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The Theory of Pulsating Flow in Conical Nozzles

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A knowledge of the dynamic characteristics of nozzles and orifices is important in many control and stability analyses of engineering devices. It is usual to assume that the instantaneous flowrate, for a given set of inlet conditions and outlet pressure, is the same as the nontransient value for the same operating conditions. Recently, in connection with the stability analysis of an externally pressurized thrust bearing, the validity of this assumption was questioned. The analysis presented in this paper was undertaken to provide an answer. The present analysis applies to any fluid, liquid, or gas flowing into a simple conical nozzle. The amplitude and phase of the mass-flux response to a sinusoidally time-varying pressure fluctuation at the nozzle exit are determined. An approximate formula is given for these quantities in terms of the nozzle throat area, the solid angle subtended by the cone, the velocity of the fluid at the nozzle throat, the acoustic velocity at the throat, and the frequency of the pressure fluctuation.

Nozzles and orifices have innumerable applications in engineering devices. Frequently a knowledge of the performance of such nozzles and orifices under dynamic conditions becomes important. For example, one may be concerned with a control or a stability analysis in which some relation between flow fluctuations and exit pressure fluctuations must be incorporated. The usual analytical procedure in these circumstances is to postulate a quasi-steady relation between flow rate and exit pressure: that is, the flow rate is taken, at every time instant, to correspond to the instantaneous exit pressure. This procedure cannot be adopted without some misgivings as to its validity, especially if the frequency of the exit pressure fluctuations is high.

Because of the importance of flow fluctuations through nozzles and orifices, considerable literature on the subject has developed. Much of this literature has been summarized in a survey article by Oppenheim [1].[2] The majority of investigations has been experimental in nature, and most of these investigations have been devoted to determining the effect of pressure fluctuations on *average* flow. However, Schultz-Grunow [2] reports that if the so-called "Strouhal number," $\mathbf{S} = \omega D/U$, is less than 0.01, quasi-steady pressure-flow relations are valid. (Here ω is the angular velocity of the fluctuating exit pressure, D is the throat diameter, and U is the fluid velocity at the throat.) Corresponding theoretical work appears to have been devoted principally to a study of the behavior of individual pulses, rather than of frequency response.[3]

The purpose of the present work is to predict the amplitude and phase relations between the instantaneous flowrate and a sinusoidally fluctuating exit pressure in a conical nozzle, Fig. 1. Although the chosen geometry is, of course, extremely simple compared with many in use, it is believed that the essential characteristics of a compressible fluid expanding and accelerating from a

[3] A paper using reflection analysis to deal with problems akin to those treated here is that of Powell [5].

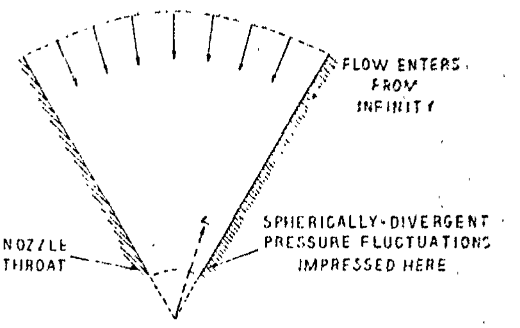

Fig. 1 Conical nozzle showing converging spherically symmetric inflow from stagnation conditions at infinity

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Nomenclature

a = local acoustic velocity
A = local flow area
A^* = flow area where $M = 1$
F = amplitude of oscillation
h = fluid enthalpy
H = stagnation enthalpy, $h + v^2/2$
$\mathcal{H}$ = dimensionless perturbed stagnation enthalpy, H'/a_*^2
$\mathbf{M}$ = local Mach number, v/a
N = amplitude factor
p = fluid pressure
P = amplitude of p'/p oscillations
Q = amplitude of $(\rho v)'/(\rho v)$ oscillations
r = radius from cone apex
r = also, residual function
R = radius, r, where $\mathbf{M} = 1$ in conical flow
S = fluid entropy
$\mathbf{S}$ = also, Strouhal number, $\omega D/v$
$\mathbf{S}^*$ = modified Strouhal number, $[\omega(A/\Omega)^{1/2}]/v$
t = time
T = absolute temperature
U = fluid velocity at nozzle throat
v = local fluid velocity (vectorial in bold face)
$\mathcal{V}$ = auxiliary function
w = dimensionless angular velocity of oscillation

stagnation reservoir are well enough realized for the results to have semi-quantitative practical significance. It is recognized that near the aperture of any conical nozzle the flow will cease to be spherically symmetrical but, again, this effect is not deemed large enough to defeat the purpose of the investigation.

