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Effect of Index of Refraction on Radiation Characteristics in a Heated Absorbing, Emitting, and Scattering Layer

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Nomenclature

a = absorption coefficient of layer, m^{-1}
D = thickness of plane layer, m
E_1, E_2, E_3 = exponential integral functions, $E_m(x) = \int_0^1 \mu^{m-2} \exp(-x/\mu) d\mu$
n = index of refraction
q = heat flux, $\mathrm{W/m^2}$
T = absolute temperature, K
T_{s1}, T_{s2} = temperatures of surrounding radiating regions, K
x = coordinate normal to boundary of plane layer, m; $X = x/D$
κ = optical depth, $(a+\sigma_s)x$; κ_D, optical thickness, $(a+\sigma_s)D$
σ = Stefan-Boltzmann constant, $\mathrm{W/m^2 \cdot K^4}$
σ_s = scattering coefficient of layer, m^{-1}
ρ^i = reflectivity of interface for internally incident radiation
τ^o = transmissivity of surface for externally incident radiation
Φ = dimensionless temperature distribution
Ψ = dimensionless radiative heat flux

Subscripts

i, o = incoming and outgoing radiation
r = radiative quantity
1, 2 = the hotter and cooler surroundings of the layer

Introduction

The index of refraction can considerably influence the temperature distribution and radiative heat flow in semitransparent materials such as some ceramics. For external radiant heating, the refractive index influences the amount of energy transmitted into the interior of the material. Emission within a material depends on the *square* of its refractive index, and hence this emission can be many times that for a blackbody radiating into a vacuum. Since radiation exiting through an interface into a vacuum cannot exceed that of a blackbody, there is extensive reflection at the internal surface of an interface, mostly by total internal reflection. This redistributes energy within the layer and tends to make its temperature distribution more uniform.

The purpose of the present analysis is to show that, for radiative equilibrium in a gray layer with diffuse interfaces, the temperature distribution and radiative heat flux for any index of refraction can be obtained very simply from the results for an index of refraction of unity. For the situation studied here, the layer is subjected to external radiative heating incident on each of its surfaces. The material emits, absorbs, and isotropically scatters radiation. For simplicity the index of refraction is unity in the medium surrounding the layer.

The surfaces of the layer are assumed diffuse. This is probably a reasonable approximation for a ceramic layer that has not been polished. When transmitted radiation or radiation emitted from the interior reaches the inner surface of an interface, the radiation is diffused and some of it thereby placed into angular directions for which there is total internal reflection. This provides a trapping effect for retaining energy within the layer and tends to equalize its temperature distribution.

An analysis of temperature distributions in absorbing-emitting layers, including index of refraction effects, was developed by Gardon (1958) to predict cooling and heat treating of glass plates. The interfaces were optically smooth; the resulting specular reflections were computed from the Fresnel reflection laws. This provides a somewhat different behavior than for diffuse interfaces. A similar application was for heating that occurs in a window of a re-entry vehicle (Fowle et al., 1969). A number of recent papers (Rokhsaz and Dougherty, 1989; Ping and Lallemand, 1989; Crosbie and Shieh, 1990) further examined the effects of Fresnel boundary reflections and nonunity refractive index. Other examples of analyses of both steady and transient heat transfer to single or multiple plane layers (Amlin and Korpela, 1979; Tarshis et al., 1969) have used diffuse assumptions at the interfaces as in the present study.

