Structural Dynamic Analysis of Rocket Engine Turbomachinery

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ER41/Propulsion Structures & Dynamic Analysis

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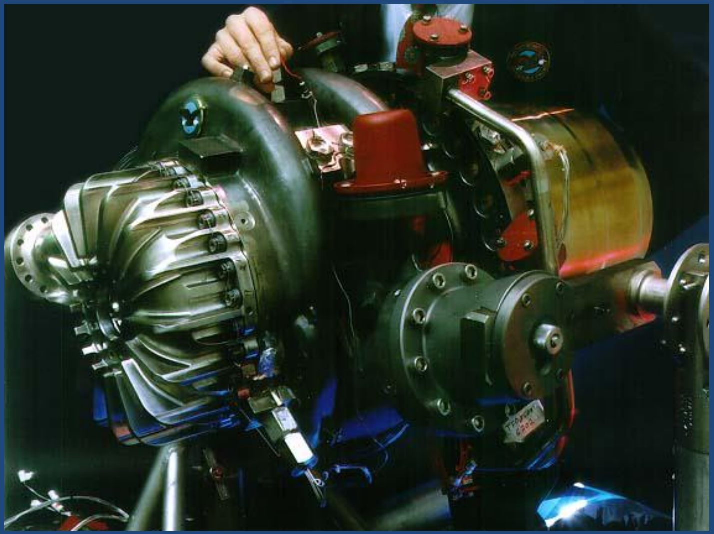
Agenda

- Motivation for Structural Dynamic Analysis of Turbomachinery
- How Turbomachinery is used in Rocket Engines
- Overview and Introduction to Structural Dynamics
  1. SDOF Systems
  2. MDOF Systems
- Application to Turbomachinery
  1. Fourier Characterization of Excitation
  2. Temporal Fourier series
  3. Modeling and Modal Analysis of Displacements and Stresses
  4. Campbell Diagrams
  5. Damping
  6. Forced Response Analysis in the Frequency Domain
- Some Additional Necessary Details
  1. Cyclic Symmetry
  2. Spatial Characterization of Excitation, Orthogonality with Mode Shape
  3. Mistuning
- Conclusion

How turbomachinery is used in Rocket Engines



MSFC Fastrac engine

- Liquid Fuel (LH2, Kerosene) and Oxidizer (LO2) are stored in fuel tanks at a few atmospheres.
- Turbines, driven by hot gas created by mini-combustors, tied with shaft to pump, which sucks in propellants and increases their pressures to several hundred atm.
- High pressure propellants sent to Combustion Chamber, which ignites mixture with injectors.
- Hot gas directed to converging/diverging nozzle to give flow very high velocity for thrust.

Motivation: Avoid High Cycle Fatigue Cracking in Turbomachinery

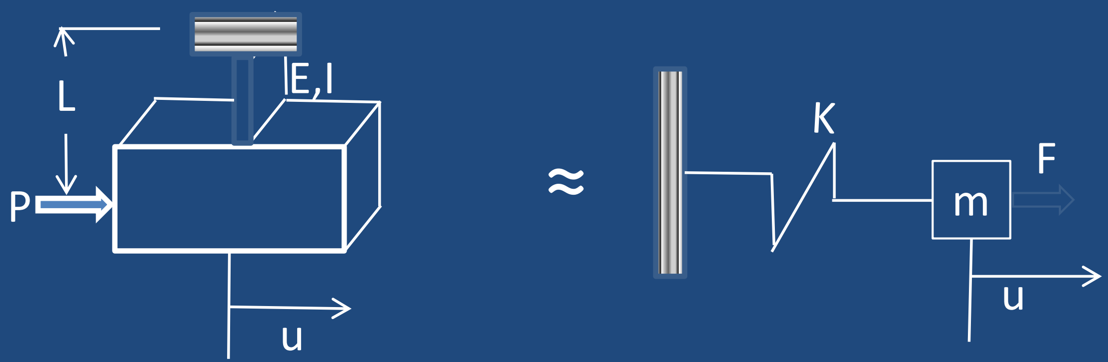
- Crack found during ground-test program can stop engine development
  - If crack propagates, it could liberate a piece, which at very high rotational speeds could be catastrophic (i.e., engine will explode)

Structural Dynamics Basics

Modeling and Free Vibration of SDOF Systems

A) Equation of Motion for Undamped System

1) Model Spring-Mass System to represent real structure



For cantilever beam, $\delta = \dfrac{PL^3}{3EI}$

Derive k_{beam} (in class):

Here, $P=k\delta$ is analogous to typical $F=kx$, so $k=p/\delta$

$$\boxed{\begin{aligned} \therefore k &= \frac{p}{pl^3/3EI} \\ k_{beam} &= \frac{3EI}{L^3} \end{aligned}}$$

2) Generate Equation of Motion (EoM) using Newton's 2nd Law

$$\Sigma F_x = m\ddot{u}$$

$$-ku + F_{external} = m\ddot{u}$$

$$m\ddot{u} + ku = F_{external}$$

Solution to Equation of Motion (2nd Order ODE)

$$u(t)_{total} = u_{particular,\ steady\ state,\ nonhomogeneous} + u_{complimentary,\ transient,\ homogeneous}$$

Where

- $u_{complimentary,\ transient,\ homogeneous}$ is the component of the solution when the RHS of the EoM is zero.
  - In physical terms, this is the response due to the internal dynamic characteristics of the structure, and comes about when there are non-zero Initial conditions (I.C.'s).

- $u_{particular,\ steady\ state,\ nonhomogeneous}$ is the component of the solution when the RHS of the EoM is non-zero.
  - In physical terms, this is the response due to external forcing functions.

Solution Methods for Homogeneous Component

1) simplest, worth remembering:

- Assume solution *u=u(t)* is of form

$$u(t) = A\cos(\omega t)$$
$$\dot{u}(t) = -A\omega\sin(\omega t)$$
$$\boxed{\ddot{u}(t) = -A\omega^2\cos(\omega t)}$$

- Now plug these equalities into eq of motion:

$$\boxed{\begin{aligned} &m(-A\omega^2\cos\omega t) + k(A\cos\omega t) = 0 \\ &A\cos\omega t(k - \omega^2 m) = 0 \end{aligned}}$$

For *A* cosωt = *0*, A has to = 0 , i.e., no response ("trivial solution")

Therefore,

$$k - \omega^2 m = 0$$

$$\omega^2 = \frac{k}{m} \quad \Rightarrow \quad \boxed{\omega = \sqrt{\frac{k}{m}} \ Rad/sec}$$

Prize question: is the natural frequency of a system the same on the moon as on earth?

Yes

Define $\lambda \equiv$ Eigenvalue = $\omega^2 \equiv$ Natural Frequency^2

So, solution for u= u(t) is $\boxed{u(t) = A\cos(\sqrt{\frac{k}{m}}\,t)}$ where A depends on the initial conditions (IC)

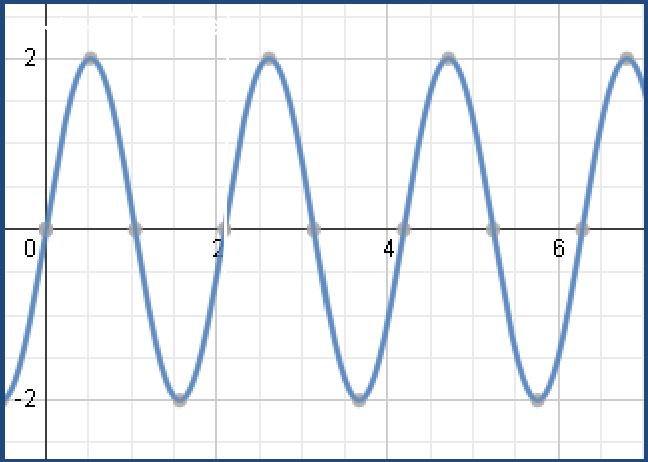
Natural Frequency Units, Period Relationship

Question: What is relationship of the natural frequency ω (sometimes called "circular natural frequency) to the natural frequency *f* in hz?

$$f\ \mathrm{hz} = \frac{\omega\ {}^{rad}\!/_{\sec}}{2\pi} hz$$

Question: What is relationship of the period *T* to the natural frequency *f* and the circular natural frequency ω?

$$T = \frac{1}{f\ \mathrm{hz}} \sec = \frac{2\pi}{\omega} \sec$$



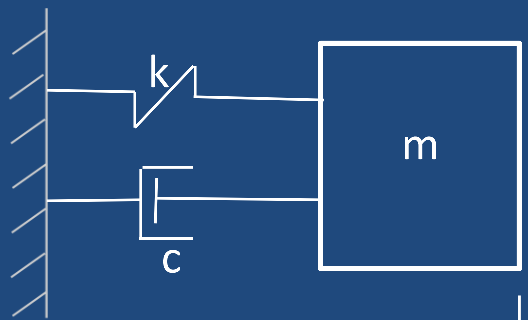
$$E.G.: u = 2\sin(3t)$$

$$\omega = 3\, {}^{Rad}\!/_{\sec}$$

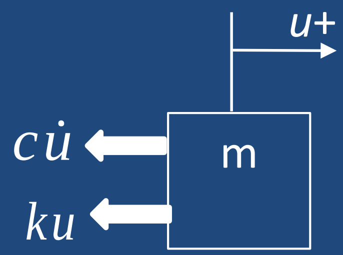
$$f = \frac{3\, {}^{Rad}\!/_{\sec}}{2\pi} hz = 0.477hz \Rightarrow u = 2\sin\left(2\pi(.477)t\right)$$

$$T = \frac{1}{f} \sec = \frac{2\pi}{\omega} \sec = \frac{2\pi}{3\, {}^{Rad}\!/_{\sec}} s = 2.094s$$

Damped Free Vibration of SDOF Systems



Damper has parameter c $\frac{lb}{\frac{in}{sec}}$



$$\sum F_{external} = m\ddot{u}$$

$$-c\dot{u} - ku = m\ddot{u}$$

$$m\ddot{u} + c\dot{u} + ku = 0$$

Damping Categories

- 3 cases of solutions:
  1) Critical Damping

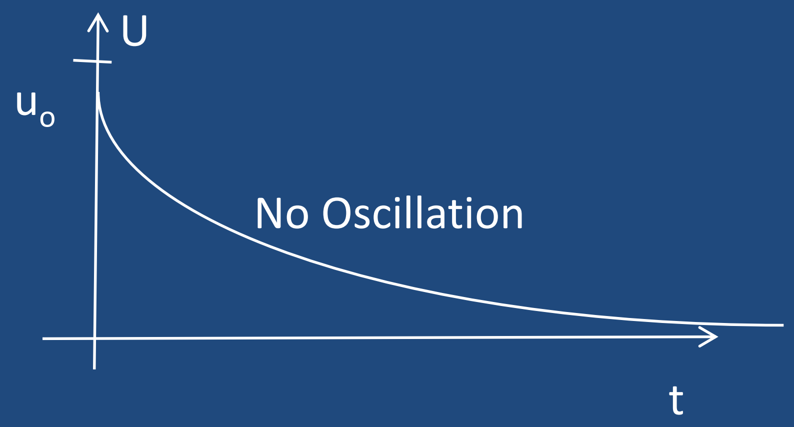
$$c = 2\sqrt{km} \equiv c_{critical}$$

  2) Overdamped

$$c > 2\sqrt{km}$$

$$c > c_{critical}$$

If we give a critically damped or overdamped SDOF system an initial displacement u_o, we get



Underdamped SDOF Equation of Motion for Free Vibration

3) Underdamped

$$c < 2\sqrt{km}$$

$$c < c_{critical}$$

Now, to talk about underdamped systems, go back to the damped eq. of motion:

$$m\ddot{u} + c\dot{u} + ku = 0$$

divide by m

$$\ddot{u} + \tfrac{c}{m}\dot{u} + \tfrac{k}{m}u = 0$$

Now define viscous damping ratio ζ as a percentage of critical damping

$$\boxed{\zeta = \frac{c}{c_{critical}}}$$

Using $\omega^2 = \frac{k}{m}$, we get

$$\ddot{u} + \frac{\zeta c_{cr}}{m}\dot{u} + \omega^2 u = 0$$

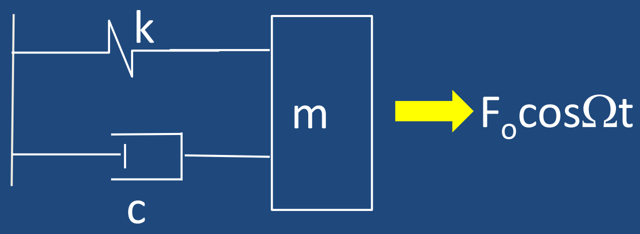
Use $c_{cr} = 2\sqrt{km}$ to get

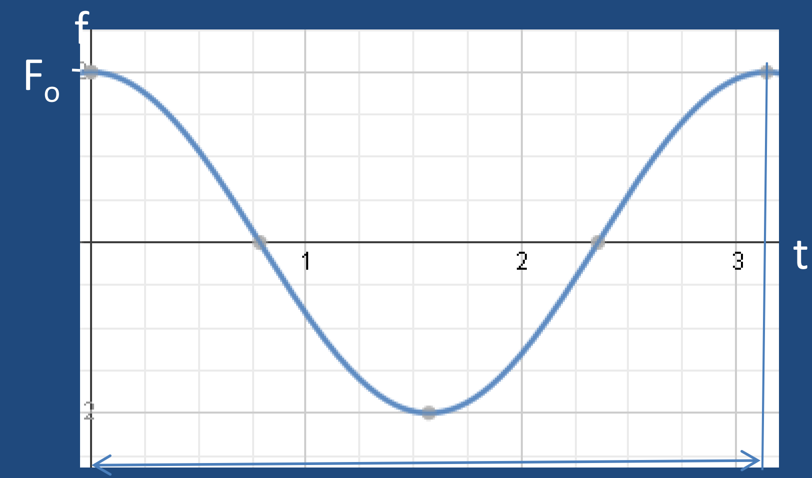
$$\ddot{u} + \frac{2\zeta\sqrt{km}}{m}\dot{u} + \omega^2 u = 0$$

$$\boxed{\ddot{u} + 2\zeta\omega\,\dot{u} + \omega^2 u = 0}$$

VERY IMPORTANT FORM, ALL YOU NEED ARE DYNAMIC CHARACTERISTICS!

Response to Harmonic Excitation





$$T = \frac{2\pi}{\Omega}$$

Ω = Excitation Frequency
f = Excitation Force (NOT enforced displacement or acceleration, but e.g. aerodynamic force)
ω = System Natural Frequency = $\sqrt{\frac{k}{m}}$

eq. of motion: $\boxed{m\ddot{u} + c\dot{u} + ku = F_o \cos \Omega t}$

Solution: $u(t) = u_t(t) + u_{ss}(t)$

where

$u_t(t) =$ transient solution $\equiv$ homogeneous (rhs=0) sol'n

$$u(t) = e^{-\zeta\omega t}(u_o \cos \omega_d t + \frac{\dot{u}_o + \zeta\omega u_o}{\omega_d} \sin \omega_d t)$$

which decays to zero after a few cycles,

If the excitation frequency stays constant or slowly varies.

Steady State Solution for Non-Homogeneous Component

- Response U as a function of excitation frequency is

$$\bar{U}(\Omega) = \bar{H}(\Omega)\bar{U}_{st}$$

where we define the "Complex Frequency Response"

$$\bar{H}(\Omega) = \frac{\text{Dynamic Response } \bar{U}}{\text{Static Response } \bar{U}_{st}}$$

and where we determine the static response U_{st} to force F_o using

$$kU_{st} = F_o \quad \rightarrow \quad U_{st} = F_o / k$$

After a long derivation, we get

$$\left|\bar{H}(\Omega)\right| = \sqrt{\frac{1}{\left(1-r^2\right)^2 + \left(2\zeta r\right)^2}}$$ and

phase lag $\phi = \tan^{-1}\left(\frac{-2\zeta r}{1-r^2}\right)$

where we define the Frequency Ratio $r = \frac{\Omega}{\omega}$

Resonance is defined when $\Omega = \omega$, ie, r=1.