In the analysis, the flow is presumed to be adiabatic and friction-free. Such assumptions are well known to be in close accord with the facts. Under such circumstances, Kelvin's theorem also assures that the flow is irrotational, since it originates in a reservoir of stagnant fluid. The fluid itself is assumed to exist in thermodynamic equilibrium and in a single phase. Otherwise, the nature of the fluid is practically unrestricted; i.e., it may be a liquid, vapor or gas.

Basic Propagation Equation

To start the present analysis, we write down Euler's equation, the mass continuity equation, and the so-called "TdS" equation of thermodynamics. Thus

$$\frac{D\mathbf{v}}{Dt} = -\frac{1}{\rho}\nabla p \tag{1}$$

$$\frac{D \ln \rho}{Dt} = -\nabla \cdot \mathbf{v} \tag{2}$$

$$TdS = dh - \frac{1}{\rho}dp \tag{3}$$

Here $D/Dt = \partial/\partial t + \mathbf{v}\cdot\nabla$ is the Stokes or Lamb operator, and the other symbols, which are standard, are defined in the Nomenclature.

For an irrotational flow:

$$(\mathbf{v}\cdot\nabla)\mathbf{v} = \nabla \frac{v^2}{2} \tag{4}$$

so that equation (1) can be rewritten as

$$\frac{\partial \mathbf{v}}{\partial t} = -\frac{1}{\rho}\nabla p - \nabla \frac{v^2}{2} \tag{5}$$

But since also, from equation (3)

$$\nabla h = \frac{1}{\rho}\nabla p \tag{6}$$

in an isentropic flow, we obtain by combining equations (5) and (6) the following useful result:

$$\boxed{\frac{\partial \mathbf{v}}{\partial t} = -\nabla H} \tag{7}$$

where H is the stagnation enthalpy; i.e., $H = h + v^2/2$.

Another result involving the stagnation enthalpy is obtained by first taking the scalar product of equation (1) with the velocity $\mathbf{v}$. Then

$$\frac{D\frac{v^2}{2}}{Dt} = -\frac{1}{\rho}\mathbf{v}\cdot\nabla p = -\mathbf{v}\cdot\nabla h = -\frac{Dh}{Dt} + \frac{\partial h}{\partial t} \tag{8}$$

where use has been made of equation (6). Finally, rearrangement of equation (8), together with the use of the isentropic relation between pressure and enthalpy, gives

$$\boxed{\frac{DH}{Dt} = \frac{1}{\rho}\frac{\partial p}{\partial t}} \tag{9}$$

Now the velocity of sound in an isentropic medium is

$$a = [(\partial p/\partial \rho)_S]^{1/2} \tag{10}$$

As a consequence, we may write

$$\frac{1}{a^2}\frac{DH}{Dt} = \frac{1}{\rho}\frac{\partial \rho}{\partial t} = \frac{\partial \ln \rho}{\partial t} \tag{11}$$

Taking the Stokesian derivative of equation 11, we obtain

$$\frac{D}{Dt}\frac{1}{a^2}\frac{DH}{Dt} = \frac{D}{Dt}\frac{\partial \ln \rho}{\partial t} \tag{12}$$

But:

$$\frac{D}{Dt}\frac{\partial \ln \rho}{\partial t} = \frac{\partial}{\partial t}\frac{D \ln \rho}{Dt} - \frac{\partial \mathbf{v}}{\partial t}\cdot\nabla \ln \rho \tag{13}$$

Substitutions into equation (13) from equations (2) and (7) yield the following "propagation equation" for the stagnation enthalpy; thus:

$$\boxed{\frac{D}{Dt}\frac{1}{a^2}\frac{DH}{Dt} = \nabla^2 H + \nabla H\cdot\nabla \ln \rho} \tag{14}$$

Differential Equation for Spherically Symmetrical Small Disturbance

Let us now suppose that the flow through the nozzle deviates only very slightly (and in a spherically symmetrical manner) from a steady, conical inflow. Moreover, in what follows let us denote as primed quantities all fluctuations from steady-state values, reserving the unprimed status for the quantities referring to the "base" conical flow. To first order in small quantities, equation (14) becomes

$$\frac{D}{Dt}\frac{1}{a^2}\frac{DH'}{Dt} = \nabla^2 H' + \nabla H'\cdot\nabla \ln \rho \tag{15}$$

because, in the steady-state conical flow, the stagnation enthalpy is everywhere uniform. This last equation may be further simplified by noting that

$$\ln(\rho v r^2) = \text{const} \tag{16}$$

Then, for spherically symmetric flow

Nomenclature

α = ratio of local velocity to acoustic velocity far upstream

β = ratio of acoustic velocity far upstream to local acoustic velocity, a_0/a

γ_s = local fluid isentropic expansion exponent, $(\partial \ln p/\partial \ln \rho)_S$

