The Dynamic Response of a Large Airplane to Continuous Random Atmospheric Disturbances

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SUMMARY

The statistical approach to the gust-loads problem, which consists in considering flight through turbulent air to be a stationary random process, is extended by including the effect of lateral variations of the instantaneous gust intensity on the aerodynamic forces and on the resultant motions and stresses of rigid and flexible airplanes. By means of some calculations of normal and rolling accelerations, as well as of the root bending moment, it is shown that these effects may be significant for large airplanes.

SYMBOLS

A = aspect ratio
b = span
BM = root bending moment
$\bar{c}$ = average chord, S/b
$C(k)$ = Theodorsen's unsteady lift function
$\tilde{C}(k)$ = generalized Theodorsen function
C_{L_α} = wing lift-curve slope
C_{l_p} = coefficient of damping in roll
cc_l = section loading coefficient
EI = bending stiffness
g = acceleration due to gravity
$h(t)$ = response to unit impulsive gust or other input
$h(t, y)$ = response influence function
$H(\omega)$ = complex amplitude of response to sinusoidal gust or other input of unit amplitude, Fourier transform of $h(t)$
$H(\omega, y)$ = complex amplitude of response influence function, Fourier transform of $h(t, y)$ with respect to t
$\tilde{H}(\omega, \eta)$ = autoconvolution function for $H(\omega, y)$
$\hat{H}(\omega, \omega')$ = Fourier transform of $H(\omega, y)$ with respect to y
I_x = mass moment of inertia about the roll axis
J_1 = Bessel function of the first kind, order 1
k = reduced frequency, $\omega\bar{c}/2U$
k' = dimensionless frequency, $\omega L^*/U$
K_0, K_1 = modified Bessel functions of the second kind
l = lift per unit span
L = lift
L' = rolling moment
L'' = generalized lift associated with first free-free bending mode
L^* = scale of turbulence
m = mass per unit span
M = airplane mass
M_w = wing mass
M' = generalized mass associated with the first free-free bending mode
N = number of peaks per unit time above a given level
Δn = load-factor increment
p = rate of roll
q = dynamic pressure
S = wing area
t = time
U = flying speed
w = vertical component of gust velocity
x = longitudinal displacement
y = lateral displacement
y^* = dimensionless lateral displacement, $y/(b/2)$
$\bar{y}$ = lateral ordinate of center of pressure
$\bar{\bar{y}}$ = lateral ordinate of center of wing mass
z = normal displacement
z^0 = normal displacement of nodal point
z_1 = generalized normal displacement for the first free-free bending mode
z^* = dimensionless normal displacement, $z/(\bar{c}/2)$
α = local angle of attack
$\gamma(y)$ = spanwise distribution of the influence function $h(t, y)$
$\gamma_L(y)$ = lift influence function, $[cc_l/\bar{c}C_{L_\alpha}]_{\alpha=1}$
$r_L'(y)$ = rolling moment influence function, $[cc_l/\bar{c}(-C_{l_p})]_{\alpha=y^*}$
$\Gamma(\eta)$ = autoconvolution of $\gamma(y)$
η = lateral space displacement, Δy
η^* = dimensionless lateral space displacement, $\eta/(b/2)$
ζ = mode shape of first free-free bending mode
κ = airplane mass parameter, $8M/C_{L\alpha}\rho S\bar{c}$
κ_r = airplane rolling-inertia parameter, $8I_x/(-C_{l_p})\rho S^2 b$
ρ = air density
σ = stress
τ = time displacement, Δt
ψ = one-dimensional correlation function
$\tilde{\psi}$ = two-dimensional correlation function
φ = one-dimensional power spectrum
$\tilde{\varphi}$ = two-dimensional power spectrum
$\tilde{\tilde{\varphi}}$ = double Fourier transform of $\tilde{\psi}$
$\hat{\varphi}$ = double Fourier transform of $\tilde{\psi}$ for axisymmetric case
ϕ = Sears' unsteady lift function for gust penetration
ω = frequency

INTRODUCTION

The local air velocity fluctuations sensed by an airplane flying through atmospheric turbulence are functions of time defined in only a statistical sense—that is, they constitute a stochastic or random process. Consequently, the responses of the airplane, such as the motions or stresses, can also be known as functions of time in only a statistical sense.

The problem of relating the statistical characteristics of the input of a dynamic system to those of the

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output gives rise to the relatively new field of statistical dynamics. (Some of the fundamental papers in this field are compiled in reference 1.) The techniques of this field, in particular those related to the generalized harmonic analysis of stationary random processes, have been applied to the gust-loads problem and related problems in aeronautics in recent years to an ever increasing extent.[2–6]

In these analyses the assumption has been made that the variation of the instantaneous gust intensity along the span can be disregarded. The type of gust structure envisioned in this approach is thus, in effect, the one shown in the *upper* part of Fig. 1, which will be referred to hereafter as the one-dimensional gust. The purpose of this paper is to take into account the variation along the span and to treat the more realistic gust structure represented in the *lower* part of Fig. 1, which will be referred to as the two-dimensional gust.

Thus, the problem is essentially one of determining the response of a linear system—namely, the airplane—to a multidimensional stationary random process—namely, atmospheric turbulence. (The assumptions of linearity and stationarity have been made in references 2 to 6 and are valid for many practical purposes connected with the gust-loads problem.) The somewhat similar problem of a finite-length correction for a hot-wire anemometer has been treated in references 7 and 8; some phases of the present problem have been considered in reference 9; and a fairly extensive analysis of the problem is given in reference 10, which forms the basis of the present paper.

The solution to the problem will be outlined briefly in the following, and the results of some calculations of the accelerations and bending moments in continuous random turbulence will then be presented in an effort to indicate the possible magnitude of the effects under consideration. The derivation of the solution presented here and some of the details of the calculations are given in the Appendixes.

The Basic Relations Between the Input and Output of a System Subjected to a Random Input

Summary of the Relations Appropriate to a One-Dimensional Gust Structure

Before discussing the analysis of the loads and motions corresponding to a two-dimensional random gust structure, a brief outline of the analysis for a one-dimensional random gust structure may be in order.

The bridges which link the statistical characteristics of the input and the output of a dynamic system subjected to a stationary random input are certain relations between the correlation functions or the power spectra of the input and output, respectively. For the present case the input is the gust intensity w, and its correlation function is defined by

$$\psi_w(\Delta x) = \overline{w(x)w(x + \Delta x)} \quad (1)$$

where the bar designates a time average. The correlation function is a statistical characteristic which is a

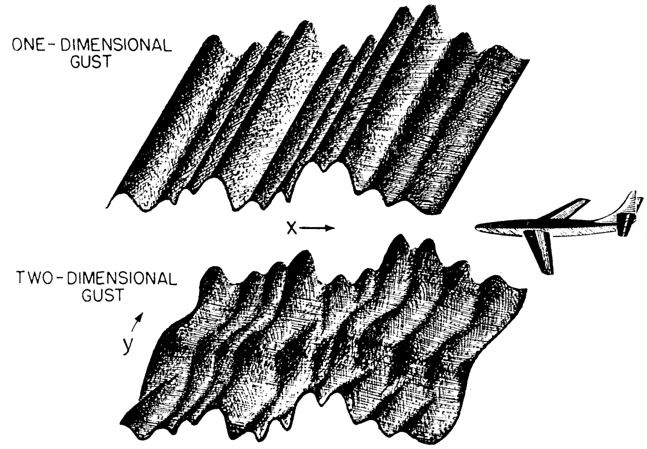


FIG. 1. One- and two-dimensional gusts.

measure of the extent to which the value of one random variable, in this case $w(x + \Delta x)$, can be predicted from a knowledge of that of another random variable, in this case $w(x)$; it contains the mean square value of the quantity of interest, inasmuch as

$$\overline{w^2} = \psi_w(0) \quad (2)$$

and contains certain additional information. For instance, the mean square value of the derivative of w is given by

$$\overline{(dw/dx)^2} = -\left\{[d^2/d(\Delta x)^2]\,\psi_w(\Delta x)\right\}_{\Delta x = 0} \quad (3)$$

A possibly even more useful statistical characteristic is the Fourier transform of the correlation function—namely,

$$\varphi_w(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} e^{-i\omega(\Delta x/U)}\,\psi_w(\Delta x)\,d\left(\frac{\Delta x}{U}\right) \quad (4)$$

(The flying speed U is introduced here because the airplane senses the turbulent excitation fundamentally as a function of time, whereas the statistical characteristics of turbulence of interest here are functions primarily of space displacements. These space displacements have to be identified with equivalent time displacements in calculating the response of the airplane.) The function $\varphi_w(\omega)$ is referred to as the power spectrum of w. It represents the part of the mean square value associated with various frequencies. In other words, if the random process w were passed through a filter which permitted only frequencies within a band of unit width about the frequency ω to pass, the mean square value of the filtered process would be $\varphi_w(\omega)$. Thus, the information it contains includes the mean square value of the process,

$$\overline{w^2} = \int_0^{\infty} \varphi_w(\omega)\,d\omega \quad (5)$$

as well as the mean square value of the time derivative of the process,

$$\overline{\dot{w}^2} = \int_0^{\infty} \varphi_w(\omega)\,\omega^2\,d\omega \quad (6)$$

The correlation function and power spectrum of the output—say the load factor increment Δn—are defined in a similar manner. The input-output relation for stationary random processes then becomes in this case

$$\varphi_{\Delta n}(\omega) = \left| H_{\Delta n}{}^{w}(\omega) \right|^2 \varphi_w(\omega) \tag{7}$$

where $H_{\Delta n}{}^{w}(\omega)$ is the transfer function of the system, which represents the complex amplitude of the load factor response of the airplane to sinusoidal gusts of unit amplitude and of wave length $2\pi U/\omega$.