At r=1, $\left|\bar{H}(\Omega)\right| = \frac{1}{2\zeta} \equiv$ Quality Factor Q

Magnification– Example

Example:

F=2; c=0.6; m=1; k=9

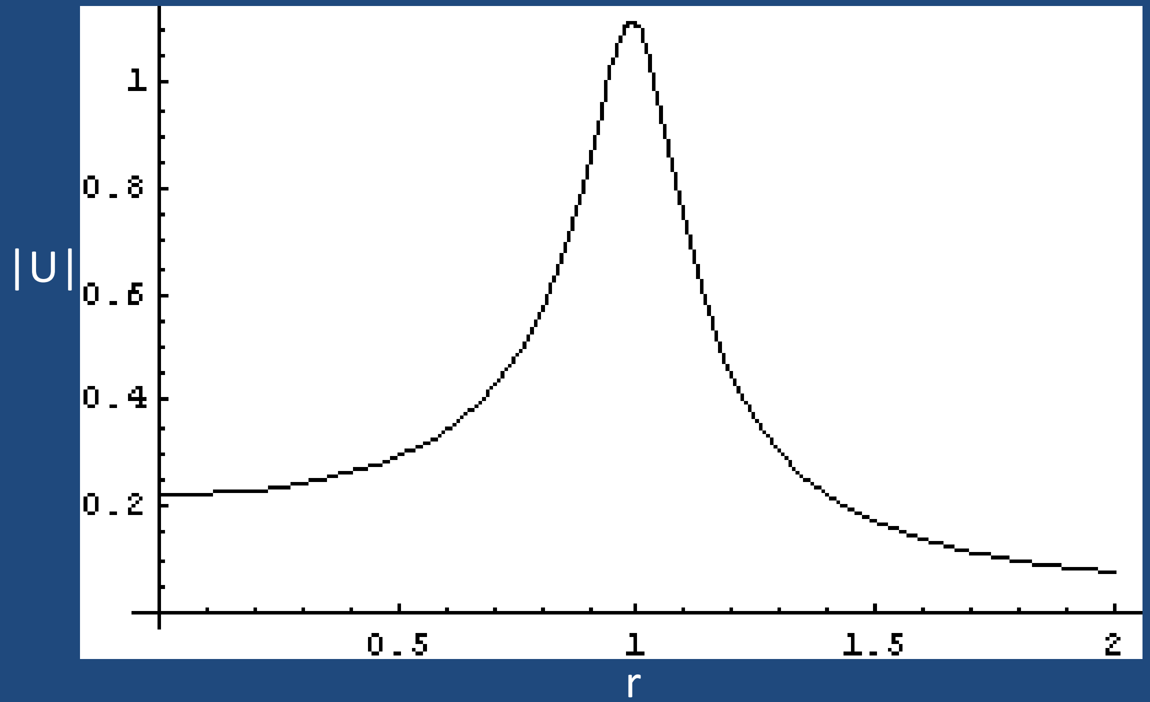
$\omega = \sqrt{\frac{k}{m}} = 3$

$\zeta = \frac{c}{2\sqrt{km}} = 0.1$

$U_{static} = \frac{F_o}{k} = 0.222$

At resonance, $|U| = Q\, U_{static} =$

$$|U| = \frac{1}{2\zeta}(.2222) = 1.111$$

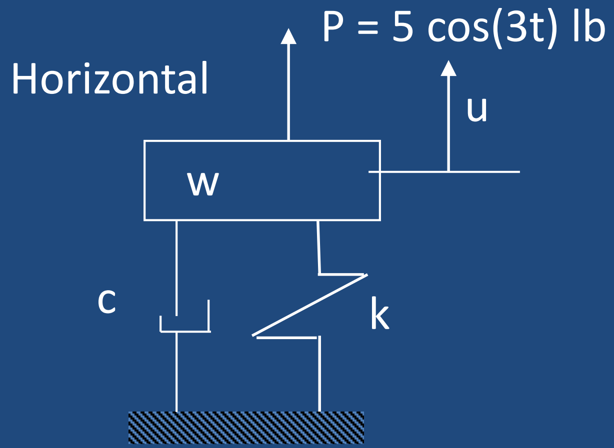


For $\Omega = 2.8$, $r = \Omega/\omega = \frac{2.8}{3} = .9333$, so

$$\left|\bar{H}(\Omega)\right| = \sqrt{\frac{1}{\left(1-r^2\right)^2 + \left(2\zeta r\right)^2}} = \sqrt{\frac{1}{\left(1-0.93333^2\right)^2 + \left(2*.1*0.93333\right)^2}} = 4.408$$

$$\phi = \tan^{-1}\left[\frac{-2\zeta r}{1-r^2}\right] = \tan^{-1}\left[\frac{-2(.1).9333}{1-.9333^2}\right] = \tan^{-1}\left(-\frac{0.01866}{0.12889}\right) = -.9665$$

Given Damped system shown below:



W= 386.1 lb
K= 4 lb/in
C= 0.3 lb-sec/in

$$c_{critical} = 2\sqrt{km}$$

a) What is the frequency of the steady-state dynamic response u(t)?

3 Rad/sec

b) If the load is applied statically, what is the displacement?

$$u_{static} = \frac{F}{k} = \frac{5}{4} = 1.25''$$

c) What is the approximate maximum response of the mass for any frequency of excitation?

$$c_{cr} = 2\sqrt{km} = 2\sqrt{4*1} = 4$$

$$\varsigma = {}^{c}\!/_{c_{cr}} = 0.3/4 = 0.075 \Rightarrow Q \cong \frac{1}{2\zeta} = \frac{1}{2*0.075} = 6.667$$

$$u_{dynamic} = Qu_{static} = 6.6667*1.25'' = 8.333''$$

d) At what approximate frequency does this occur?

$$\text{at } \omega; \quad \omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{4}{1}} = 2\,{}^{rad}\!/_{s}$$

e) Are there any excitation frequencies where the response would be close to the static response, and if so, what would be one of these frequencies?

Yes, frequencies < 0.5*ω, therefore < 1 rad/s

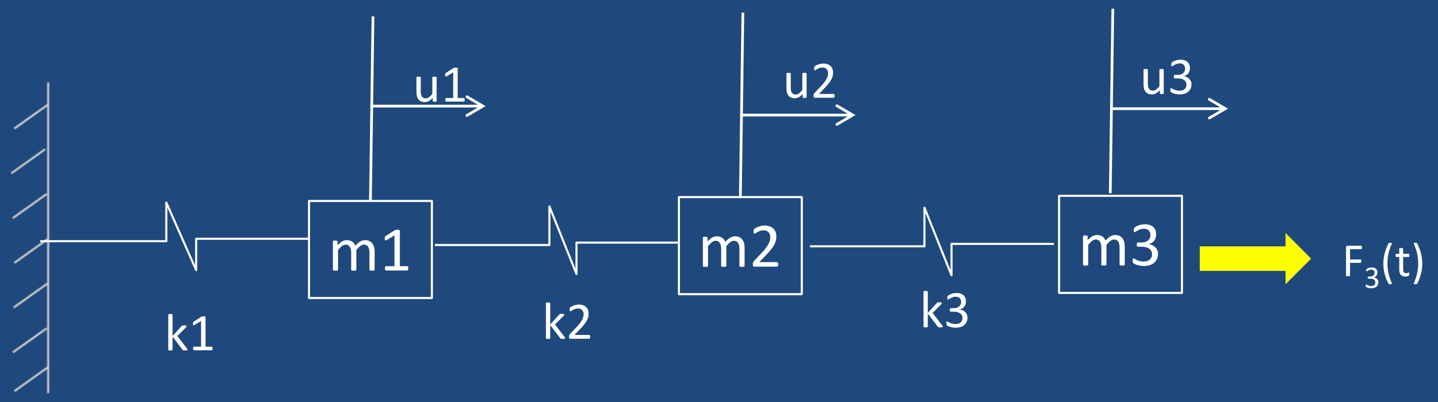
2) The Complex Frequency Response H is the ratio of the dynamic response to the___static response

as a function of _frequency ratio r=Ω/ω.

Multi-Degree-of-Freedom Modeling

- Structures discretized using finite elements (rigorous mathematical representation of little piece of a solid)
- Machines, other mechanical systems can be modeled fairly accurately with lumped parameter MDOF models (i.e., lumped rigid masses, massless springs & dampers).

- Equations of Motion (EoM)
  - Newton method: easiest to use for translational systems, but very difficult for rotational motion



Modal Analysis – Obtain Natural Frequencies and Modes

Solutions for Undamped, Free Vibration of MDOF Systems with N dof's.

$$[M]\{\ddot{u}\}+[K]\{u\}=\{0\}$$

Assume solution of form

$$\{u\}_m=\{\phi\}_m e^{i(\omega_m t+\alpha_m)}$$

m=1,M where M$\leq$N (can choose to "use" less than N solutions)

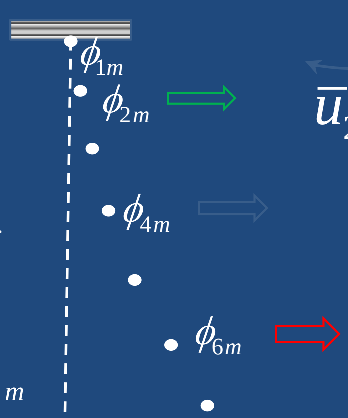
where m spatial solutions $\equiv$ Eigenvectors $\equiv$ Mode Shapes

Discrete MDOF modes

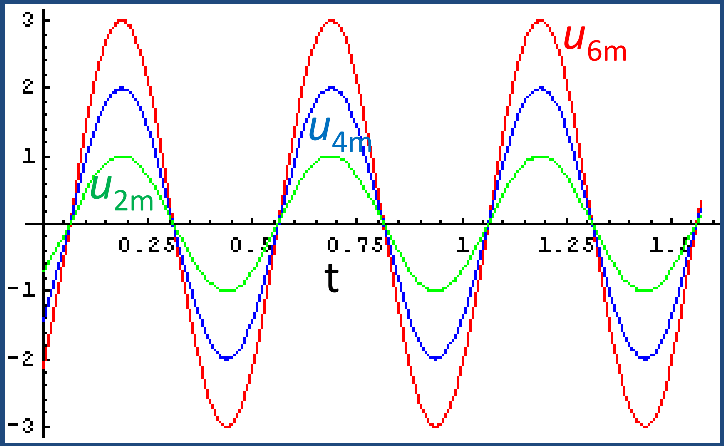
$\{\phi\}_m$

Alternate nomenclature: ϕ_{ij}, where *i* is dof, *j* is mode number

$$\begin{Bmatrix}\phi_1\\ \phi_2\\ \phi_3\\ \vdots\\ \phi_N\end{Bmatrix}_m$$



$$\bar{u}_{2m}(t)=\phi_{2m}e^{i(\omega_m t+\alpha_m)}$$

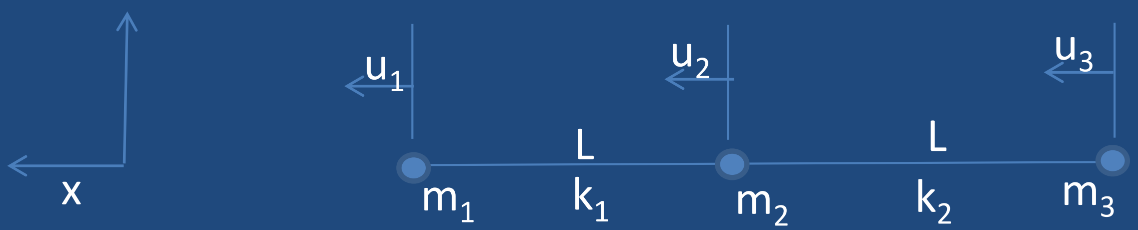


Example: Natural Frequencies and Modes of Axial Vibration of Cantilever Bar

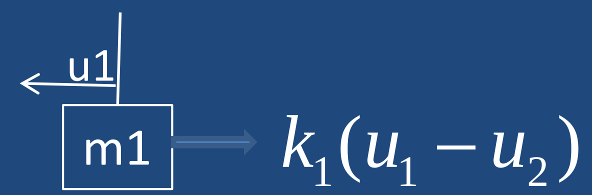
$E = 2.9e7$ lb/in^2

$$\rho = 0.1 \frac{\frac{lb-\sec^2}{in}}{in^3}$$

A) Discretize into 2 finite elements, draw coordinate system:



Write Eq’s of Motion:



$$-k_1(u_1 - u_2) = m_1\ddot{u}_1$$
$$m_1\ddot{u}_1 - k_1u_2 + k_1u_1 = 0$$
$$m_1\ddot{u}_1 + k_1u_1 - k_1u_2 = 0$$

Lump mass at nodes: $m_{element} = \rho A L$

So

$m_1 = \rho AL/2$

$m_2 = \rho AL$

$m_3 = \rho AL/2$

Element stiffness:

$k_1 = k_2 = AE/L$

$$k_1(u_1 - u_2) - k_2(u_2 - u_3) = m_2\ddot{u}_2$$
$$m_2\ddot{u}_2 + k_1u_2 - k_1u_1 - k_2u_3 + k_2u_2 = 0$$
$$m_2\ddot{u}_2 - k_1u_1 + (k_1 + k_2)u_2 - k_2u_3 = 0$$

$$k_2(u_2 - u_3) = m_3\ddot{u}_3$$
$$m_3\ddot{u}_3 - k_2(u_2 - u_3) = 0$$
$$m_3\ddot{u}_3 - k_2u_2 + k_2u_3 = 0$$

Finite Element Method Derivation p. 3

Equation of Motion in Matrix Form:

$$\frac{\rho AL}{2}\begin{bmatrix}1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{Bmatrix}\ddot{u}_1\\ \ddot{u}_2\\ \ddot{u}_3\end{Bmatrix}+\frac{AE}{L}\begin{bmatrix}1 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 1\end{bmatrix}\begin{Bmatrix}u_1\\ u_2\\ u_3\end{Bmatrix}=\begin{Bmatrix}0\\ 0\\ 0\end{Bmatrix}$$

Apply Boundary Conditions (BC's): Since u_3=0, can cross out corresponding row & column.

Write system matrix [D]=([K]-ω^2[M]), let λ=ω^2

$$[D]=\left(\frac{AE}{L}\begin{bmatrix}1 & -1\\ -1 & 2\end{bmatrix}-\lambda\frac{\rho AL}{2}\begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}\right)a$$

4) Divide through by AE/L;

Let

$$\mu=\frac{\frac{\rho AL}{2}}{\frac{AE}{L}}\lambda=\frac{\rho L^2}{2E}\lambda$$

then $[D]=\begin{bmatrix}1-\mu & -1\\ -1 & 2-2\mu\end{bmatrix}$

5) Set |D|=0

$$\begin{vmatrix} 1-\mu & -1 \\ -1 & 2-2\mu \end{vmatrix} = 0$$

$$(1-\mu)(2-2\mu)-1=0$$

$$2\mu^2-4\mu+1=0$$

6) Solve for roots μ_m; solve for ω_m

$$\mu_{1,2} = \frac{4\pm\sqrt{16-(4*2*1)}}{2*2} = 1\pm\frac{\sqrt{2}}{2}$$

$$\mu_1 = .2928 \Rightarrow \lambda_1 = \frac{2E}{\rho L^2}\mu = \frac{2(2.9e7\,\frac{lb}{in^2})}{0.1\frac{lb\sec^2}{in^4}(480in)^2}0.2928 = 737.1 \Rightarrow \omega_1 = 27.15\frac{Rad}{\sec}$$

$$\mu_2 = 1.707 \Rightarrow \lambda_2 = \frac{2(2.9e7\,\frac{lb}{in^2})}{0.1\frac{lb\sec^2}{in^4}(480in)^2}1.707 = 4297.5 \Rightarrow \omega_2 = 65.5\frac{Rad}{\sec}$$

7) Plug μ_i in to $[D]\{\phi\}=0 \Rightarrow \{\phi\}$, use "max" normalization

$$\mu_1 : (1-0.2928)\phi_{11} - \phi_{21} = 0$$

$$.7071\phi_{11} - \phi_{21} = 0 \Rightarrow \phi_{21} = .7071\phi_{11}$$

$$\text{Let } \phi_{11} = 1 \Rightarrow \phi_{21} = .7071 \Rightarrow \{\phi\}_1 = \begin{Bmatrix} 1 \\ .7071 \end{Bmatrix}$$

2nd equation in $[D]\{\phi\}=0$ gives same thing (not linearly independent)

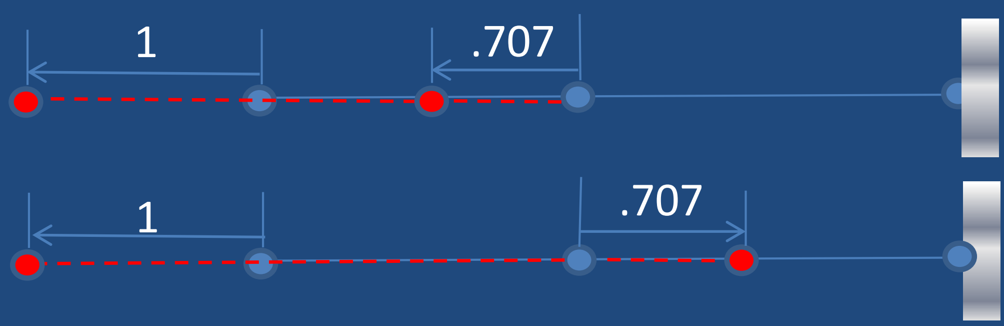
$$\mu_2 : (1-1.7071)\phi_{12} - \phi_{22} = 0$$

$$-.7071\phi_{12} = \phi_{22}$$

$$\text{Let } \phi_{12} = 1 \Rightarrow \phi_{22} = -.7071 \Rightarrow \{\phi\}_2 = \begin{Bmatrix} 1 \\ -.7071 \end{Bmatrix}$$

$$\therefore [\Phi] = \begin{bmatrix} 1 & 1 \\ .7071 & -.7071 \end{bmatrix}$$

Max normalized mode shapes are useful for visualization



Mode 1, both dof's IN-PHASE

Mode 2, both dof's OUT-OF-PHASE (slinky)

8) If desired, obtain Mass Normalized Mode Shapes

s_1= 0.10206 and s_2=0.10206, so

$$[\Phi]_{mass} = \begin{bmatrix} .10206 & .10206 \\ .07216 & -.07216 \end{bmatrix}$$