γ^* = dimensionless fluid coefficient, $\gamma^* - 1 = (\partial/\partial \ln \rho)_S \ln(\partial p/\partial \rho)_S$

ζ = ratio of local radius to that where $\mathsf{M} = 1$ in conical flow, r/R

η = auxiliary function

ξ = modified radial variable, $\int \frac{\beta}{1 - \mathsf{M}^2}\, d\zeta$

π = dimensionless perturbed pressure, p'/p

π = also, 3.14159

ρ = fluid density

τ = dimensionless time, $a_0 t/R$

ψ = phase angle

ω = angular velocity of oscillation

Ω = solid angle subtended by cone

NOTE: Primes denote perturbed variables.
Subscript zero refers to conditions far upstream.

$$\nabla^2 H' + \nabla H' \cdot \nabla \ln \rho = \frac{1}{r^2}\frac{\partial}{\partial r}\frac{r^2 H'}{\partial r} - \frac{\partial H'}{\partial r}\left\{\frac{\partial \ln v}{\partial r} + \frac{2}{r}\right\}$$

$$= v\frac{\partial}{\partial r}\frac{1}{v}\frac{\partial H}{\partial r} \tag{17}$$

The partial differential equation for the fluctuation of the stagnation enthalpy is, therefore,

$$\frac{D}{Dt}\frac{1}{a^2}\frac{DH'}{Dt} = v\frac{\partial}{\partial r}\frac{1}{v}\frac{\partial H'}{\partial r} \tag{18}$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + v\frac{\partial}{\partial r} \tag{19}$$

It is convenient to render equation (19) dimensionless by introducing the following new variables:

$$\mathcal{H} = \frac{H'}{a_0^2}; \qquad \tau = \frac{a_0 t}{R}; \qquad \zeta = r/R \tag{20}$$

where a_0 is the acoustic velocity for the fluid in the reservoir and R is the radius (measured from the apex of the cone) where a Mach number of unity would be obtained if the conical flow actually persisted up to that radius. In terms of these variables, equation (18) becomes

$$\frac{D}{D\tau}\left(\frac{a_0}{a}\right)^2\frac{D\mathcal{H}}{D\tau} = \left(\frac{v}{a_0}\right)\frac{\partial}{\partial \zeta}\left(\frac{a_0}{v}\right)\frac{\partial \mathcal{H}}{\partial \zeta} \tag{21}$$

Further progress is aided by performing a frequency analysis in time. Therefore we assume that $\mathcal{H}$ can be represented in the form:

$$\mathcal{H} = F(\zeta)e^{i\omega\tau} \tag{22}$$

Upon insertion of this form for $\mathcal{H}$ in equation (18), we obtain the following ordinary differential equation for F:

$$F'' - F'\left[\frac{d}{d\zeta}\ln\left(\frac{\alpha}{1-\mathbf{M}^2}\right) + \frac{2i\mathbf{M}w\beta}{1-\mathbf{M}^2}\right]$$

$$+ F\left[\frac{(w\beta)^2}{1-\mathbf{M}^2} - \frac{iw\mathbf{M}}{1-\mathbf{M}^2}\frac{d\ln\beta^2}{d\zeta}\right] = 0 \tag{23}$$

where

$$\mathbf{M} = v/a; \qquad \alpha = v/a_0; \qquad \beta = a_0/a \tag{24}$$

WKB Solution for $\mathcal{H}$

We now proceed to obtain solutions to equation (23). Several transformations of variable are necessary to put this equation in more tractable form. First, let

$$F = \left(\frac{\alpha}{1-\mathbf{M}^2}\right)^{1/2}\exp\left\{iw\int_1^\zeta \frac{\beta\mathbf{M}}{1-\mathbf{M}^2}d\zeta\right\}\mathcal{V}(\zeta) \tag{25}$$

Then

$$\mathcal{V}''(\zeta) + I\mathcal{V} = 0 \tag{26}$$

where

$$I = \left(\frac{w\beta}{1-\mathbf{M}^2}\right)^2 + \frac{1}{2}\frac{d^2}{d\zeta^2}\ln\left(\frac{\alpha}{1-\mathbf{M}^2}\right)$$

$$- \frac{1}{4}\left\{\frac{d}{d\zeta}\ln\left(\frac{\alpha}{1-\mathbf{M}^2}\right)\right\}^2 \tag{27}$$