Analysis

A plane layer of ceramic material has thickness D as shown in Fig. 1. It absorbs, emits, and isotropically scatters radiation. The limiting case is considered here where the layer temperature distribution is dominated by radiation so heat conduction is neglected. The material has a constant index of refraction; it is the effect of the index of refraction that is investigated here. Because of the large amount of scattering in many ceramic materials, the interfaces between the ceramic and the surrounding air or vacuum are assumed to be diffuse. The refractive index of the surroundings is unity. As shown in Fig. 1, the layer is subjected to radiation from the surroundings at T_{s1} and T_{s2} on the two sides $x=0$ and $x=D$. It is assumed that the surroundings act as black environments so the incident energies on the two sides are $q_{r1}(0) = \sigma T_{s1}^4$ and $q_{r2}(D) = \sigma T_{s2}^4$. Inside the layer there are outgoing and incoming fluxes, q_o and q_i, at each interior surface, as shown on the figure. Since scattering is included, the local optical depth is related to the x coordinate by $\kappa = (a+\sigma_s)x$.

The temperature distribution inside the layer is governed by the integral equation given by Siegel and Howell (1981), modified with the index of refraction factor as

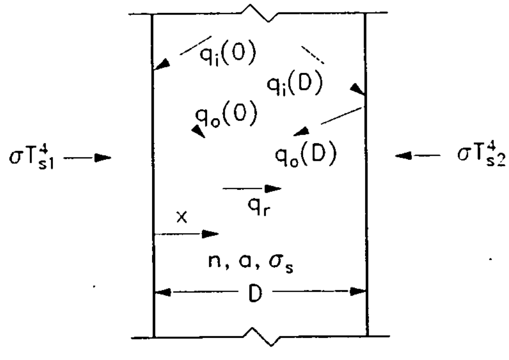

Fig. 1 Layer geometry, coordinate system, and nomenclature of heat fluxes at boundaries

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$$n^2\sigma T^4(\kappa)=\frac{1}{2}\,[q_o(0)E_2(\kappa)+q_o(\kappa_D)E_2(\kappa_D-\kappa)] + \frac{n^2}{2}\int_0^{\kappa_D}\sigma T^4(\kappa^*)E_1(|\kappa-\kappa^*|)d\kappa^* \quad (1)$$

The radiative flux, which is a constant through the layer for the present conditions of radiative equilibrium, can be found from the temperature distribution as (Siegel and Howell, 1981),

$$q_r=q_o(0)-2q_o(\kappa_D)E_3(\kappa_D)-2n^2\int_0^{\kappa_D}\sigma T^4(\kappa)E_2(\kappa)d\kappa \quad (2)$$

The following dimensionless groups are now defined:

$$\Phi(\kappa)=\frac{n^2\sigma T^4(\kappa)-q_o(\kappa_D)}{q_o(0)-q_o(\kappa_D)}; \quad (3a)$$

$$\Psi=\frac{q_r}{q_o(0)-q_o(\kappa_D)} \quad (3b)$$

Equations (1) and (2) then become (note that $X=x/D=\kappa/\kappa_D$)

$$\Phi(X)=\frac{1}{2}\,E_2(\kappa_D X)+\frac{1}{2}\,\kappa_D\int_0^1\Phi(X^*)E_1(\kappa_D|X-X^*|)dX^* \quad (4)$$

$$\Psi=1-2\kappa_D\int_0^1\Phi(X)E_2(\kappa_D X)dX \quad (5)$$

In Eqs. (4) and (5) Φ and Ψ are not functions of n. Hence to obtain Φ and Ψ for all $n\geq 1$ it is necessary to solve Eq. (4) only once for each κ_D and use each result to evaluate Eq. (5). For the special case when the index of refraction of the layer is 1, there are no reflections at the interfaces so that $q_o(0)=\sigma T_{s1}^4$, $q_o(\kappa_D)=\sigma T_{s2}^4$, and the dimensionless groups become

$$\Phi(\kappa)=\frac{T^4(\kappa)-T_{s2}^4}{T_{s1}^4-T_{s2}^4}; \quad (6a)$$

$$\Psi=\frac{q_r}{\sigma T_{s1}^4-\sigma T_{s2}^4}; \quad (6b)$$

For $n>1$, however, the Φ and Ψ in Eq. (3) contain the outgoing boundary fluxes $q_o(0)$ and $q_o(\kappa_D)$ that are unknown, so the solution has not yet provided $T(\kappa)$ and q_r for the cases when $n>1$. In order to find these quantities the $q_o(0)$ and $q_o(\kappa_D)$ must be obtained in terms of known quantities. This is accomplished by looking at the boundary conditions in detail.