Natural Frequencies, Modes, & Modal Matrix

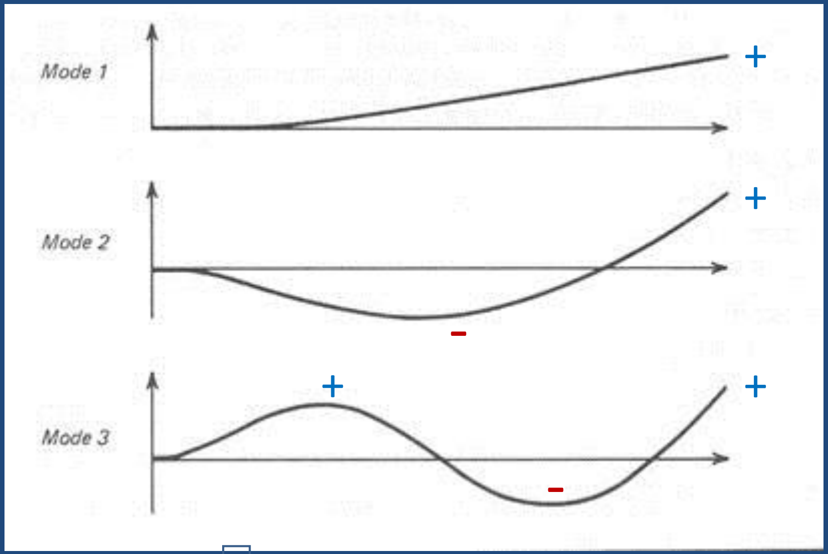
See great animations of MDOF systems by Dr. Dan Russell, Graduate Program in Acoustics, Penn State. Or https://www.youtube.com/watch?v=kvG7OrjBirI

Eigenvalue λ = natural frequency ω^2

$$\lambda_1 = \omega_1^{\ 2} \Rightarrow \{\phi\}_1$$

$$\lambda_2 = \omega_2^{\ 2} \Rightarrow \{\phi\}_2$$

$$\lambda_3 = \omega_3^{\ 2} \Rightarrow \{\phi\}_3$$



Displacements for all locations of mode shape are either in-phase or 180^{o} out-of-phase with each other, but have phase relationship of α_i with excitation.

$$\text{Modal Matrix } \Phi = \left[\{\phi\}_1 \quad \{\phi\}_2 \quad \{\phi\}_3 \right] = \begin{bmatrix} \phi_{11} & \cdots & \phi_{13} \\ \vdots & \ddots & \vdots \\ \phi_{n1} & \cdots & \phi_{n3} \end{bmatrix}$$

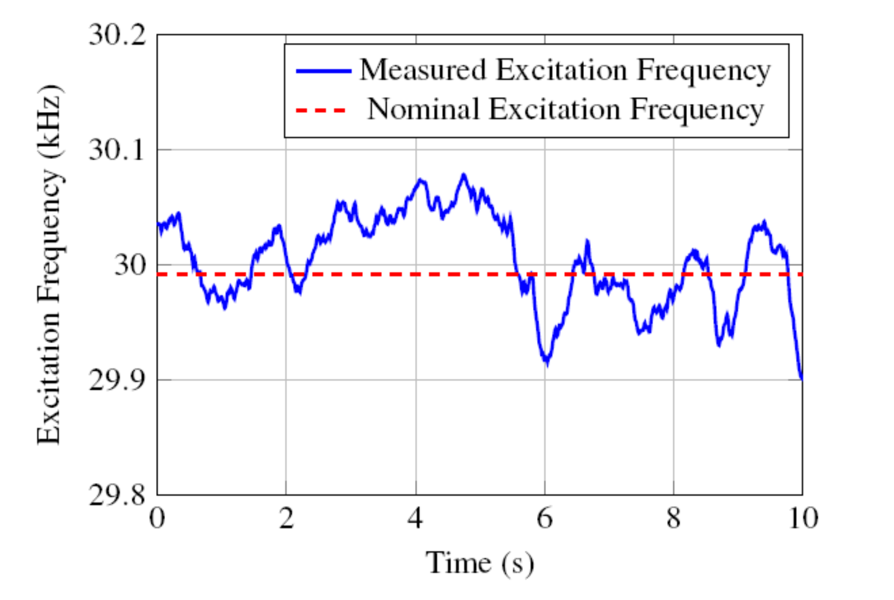
Application of Structural Dynamics to Turbomachinery

Characterization of Excitations – Speed Range

- First obtain speed range of operation from performance group.
  - For Rocket Engines, there are generally several “nominal” operating speeds dependent upon phase of mission (e.g., reduce thrust during “Max Q”).
  - However, since flow is the controlling parameter, actual rotational speeds are uncertain (especially during design phase)
  - For new LPS engine being built at MSFC, assuming possible variation +/-5% about each of two operating speeds.

Rated Power Level	70%	100%
Low Range	20759.4	26125
Nominal	21852	27500
High Range	22944.6	28875

In addition, speed generally isn’t constant, but instead “dithers”.†

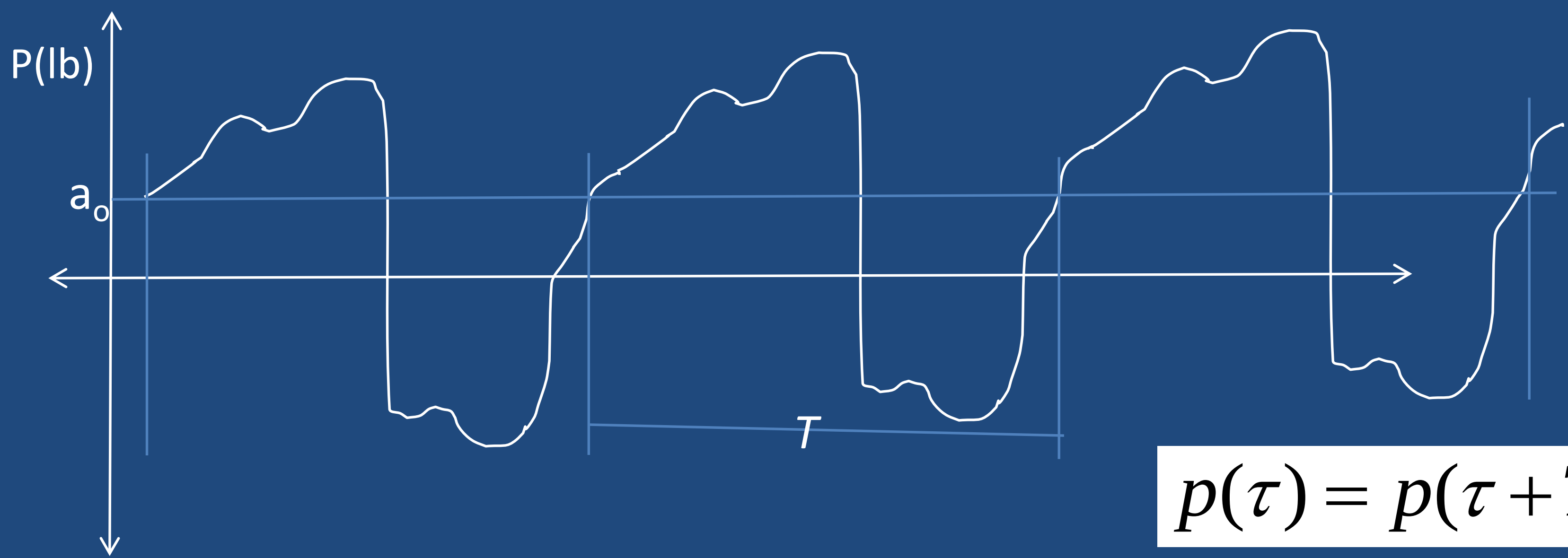


†**Implementation of Speed Variation in the Structural Dynamic Assessment of Turbomachinery Flow Path Components**
Andrew M. Brown, R. Benjamin Davis and Michael K. DeHaye
J. Eng. Gas Turbines Power 135(10), 102503 (Aug 30, 2013) Paper No: GTP-13-1206; doi: 10.1115/1.4024960

Quantify Engine Forces using Fourier Analysis

- Forces are not, in general, perfect sine waves (although sometimes they're close)
- We can deal with these in two ways:
  - Represent forces as sum of Sines (Spectral, Frequency Domain Approach), sum response to each Sine
  - Calculate response to actual temporal (time history) loading using "impulse response function"

- Spectral Approach: given a periodic but non-harmonic excitation



$$p(\tau) = p(\tau + T)$$

Fourier Analysis

- Jean Fourier realized we can write loading p(t) as sum of average, cosines, & sines:

$$p(t) = a_o + \sum_{n=1}^{\infty}[a_n \cos(n\Omega_1 t) + b_n \sin(n\Omega_1 t)]$$

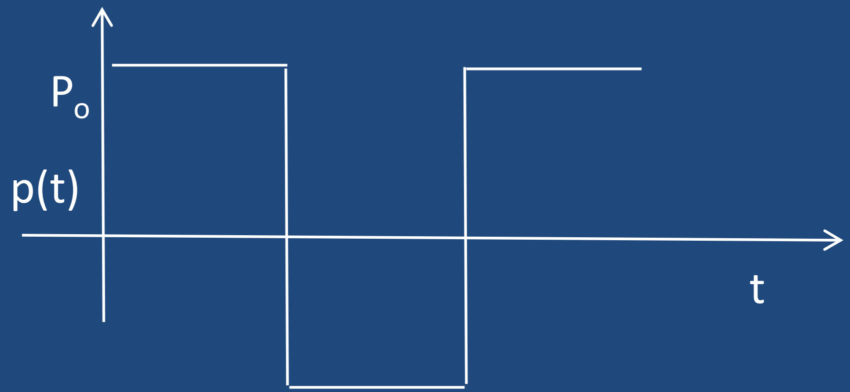
where

$$a_0 = \frac{1}{T_1}\int_{\tau}^{\tau+T} p(t)dt = \text{avg value of p(t)}$$

$$a_n = \frac{2}{T_1}\int p(t)\cos(n\Omega_1 t)dt$$

$$b_n = \frac{2}{T_1}\int p(t)\sin(n\Omega_1 t)dt$$

- Textbook Example: Using Fourier Series, represent square wave excitation as:

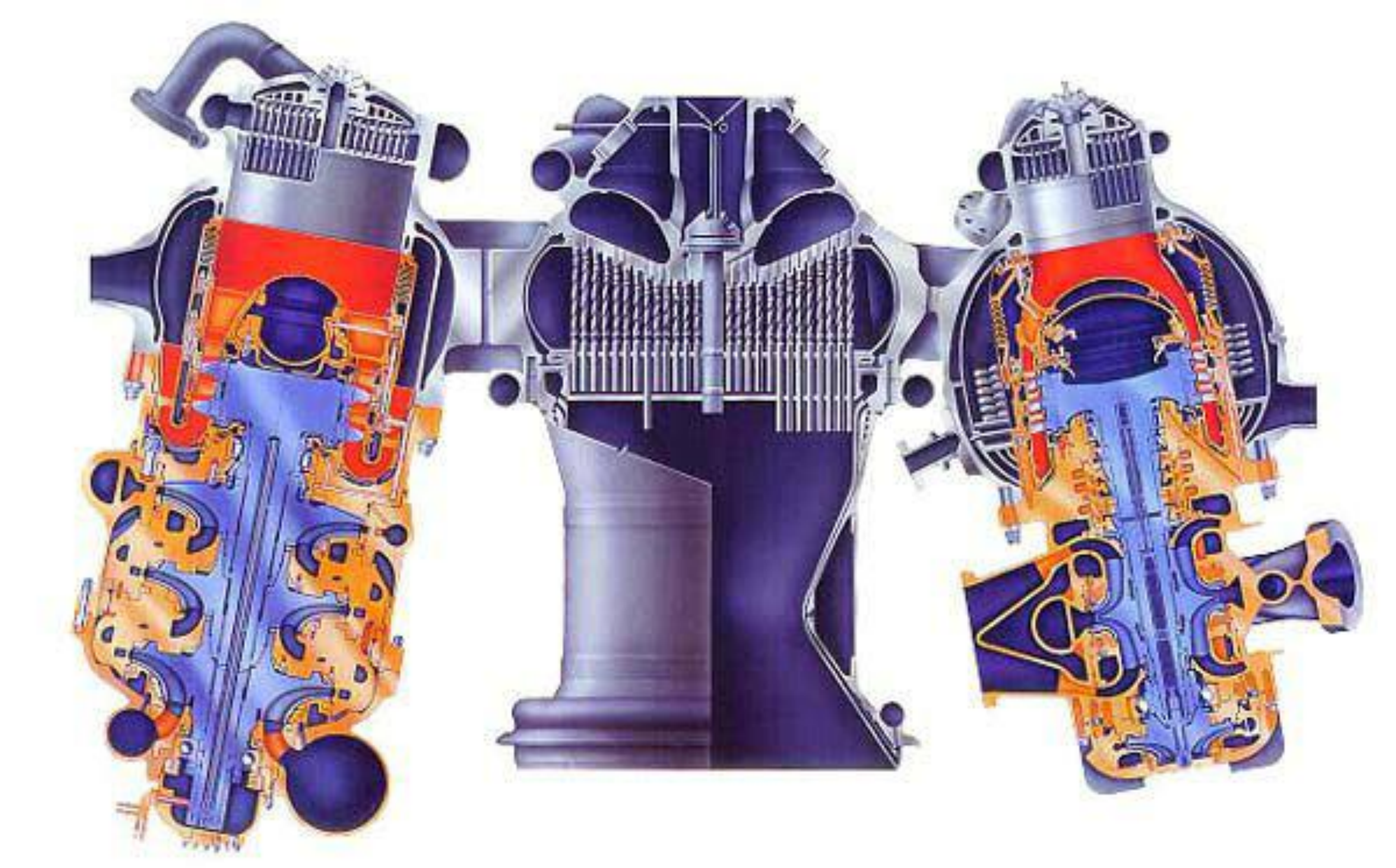


$$p(t) = \frac{4p_0}{\pi}\sum_{n=1,3,5,\ldots}\left(\frac{1}{n}\right)\sin(n\Omega_1 t)$$

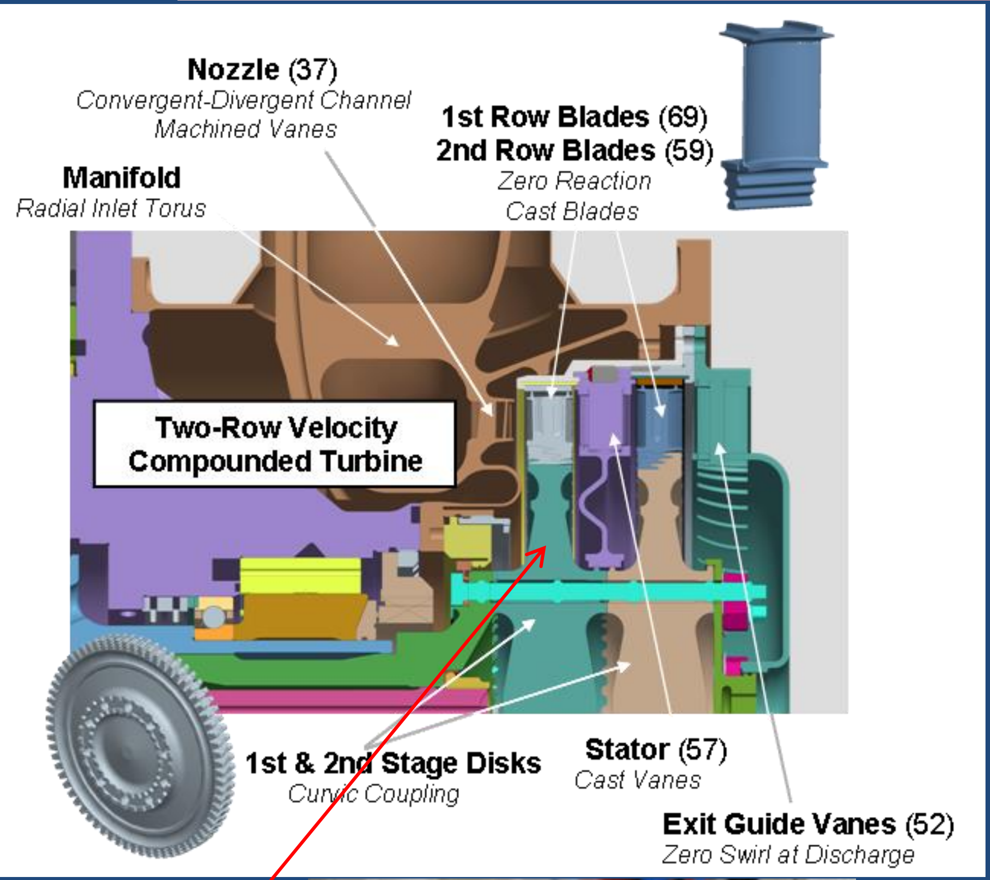
Characterization of Mechanical Excitation due to Unbalance

- Since all real machinery has some level of unbalance and shaft whirl, sinusoidal ("harmonic") loads at 1 and 2 times rotational speed ("1 and 2N") will be generated, along with up to their 3rd multiples (also called "harmonics") ➡ 3-6N throughout turbopump.