From the power spectrum of the output such statistical quantities as the mean square value and the mean square derivative can then be obtained directly by the equivalent of Eqs. (5) and (6). From these quantities, in turn, other statistical characteristics of interest can be obtained on the basis of certain assumptions. For instance, if the probability distribution of the process is Gaussian, the expected number N of peaks in the load factor exceeding a given level Δn per unit time can be obtained from Rice's asymptotic expression given in reference 1—namely,

$$N = (1/2\pi) \sqrt{\overline{\Delta \dot{n}^2}/\overline{\Delta n^2}}\, e^{-(1/2)[(\Delta n)^2/\overline{\Delta n^2}]} \tag{8}$$

valid for $\Delta n > 2\sqrt{\overline{\Delta n^2}}$. This expression, which also furnishes an estimate of the expected time $1/N$ required to encounter a peak load factor increment equal to or greater than Δn, is shown plotted in Fig. 2. The ratio $\sqrt{\overline{\Delta \dot{n}^2}}/\sqrt{\overline{\Delta n^2}}$ has the dimensions of a frequency and can be considered to be a representative or predominant frequency of the time history of Δn. If the transfer function has the characteristics of a narrow band-pass filter, which may be the case if the airplane is on the verge of an instability condition (due to loss of aerodynamic damping or aeroelastic action), this representative frequency is the frequency ω_0 at which the airplane or wing tends to oscillate.

The Input-Output Relation for a Two-Dimensional Gust Structure

For the two-dimensional gust, the progression from input to output is fundamentally the same. However, some of the concepts referred to in the preceding paragraphs have to be generalized. A two-dimensional correlation function for the input can be defined as

$$\tilde{\psi}_w(\Delta x, \Delta y) = \overline{w(x, y)\, w(x + \Delta x, y + \Delta y)} \tag{9}$$

This correlation function relates the gust velocities at points x, y and $x + \Delta x$, $y + \Delta y$. In most cases its value should be independent of the orientation of the two points relative to each other and depend only on the distance between them, so that an airplane flying from east to west senses the same gust variation, in a statistical sense, as an airplane flying north to south, or in any other direction. In this case, the two-dimensional correlation function can readily be expressed in terms of the one-dimensional function,

$$\tilde{\psi}_w(\Delta x, \Delta y) = \psi_w(\sqrt{\Delta x^2 + \Delta y^2}) \tag{10}$$

This property of the turbulence is referred to as axisymmetry (with respect to a vertical axis) and represents a less restrictive assumption than that of isotropy, which is usually assumed in studies of turbulence. In the following, the turbulence under consideration will be assumed to have this property.

A two-dimensional power spectrum may then be defined as the Fourier transform with respect to Δx of the two-dimensional correlation function,

$$\tilde{\varphi}_w(\omega, \Delta y) = \frac{1}{\pi} \int_{-\infty}^{\infty} e^{-i\omega(\Delta x/U)}\, \tilde{\psi}_w(\Delta x, \Delta y)\, d\left(\frac{\Delta x}{U}\right) \tag{11}$$

In deriving an input-output expression which relates the power spectrum of some output, say, the stress σ at some point in the wing, to this two-dimensional input spectrum, a convenient starting point is the superposition integral which relates the instantaneous value of the output to the past history of the input at various stations on the wing,

$$\sigma(t) = \int_{-\infty}^{\infty} \int_{-(b/2)}^{b/2} h(t_1, y)\, w(t - t_1, y)\, dy\, dt_1 \tag{12}$$

The required response function $h(t, y)$ is actually an influence function, which defines the influence of the gust intensity at a station y and a time t on the output. This function represents the response, say the stress σ, to a very narrow impulsive gust which at time $t = 0$ impinges on the wing at station y. In principle, such a response function could be calculated if the indicial pressure or lift distribution on the wing were known for impulsive gusts impinging on the wing over a very narrow front. For a given location y of gust impingement and a given time t, such a lift-distribution function would tend to look similar to the one indicated by cross hatching in the upper part of Fig. 3. The desired influence function $h(t, y)$ could then be obtained by calculating the integrals and moments and, hence, the stresses associated with this lift distribution for each value of t and y. For a stress which is propor-

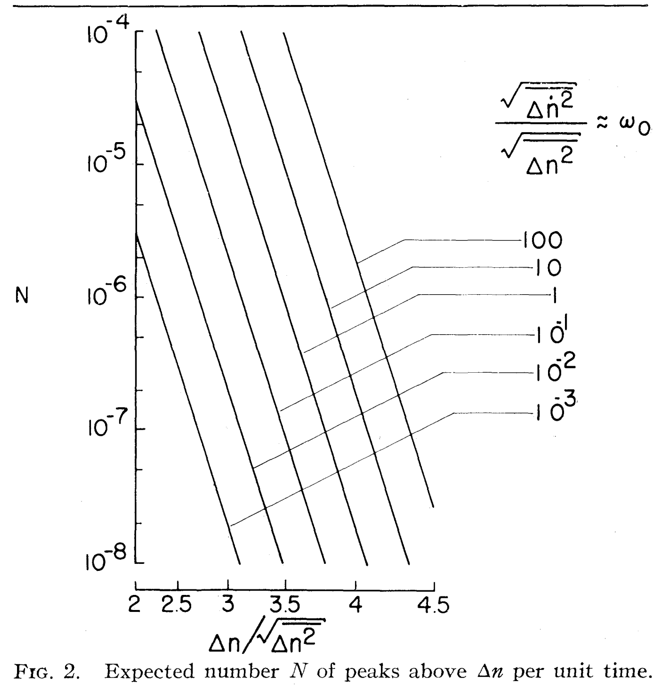


FIG. 2. Expected number N of peaks above Δn per unit time.

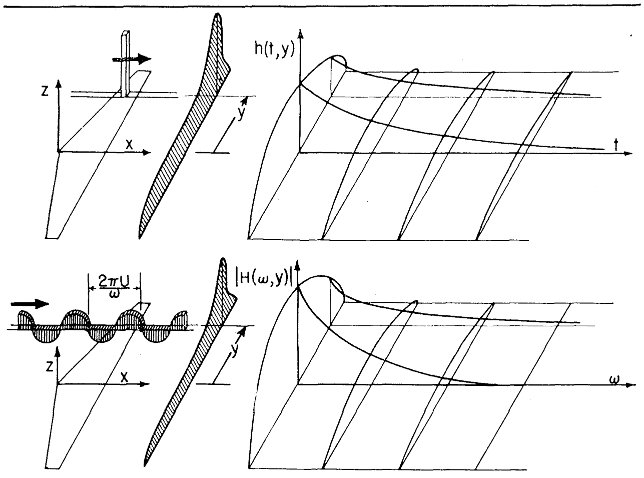


FIG. 3. Influence functions for two-dimensional gust.

tional to the total lift or vertical shear, such a function is shown at the right in the upper part of Fig. 3.

For practical purposes, this direct approach suffers from the fact that the required indicial pressure- and lift-distribution functions are very difficult to calculate. However, the integrals and moments of these functions and, hence, the desired function $h(t, y)$ can be identified with relatively easily calculated lift distributions on the same wing in reverse flow by means of the reciprocity theorem of linearized lifting-surface theory.[11]

The Fourier transform $H(\omega, y)$ of $h(t, y)$ represents the complex amplitude of the response to sinusoidal gusts of unit amplitude and with wave length $2\pi U/\omega$ impinging on the wing at station y, as indicated in the lower part of Fig. 3. The direct calculation of this function again requires a knowledge of pressure or lift distributions due to very narrow sinusoidal gusts, which are very difficult to calculate, and, for a given y and ω, would look like the one indicated by cross-hatching. But again, the required integrals and moments can be identified with certain lift distributions on the wing in reverse flow. The specific nature of these lift distributions for the various cases considered here is indicated in Appendix B.

In terms of the two-dimensional input spectrum $\tilde{\varphi}_w(\omega, \Delta y)$ and the influence function $H_\sigma^w(\omega, y)$, the output spectrum can then be expressed as follows:

$$\varphi_\sigma(\omega) = \int_{-(b/2)}^{b/2} \int_{-(b/2)}^{b/2} H_\sigma^w(\omega, y_2)\, [H_\sigma^w(\omega, y_1)]^* \times \tilde{\varphi}_w(\omega, y_2 - y_1)\, dy_1 dy_2 \quad (13)$$

where the asterisk designates that the complex conjugate of $H_\sigma^w(\omega, y_1)$ is to be taken. This expression then represents the input-output relation for the two-dimensional gust, analogous to Eq. (7) for the one-dimensional gust.

Once the power spectrum of σ has been obtained in this manner, the mean square value and the value of the mean square derivative are obtained by integration [see Eqs. (5) and (6)], and other statistical characteristics, such as the expected number of peaks, can then be calculated from these values, as before.

RESULTS OF CALCULATIONS

Spectrum of the Lift Due Directly to Turbulence

One of the basic quantities in any calculation of the dynamic response of an airplane to turbulence is the lift produced directly by the turbulence. In order to indicate how this lift is affected by the averaging effect of the span, the spectrum $\varphi_{w_e}(\omega)$—in dimensionless form—of the gust velocity averaged over the span (with a weighting factor which depends on the lift-producing capabilities of the various stations along the span, and which is here taken as unity) is shown in Fig. 4. The abscissa is a reduced frequency and the parameter a dimensionless span, both of which contain a length L^* which will be referred to as the scale of turbulence. The scale of turbulence is here defined as twice the integral of the function $\psi_w(\Delta x)$ divided by $\overline{w^2}$—that is,

$$L^* = 2 \int_0^\infty \frac{\psi_w(\Delta x)}{\overline{w^2}}\, d(\Delta x) \quad (14)$$

or

$$L^* = \pi U[\varphi_w(0)/\overline{w^2}] \quad (15)$$

Intuitively, it may be thought of as a distance such that the instantaneous gust velocities at points separated by less than that amount tend to be similar, whereas those at points separated by a greater amount tend to be substantially independent of each other in a statistical sense.