At the diffuse interfaces the internal fluxes are related to the transmission of external flux and the reflection of internal flux by

$$q_o(0)=\sigma T_{s1}^4\tau^o+q_i(0)\rho^i \quad (7a)$$

$$q_o(\kappa_D)=\sigma T_{s2}^4\tau^o+q_i(\kappa_D)\rho^i \quad (7b)$$

At the inside surfaces of the two boundaries there are the following relations between the radiative flux and the outgoing and incoming fluxes:

$$q_r(0)=q_o(0)-q_i(0) \quad (8a)$$

$$q_r(\kappa_D)=-q_o(\kappa_D)+q_i(\kappa_D) \quad (8b)$$

Equations (7a) and (8a) are combined to eliminate $q_i(0)$ and similarly for Eqs. (7b) and (8b) to eliminate $q_i(\kappa_D)$. This yields

$$q_o(0)=\frac{1}{1-\rho^i}\,(\tau^o\sigma T_{s1}^4-q_r\rho^i) \quad (9a)$$

$$q_o(\kappa_D)=\frac{1}{1-\rho^i}\,(\tau^o\sigma T_{s2}^4+q_r\rho^i) \quad (9b)$$

These relations are substituted into Eq. (3b) to eliminate the q_o's. This yields the radiative flux for any index of refraction in terms of the value of Ψ. Use is also made of the relation at an interface (Richmond, 1963) that $\tau^o=(1-\rho^i)n^2$. The result for q_r *for any n* is,

$$\frac{q_r}{\sigma(T_{s1}^4-T_{s2}^4)}=\frac{n^2\Psi}{1+\dfrac{2\rho^i}{1-\rho^i}\,\Psi} \quad (10)$$

Following the same procedure, the temperature distribution is found by starting with Eq. (3a), using Eq. (9) to eliminate the q_o's and then using Eq. (10) to eliminate q_r. This yields the $T(X)$ *for any n* as

$$\frac{T^4(X)-T_{s2}^4}{T_{s1}^4-T_{s2}^4}=\frac{\Phi(X)+\dfrac{\rho^i}{1-\rho^i}\,\Psi}{1+\dfrac{2\rho^i}{1-\rho^i}\,\Psi} \quad (11)$$

To use these relations values of ρ^i are needed for various refractive indexes. The externally incident radiation is diffuse. Although the interfaces are probably not optically smooth, it is assumed that each bit of roughness acts as a smooth facet so that the reflectivity can be obtained from the usually used interface relations for a nonabsorbing dielectric medium. Then by integrating the reflected energy over all incident directions the relation for $\rho^i(n)$ is (Richmond, 1963),

$$\rho^i(n)=1-\frac{1}{n^2}\left\{\frac{1}{2}-\frac{(3n+1)(n-1)}{6(n+1)^2}-\frac{n^2(n^2-1)^2}{(n^2+1)^3}\ln\left(\frac{n-1}{n+1}\right) + \frac{2n^3(n^2+2n-1)}{(n^2+1)(n^4-1)}-\frac{8n^4(n^4+1)}{(n^2+1)(n^4-1)^2}\ln(n)\right\} \quad (12)$$

As discussed by Cox (1965), in the fairly transparent spectral regions of ceramic materials, the extinction coefficient in the complex index of refraction is usually not large enough to affect the surface reflectivity significantly, so Eq. (12) for nonattenuating dielectrics gives good results. The extinction

