output and input signals with a small transient error. System's sinusoidal response is presented in Figure 2. Good tracking performance can be observed in this case as well.

## VIII. Concluding remarks

We have presented a certainty equivalence adaptive backstepping control method for nonlinear systems with unmatched uncertainties and unknown virtual control coefficients without over-parametrization. The approach uses a fast identification model, which is independent of the control design, and the command filtered backstepping method. This separation of the estimation and control design is an important feature that enables the designer to achieve desired transient and steady state properties by the proper choice of the control parameters.

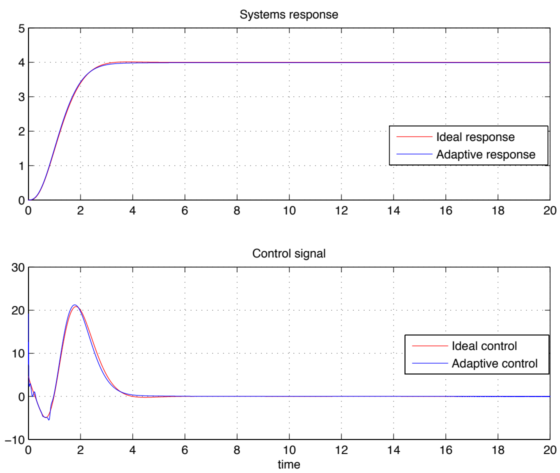


Fig. 1. Step response and the corresponding control signal history.

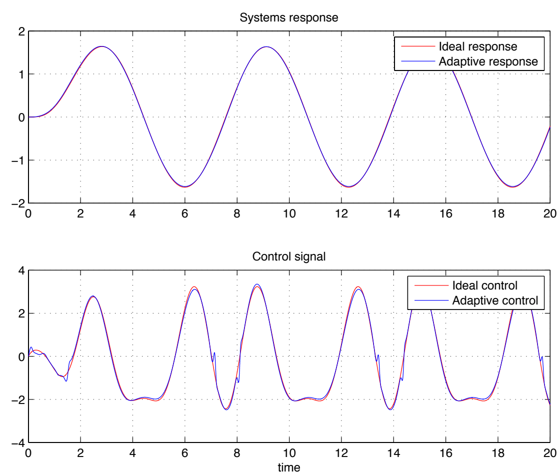


Fig. 2. Sinusoidal response and the corresponding control signal history.

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