The correlation function $\psi_w(\Delta x)$ to which these calculations pertain is indicated at the right of Fig. 4; this correlation function is a simple analytical expression suggested by measurements of turbulence in wind tunnels; it agrees fairly well with the available knowledge of the correlation function of atmospheric turbulence[2, 5] and has been used previously for gust-loads calculations in reference 4. On the basis of the available knowledge the scale of atmospheric turbulence appears to be in the order of at least several hundred feet, although near the ground it may be somewhat smaller.

When the span is very small, there is no averaging effect, and $\varphi_{w_e}(\omega)$, the spectrum of the averaged gust intensity, becomes $\varphi_w(\omega)$, the spectrum of the unaveraged gust intensity. This function is the one

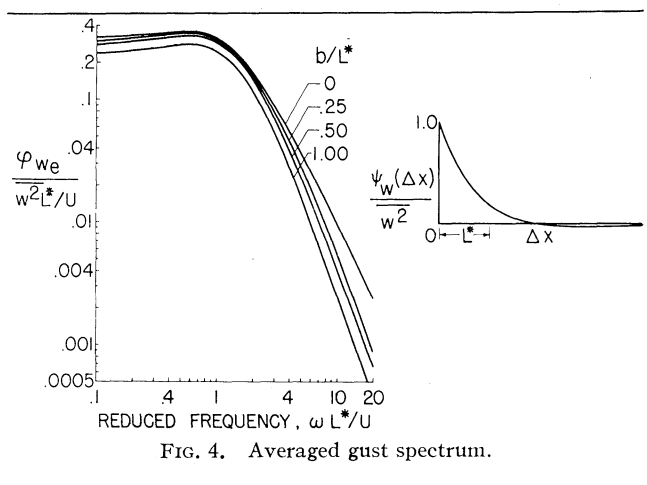


FIG. 4. Averaged gust spectrum.

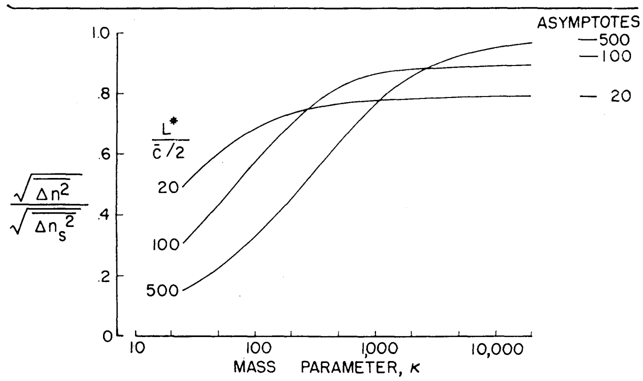


FIG. 5. Effect of airplane mass on normal acceleration.

shown for $b/L^* = 0$. As the span increases, the averaging effect becomes more and more pronounced and serves to attenuate the spectrum at all frequencies, but particularly at high frequencies, as may be expected since the tendency for gusts of very short wave lengths (or high frequencies) to cancel each other is much greater than that of gusts of larger wave lengths.

The spectrum of the lift is obtained essentially by multiplying the spectrum for the averaged gust velocity by the absolute square of the unsteady lift function for gust penetration (the so-called Sears function in the case of two-dimensional incompressible flow), and the mean square of the lift is then obtained by integrating the lift spectrum. Inasmuch as this unsteady lift function is unaffected by the averaging process, the effects of averaging on $\varphi_w(\omega)$ are reflected in similar effects on the lift spectrum and, hence, on the mean square lift.

Mean Square Normal Load Factor Increment

In calculating the loads produced by the lift discussed in the preceding section, the motions of the airplane must be taken into account. Before considering the averaging effects of the span on these loads, however, a brief discussion of some results obtained for the one-dimensional gust case may be in order.

If the airplane is considered to be subjected to continuous random turbulence with the spectrum $\varphi_w(\omega)$ given by the curve for $b/L^* = 0$ in Fig. 4, and if the airplane is permitted to move vertically only, the mean square load factor can be calculated readily.[4] Some typical results are presented in Fig. 5. The ordinate is the ratio of the root mean square load factor calculated in this manner to the root mean square load factor given by the sharp-edged gust formula and, thus, represents an alleviation factor for the value obtained from this formula; the abscissa is an airplane mass parameter

$$\kappa \equiv 8M/C_{L_\alpha}\rho S\bar{c}$$

The results indicate, as may be expected, that the sharp-edged gust formula which, in effect, disregards the motions of the airplane and unsteady-lift effects, is most accurate for large values of the mass parameter, which imply small motions on the part of the airplane. Also, the alleviation factor and, hence, the mean square load factor are seen to depend on the ratio of the scale of turbulence to the mean chord.

Now if a two-dimensional gust input is to be used, the result will depend on the ratio of the span to the scale of turbulence, as was the case in Fig. 4, as well as on the ratio of the mean chord to the scale of turbulence, as in the case in Fig. 5. However, for a given airplane these two ratios are not independent, since the ratio of the span to the average chord is fixed. Therefore, it was felt that, for a realistic appraisal of the averaging effect of the span, the two ratios should be varied simultaneously in such a way as to maintain a fixed airplane geometry.

The results of such an analysis are indicated in Fig. 6, where the ordinate is the ratio of the mean square load factor obtained from an analysis using a two-dimensional gust structure to the value obtained for a one-dimensional gust for a given value of κ—namely, 100. The abscissa is the span ratio b/L^*. Curves are shown for three aspect ratios. The value of the ratio of the scale of turbulence to one-half of the average chord can be determined for each point on the curves from the relation

$$L^*/(\bar{c}/2) = 2A/(b/L^*)$$

The results presented in Fig. 6 indicate that, if the variation of gust intensity along the span is taken into account, the mean square normal load factor is reduced by an amount which depends on the aspect ratio of the wing and the ratio of the span to the scale of turbulence. For wings of high aspect ratio and spans of the order of one-quarter or more of the scale of turbulence, this reduction appears to be quite substantial. However, as will be indicated presently, the results presented in this Figure do not imply that the stresses are necessarily lower.

Mean Square Rolling Acceleration

Fig. 7 pertains to a problem which can be analyzed only by taking into account the spanwise variation of gust intensity—namely, the problem of rolling response to vertical gusts—because if the gust intensity is the same along the span there is no tendency to roll. The mean square rolling acceleration has been calculated

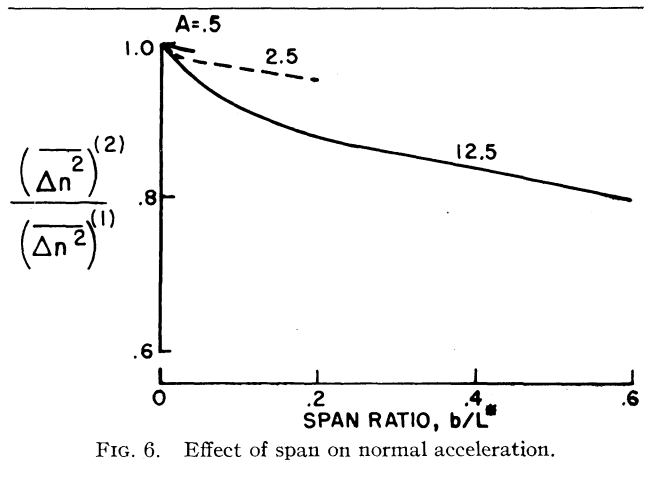


FIG. 6. Effect of span on normal acceleration.

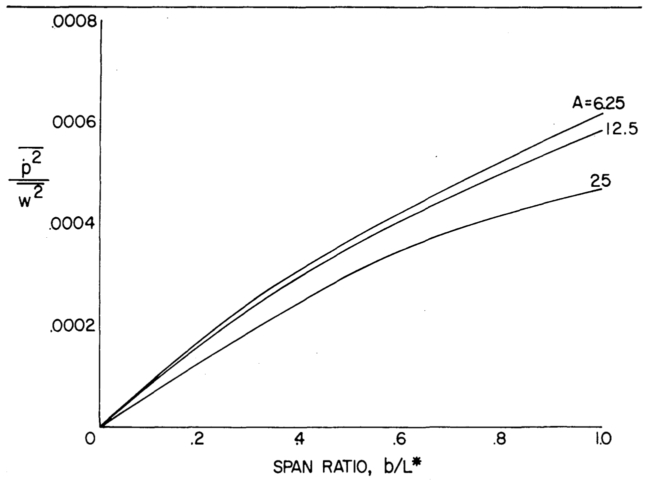


FIG. 7. Effect of span on rolling acceleration.

for one value of a rolling moment of inertia parameter analogous to the mass parameter—i.e.,

$$\kappa_r \equiv 8I_x/(-C_{l_p})\rho S^2 b$$

which has a value of 25.75 for the calculations represented in Fig. 7 and for several values of the chord and span ratios.

As may be expected, as the span goes to zero, so does the effect of variations in gust intensity along the span and, hence, the tendency to roll. (If the span were much larger than the scale of turbulence, and, therefore, beyond the range of practical interest in connection with the gust-loads problem, the local values of the gust intensity along the span would tend to cancel each other and again produce no tendency to roll, although the curves are not shown here for sufficiently large values of the span ratio to demonstrate this point.)

In considering these results the fact that the airplane also rolls due to side gusts acting on the vertical tail should be kept in mind. Some estimates indicate that for large airplanes the effect considered here is likely to predominate. In any event, the two contributions to the rolling motion are statistically independent if the turbulence is isotropic, so that the spectra can be added directly to obtain the spectrum of the total rolling acceleration.