Table 1 Dimensionless temperature distribution, Φ

	Optical Thickness, κ_D						
X	0.1	0.3	1.0	3.0	10	30	100
0	.5710	.6419	.7582	.8693	.9495	.9819	.9948
.05	.5619	.6229	.7230	.8211	.8966	.9304	.9456
.10	.5541	.6072	.6946	.7819	.8511	.8828	.8974
.20	.5397	.5786	.6429	.7088	.7627	.7876	.7994
.30	.5262	.5517	.5942	.6384	.6750	.6920	.6999
.40	.5130	.5257	.5468	.5690	.5874	.5961	.6002
.50	.5000	.5000	.5000	.5000	.5000	.5000	.5000
.60	.4870	.4743	.4532	.4310	.4125	.4038	.4002
.70	.4738	.4483	.4058	.3616	.3249	.3079	.3004
.80	.4603	.4214	.3571	.2912	.2372	.2123	.2007
.90	.4459	.3928	.3054	.2181	.1488	.1171	.1026
.95	.4382	.3771	.2770	.1789	.1034	.0695	.0543
1.0	.4290	.3581	.2419	.1307	.0505	.0181	.0052

Table 2 Dimensionless heat flux, Ψ

Optical Thickness κ_D	Ψ
0.1	.9157
0.3	.7936
1.0	.5534
3.0	.3017
10	.1168
30	.0419
100	.0122

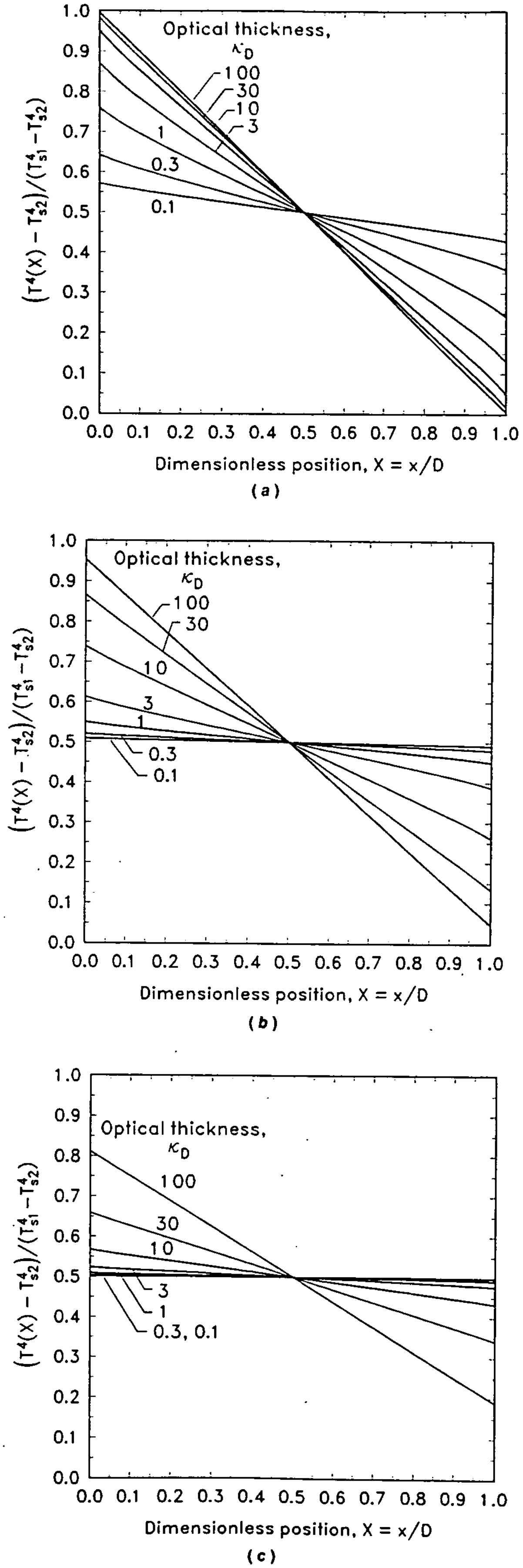

Fig. 2 Effect of index of refraction on dimensionless temperature distributions for various optical thicknesses: (*a*) index of refraction, $n=1$; (*b*) index of refraction, $n=2$; (*c*) index of refraction, $n=4$