Space Shuttle Main Engine Powerhead Cross-Section

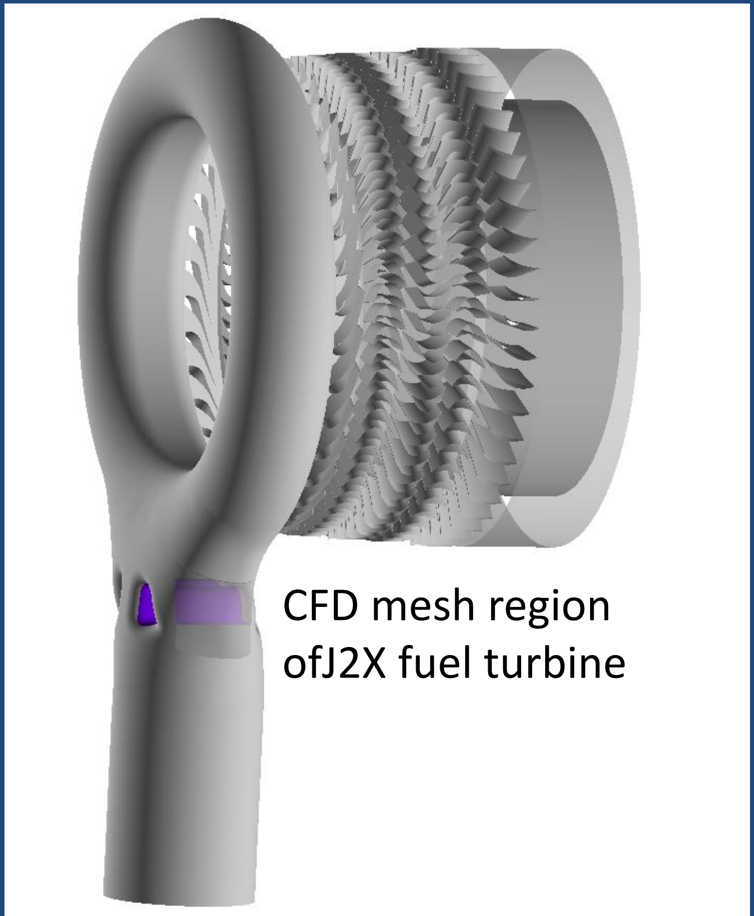


Characterization of Fluid Excitation



Bladed disk

- Harmonic excitation at engine order = Number of flow distortions and up to their 3rd multiples arising from adjacent upstream and downstream blade and vane counts.
  - Use CFD to generate Loading



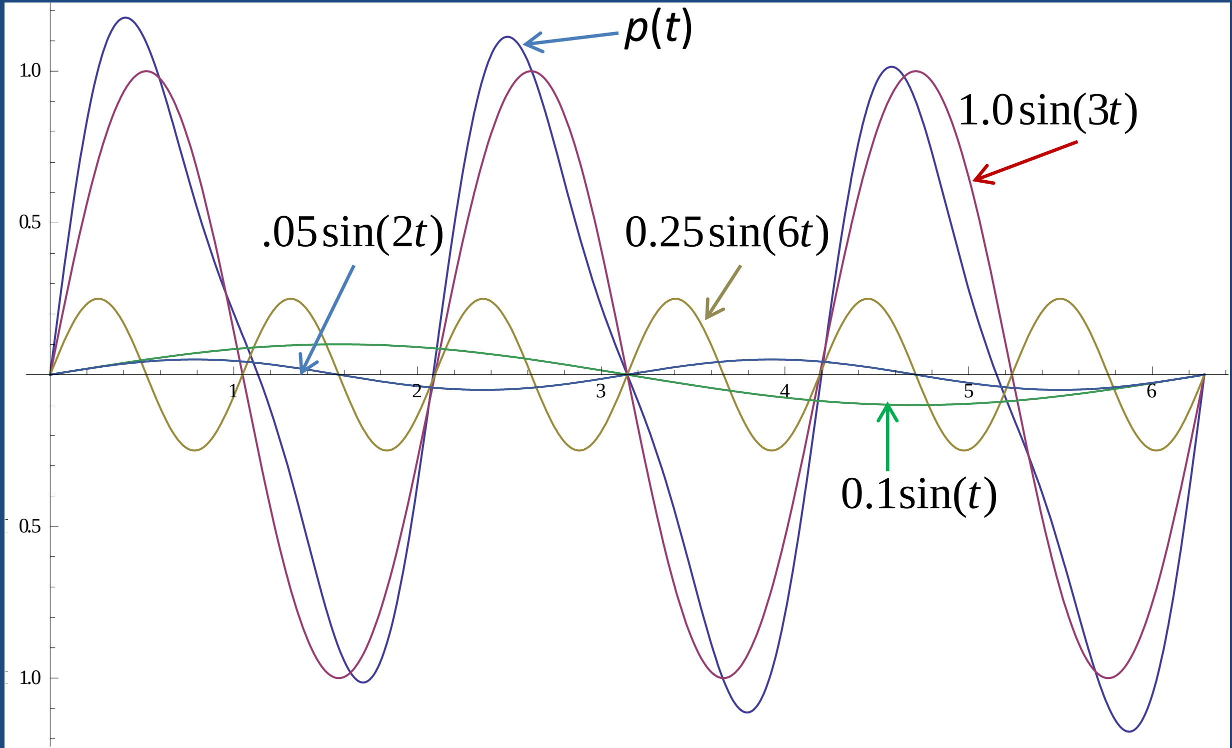
CFD mesh region ofJ2X fuel turbine

Engine Example of Application of Fourier Series

- Excitation wave based upon a pump with 3 primary distortions (e.g. diffusers), within slightly asymmetric overall field.
- Let primary excitation at 3N have an amplitude of 1, and asymmetric primary distortion have an amplitude of 0.1.
  - each of these will have a harmonic, since they aren't perfect sinusoidal distortions, such that the harmonic of the asymmetric is 0.05, and the amplitude of the harmonic of the primary distortion is 0.25.
  - So $p(t) = b_1 Sin(t) + b_2 Sin(2t) + b_3 Sin(3t) + b_6 Sin(6t)$

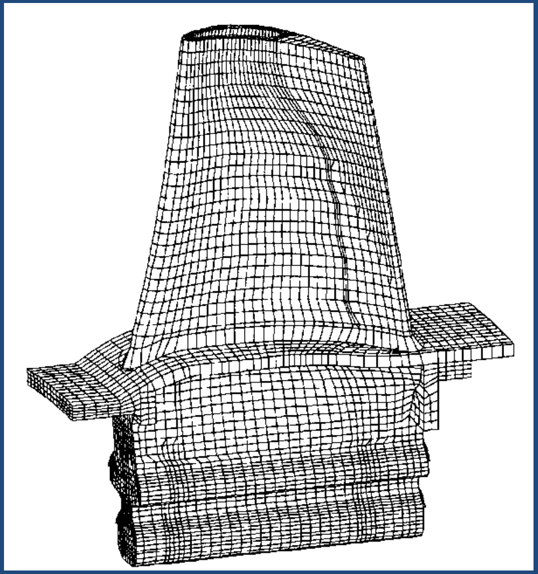
Where
b_1=0.1, b_2=0.05,
b_3=1.0, b_6=0.25

- Have to assess dynamics for each frequency component of excitation.

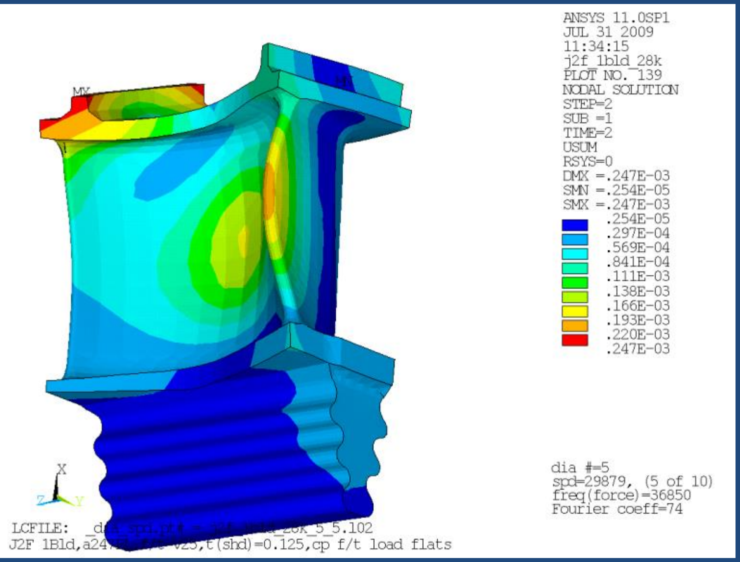


Now Structure: Create FEM of component, Modal Analysis

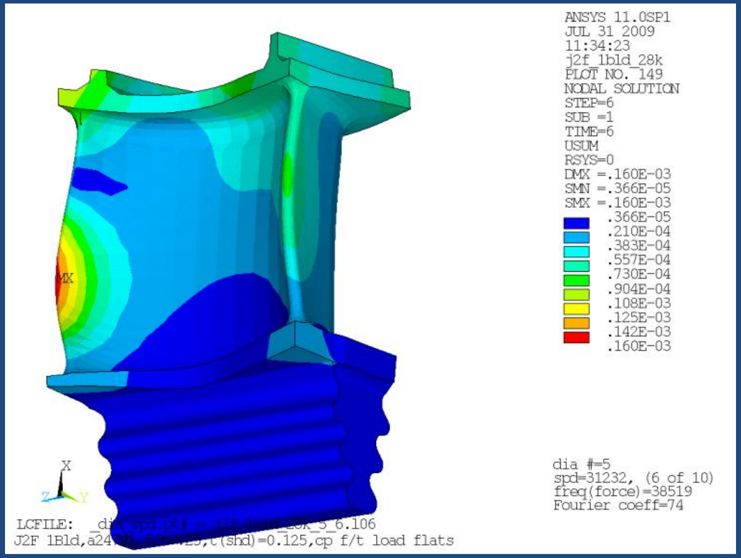
Example: Turbine Blades



Mode 12 at 36850 hz



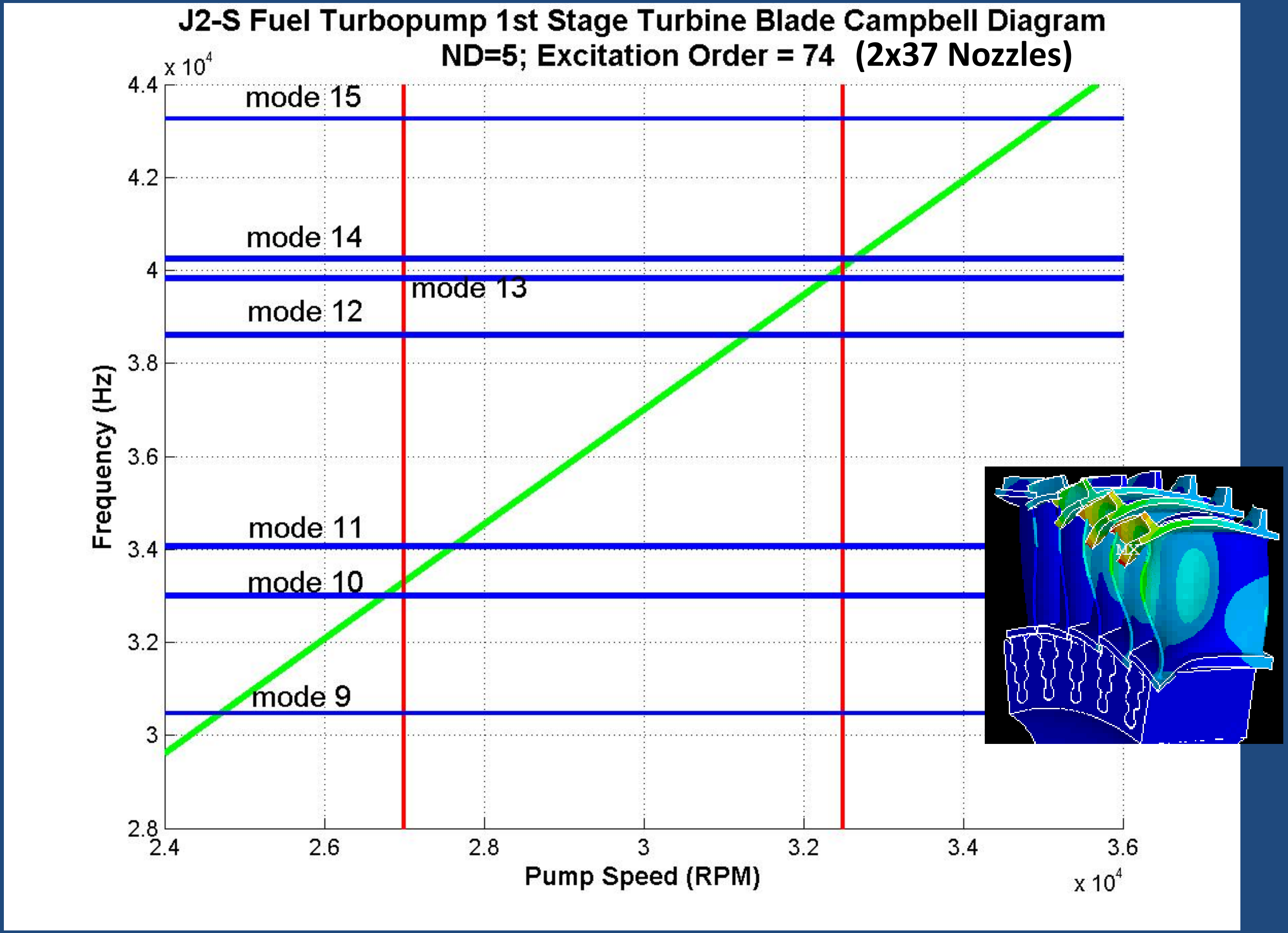
Mode 13 at 38519 hz



<u>Modal Animations very useful</u> for identifying problem modes, optimal damper locations

Create "Campbell Diagram"

- Simplest Version of Campbell Diagram is just a glorified Resonance Chart.

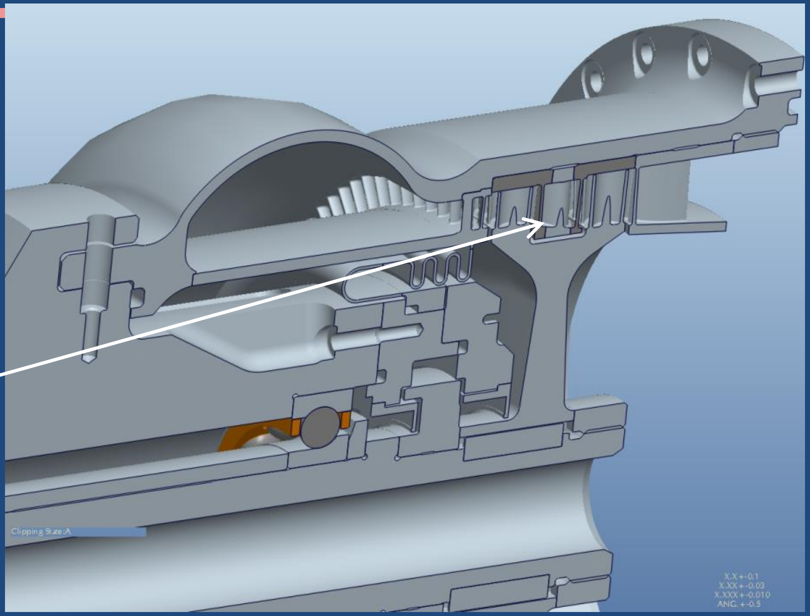


Modal Analysis has Multiple Uses

- Redesign Configuration to move excitations ranges away from natural frequencies
- Redesign component to move resonances out of operating range.
- Put in enough damping to significantly reduce response
- Use as first step in “Forced Response Analysis” (applying forces and calculating structural response).

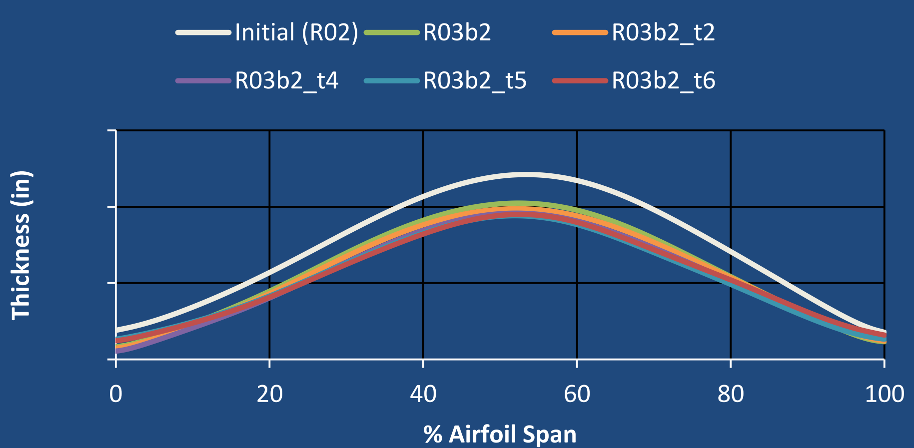
LPSP Turbine Stator Redesign to Avoid Resonance

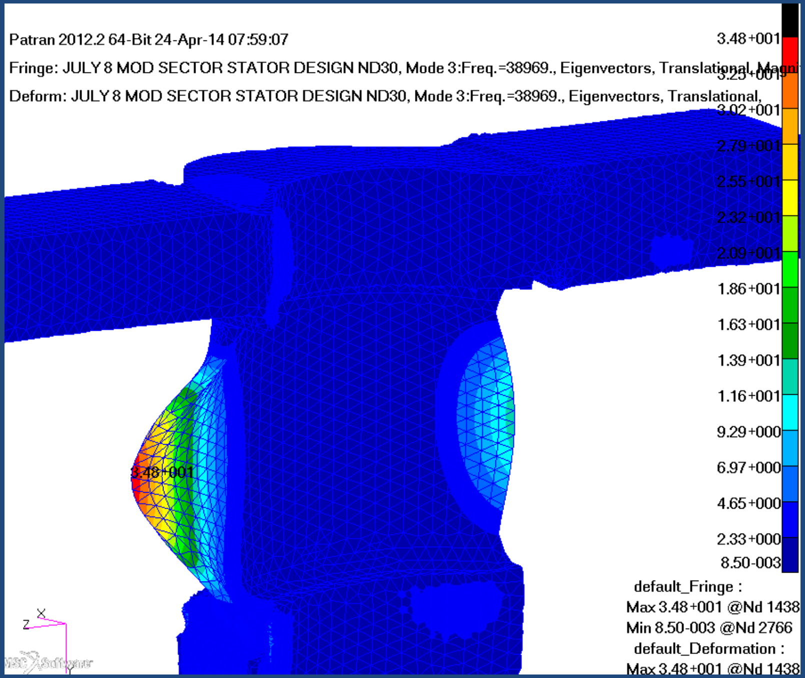
- Modal analysis of original design indicated resonance with primary mode by primary forcing function.
  - Since excitation simultaneously from upstream and downstream blades, critical to change design to avoid resonance.
  - Extensive optimization effort performed to either move natural frequency out of range and/or change count of turbine blades to move excitation.