Mean Square Root Bending Moment of a Rigid Wing

In order to obtain a more direct measure of the effect of spanwise variation in gust intensity on the stresses in an airplane the mean square root bending moment has been calculated for a rigid and a flexible wing. Fig. 8 pertains to an airplane with a rigid wing, which is permitted to move vertically only. The mass parameter is 100. The ratio of the mean square bending moment to a mean square bending moment calculated from a sharp-edge gust formula, which includes no inertia or unsteady-lift effects, is shown as a function of the ratio of the wing mass to the total mass of the airplane. The curve labeled $b/L^* = 0.25$ represents the results calculated by taking spanwise variations of the gust intensity into account, whereas the curve labeled $b/L^* = 0$ represents the results of calculations in which these variations are ignored. The aspect ratio of the wing considered here is 12.5, and the scale ratio $L^*/(\bar{c}/2)$ is 100 in both cases.

The effect of taking spanwise variations into account is to decrease the mean square bending moment slightly if most of the mass of the airplane is contained in the fuselage, but to increase it if much or most of the weight is in the wing. For the airplane with most of the weight in the fuselage the decreased moment reflects to a large extent the decreased normal load factor. On the other hand, for the extreme case of a flying wing, which contains its entire mass in the wing, the net root bending moment is zero if the gust is uniform along the span, provided the mass distribution and lift distribution have the same lateral centroid location, because the bending moments due to the turbulence directly, due to the motion of the airplane, and due to the inertia effects then cancel each other. If, for the same case, the variation of the gust intensity along the span is considered, however, a net bending moment does exist. Thus, the mean square bending moment shown in the Figure for a mass ratio of unity is due entirely to the spanwise variation of gust intensity, and, at mass ratios between one-half and one, this effect results in large increases in the mean square bending moment.

Mean Square Root Bending Moment of a Flexible Wing

The mean square root bending moment of an airplane with a flexible wing and a higher value of the mass parameter κ—namely, 175—is shown in Fig. 9. The airplane is now considered to be free to move vertically, and the wing is assumed to distort in the first symmetrical free-free bending mode. The fact that the fundamental bending frequency and the mode shape change as the mass is redistributed from the fuselage to the wing is taken into account.

For this case a one-dimensional gust produces a bending moment even when all the mass is in the wing. The effect of taking the spanwise variation of gust intensity into account is again to reduce the mean square bending moment when most of the mass is in the fuselage and to increase it if most of the mass is in

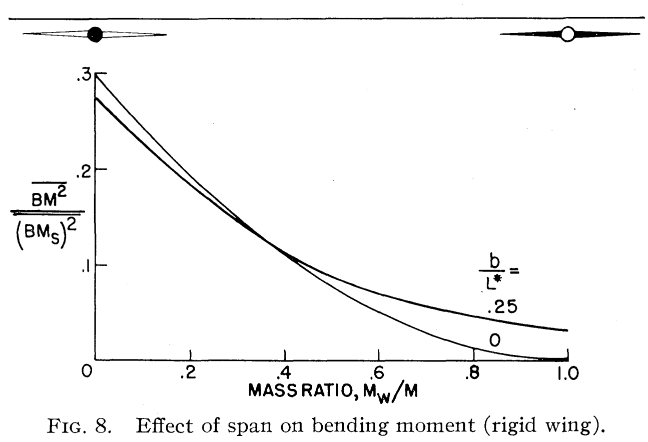


FIG. 8. Effect of span on bending moment (rigid wing).

the wing; however, the magnitude of the effect appears to be considerably greater than it was for the rigid wing. (In view of the different mass parameters the results presented in Figs. 8 and 9 should not be compared with each other in an attempt to deduce the effects of wing flexibility.)

Discussion

Some Limitations of Analyses of Airplane Response to Continuous Random Excitation

The purpose of the analysis presented here has been to determine how the results of an analysis based on a one-dimensional random gust are modified if a two-dimensional random gust is considered. Before discussing the results presented, a short discussion of some of the limitations and implications of this type of analysis may be in order.

The assumptions of linearity and stationarity commonly made in analyses based on a one-dimensional gust structure have been carried over. Linearity primarily implies small motions superimposed on a steady mean motion. Stationarity, in a statistical sense, implies that the statistical characteristics of the turbulence along the flight path remain substantially invariant for a sufficiently long period of time—that is, a period of time several times longer than either the time required for any transient aerodynamic and dynamic effects to subside or the time required to travel a distance equal to the scale of turbulence, whichever is greater. This condition is likely to be satisfied in turbulence of low or medium intensity; however, whether or not the turbulence encountered in thunderstorms satisfies this condition is not known at present.

The present study of the response to two-dimensional turbulence, as well as the previous studies of the response to one-dimensional turbulence, are thus concerned with only a part of the gust-loads picture. They furnish an estimate of the expected number of peak loads per unit time above a given level for any *specified* mean square intensity of the turbulent input, but this information must be combined with the *probability of encountering a given input intensity* in a given period of time, which represents a meteorological and operational problem, in order to arrive at an estimate of the time required to exceed a given peak load in a *given type of operation*. This problem is discussed in reference 12.

The additional assumption of axisymmetry made in this paper is valid whenever the turbulence has no preferred direction in the horizontal plane. This condition is likely to be satisfied in general, except near the ground (where the turbulence may be the result of obstacles on the ground which have a definite orientation, such as a mountain range) and possibly at the edges of the jet stream. If the turbulence does not have this property, the analysis of this paper can readily be extended to cover this situation; however, more knowledge, in the form of two-dimensional correlation

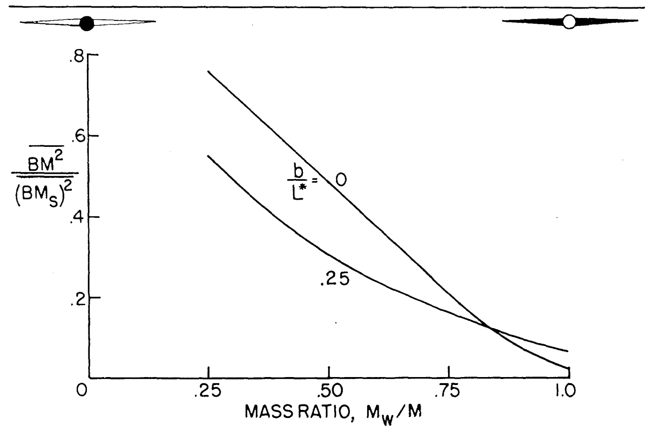


FIG. 9. Effect of span on bending moment (flexible wing).

functions or the corresponding power spectra, would then be required than is now available.

A characteristic of atmospheric turbulence which appears to play a prominent part in the calculation discussed here is the scale of turbulence, although if an experimentally obtained input spectrum had been used instead of the analytical expressions used here this quantity would not have appeared explicitly. It is very difficult to obtain experimentally, because its measurement requires long periods of time, during which the assumption of stationarity is likely to be violated. Therefore, if this quantity is to be obtained from experimental results, the best procedure may be to fit an experimentally obtained correlation function or spectrum by an analytic expression and then to deduce the scale of turbulence from this expression, using Eq. (14) or (15).

Appraisal of the Calculated Results

As pointed out in this paper, the lateral variation of gust intensity may affect the response to turbulence to various extents. In some problems this effect constitutes a refinement of an analysis based on a one-dimensional gust structure, as in the case for the normal load factor, for instance; in others this effect is the primary cause of the response, as in the case for the rolling accleration or the root bending moment on a rigid flying wing.

On the basis of the results of the calculations, the effects considered here may be significant if the span of the airplane is one-tenth or more of the scale of turbulence, that is, for airplanes with a span as low as 50 ft. in some cases. However, in other problems these effects may be significant only for the very largest airplanes. As was demonstrated, they may serve to increase or decrease the loads and motions, although some of the problems where the increase is very large, such as the root bending moment on a flying wing, may be those for which gust loads are not likely to be critical. The effects considered here appear to be more significant for relatively flexible than for relatively rigid airplanes, and on the basis of calculations performed elsewhere[13] they appear to affect the response in the higher modes to an even larger extent than the response in the lower modes considered in the calculations of the present paper.

Although only calculations of mean square values have been presented, the values of the mean square derivatives could have been presented as well, inasmuch as they can be obtained directly from the spectra which were used to obtain the mean square values. Hence, the expected number of peaks could have been calculated. It is not expected that such calculations would alter the conclusions reached here.

Application to Other Problems

The effects discussed here may find application to other problems. For instance, some phases of the buffeting problem may be amenable to analysis on the basis of assumptions of linearity and stationarity. If so, the effects of spanwise variation of input intensity are likely to be very much larger than in the gust-loads problem, because the scale of turbulence is small compared to the wing span in this problem. Whether the assumption of axisymmetry will yield useful results or whether two-dimensional input correlation functions and spectra will have to be obtained remains to be seen.

Concluding Remarks

An approach has been outlined for taking into account the spanwise variations in instantaneous intensity of a continuous random gust structure. The results of some calculations have been presented which indicate that this effect, which may serve to decrease or increase the loads and motions, depending on the individual case, may be significant even on medium-sized airplanes for certain gust responses. One such response is the root bending moment, particularly if the wing is flexible, and another is the rolling acceleration. However, for other responses, such as the normal acceleration, this effect is likely to be small even for the larger airplanes. Thus, in quantitative studies of some, but not all, responses to continuous random turbulence, the effects of spanwise variation of gust intensity will have to be taken into account for large airplanes.