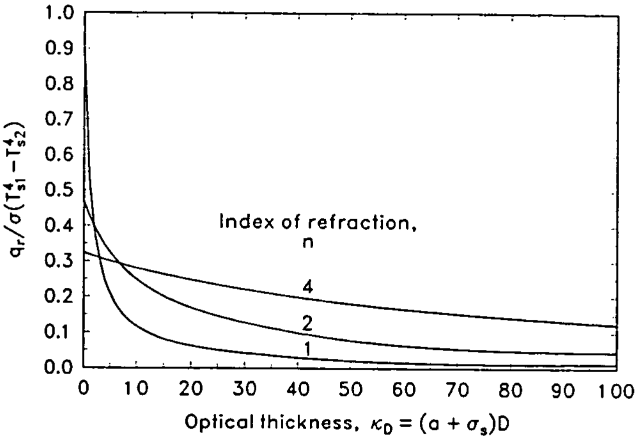

Fig. 3 Effect of index of refraction on radiative heat flux through the layer as a function of optical thickness

effect can be important in highly absorbing wavelength bands. General property relations as given by Hering and Smith (1968) would then be required to estimate average surface properties for use in the present gray analysis.

Equations (10)–(12) provide the means for obtaining the temperature distributions and radiative heat fluxes for a layer with $n>1$ from the results for $n=1$. The solutions of Eqs. (4) and (5) for Φ and Ψ were obtained in another context by Heaslet and Warming (1965). In connection with some of our work (Siegel, 1987), a computer program was available; solutions were obtained from that program to have results in a convenient tabular form for the desired optical thicknesses. As discussed in that work the program uses a Gaussian integration subroutine, and 80 grid points were used across the layer. Cubic spline interpolation is used to obtain the irregularly spaced integration points for the Gaussian routine. To deal with the singularity $E_1(0) \rightarrow \infty$, the integral in Eq. (4) was evaluated analytically for X^* very close to X by assuming $\Phi(X^*) \approx \Phi(X)$ so that $\Phi(X)$ could be taken out of the integral for very small $|X-X^*|$ (see Siegel, 1987). The results were in agreement with those of Heaslet and Warming (1965).

Results and Discussion

The Φ and Ψ were evaluated from Eqs. (4) and (5). The results are in excellent agreement with the plotted values of Heaslet and Warming (1965) and are given for convenience in Tables 1 and 2 for $0.1 \le \kappa_D \le 100$. Equations (10) and (11) were then used to obtain dimensionless radiative heat fluxes and temperatures for $n>1$. The required ρ^i were obtained from Eq. (12).

Figure 2 shows the dimensionless temperature distributions for $n=1, 2$, and 4. For small optical thickness the dimensionless distribution approaches 0.5, and as $\kappa_D \rightarrow \infty$ the profile becomes linear extending from 1.0 to 0. The effect of increasing n is to decrease the range of the temperature distributions, and for a fixed κ_D they move closer to 0.5 as n is made larger. The dimensionless profiles are all rather linear. The fact that the profiles become more uniform with increasing n is the result of increasing internal reflections within the absorbing and scattering layer and decreasing transmission into the layer. Since each element in the layer is in radiative equilibrium all locally absorbed radiation must be reemitted. Since scattering is assumed isotropic, local scattering is added to the local emission. The large amount of internal reflection tends to equalize the energy throughout the layer and flatten the temperature distributions. The effect of n is calculated very easily from the simple relation given by Eq. (11), where ρ^i is a function of n.