Courtesy D. O'N

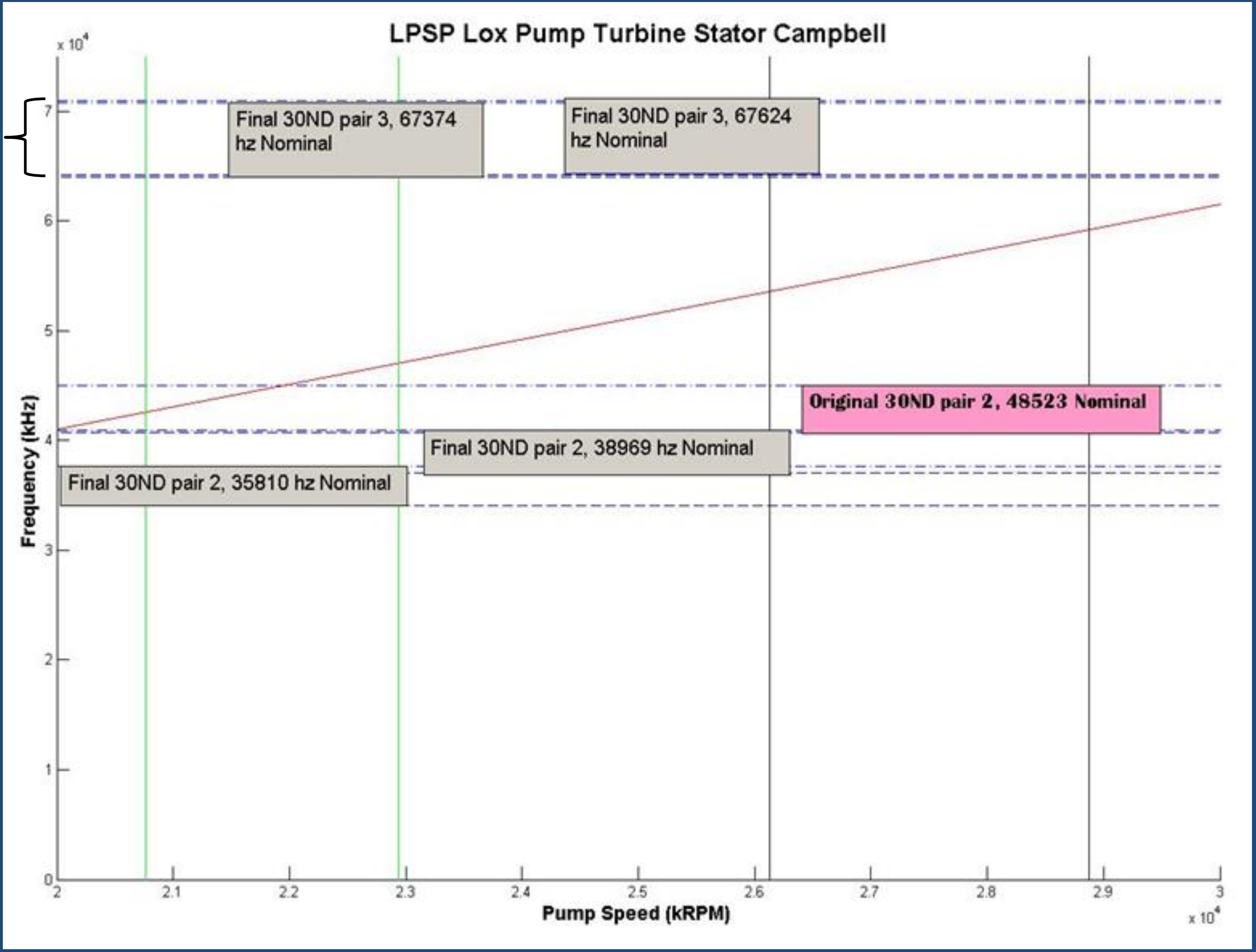
Stator Airfoil Thickness Changes





Final and Original Campbell of Modes for Stator Vane 30ND Family

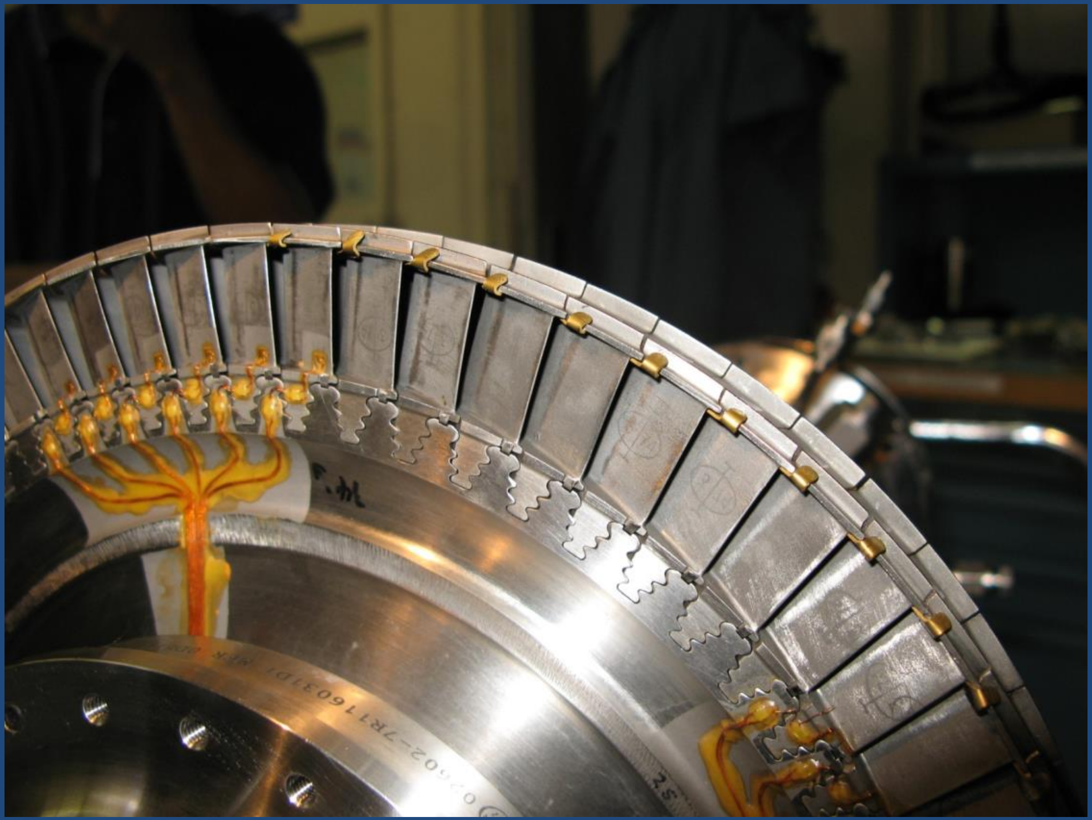
Range of +/- 5% on natural frequencies to account for modeling uncertainty



Damping

A. Brown
MSFC Propulsion
Structural Dynamics

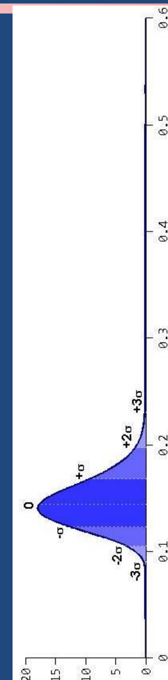
- Damping is critical parameter for forced response prediction, so "whirligig" test program used to obtain data.
- Whirligig is mechanically-driven rotor with bladed-disk excited by pressurized orifice plate simulate blade excitation.
- Key assumption is that this reflects true configuration.
- SDOF Curve fit technique applied to selected top-responding blades to derive damping from response.

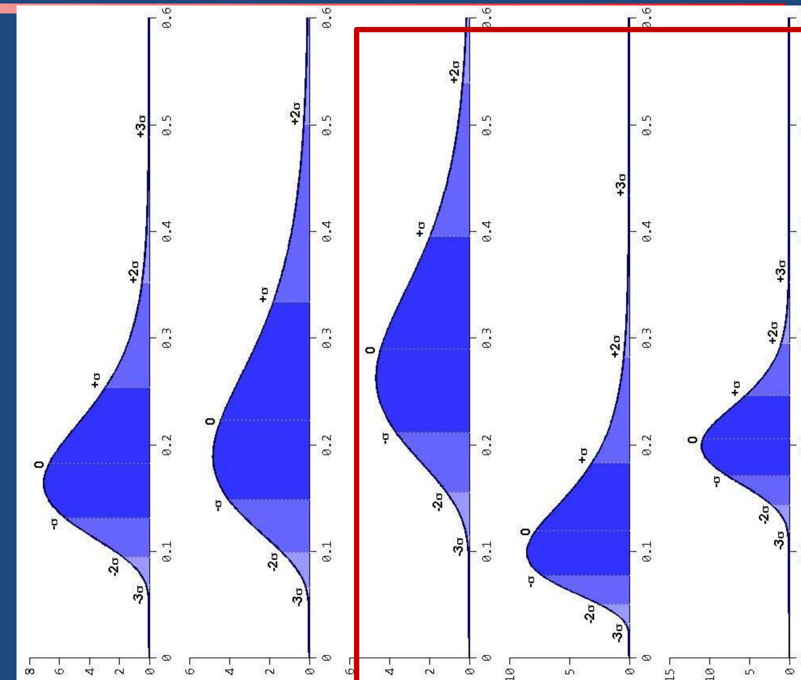


A. Brown
MSFC Propulsion
Structural Dynamics

Damping Results from Whirligig

- Data shows wide-variation in damping, but reasonable population (15-20 acceptable samples) for statistical characterization.
- Lognormal distribution fits obtained for each mode.

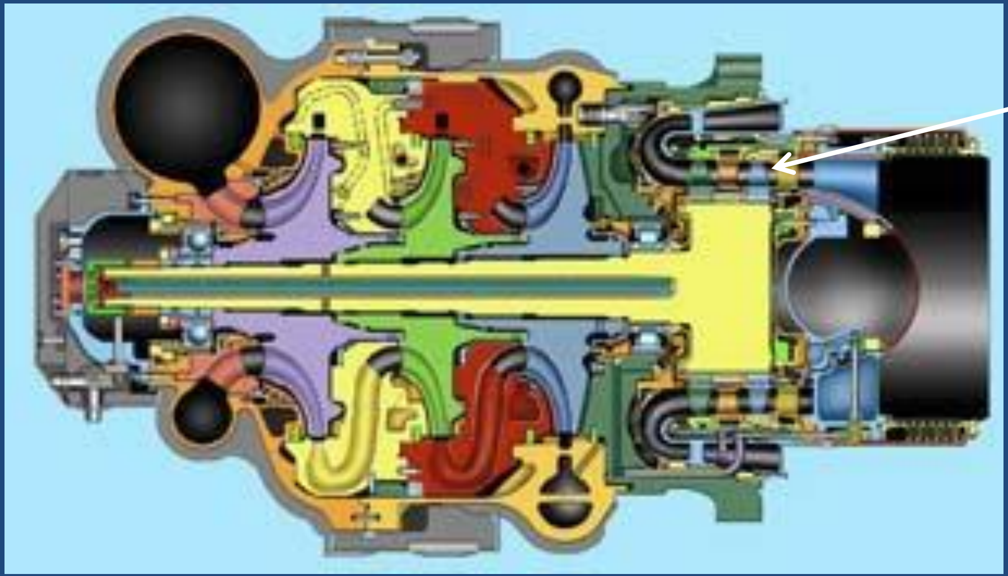




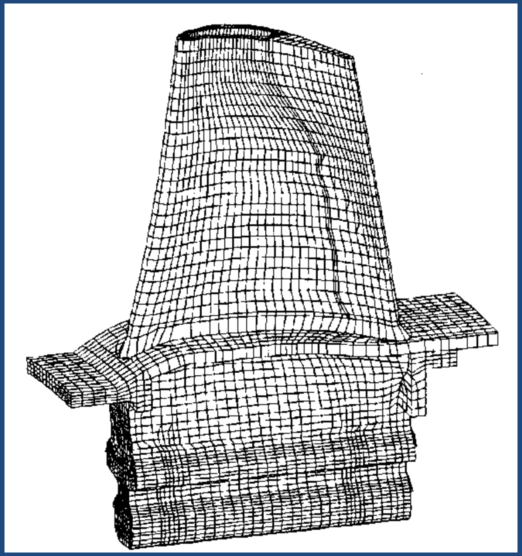
	Nodal Diameter	5	5	5	5	5	5	5	5	5	5
	Mode	3	4	5	6	7	8	9	10	11	12
	Samples	18	17		17		14	12	8	16	20
Amp	Mean	15.6	7.8		20.7		18.9	13.5	6.0	43.5	15.4
	Sigma	3.2	1.9		9.2		18.6	8.4	0.9	17.7	3.2
	Min	9.9	5.0		7.4		5.4	6.1	5.0	23.8	12.4
	Max	20.3	11.2		35.4		54.2	33.6	7.7	87.7	24.1
Freq	Mean	10967	13831		23068		28867	30588	32998	34643	37191
	Sigma	17	69		282		345	211	256	220	132
	Min	10936	13695		22921		28446	30165	32497	34357	37056
	Max	10997	13908		23816		29662	30907	33311	35013	37346
Zeta	Mean	0.404	0.702		0.146		0.193	0.242	0.304	0.131	0.209
	Sigma	0.103	0.163		0.023		0.065	0.102	0.097	0.059	0.038
	Min	0.314	0.520		0.106		0.116	0.139	0.162	0.078	0.153
	Max	0.720	0.976		0.191		0.348	0.450	0.423	0.325	0.293
	LogNormal Dist.:										
	0σ Equivalent	0.391	0.684		0.144		0.183	0.223	0.290	0.119	0.206
	-σ Equivalent	0.305	0.544		0.123		0.132	0.149	0.212	0.078	0.172
	-2σ Equivalent	0.237	0.433		0.105		0.095	0.099	0.155	0.051	0.143
	-3σ Equivalent	0.184	0.343		0.090		0.068	0.066	0.113	0.033	0.119

Can Also Use Modal Analysis in Failure Investigations

- Examination of *Modal Stress* Plots provides link to location of observed cracking.