Appendix A—Derivation of the Input-Output Relation and Various Transfer Functions for a One-Dimensional Gust Structure

In this Appendix the relation between the power spectra of the input and output of a linear dynamic system subjected to a stationary random input will be derived in a form which can readily be generalized to the analyis of a two-dimensional gust structure. (See references 3, 4, and 6 for greater detail.) The transfer functions required in the calculations discussed in this paper will then be derived for the case of a one-dimensional gust structure.

The Input-Output Relation

The correlation function of a stationary random process $f(t)$ will be defined as

$$\psi_f(\tau) = \overline{f(t)f(t+\tau)} \tag{A-1}$$

where the bar designates a time average,

$$\overline{(\;)} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} (\;)\, dt$$

For many purposes the Fourier transform of this function is very useful. This transform will be referred to as the power spectrum of $f(t)$ and defined by

$$\begin{aligned}\varphi_f(\omega) &= \frac{1}{\pi}\int_{-\infty}^{\infty} \psi_f(\tau)\, e^{-i\omega\tau}\, d\tau \\ &= \frac{2}{\pi}\int_{0}^{\infty} \psi_f(\tau) \cos \omega\tau\, d\tau\end{aligned} \tag{A-2}$$

If now this process represents the input to a linear dynamic system, the output of which is $g(t)$, the functions $f(t)$ and $g(t)$ can be related by the superposition integral

$$g(t) = \int_{-\infty}^{\infty} h(t_1)\, f(t - t_1)\, dt_1 \tag{A-3}$$

where $h(t)$ is the response of the given system to a unit impulse.

The correlation function for $g(t)$ can be defined in the same way as the one for $f(t)$; upon substituting Eq. (A-3) into this defining equation and making use of Eq. (A-1), this correlation function can be written as

$$\psi_g(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(t_1)\, h(t_2)\, \psi_f(\tau + t_1 - t_2)\, dt_1 dt_2 \tag{A-4}$$

and upon defining a power spectrum for $g(t)$ in the same way as the one for $f(t)$, substituting Eq. (A-4) into this defining relation and using Eq. (A-2), this spectrum can be written as

$$\varphi_g(\omega) = |H(\omega)|^2\, \varphi_f(\omega) \tag{A-5}$$

where $H(\omega)$ is the Fourier transform of $h(t)$,

$$H(\omega) = \int_{-\infty}^{\infty} h(t) e^{-i\omega t}\, dt \tag{A-6}$$

and represents the complex amplitude of the output of the system when subjected to a sinusoidal input of frequency ω and unit amplitude.

When the system is an airplane flying at a mean speed U and the input to the system is the gust intensity w, cognizance should be taken of the fact that along the flight path w is a function of only one variable, either the distance traveled along the flight path, x, or the time required to travel that distance, namely $t = x/U$. In the body of this paper w has been considered as a function of distance, with the result that the definitions of ψ_w and φ_w given in the body take special forms of those given in this Appendix.

The reason for this convention is that if an airplane flies sufficiently rapidly the gust velocity at any point of its flight path does not vary significantly during the time the airplane is in the vicinity of the point, so that the statistical characteristics, in particular the correlation function, are functions of space displacements only. (The term "vicinity" is used here to designate the space surrounding the point and of such extent that there is a strong correlation between the gust velocities at all points of that space with those at the given point.) The assumption has been made in previous analyses of the response of an airplane to continuous random atmospheric turbulence, and will be made here, that this condition, which is equivalent to Taylor's hypothesis made in studies of turbulence in wind tunnels, is satisfied. From available information concerning the properties of the atmosphere the required speed appears to be in the order of about 100 or 200 ft. per sec., so that for all higher flying speeds time displacements τ can be identified with space displacements $\Delta x = U\tau$, and vice versa.

In the calculations of this paper the following expression for the correlation function of atmospheric turbulence has been used.[4,6]

$$\psi_w(\Delta x) = \overline{w^2}[1 - (|\Delta x|/2L^*)]e^{-(|\Delta x|/L^*)} \quad \text{(A-7)}$$

The power spectrum corresponding to this correlation function is

$$\varphi_w(\omega) = (\overline{w^2}L^*/\pi U)\ [(1 + 3k'^2)/(1 + k'^2)^2] \quad \text{(A-8)}$$

where $$k' \equiv \omega L^*/U$$

The Transfer Functions for the Normal-Load-Factor Increment and the Root Bending Moment of a Rigid Airplane

The equation of motion of an airplane subjected to sinusoidal gusts and free to move vertically only can be written as

$$M\ddot{z} = -(C_{L_\alpha}qS/U)\ \tilde{C}(k)\dot{z} + L_g \quad \text{(A-9)}$$

where $L_g = C_{L_\alpha}qS(w/U)\phi(k)$; and $\tilde{C}(k)$ is an unsteady-lift function for changes in angle of attack, which for incompressible two-dimensional flow is the Theodorsen function $C(k)$ (see reference 14) plus an apparent-mass term $ik/2$; the function $\phi(k)$ is an unsteady-lift function for gust penetration, and is for two-dimensional incompressible flow the function first given in reference 15 and usually referred to as the Sears function. Hence, if a normal load factor increment and a mass parameter are defined by

$$\Delta n = \ddot{z}/g$$

and $$\kappa = 8M/C_{L_\alpha}\rho S\bar{c}$$

this equation can be written as

$$\Delta n = \{1/(\kappa/2) + [\tilde{C}(k)/ik]\}\ (2U^2/g\bar{c})\phi(k)\ (w/U) \quad \text{(A-10)}$$

The factor multiplying w on the right side of Eq. (A-10) constitutes the required transfer function. Using a refrence value of the load factor increment defined by the sharp-edge gust formula

$$\Delta n_s = (C_{L_\alpha}qS/Mg)\ (w/U) = (2/\kappa)\ (2U^2/g\bar{c})\ (w/U) \quad \text{(A-11)}$$

and Eqs. (5) and (A-8), the mean square value of Δn can be written as

$$\frac{\overline{(\Delta n)^2}}{\overline{(\Delta n_s)^2}} = \int_0^\infty \frac{|\phi(k)|^2}{|1 + [2\tilde{C}(k)/ik\kappa]|^2}\frac{1}{\pi}\frac{1 + 3k'^2}{(1 + k'^2)^2}\,dk'$$

where $\overline{(\Delta n_s)^2}$ represents the mean square value of Δn as calculated by the sharp-edge gust formula. In the calculation of $\overline{(\Delta n)^2}$ discussed in this paper the quasi-steady value of $\tilde{C}(k) \cong 1$ and the approximation used in references 4 and 6 for $|\phi(k)|^2$—namely,

$$|\phi(k)|^2 \cong 1/(1 + 2\pi k)$$

have been used.

If the gust is uniform along the span, the aerodynamic loads due directly to the gust and due to the motion of the airplane have the same lateral center of pressure. Hence, the bending moment due to sinusoidal gusts can be written as

$$BM = (C_{L_\alpha}qS/U)\ (\bar{y}/2)\ [\phi(k)w - \tilde{C}(k)\dot{z}] - (M_w/2)\bar{\bar{y}}\ddot{z} \quad \text{(A-12)}$$

However, using Eq. (A-9) this equation can be simplified to

$$BM = [(\bar{y}/2)M - (\bar{\bar{y}}/2)M_w]\ddot{z} = (\bar{y}/2)M[1 - (\bar{\bar{y}}/\bar{y})\ (M_w/M)]\ddot{z}$$

Therefore, the transfer function for BM is equal to the transfer function for Δn multiplied by the factor

$$(\bar{y}/2)M[1 - (\bar{\bar{y}}/\bar{y})\ (M_w/M)]g$$

In the calculations of the bending moments discussed in this paper the values of $\tilde{C}(k)$ and $\phi(k)$ for two-dimensional incompressible flow (as corrected for finite-span effects by using the appropriate value of C_{L_α} instead of 2π) were used. A reference value of the bending moment as calculated from the equivalent of the sharp-edge gust equation,

$$BM_s = (1/2)C_{L_\alpha}qS\bar{y}(w/U) \quad \text{(A-13)}$$

was used to reduce the results to dimensionless form. The values of $\bar{\bar{y}}$ and $\bar{y}$ were assumed to be identical.

Transfer Function for the Root Bending Moment of a Flexible Wing

The method of calculating the transfer function for the root bending moment of a flexible wing is based on the modal approach of reference 16.