The effect of n and κ_D on the dimensionless radiative heat flux through the layer is in Fig. 3. The heat flux decreases as the optical thickness increases, and the effect of n is quite pronounced in altering the heat flux. The curve becomes very flat for $n = 4$. This is because interface reflections and increased internal emission (as a result of the n^2 factor) have become quite strong in regulating the heat transfer. The effect of optical thickness is thereby suppressed. At large optical thicknesses this increases the radiative flux as compared with the results for $n = 1$. As the layer becomes transparent $\kappa_D \rightarrow 0$, $\Psi \rightarrow 1$ and the dimensionless flux from Eq. (10) approaches $n^2(1-\rho^i)/(1+\rho^i)$.

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The Prandtl Number Effect on Melting Dominated by Natural Convection

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Nomenclature

B = constant, Eq. (13)
c_P = liquid specific heat
Fo = Fourier number $= \alpha t/H^2$
g = gravitational acceleration
Gr = Grashof number $= g\beta\Delta T H^3/\nu^2$
h_{sf} = latent heat of melting
H = height
k = liquid thermal conductivity
Nu = Nusselt number $= q'(k\Delta T)$
Pr = Prandtl number $= \nu/\alpha$
q' = overall heat transfer rate, W/m
Ra = Rayleigh number $= g\beta\Delta T H^3/(\alpha\nu)$
Ra_z = Rayleigh number $= g\beta\Delta T z^3/(\alpha\nu)$
s = thickness of conduction zone
Ste = Stefan number $= c_P\Delta T/h_{sf}$
T_h = temperature of isothermally heated wall
T_m = melting point
ΔT = temperature difference $= T_h - T_m$
z = height of convection zone
α = liquid thermal diffusivity
β = liquid coefficient of volumetric thermal expansion
δ = thermal boundary layer thickness
ν = kinematic viscosity
$()_c$ = critical

Objective

In a recent paper, Gobin and Benard (1990) considered the task of correlating the heat transfer data for melting in the presence of natural convection when the Pr value of the liquid phase is considerably smaller than 1. Earlier correlations were reported by Webb and Viskanta (1986), and Beckermann and Viskanta (1989). Additional low-Pr studies were conducted by Webb and Viskanta (1985), Gau and Viskanta (1986), Wolff and Viskanta (1987), and Beckermann (1989). To correlate the low-Pr data is an important and timely task, especially in view of the voluminous work that has been dedicated to situations in which Pr is greater than 1.

For the convection-dominated regime known also as quasi-stationary melting, Gobin and Benard (1990) correlated their low-Pr numerical data with the formula:

$$\mathrm{Nu} = 0.29\,\mathrm{Ra}^{0.27}\mathrm{Pr}^{0.18} \tag{1}$$

They noted that this correlation does not agree with the $\mathrm{Nu} \sim (\mathrm{RaPr})^{1/4}$ trend that might be expected from the single-phase natural convection scales for low Prandtl numbers (Bejan, 1984). They concluded that:

1 The relevance of the group (RaPr) is not verified by their numerical results for convection-dominated melting, and

2 Further work is required to determine the scaling laws that govern the transition from the initial (conduction) regime to the final (convection) regime of the process of melting by side heating.

These two conclusions defined the work presented in this note. In it we report the correct scales of natural convection melting when the Prandtl number is small. We then construct a scaling-correct heat transfer correlation that spans the entire range of Prandtl numbers.

Scale Analysis

The scales of the natural convection melting process can be determined by extending Jany and Bejan's (1988) high-Pr theory to the range of low Prandtl numbers represented by liquid metals. The theory is based on the geometric fact that during the transition from conduction to convection-dominated melting the melt region contains two distinct zones. As shown in Fig. 1, the upper zone of height z is ruled by convection (namely, distinct boundary layers, δ), while the lower zone of uniform thickness s and height $(H - z)$ is governed by horizontal conduction. It is assumed that the flow is laminar, and that $\mathrm{Ste} \ll 1$.

The conduction thickness is well known,

$$s \sim H(\mathrm{SteFo})^{1/2} \tag{2}$$

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