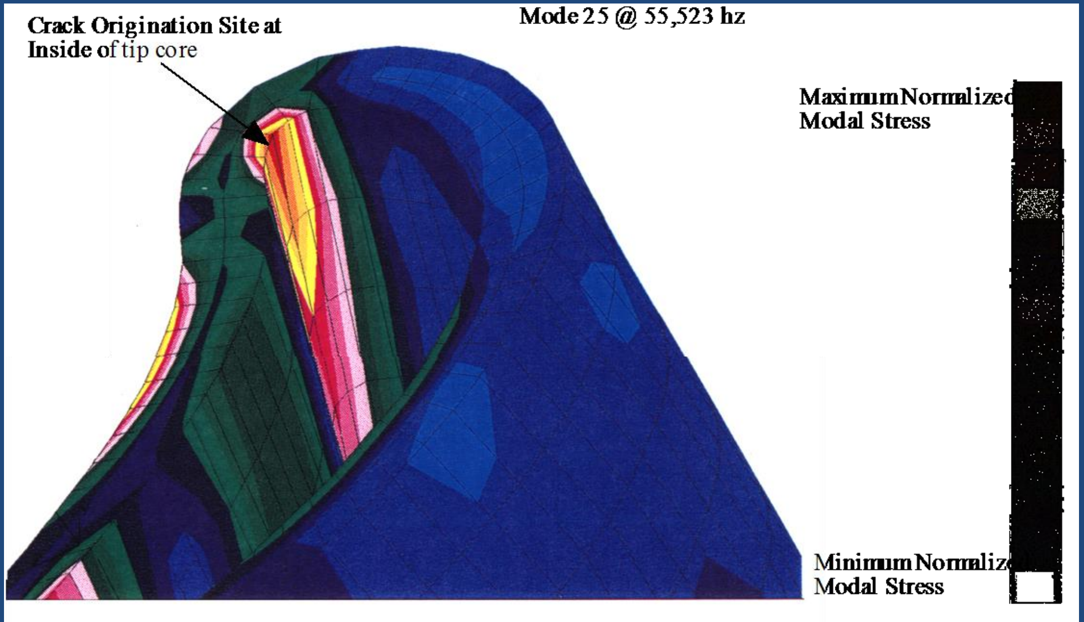
SSME HPFTP 1st Stage Turbine Blade



Modal displacement

Modal stress

$$\phi^m = \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_N \end{Bmatrix}^m \rightarrow \phi_\sigma^m = \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \vdots \\ \sigma_N \end{Bmatrix}^m$$



Generalized Coordinates/Modal Superposition

- Frequency and Transient Response Analysis uses Concept of Modal Superposition using Generalized (or Principal Coordinates).

$$[M]\{\ddot{u}\}+[C]\{\dot{u}\}+[K]\{u\}=\{P(t)\}$$

- <u>Mode Superposition Method</u> – transforms to set of uncoupled, SDOF equations that we can solve using SDOF methods.
- First obtain $[\Phi]_{mass}$. Then introduce coordinate transformation:

$$\{u\} = {}_{N}\overset{M}{[\Phi]}\{\eta\}_{M}$$

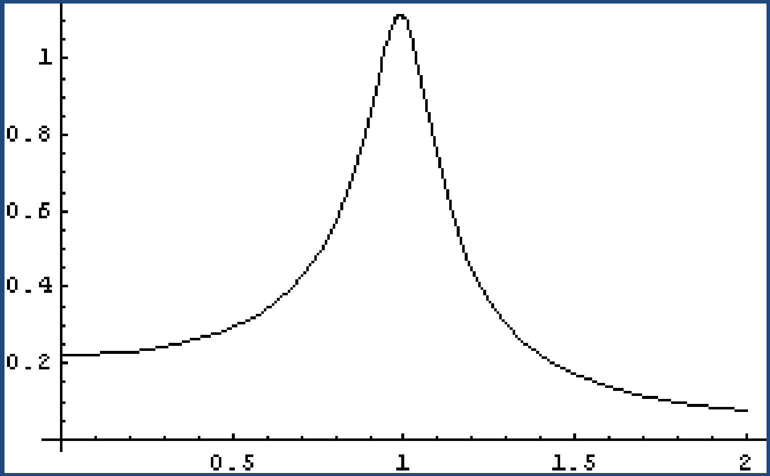
$$[M][\Phi]\{\ddot{\eta}\}+[C][\Phi]\{\dot{\eta}\}+[K][\Phi]\{\eta\}=\{P(t)\}$$

$$\left[{}^{\backslash} I_{\backslash}\right]\{\ddot{\eta}\}+\left[{}^{\backslash} \mathcal{C}_{\backslash}\right]\{\dot{\eta}\}+\left[{}^{\backslash} \kappa_{\backslash}\right]\{\eta\}=[\Phi]^{T}\{P(t)\}.$$

for the SDOF equation of motion,

$$m\ddot{u}+c\dot{u}+ku=F \;\rightarrow \ddot{u}+2\varsigma\omega\dot{u}+\omega^2 u=F$$

$$\left|U(\Omega)\right| = {Fo}/{k}\sqrt{\frac{1}{\left(1-\left(\frac{\Omega}{\omega}\right)^2\right)^2+\left(2\zeta\left(\frac{\Omega}{\omega}\right)\right)^2}}$$



$$r=\frac{\Omega}{\omega}$$

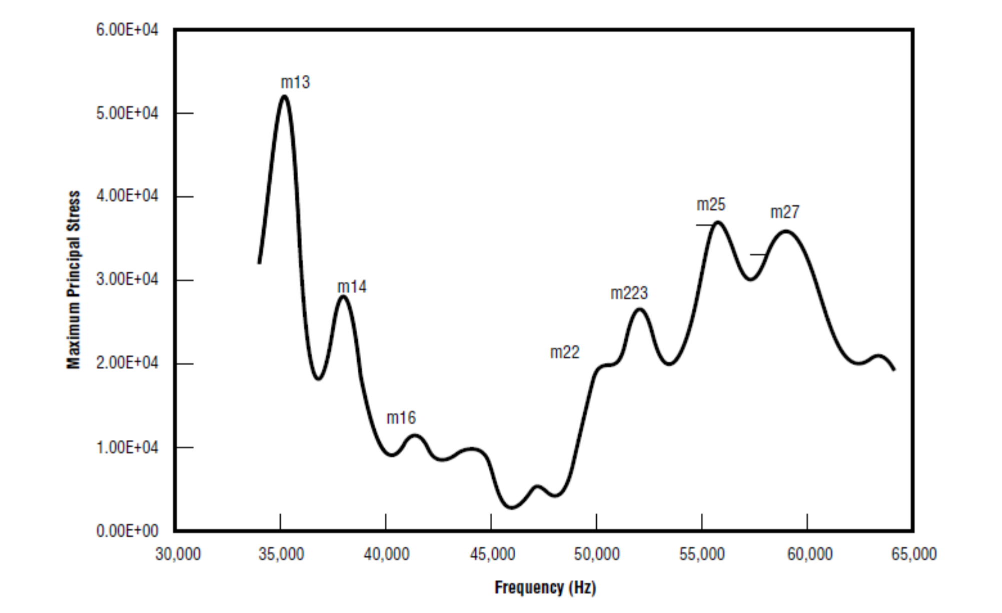
So we get the same equations in η:

$$\ddot{\eta}_m+2\zeta_m\omega_m\dot{u}_m+\lambda_m\eta_m=\{\phi\}_m^T\{P(t)\}$$

$$\left|\eta_m(t)\right| = \frac{\{\phi\}_m^T\{F\}}{\lambda_m}\frac{1}{\sqrt{\left(1-\left(\frac{\Omega}{\omega_m}\right)^2\right)^2+\left(2\zeta_m\frac{\Omega}{\omega_m}\right)^2}}$$

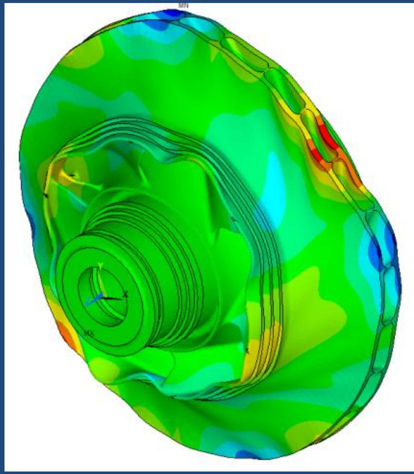
- For "Frequency Response" Analysis, apply Fourier coefficients coming from CFD such that excitation frequencies match Campbell crossovers.

SSME HPFTP 1st Blade Frequency Response

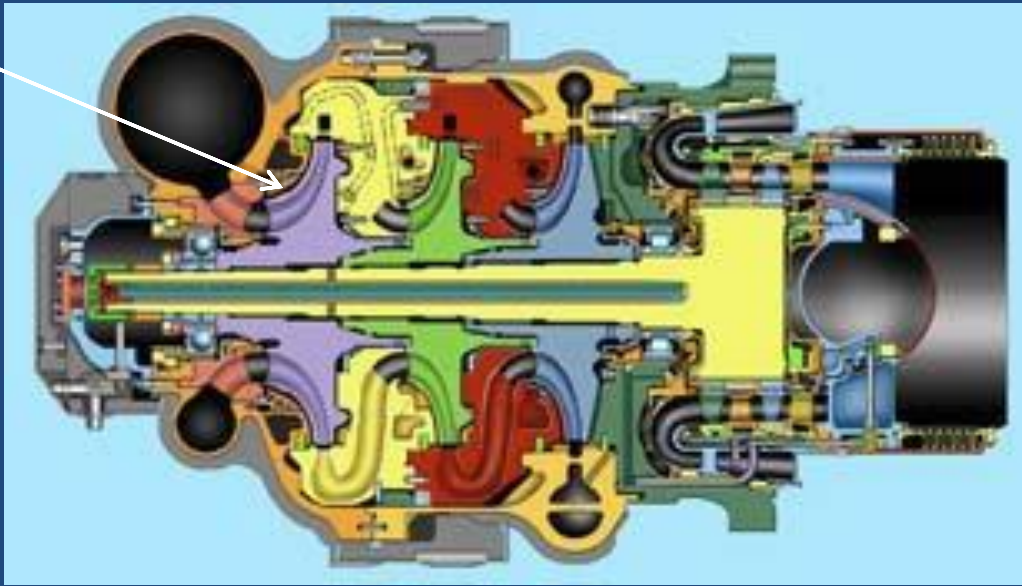


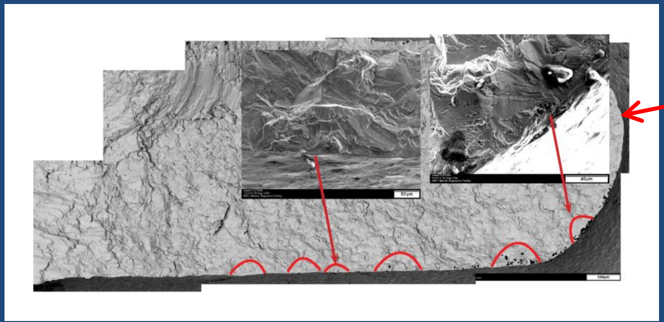
Forced Response Analysis in Failure Investigations

- SSME HPFTP 1st Stage Impeller.



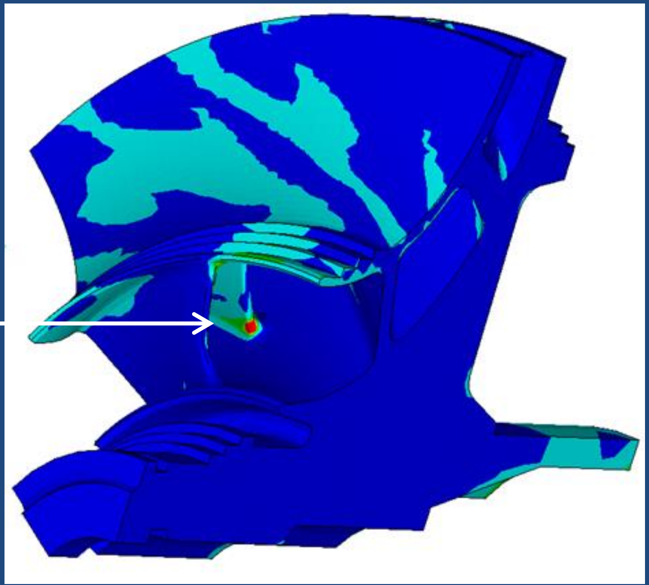
Mode shape





Crack location 1st splitter

Frequency Response Analysis



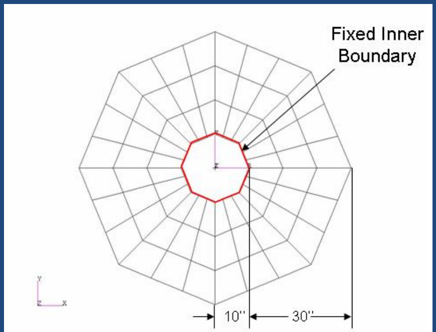
Cyclic Symmetry in Turbine Components

- Many structures possess some kind of symmetry that can be used to simplify their analysis.
- A cyclically symmetric structure possesses rotational symmetry, i.e., the original configuration is obtained after the structure is rotated about the axis of symmetry by a given angle.
- Instead of modeling entire structure, only model one sector.
- For turbomachinery structures, structural analysis is generally <u>only possible</u> by taking advantage of huge reduction in model size by using cyclic symmetry.

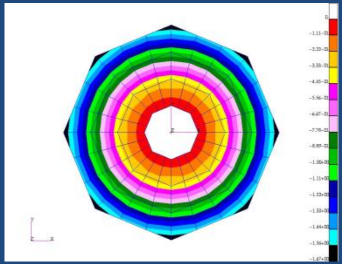
Section courtesy of Dr. Eric Christensen, DCI Inc.

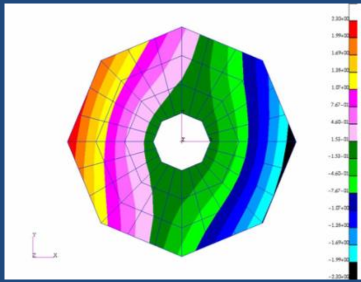
Characteristics of Cyclic Symmetric Modes

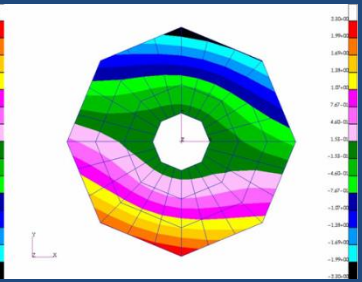
- Most Nodal Diameter modes exist in pairs, same shape but rotated by π/ND
- For <u>purely cyclic symmetric sections</u>, highest Nodal Diameter possible is N/2 for even # sectors, (N-1)/2 for odd # sectors.
- First family of modes:
  - Has unique eigenvalues
  - Has unique eigenvectors
  - All segments have same mode shape
- The next family of modes:
  - Pairs of degenerate (identical) eigenvalues
  - Non-unique eigenvectors
- The last family of modes:
  - Only exist if N is even
  - Has unique eigenvalues & eigenvectors.

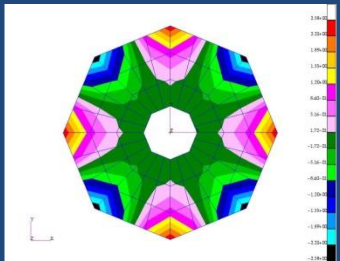


Flat Plate Example N = 8 Segments



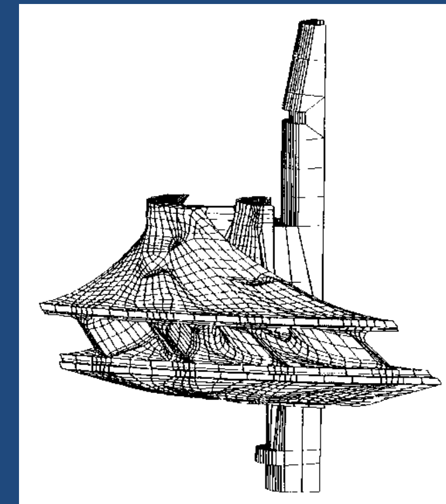






Example - SSME Fuel Turbopump 3rd Stage Impeller

- Cyclically symmetric sections have N = 6, so max ND=N/2=3 for those sections.
- However, much of impeller is disk-like, with axisymmetry, having infinite # ND's.



- Codes generate "Harmonic Families" of modes which only contain Nodal Diameter modes according to following stair-step pattern:
  - H0 contains ND= 0, 7, 14, …
  - H1 contains ND= 1, 6 ,8, 13, 15,…
  - H2 contains ND= 2, 5, 9, 12, 16,…
  - H3 contains ND= 3, 4 ,10,11, 17,

Mode	Natural Frequencies (Hz)			
	Harmonic 0	Harmonic 1	Harmonic 2	Harmonic 3
1	3,710	4,210	5,180	4,900
2	8,360	4,210	5,180	6,530
3	9,780	7,190	5,750	6,850
4	10,100	7,190	5,750	10,400
5	10,500	8,740	7,900	11,500
6	11,200	8,740	7,900	13,200
7	12,100	9,670	10,000	14,300
8	13,500	9,670	10,000	14,900
9	16,200	11,200	11,500	15,200
10	17,500	11,200	11,500	15,900
11	17,800	12,500	13,300	16,000
12	18,000	12,500	13,300	16,500
13	18,600	13,000	14,000	18,900
14	18,900	13,000	14,000	19,300
15	19,500	14,700	14,400	19,600

Implications of Cyclic Symmetry - Generalized Force

- Generalized (or Modal) Force defined as

$$\{\mathfrak{F}\}_m = \{\Phi\}_m^T \{F\}.$$

- This is just the dot product of each mode with the excitation force vector and means that the response is directly proportional to the similarity of the spatial shape of each mode with the spatial shape of the force.

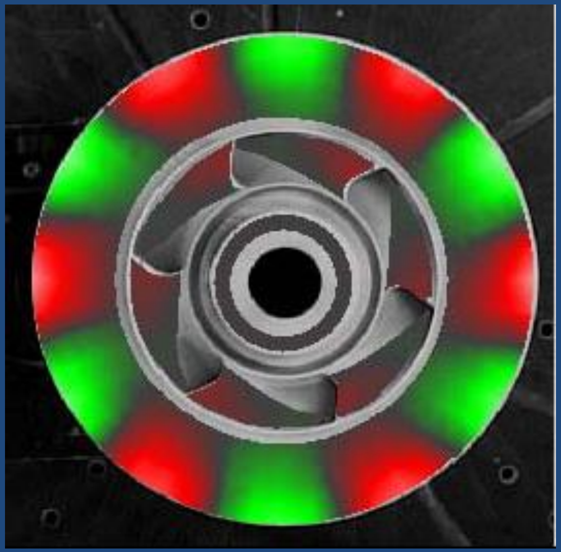
- For pure harmonic waves, the "Orthogonality Principle" states

$$\int_{-\pi}^{\pi} \sin(n\theta)\sin(m\theta) = \begin{cases} \pi \text{ when n=m} \\ 0 \text{ otherwise} \end{cases}$$

- Think of the $\{\Phi\}$ as a continuous function, and the force the same way.
  - Then the Dot Product is the same as an integration of the product of the two functions.
  - So this says that the only non-zero result of an excitation wave shaped like a Sine and a mode shaped like a Sine is for the components of those waves that have the same wave number!