The equation for the bending distortion of a flexible wing can be written as

$$(d^2/dy^2)\ [EI(d^2z/dy^2)] = l_g + l_m + l_i \quad \text{(A-14)}$$

where l_g and l_m are, respectively, the aerodynamic loads per unit span due to the gust directly and due to the airplane motion, and where l_i is the load per unit span due to inertia effects. If the airplane is restricted to vertical motion and to distortion in the first free-free bending mode, the function $z(y)$ can be written as

$$z(y) = z_0 + z_1\,\zeta(y) \quad \text{(A-15)}$$

The mode shape $\zeta(y)$ must satisfy the equation

$$(d^2/dy^2)\ [EI(d^2\zeta/dy^2)] = m\omega_0^2\,\zeta \qquad \text{(A-16)}$$

where ω_0 is the frequency of the oscillation in the first free-free bending mode, as well as the orthogonality condition

$$\int_{-(b/2)}^{b/2} m\,\zeta\,dy = 0 \qquad \text{(A-17)}$$

As a result of Eqs. (A-15) and (A-16), Eq. (A-14) can, for sinusoidal gusts of frequency ω, be written as

$$z_1\omega_0^2 m\zeta = l_g - (\dot z_0/U)l_1 - (\dot z_1/U)l_\zeta + \omega^2 m(z_0 + z_1\zeta) \qquad \text{(A-18)}$$

where l_1 and l_ζ are the lift distributions due to an angle of attack equal to unity and an angle of attack distribution equal to ζ, respectively. Integrating this equation and using Eq. (A-17) and then multiplying it by ζ first and then integrating yields the pair of equations

$$\left.\begin{aligned} 0 &= L_g - (\dot z_0/U)L_1 - (\dot z_1/U)L_\zeta + \omega^2 M z_0 \\ z_1\omega_0^2 M' &= L_g'' - (\dot z_0/U)L_1'' - (\dot z_1/U)L_\zeta'' + \omega^2 M' z_1 \end{aligned}\right\} \qquad \text{(A-19)}$$

where M' is the generalized mass associated with the first free-free bending mode,

$$M' = \int_{-(b/2)}^{b/2} m\,\zeta^2\,dy$$

and where the actual and generalized lifts due to motion are defined by

$$L_1 = C_{L_\alpha} qS\tilde C(k)$$

$$L_\zeta = C_{L_\alpha} qS\tilde C(k)K$$

$$L_1'' = C_{L_\alpha} qS\tilde C(k)\boldsymbol{K}$$

$$L_\zeta'' = C_{L_\alpha} qS\tilde C(k)K'$$

and

$$\left.\begin{aligned} K &= \int_0^1 \left(\frac{cc_l}{\bar c C_{L_\alpha}}\right)_{\alpha=\zeta} dy^* \\ \boldsymbol{K} &= \int_0^1 \left(\frac{cc_l}{\bar c C_{L_\alpha}}\right)_{\alpha=1} \zeta\,dy^* \\ K' &= \int_0^1 \left(\frac{cc_l}{\bar c C_{L_\alpha}}\right)_{\alpha=\zeta} \zeta\,dy^* \end{aligned}\right\} \qquad \text{(A-20)}$$

For an unswept wing, which is the only case considered in the following, the constants K and $\boldsymbol{K}$ are identical by virtue of the reciprocity relations of linearized lifting surface theory.[11] Hence, both will be designated by K in the following.

For subsonic flow the required lift distributions can be obtained from reference 17. In the calculations discussed in the present paper the mode shape ζ was taken as

$$\zeta = -a + (1 + a)y^{*2} \qquad \text{(A-21)}$$

and the lift distributions given in reference 17 for a wing of aspect ratio 12, taper ratio 1/2 with uniform angle of attack and with parabolic twist were used.

Similarly, for a gust which is uniform along the span,

$$\left.\begin{aligned} L_g &= C_{L_\alpha} qS\phi(k)\ (w/U) \\ L_g'' &= C_{L_\alpha} qS\phi(k)K(w/U) \end{aligned}\right\} \qquad \text{(A-22)}$$

In terms of these lifts, Eq. (A-19) can be written in dimensionless form as

$$\begin{bmatrix} -\frac{\kappa}{2}k^2 + ik\tilde C(k) & ikK\tilde C(k) \\ ikK\tilde C(k) & (k_0^2 - k^2)\frac{\kappa'}{2} + ikK'\tilde C(k) \end{bmatrix} \begin{Bmatrix} z_0^* \\ z_1^* \end{Bmatrix} = \phi(k) \begin{Bmatrix} \dfrac{w}{U} \\ K\dfrac{w}{U} \end{Bmatrix} \qquad \text{(A-23)}$$

where $\qquad k_0 = \omega_0\bar c/2U$

and $\qquad \kappa' = 8M'/C_{L_\alpha}\rho S\bar c$

Now, the root bending moment can be expressed as

$$\begin{aligned} BM &= EI_0(d^2z/dy^2)_{y=0} \\ &= EI_0[(\bar c/2)/(b/2)^2]z_1^*\zeta''_0 \end{aligned}$$

where $\qquad \zeta''_0 \equiv [d^2\zeta/dy^{*2}]_{y^*=0}$

so that $\qquad BM = H_1(w/U) + H_2K(w/U) \qquad \text{(A-24)}$

where the transfer functions H_1 and H_2 are obtained by solving Eq. (A-23) for z_1^* and are defined by

$$\left.\begin{aligned} H_1 &= EI_0\frac{\bar c/2}{(b/2)^2}\zeta''_0\frac{-ikK\tilde C(k)}{D(k)}\phi(k) \\ H_2 &= EI_0\frac{\bar c/2}{(b/2)^2}\zeta''_0\frac{-(\kappa/2)k^2 + ik\tilde C(k)}{D(k)}\phi(k) \end{aligned}\right\} \qquad \text{(A-25)}$$

where

$$D(k) = (k^2 - k_0^2)k^2\frac{\kappa}{2}\frac{\kappa'}{2} - ik\tilde C(k) \times \left\{(k^2 - k_0^2)\frac{\kappa'}{2} + k^2\frac{\kappa}{2}K'\right\} + (K^2 - K')k^2\tilde C^2(k)$$

Hence, for the one-dimensional gust structure, the desired transfer function from the gust to the root bending moment is

$$H_{BM}{}^w = (1/U)\ [H_1 + KH_2]$$

Appendix B—Derivation of Output Spectra for a Two-Dimensional Gust Structure

The General Input-Output Relations

In analogy with Eq. (A-3), a superposition integral for an output, say $\sigma(t)$, of a system subjected to a two-dimensional input can be written as

$$\sigma(t) = \int_{-\infty}^{\infty}\int_{-(b/2)}^{b/2} h(t_1, y)\,w[U(t - t_1), y]\,dy\,dt_1 \qquad \text{(B-1)}$$

where the function $h(t, y)$ has been discussed in the body of the paper. If now the correlation function for σ is written in terms of this integral, a time average of the type $\overline{w(x, y)\, w(x + \Delta x, y + \Delta y)}$ occurs on the right side of the equation. This function will be defined as the two-dimensional correlation function $\tilde{\psi}_w(\Delta x, \Delta y)$ for the vertical component of the gust velocity.

As mentioned in the body of this paper, by assuming the turbulence to be axisymmetric this function can be expressed in terms of the one-dimensional correlation function as follows:

$$\tilde{\psi}_w(\Delta x, \Delta y) = \psi_w(\sqrt{\Delta x^2 + \Delta y^2}) \tag{B-2}$$

This relation will be assumed to be valid in the following.

The correlation function for σ can then be written as follows:

$$\psi_\sigma(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-(b/2)}^{b/2}\int_{-(b/2)}^{b/2} h(t_1, y_1)\, h(t_2, y_2) \times \psi_w\,[\sqrt{U^2(\tau + t_1 - t_2)^2 + (y_2 - y_1)^2}]\, dy_1\, dy_2\, dt_1\, dt_2 \tag{B-3}$$

By taking the Fourier transform with respect to τ of both sides of this equation, the equivalent relation for the power spectrum of σ is obtained,

$$\varphi_\sigma(\omega) = \int_{-(b/2)}^{b/2}\int_{-(b/2)}^{b/2} H(\omega, y_2)\, [H(\omega, y_1)]^* \times \tilde{\varphi}_w(\omega, y_2 - y_1)\, dy_1\, dy_2$$
$$= \int_{-(b/2)}^{b/2}\int_{-(b/2)}^{b/2} \mathcal{R}\{H(\omega, y_2)\, [H(\omega, y_1)]^*\} \times \tilde{\varphi}_w(\omega, y_2 - y_1)\, dy_1\, dy_2 \tag{B-4}$$

where $\mathcal{R}\{\ \}$ designates that the real part is to be taken, and where the asterisk designates that the complex conjugate of $H(\omega, y_1)$ is to be taken. The functions $\tilde{\varphi}_w$ and H are the Fourier transforms of $\tilde{\psi}_w$ and h,

$$\tilde{\varphi}_w(\omega, \Delta y) = \frac{1}{\pi}\int_{-\infty}^{\infty} e^{-i\omega(\Delta x/U)}\, \tilde{\psi}_w(\Delta x, \Delta y)\, d\left(\frac{\Delta x}{U}\right)$$

and $$H(\omega, y) = \int_{-\infty}^{\infty} e^{-i\omega t}\, h(t, y)\, dt$$

Eq. (B-4) represents the input-output relation for a two-dimensional stationary stochastic input to a linear system.

For axisymmetric turbulence the two-dimensional spectrum $\tilde{\varphi}_w(\omega, \Delta y)$ can be obtained from the one-dimensional spectrum $\varphi_w(\omega)$ by means of the relation

$$\tilde{\varphi}_w(\omega, \eta) = \varphi_w(\omega) - |\eta| \int_{|\omega|}^{\infty} \varphi_w(\omega') \frac{\omega'}{\sqrt{\omega'^2 - \omega^2}} \times J_1(|\eta| \sqrt{\omega'^2 - \omega^2})\, d\omega' \tag{B-5}$$

where $\eta \equiv \Delta y$.