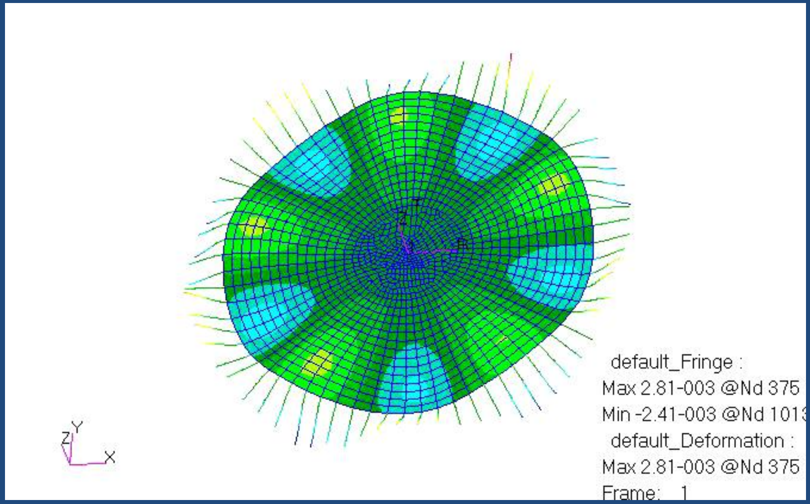
Have to determine Nodal Diameter of Modes to identify Resonance

- For disks and disk dominated modes, 5ND Traveling Wave will excite a 5ND mode

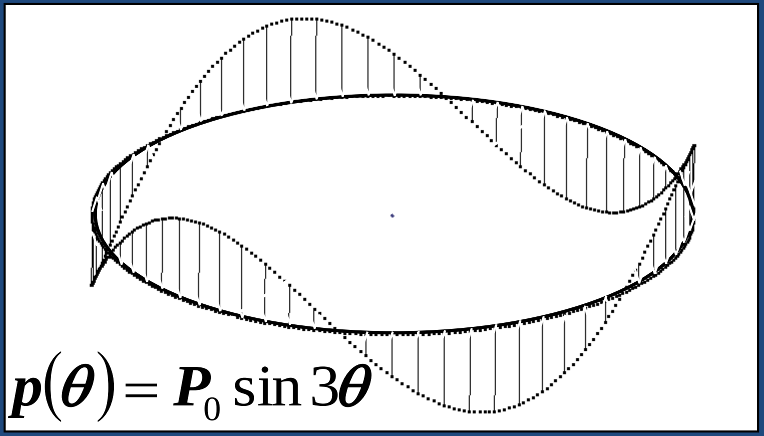


5ND standing wave mode of Impeller (modal test using holography)

5ND travelling wave Mode of Bladed-Disc

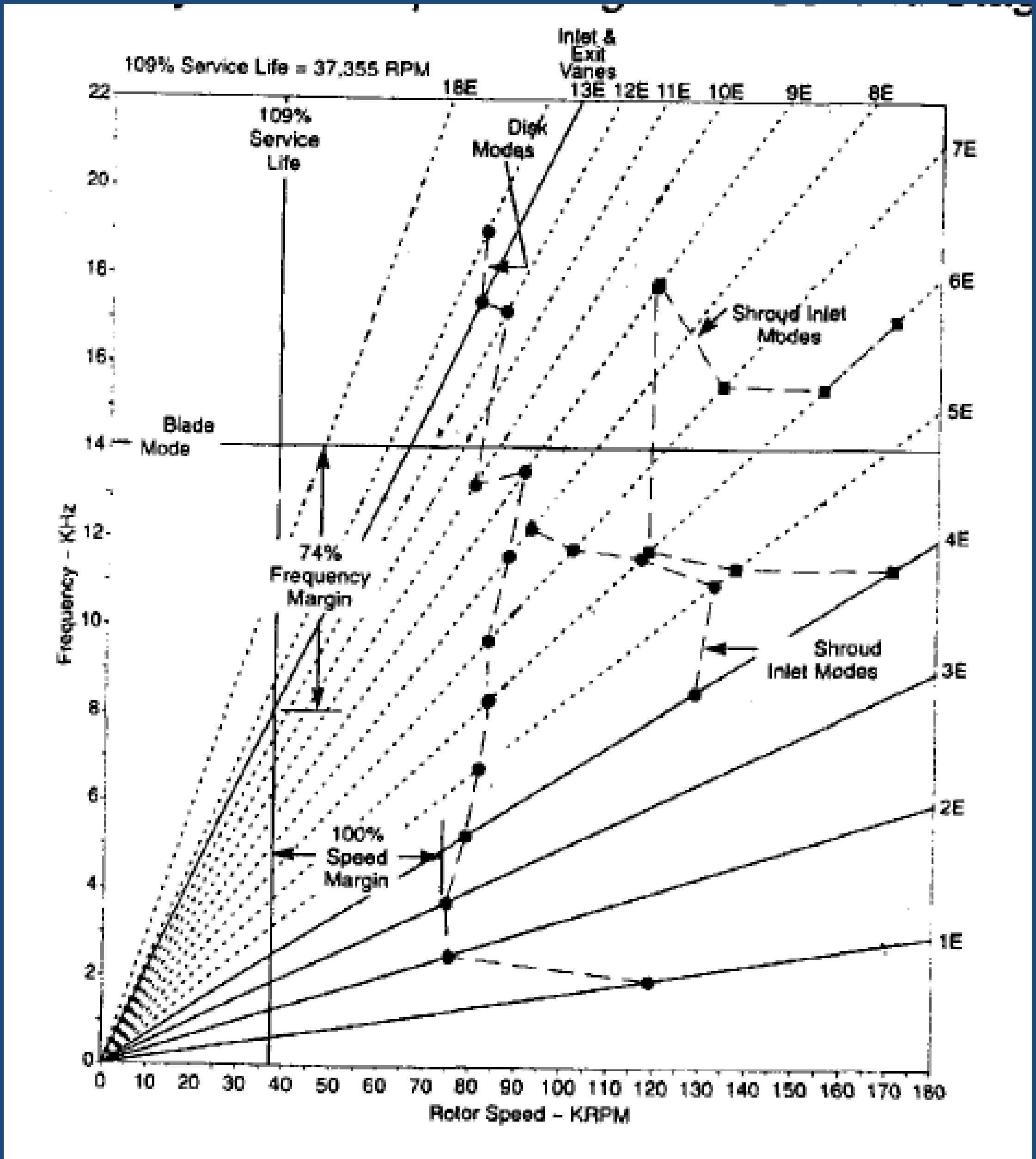


- On the other hand, 3ND excitation (perhaps from pump diffusers) will not excite a 5ND structural mode.



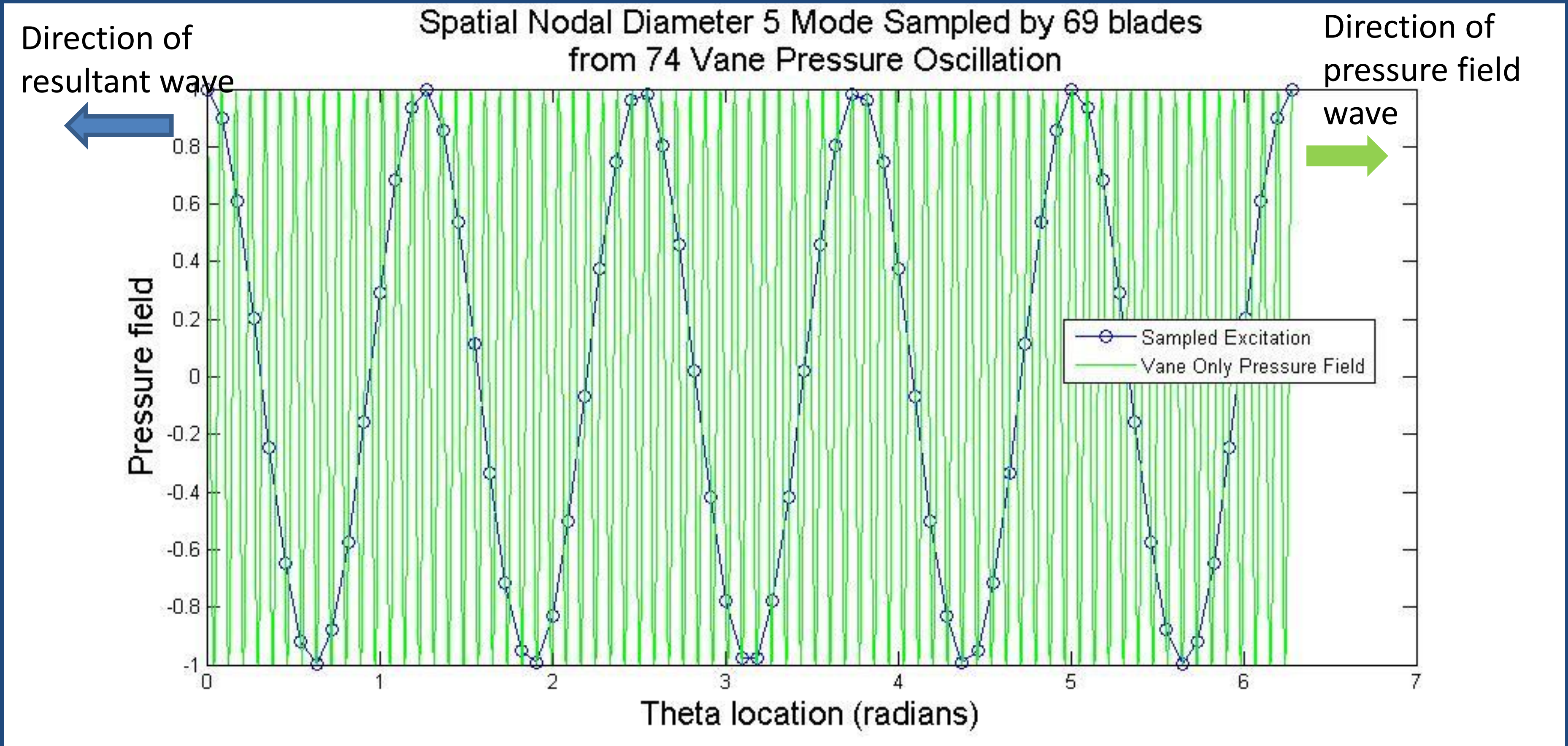
Impeller Campbell Diagram

- "Triple Crossover Points" (speed, ω, and ND) needed for resonance of pure shroud (disk) modes.
- ND mode at exact spatial number of distortions in excitation.



“Blade/Vane” Interaction causes different ND excitation

- Sampling by discrete number of points on structure of pressure oscillation results in spatial Nodal Diameter excitation at the difference of the two counts.
- E.g., a 74 wave number pressure field (coming from 2x37 vanes), exciting 69 blades results in a Nodal Diameter mode of 69-74=-5, where sign indicates direction of traveling 5ND wave (*plot courtesy Anton Gagne*).



Tyler–Sofrin Blade–Vane Interaction Charts

- Chart identifies Nodal Diameter families that can be excited

Upstream Nozzle Multiples	37	74	111	148		Downstream Stator Multiples	57	114	171	228
Blade multiples						Blade multiples				
69	32	-5	N/A	N/A		69	12	N/A	N/A	N/A
138	N/A	N/A	27	-10		138	N/A	24	-33	N/A
207	N/A	N/A	N/A	N/A		207	N/A	N/A	N/A	-21

- All modes in Campbell have to be from these families
  - E.g., Nodal Diameter 5, blade mode 3 (torsion)
- Temporal Frequency of Excitation is at the engine order of the distortion.
- Much of chart is marked “N/A – not applicable” because.....
  - Highest number of ND waves in a cyclic symmetric structure is N/2 or (N-1)/2

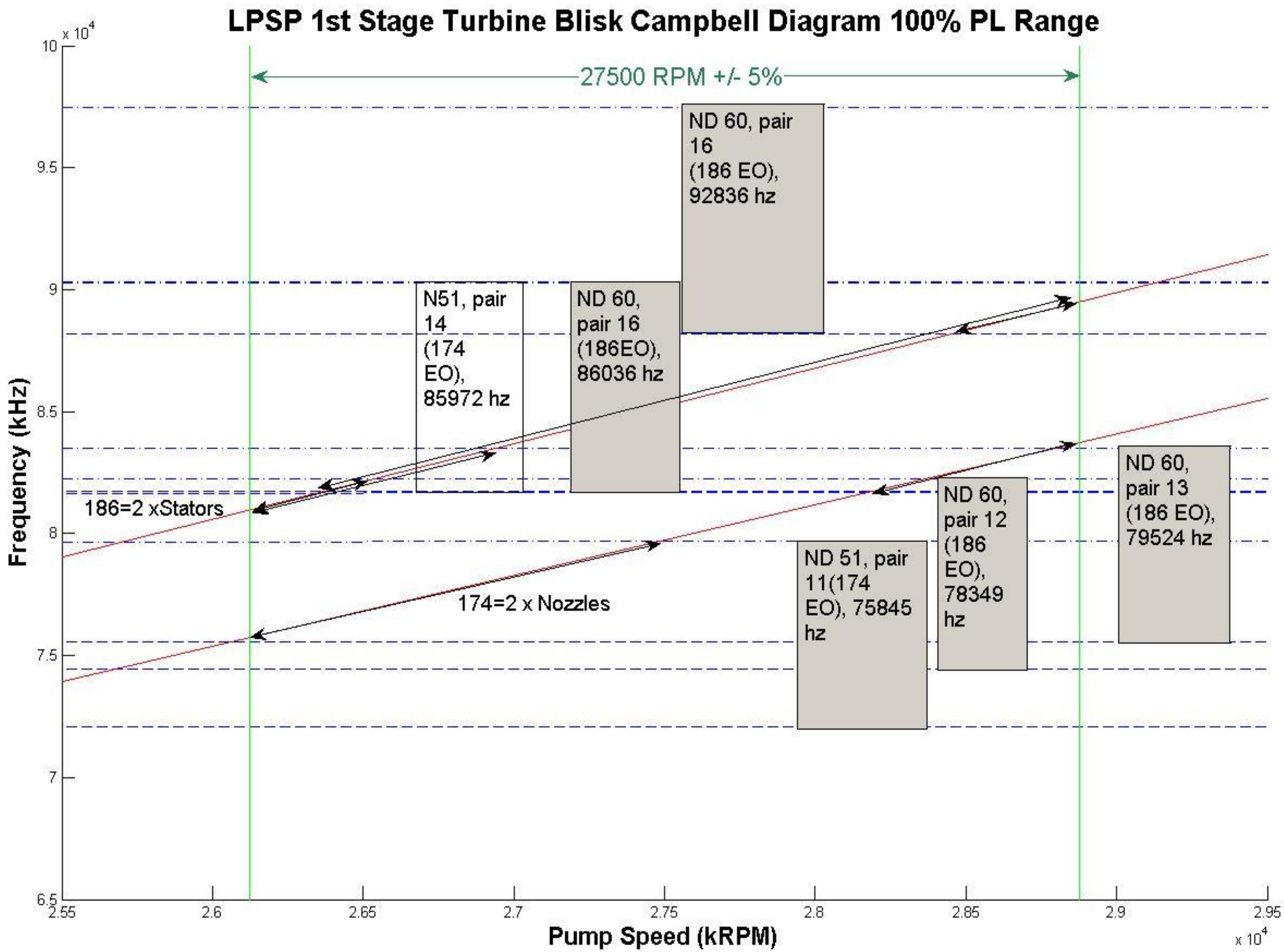
Example – LPSP Turbine Blisk Aliasing Tables, Non-Problematic Modal Evaluation

- 123 blades allow (123-1)/2=61 Nodal Diameters

	1st Stage Nozzles				2nd Stage Stators			Exit Guide Vanes		
	87	174	261		93	186	279	97	194	291
blades										
123	**36**	**51**	138		**30**	63	156	**26**	71	168
246	159	72	**15**		153	**60**	**33**	149	**52**	**45**
369	282	195	108		276	183	90	272	175	78
0	87	174	261		93	186	279	97	194	291

- Many crossings judged low risk, or acceptable risk for non-flight program
  - 15ND, 33ND, and 45ND crossing modes eliminated due to probable low 3X forcing function, high frequency
  - 52ND modes extremely complicated, high frequency
  - Many modes eliminated due to non-adjacency of forcing function
- Also have to consider mechanical excitations order 1-6N
  - In this case, no 1-6ND modes have crossings with appropriate forcing function.