For the correlation function and spectrum of Eqs. (A-7) and (A-8) the spectrum $\tilde{\varphi}_w(\omega, \eta)$ is

$$\tilde{\varphi}_w(\omega, \eta) = (\overline{w^2}L^*/\pi U)\, \{(\eta/L^*)\, [(1 + 3k'^2) \div (1 + k'^2)^{3/2}]K_1\, [(\eta/L^*) \sqrt{1 + k'^2}] - (\eta/L^*)^2\, [1/(1 + k'^2)]K_0[(\eta/L^*) \sqrt{1 + k'^2}]\} \tag{B-6}$$

The double integral of Eq. (B-4) can be reduced to a single integral by introducing an autoconvolution function of $H(\omega, y)$,

$$\tilde{H}(\omega, \eta) = 2\int_{-(b/2)}^{(b/2)-\eta} \mathcal{R}\{H(\omega, y)\, [H(\omega, y + \eta)]^*\, dy \tag{B-7}$$

The expression for $\varphi_\sigma(\omega)$ then becomes

$$\varphi_\sigma(\omega) = \int_0^b \tilde{H}(\omega, \eta)\, \tilde{\varphi}_w(\omega, \eta)\, d\eta \tag{B-8}$$

An alternative method of evaluation consists in using the double Fourier transform of the two-dimensional correlation function,

$$\tilde{\tilde{\varphi}}_w(\omega_1, \omega_2) = \frac{1}{\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i[\omega_1(\Delta x/U) + \omega_2(\Delta y/U)]} \times \tilde{\psi}_w(\Delta x, \Delta y)\, d\left(\frac{\Delta x}{U}\right) d\left(\frac{\Delta y}{U}\right)$$
$$= \frac{1}{\pi}\int_{-\infty}^{\infty} e^{-\omega_1(\Delta y/U)}\, \tilde{\varphi}_w(\omega_1, \Delta y)\, d\left(\frac{\Delta y}{U}\right)$$

If the turbulence is axisymmetric, this double Fourier transform is a function only of the frequency $\sqrt{\omega_1^2 + \omega_2^2}$—that is,

$$\tilde{\tilde{\varphi}}_w(\omega_1, \omega_2) = \hat{\varphi}(\sqrt{\omega_1^2 + \omega_2^2})$$

where $\hat{\varphi}_w$ can be obtained from ψ_w or φ_w by means of the following relations:

$$\left.\begin{aligned} \hat{\varphi}_w(\omega) &= \frac{2}{\pi}\int_0^{\infty} \frac{\Delta x}{U} J_0\left(\omega \frac{\Delta x}{U}\right) \times \psi_w(\Delta x)\, d\left(\frac{\Delta x}{U}\right) \\ \text{or } \hat{\varphi}_w(\omega) &= -\frac{2}{\pi}\int_{|\omega|}^{\infty} \frac{d\varphi_w(\omega')}{d\omega'} \times \frac{d\omega'}{\sqrt{\omega'^2 - \omega^2}} \\ &= -\frac{2}{\pi}\,\text{F.P.}\left\{\int_{|\omega|}^{\infty} \varphi_w(\omega') \times \frac{\omega' d\omega'}{(\omega'^2 - \omega^2)^{3/2}}\right\} \end{aligned}\right\} \tag{B-9}$$

where F.P. designates "the finite part of."

For the correlation function and spectrum given in Eqs. (A-7) and (A-8), the function $\hat{\varphi}_w(\omega)$ is

$$\hat{\varphi}_w(\omega) = (\overline{w^2}L^{*2}/U^2)\, (3/\pi)\, [k'^2/(1 + k'^2)^{5/2}]$$

In terms of the spectrum $\hat{\varphi}_w$ and the Fourier transform with respect to y of $H(\omega, y)$—namely,

$$\hat{H}(\omega, \omega') = \int_{-(b/2)}^{b/2} e^{-i\omega'(y/U)} H(\omega, y)\, dy \tag{B-10}$$

Eq. (B-4) can be written as

$$\varphi_\sigma(\omega) = \frac{1}{2}\int_{-\infty}^{\infty} |\hat{H}(\omega, \omega')|^2 \hat{\varphi}_w(\sqrt{\omega^2 + \omega'^2})\, d\omega' \tag{B-11}$$

which thus constitutes an alternative input-output relation.

In many cases the influence function $h(t, y)$ and, consequently, also its Fourier transform $H(\omega, y)$, have the property that they can be written as a product of functions depending on time (or frequency) and space alone—that is,

and
$$\left.\begin{aligned} h(t, y) &= \boldsymbol{h}(t)\ \gamma(y) \\ H(\omega, y) &= \boldsymbol{H}(t)\ \gamma(y) \end{aligned}\right\} \tag{B-12}$$

where $\boldsymbol{H}(\omega)$ is the Fourier transform of $\boldsymbol{h}(t)$. For this case the evaluation of the integral of the input-output relation is simplified considerably. For instance, Eq. (B-8) now becomes

$$\varphi_\sigma(\omega) = b|\boldsymbol{H}(\omega)|^2 \int_0^b \Gamma(\eta)\ \tilde{\varphi}_w(\omega, \eta)\, d\eta \tag{B-13}$$

where the function $\Gamma(\eta)$ defined by

$$\Gamma(\eta) = \frac{2}{b}\int_{-(b/2)}^{(b/2)-\eta} \gamma(y)\ \gamma(y + \eta)\, dy \tag{B-14}$$

is independent of frequency.

In many cases when $h(t, y)$ and $H(\omega, y)$ do not have forms indicated in Eqs. (B-12), they can be expressed as a sum of functions which do have those forms, that is

$$\left.\begin{aligned} h(t, y) &= h_1(t)\ \gamma_1(y) + h_2(t)\ \gamma_2(y) + \ldots \\ H(\omega, y) &= H_1(\omega)\ \gamma_1(y) + H_2(t)\ \gamma_2(y) + \ldots \end{aligned}\right\} \tag{B-15}$$

The manner in which this property can be used to advantage will be indicated in one of the examples that follow.

Spectrum of the Lift Due Directly to Turbulence

According to the reciprocity theorem of linearized lifting surface theory,[11] the lift influence function is equal to the lift distribution for unit angle of attack in reverse flow. If the wing is unswept, as is the case for the calculations described herein, there is no distinction between the lift distributions in direct and reverse flow. Inasmuch as the distribution of the lift on an oscillating unswept wing appears to be substantially independent of frequency, as indicated by the results calculated in reference 18, the influence function for the lift due directly to turbulence will be assumed to be expressible in the form indicated in Eq. (B-12), specifically,

$$H_{L_g}{}^w(\omega, y) = [(C_{L_\alpha}qS/Ub)\ \phi(k)]\ \gamma_L(y)$$

where $\gamma_L(y)$ is the section loading coefficient for unit angle of attack, $(cc_l/\bar{c}C_{L_\alpha})_{\alpha=1}$.

Hence, the function $\Gamma(\eta)$ becomes

$$\Gamma(\eta^*) = \int_{-1}^{1-\eta^*} \gamma_L(y^*)\ \gamma_L(y^* + \eta^*)\, dy^*$$

For instance, if the lift distribution is uniform across the span, $\gamma_L = 1$, so that

$$\Gamma(\eta^*) = 2 - \eta^* \tag{B-16}$$

Eq. (B-13) then becomes

$$\varphi_{L_g}(\omega) = \frac{b^2}{2}\left[\frac{C_{L_\alpha}qS}{Ub}\right]^2 |\phi(k)|^2 \int_0^2 \Gamma(\eta^*)\ \tilde{\varphi}_w(\omega, \eta)\, d\eta^* \tag{B-17}$$

In analogy with the expression for a one-dimensional random gust—namely,

$$\varphi_{L_g}(\omega) = [C_{L_\alpha}qS/U]^2\ |\phi(k)|^2\ \varphi_w(\omega)$$

Eq. (B-17) can be written as

$$\varphi(\omega)_{L_g} = [C_{L_\alpha}qS/U]^2\ |\phi(k)|^2\ \varphi_{w_e}(\omega) \tag{B-18}$$

where $\varphi_{w_e}(\omega)$, an effective spectrum for the vertical component of the gust velocity, is defined by

$$\varphi_{w_e}(\omega) = \frac{1}{2}\int_0^2 \Gamma(\eta^*)\ \tilde{\varphi}_w(\omega, \eta)\, d\eta^*$$

Using the functions $\Gamma(\eta^*)$ and $\tilde{\varphi}_w(\omega, \eta)$, defined by Eqs. (B-16) and (B-6), respectively, the function φ_{w_e} has been calculated and is shown in Fig. 4. From this spectrum, or its equivalent obtained with the distribution γ_L appropriate to any given case, the spectrum of the lift can then be obtained from Eq. (B-18).

Spectrum of the Normal Load Factor Increment

If the spanwise variation of gust intensity is taken into account, the term L_g on the right side of Eq. (A-9) is the lift discussed in the preceding section. However, the other terms in the equation are independent of the nature of the input, so that the transfer function in Eq. (A-10) is the same as before. Consequently, the power spectrum for Δn can be written in the same way as before, with φ_w replaced by the function φ_{w_e} defined in the preceding section. The results shown in Fig. 6 have been obtained by integrating the spectrum for Δn obtained in this manner.

Spectrum of the Rolling Acceleration

The equation of motion of an airplane in roll due to sinusoidal vertical gusts can be written as

$$I_x\dot{p} = C_{l_p}qSb\tilde{C}(k)\ (pb/2U) + L'_g \tag{B-19}$$

where L'_g, the rolling moment due directly to gusts, can be written as

$$L'_g = \frac{(-C_{L_p})qSb}{U}\ \phi(k)\ \frac{1}{4}\int_{-1}^{1} \gamma_{L'}(y^*)\ w(y^*)\, dy^*$$

where the rolling moment influence function $\gamma_{L'}$ is equal to the lift distribution in roll—that is, $[cc_l \div c(-C_{l_p})]_{\alpha=y^*}$. Hence, the rolling acceleration $\dot{p}$ can be expressed in terms of w as follows:

$$\frac{\dot{p}b}{2g} = \left[\frac{U}{Sg}\frac{1}{\kappa_r + [\tilde{C}(k)/ik]}\ \phi(k)\right]\frac{b}{2}\int_{-1}^{1} \gamma_{L'}(y^*) \times w(y^*)\, dy^*$$

so that the influence function relating $\dot{p}b/2g$ to w has the form indicated in Eq. (B-12), with $\boldsymbol{H}(\omega)$ being the function in brackets in the preceding equation, and $\gamma(y)$ being the function $\gamma_{L'}(y^*)$.

The spectrum of the rolling moment can thus be obtained in the manner employed previously for the lift, by calculating an "effective" gust spectrum and multiplying it by $|\boldsymbol{H}(\omega)|^2$.