Turbine Blisk Campbell for Problematic Modes in 100% PL Range



Turbine Blisk Problematic Modes Possible Resolutions

		Natural Frequencies				
ND	mode pair number	70% PL	100% PL	mode shape description	excitation source and order	potential solution
36	5	**28955.47**	28964.54	1st Torsion NOTE 12/31/13- 2ND blade -are nozzles problem?	87=1 x 1st st upstream nozzles	change # nozzles to 89 to move lower bound of 70% above mode.
51	9	**58040.23**	58048.62	1st blade chordwise 2nd bending	174= 2 x 1st st upstream nozzles	change # nozzles to 89 to move lower bound of 70% above mode.
51	11	75845.08	**75878.04**	1st blade spanwise TE 2 wave	174= 2 x 1st st upstream nozzles	CFD to determine magnitude of 2x
60	12	78335.00	**78349.16**	2nd blade bending	186 = 2 x upstream 2nd st stator	CFD to determine magnitude of 2x
60	13	79511.67	**79524.37**	2nd blade ?	186 = 2 x upstream 2nd st stator	CFD to determine magnitude of 2x
51	14	85944.41	**85972.20**	1st blade TE spanwise 1 wave	174= 2 x 1st st upstream nozzles	CFD to determine magnitude of 2x
60	14	86007.76	**86035.52**	1st blade TE 1.5 wave	186 = 2 x downstream 2nd st stator	CFD to determine magnitude of 2x
60	16	92827.38	**92835.74**	1st blade TE 0.5 wave	186 = 2 x downstream 2nd st stator	change # stators to 92, mode will be above range

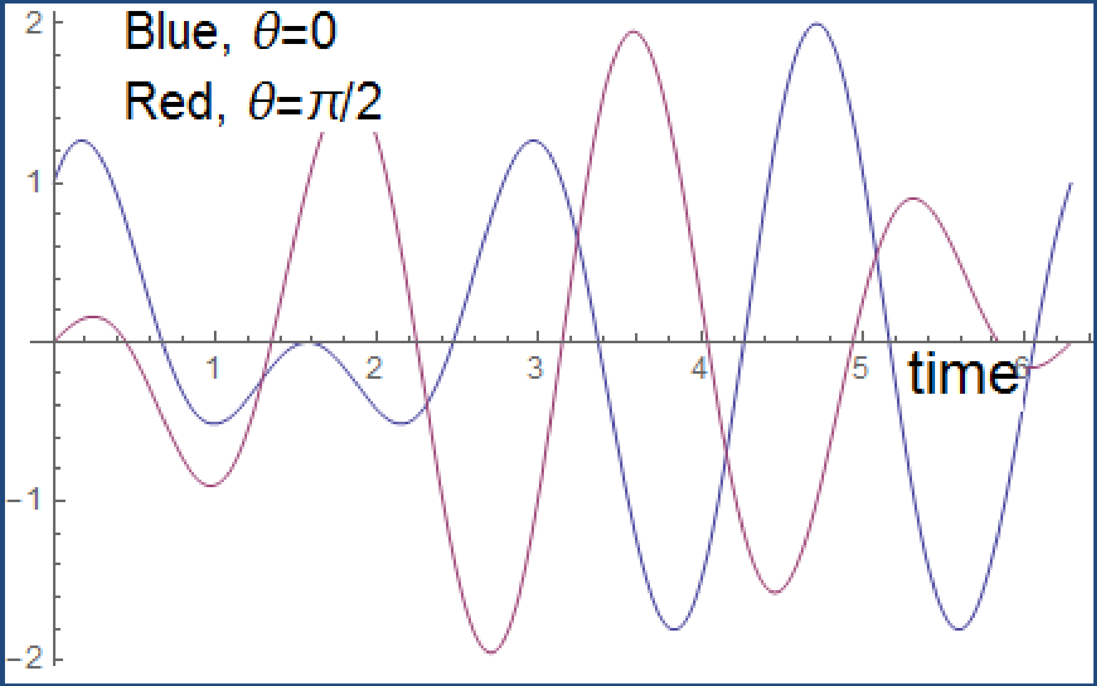
Spatial Fourier Analysis helpful to identify ND number of both excitation and modes

- Let's say we have measured pressure field

$$p(t,\theta)$$

that has the unknown temporal and spatial form:

$$s = \mathrm{Sin}[3t + 2\theta] + \mathrm{Cos}[4t + 3\theta]$$



- First perform Temporal Fourier Analysis at each value of θ, using Complex Form (more efficient than harmonic form)

$$p(t) = \sum_{n=-\infty}^{\infty} [c_n e^{in\Omega_1 t}]$$

where

$$c_n = \frac{1}{T_1}\int_0^T p(t) e^{in\Omega_1 t} dt$$

Spatial Decomposition of Each Temporal Fourier Component

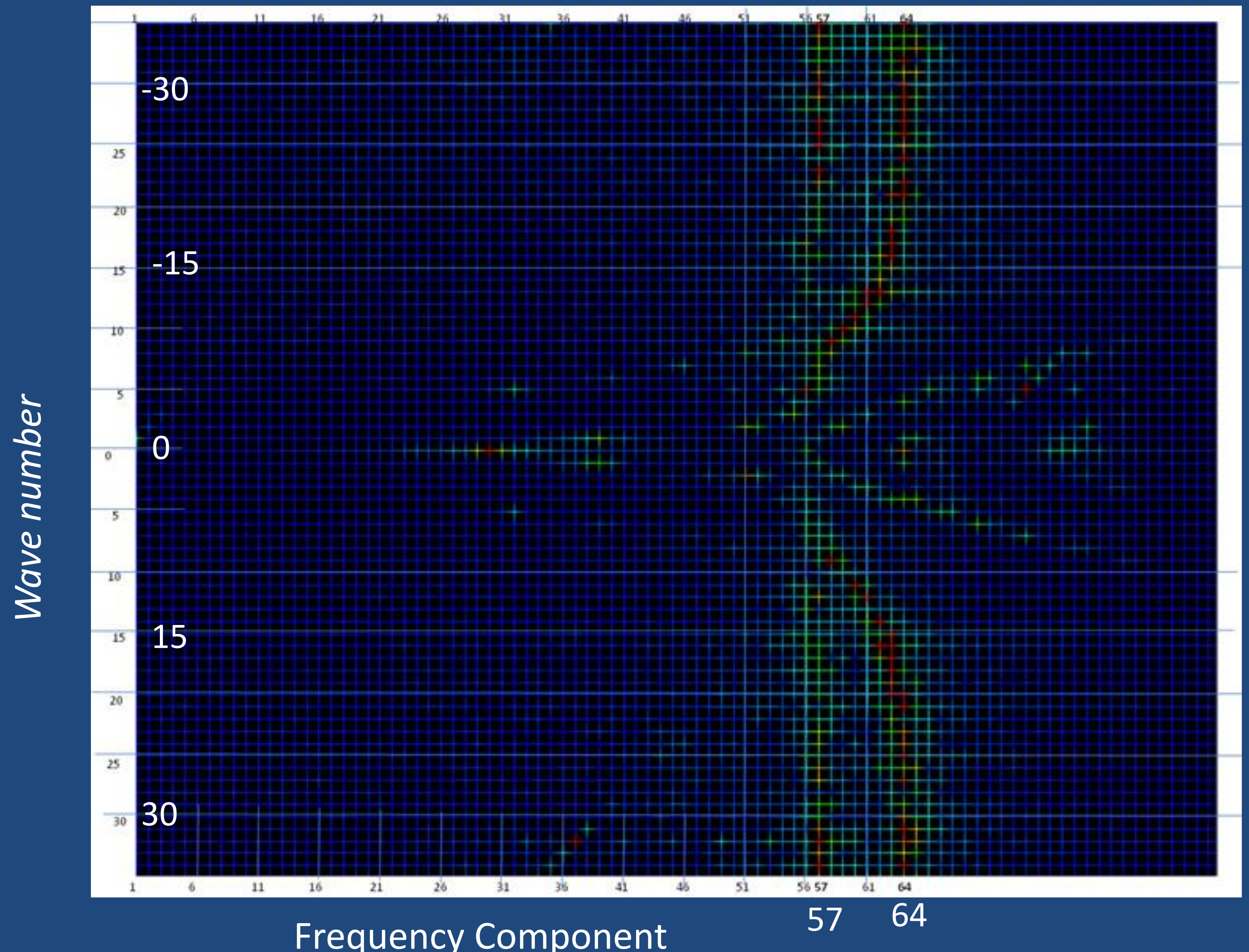
- Now look at obtaining spatial components for each of the temporal fourier components ("bins").

$$p(\theta) = \sum_{n=-\infty}^{\infty} [c_n e^{in\Omega_1\theta}]$$

where

$$c_n = \frac{1}{2\pi}\int_0^{2\pi} p(\theta)e^{in\Omega_1\theta}d\theta$$

2-D Fourier Transform Shows Spatial Complexity of J2X turbine flow field and response, used in evaluation of forced response methodologies



Mistuning

- Due to manufacturing tolerances, the turbine blades on a bladed disk will never be identical
- Result is that the bladed-disk modes (e.g., ND17, 1^{st} Torsion) will have slightly varying natural frequencies and mode shapes; typical variation is +/- 1.5%.
- Variation itself called *Mistuning*, which generates two effects in addition to bifurcation of individual modes into n*individual modes:
  - *Localization* – mode shapes warp such that maximum deflection is at a single location, rather than at every high point in a “tuned” nodal diameter mode.
  - *Amplification* – most important effect – the maximum resonant “mistuned” response is frequently up to twice as much as the “tuned” maximum resonant response.
- Probabilistic analysis techniques required since every bladed-disk will be different
- Innumerable papers and Ph.D. theses have been devoted to this topic over the last 40 years.
- Tractable techniques for predicting level of amplification for a design have only existed since 2004.

Localization due to Mistuning

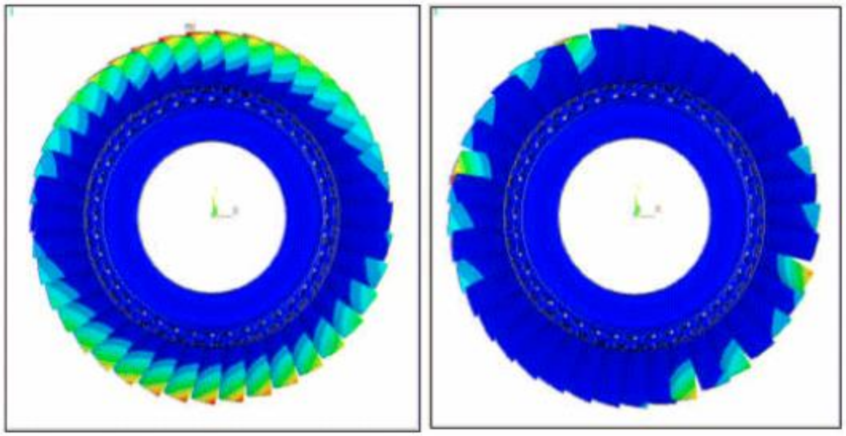


Fig. 3b One ND tuned and mistuned mode shapes

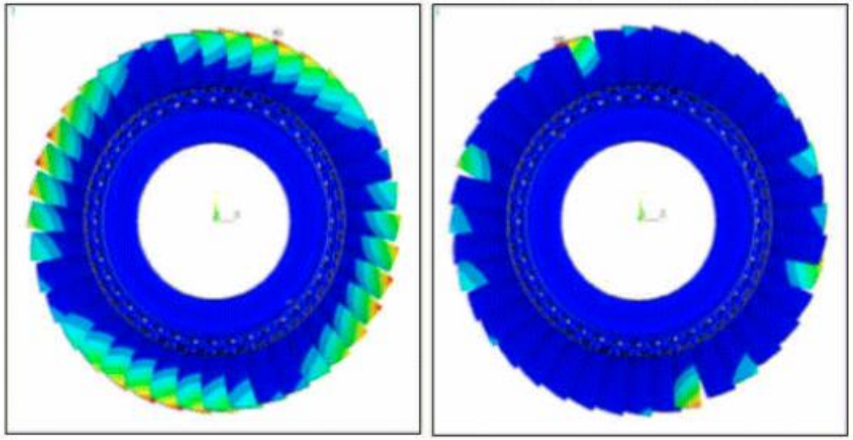
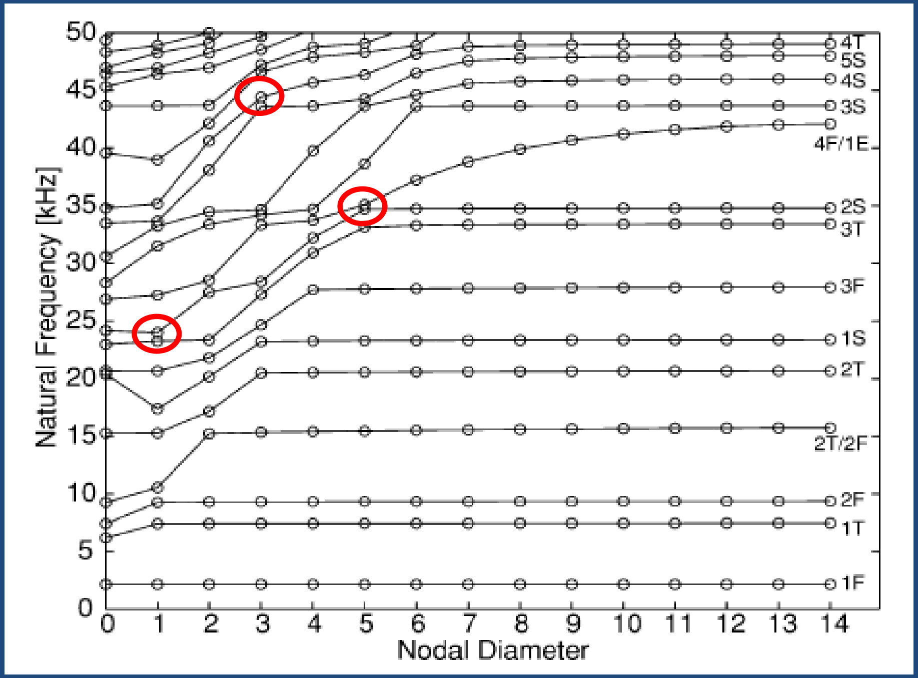


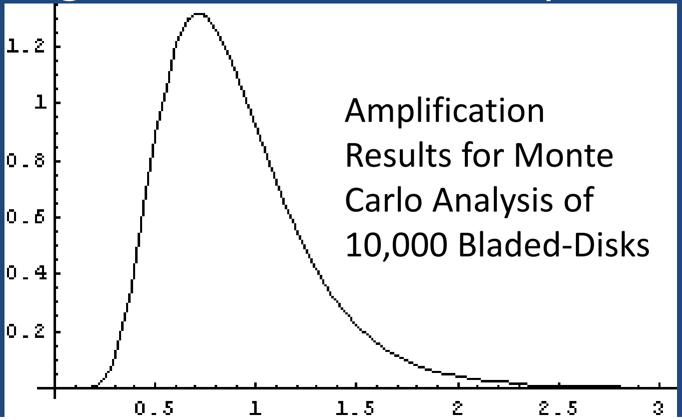
Fig. 3c Two ND tuned and mistuned mode shapes

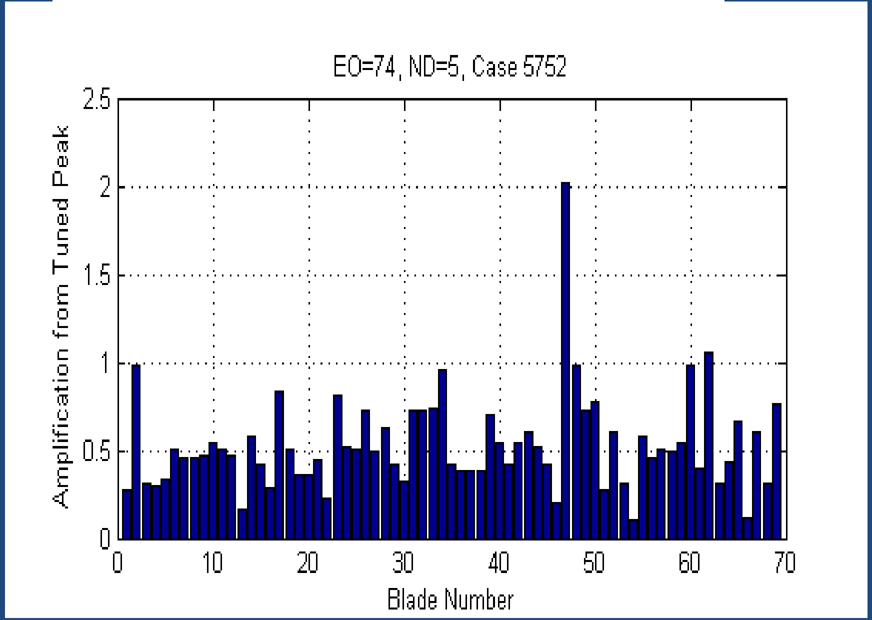
Ref: Rao, J.S., Mistuning of Bladed Disk Assembies to Mitigate Resonance, Altair Engineering, 2006

Amplification due to Mistuning

- Maxiumum amplified blade is not blade with most mistuning, appears to be most pronounced near locations of “eigenvalue veering” on Nodal Diameter plot.







- Can predict amplification for a given design using “MISER” by assuming a std deviation of frequency mistuning and generating a nodal diameter diagram for modes of interest.

Ref: Castinier, M. P., Pierre, C., Modeling and Analysis of Mistuned Bladed Disk Vibration: Status and Emerging Directions, Journal of Propulsion and Power, Vol. 22, No. 2, March-April 2006.

Conclusion

- Structural Dynamic Analysis of Turbomachinery is critical aspect of design, development, test, and failure analysis of Rocket Engines.
- Process of Analysis starts consists of modeling, modal analysis and characterization, comparison with excitation field, and forced response analysis if necessary.
- Thorough understanding of Fourier Analysis, Vibration Theory, Finite Element Analysis critical.
- Knowledge of Turbomachinery Design and Fluid Dynamics very useful.