In the calculations discussed in this paper, κ_r was taken as 25.75, U = 400 ft. per sec., and S = 1,250 sq. ft.

Spectrum for the Root Bending Moment of a Rigid Wing

If variations of the gust intensity along the span are taken into account, Eq. (A-12) becomes

$$BM = -\frac{1}{2}\frac{C_{L_\alpha}qS}{U}\bar{y}\tilde{C}(k)\dot{z} - \frac{M_w}{2}\bar{y}\ddot{z} + \frac{C_{L_\alpha}qSb}{4U}\phi(k)\int_{-1}^{1}\gamma_B(y^*)\,w(y^*)\,dy^* \quad \text{(B-20)}$$

where $\gamma_B(y^*)$ is an influence function for the bending moment and is, according to the reciprocity theorem, the lift distribution $cc_l/\bar{c}C_{L_\alpha}$ for an angle of attack distribution which is 0 on the left wing and equal to y^* on the right wing. As a result of the superposition principle, this lift distribution is equal to one-half the sum of the lift distribution due to unit symmetrical linear twist and the lift distribution due to unit antisymmetrical linear twist (damping-in-roll condition). For the calculations described herein these lift distributions, as well as $\gamma_L(y^*)$, were obtained from the results given in reference 17 for a wing of aspect ratio 12, taper ratio 1/2.

Inasmuch as $\ddot{z}$ and $\dot{z}$ can be expressed in terms of L_g by means of the transfer function discussed in connection with the normal load factor increment, the bending moment can be expressed in terms of w by means of an expression of the form of Eq. (B-15), with

$$\left.\begin{aligned} H_1(\omega) &= \frac{-(1/2)\,[C_{L_\alpha}qS/U]\,\bar{y}[\tilde{C}(k)/ik] - (M_w/2)\bar{y}}{(\kappa/2) + [\tilde{C}(k)/ik]} \times (2U/S)\,\phi(k) \\ H_2(\omega) &= \frac{C_{L_\alpha}qS}{2U}\phi(k) \\ \gamma_1(y) &= \gamma_L(y^*) \\ \gamma_2(y) &= \gamma_B(y^*) \end{aligned}\right\} \quad \text{(B-21}$$

The spectrum for BM may therefore be derived as follows. The bending moment may be written as a sum of two superposition integrals involving the Fourier transforms $h_1(t)$ and $h_2(t)$ of $H_1(\omega)$ and $H_2(\omega)$,

$$BM(t) = \int_{-\infty}^{\infty} h_1(t_1)dt_1\int_{-(b/2)}^{(b/2)}\gamma_1(y)\,w(t-t_1, y)\,dy + \int_{-\infty}^{\infty} h_2(t_2)\,dt_2\int_{-(b/2)}^{(b/2)}\gamma_2(y)\,w(t-t_2, y)\,dy$$

Upon forming the correlation function for BM, introducing the assumption of axisymmetry, and calculating the Fourier transform of the correlation function, the power spectrum for the bending moment is obtained as

$$\varphi_{BM}(\omega) = |H_1(\omega)|^2\,\varphi_{we}(\omega) + 2\Re\{H_1(\omega)\,[H_2(\omega)]^*\}\times \varphi_{we_1}(\omega) + |H_2(\omega)|^2\,\varphi_{we_2}(\omega) \quad \text{(B-22)}$$

where $\Re$ and the asterisk designate "the real part of" and call for the complex conjugate, respectively, as before; the spectrum $\varphi_{we}(\omega)$ has been defined previously, and the other spectra are defined as

$$\left.\begin{aligned} \varphi_{we_1}(\omega) &= \frac{1}{2}\int_0^2 \Gamma_1(\eta^*)\,\tilde{\varphi}_w(\omega, \eta)\,d\eta^* \\ \varphi_{we_2}(\omega) &= \frac{1}{2}\int_0^2 \Gamma_2(\eta^*)\,\tilde{\varphi}_w(\omega, \eta)\,d\eta^* \end{aligned}\right\} \quad \text{(B-23)}$$

where, in turn,

$$\Gamma_1(\eta^*) = \int_{-1}^{1-\eta^*}\gamma_L(y^*)\,\gamma'_B(y^*+\eta^*)\,dy^*$$

$$\Gamma_2(\eta^*) = \int_{-1}^{1-\eta^*}[\gamma'_B(y^*)\,\gamma'_B(y^*+\eta^*) + \gamma''_B(y^*)\,\gamma_B''(y^*+\eta^*)]\,dy^*$$

where $\gamma'_B(y^*)$ and $\gamma''_B(y^*)$ are, respectively, the symmetrical and antisymmetrical part of $\gamma_B(y^*)$ and, as was pointed out previously, are equal to one-half of the lift distributions $cc_l/\bar{c}C_{L_\alpha}$ for a unit linear symmetric twist and unit linear antisymmetric twist, respectively.

The contribution of the antisymmetric part of γ_B to Γ_2 and, hence, to the spectrum for the bending moment stems basically from the asymmetry of the instantaneous distribution of gust intensity over the span. This asymmetry gives rise to a rolling moment and, hence, to rolling motions, which were considered in the preceding section of this Appendix, and which contribute additional bending moments due to the aerodynamic and inertia loads associated with these motions. If, for the purpose of calculating the bending moment due to symmetrical flight through turbulent air, these motions are disregarded, the contribution of γ''_B to Γ_2 should be disregarded as well, so that the second part of the expression for Γ_2 is generally spurious and should be ignored. As pointed out in reference 10, the net bending moment due to the symmetrical part of the instantaneous gust distributions and the resulting vertical motions and the net bending moment due to the antisymmetrical part and the resulting rolling motions are statistically independent, so that their power spectra can be added directly to obtain the total net bending moment due to atmospheric turbulence and the resulting airplane motions.

Inasmuch as

$$\frac{1}{2}\int_0^2 \Gamma_1(y^*)\,dy^* = \frac{\bar{y}}{b/2}$$

and

$$\frac{1}{2}\int_0^2 \Gamma_2(y^*)\,dy^* = \left(\frac{\bar{y}}{b/2}\right)^2$$

the functions $\varphi_{we}(\omega)$, $\varphi_{we_1}(\omega)$ and $\varphi_{we_2}(\omega)$ reduce, respectively, to $\varphi_w(\omega)$, $[\bar{y}/(b/2)]\varphi_w(\omega)$, and $[\bar{y}/(b/2)]^2\,\varphi_w(\omega)$, as the ratio of the span to the scale of turbulence approaches 0. Consequently, by introducing these values and the definitions of $H_1(\omega)$ and $H_2(\omega)$ given in Eq. (B-21) into Eq. (B-22), the value of $\varphi_{BM}(\omega)$ given there can readily be shown to reduce the one obtainable directly from Eq. (A-12) when the span ratio approaches 0.

Spectrum for the Root Bending Moment of a Flexible Wing

The technique of the preceding section can readily be adapted to the flexible wing. The two functions $H_1(\omega)$ and $H_2(\omega)$ defined in Eq. (A-25) can now be used directly, provided that the functions $\gamma_1(y)$ and $\gamma_2(y)$ are defined as

$$\gamma_1(y) = (1/Ub)\gamma_L(y^*) \qquad \gamma_2(y) = (1/Ub)\gamma_\zeta(y^*)$$

where $\gamma_\zeta(y^*)$ is an influence function for L''_ζ, and according to the reciprocity theorem is equal to the lift distribution $cc_l/\bar{c}C_{L_\alpha}$ due to an angle of attack distribution equal to ζ—that is, the lift distribution used previously in defining K and K' in eq. (A-20).

Eqs. (B-22) and (B-23) can thus be used directly for the flexible-wing case provided Γ_1 and Γ_2 are redefined as

$$\Gamma_1(\eta^*) = \frac{1}{U^2b^2}\int_{-1}^{1-\eta^*} \gamma_L(y^*)\ \gamma_\zeta(y^* + \eta^*)\ dy^*$$

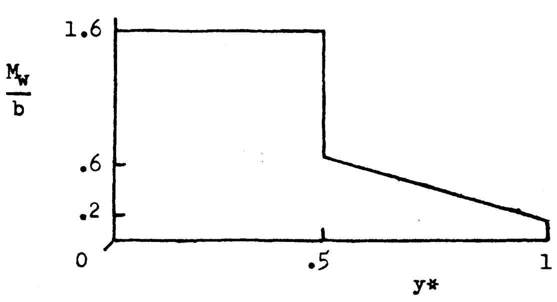
$$\Gamma_2(\eta^*) = \frac{1}{U^2b^2}\int_{-1}^{1-\eta^*} \gamma_\zeta(y^*)\ \gamma_\zeta(y^* + \eta^*)\ dy^*$$

and provided $\varphi_{we}(\omega)$ is now divided by U^2b^2. The power spectrum for the bending moment obtained in this manner can, again, be shown to reduce to the one obtainable from Eq. (A-24) when the span ratio approaches zero.

For the flexible-wing calculations discussed in this paper the following quantities were used:

$$\kappa = 175 \qquad b = 125 \text{ ft.}$$
$$EI_0 = 22 \times 10^8 \text{ lb. ft.}^2 \qquad \bar{c} = 10 \text{ ft.}$$

The wing mass was assumed to be distributed along the span in the following manner, regardless of how much of the airplane mass was in the wing:



The mode shape, in particular the constant a in Eq. (A-21), was permitted to vary with the mass ratio M_w/M in such a way as to satisfy the orthogonality condition. Consequently, the frequency ω_0 and the generalized-mass parameter κ' varied with this ratio. For a value of this ratio of 0.25 the following values were used:

$$\omega_0 = 15.4 \qquad a = 0.147 \qquad \kappa' = 8.73$$

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