Next, let

$$\xi = \int_1 \frac{\beta}{1-\mathbf{M}^2}d\zeta \tag{28}$$

with

$$\mathcal{V} = \left(\frac{1-\mathbf{M}^2}{\beta}\right)^{1/2}\eta \tag{29}$$

so that

$$F = \left(\frac{\alpha}{\beta}\right)^{1/2}\exp\left\{iw\int \mathbf{M}d\xi\right\}\eta(\xi) \tag{30}$$

The corresponding differential equation for η is

$$\eta''(\xi) + \{w^2 - r(\xi)\}\eta(\xi) = 0 \tag{31}$$

where

$$r(\xi) = -\frac{d^2}{d\xi^2}\ln\left(\frac{\alpha}{\beta}\right)^{1/2} + \left\{\frac{d}{d\xi}\ln\left(\frac{\alpha}{\beta}\right)^{1/2}\right\}^2 \tag{32}$$

Equation (31) is in a form to which the well-known WKB method [3] can be applied. The solutions so derived are, in this case, asymptotically exact in the limits of small Mach number, $\mathbf{M}$, or of high wave number, w. In fact, as shown in the Appendix, *the error of the WKB method as used here is* $0(\mathbf{M}^3/w^2)$. The approximate solution of equation (31) which has a form at $\zeta = \infty$ corresponding to an outwardly radiating wave is

$$\eta = e^{-iw\xi}e^{\frac{i}{2w}\int r(\xi)d\xi} \tag{33}$$

With the same order of accuracy we have

$$\mathcal{H} = \left(\frac{\alpha}{\beta}\right)^{1/2}\exp\left\{iw(\tau - \xi) + iw\int \mathbf{M}d\xi + \frac{i}{2w}r(\xi)d\xi\right\} \tag{34}$$

Amplitude and Phase Relations Between Flow and Pressure Disturbances

From equations (7) and (9) we obtain for the disturbed pressure and velocity the following expressions:

$$\frac{\partial v'}{\partial t} = -\frac{\partial H'}{\partial r} \tag{35}$$

$$\frac{1}{\rho}\frac{\partial p'}{\partial t} = \frac{\partial H'}{\partial t} + v\frac{\partial H'}{\partial r} \tag{36}$$

Moreover the isentropic relation between pressure and density gives

$$p' = a^2\rho' \tag{37}$$

Therefore, the perturbed mass flux is

$$(\rho v)' = \rho v' + v\rho' = \rho v' + \frac{v}{a^2}p' \tag{38}$$

$$\frac{\partial(\rho v)'}{\partial t} = \rho\frac{\partial v'}{\partial t} + \frac{v}{a^2}\frac{\partial p'}{\partial t}$$

$$= -\rho\frac{\partial H'}{\partial r} + \frac{\rho v}{a^2}\left\{\frac{\partial H'}{\partial t} + v\frac{\partial H'}{\partial r}\right\} \tag{39}$$

In dimensionless form, equation (36) becomes

$$\frac{1}{\gamma_e\beta^2}\frac{\partial \pi}{\partial \tau} = \frac{\partial \mathcal{H}}{\partial \tau} + \frac{\mathbf{M}}{1-\mathbf{M}^2}\frac{\partial \mathcal{H}}{\partial \xi} \tag{40}$$

where

$$\pi = \frac{p'}{p}; \quad \gamma_e \equiv \left(\frac{\partial \ln p}{\partial \ln \rho}\right)_s \tag{41}$$

Correspondingly, equation (39) becomes

$$\frac{\alpha}{\beta}\frac{\partial\frac{(\rho v)'}{\rho v}}{\partial\tau} = \mathsf{M}\frac{\partial\mathcal{H}}{\partial\tau} - \frac{\partial\mathcal{H}}{\partial\xi} \tag{42}$$

Taking the ratio of equations (40) and (42), we obtain

$$\frac{\frac{\partial\pi}{\partial\tau}}{\frac{\partial\frac{(\rho v)'}{\rho v}}{\partial\tau}} = \gamma_e\mathsf{M}\,\frac{\frac{\partial\ln\mathcal{H}}{\partial\tau} + \frac{\mathsf{M}}{1-\mathsf{M}^2}\frac{\partial\ln\mathcal{H}}{\partial\xi}}{\mathsf{M}\frac{\partial\ln\mathcal{H}}{\partial\tau} - \frac{\partial\ln\mathcal{H}}{\partial\xi}} \tag{43}$$

The WKB form of $\mathcal{H}$, given by equation (34), is well adapted to evaluating this last expression. After some simplification, the following form is found; thus

$$\frac{\frac{\partial\pi}{\partial\tau}}{\frac{\partial\frac{(\rho v)'}{\rho v}}{\partial\tau}} = -\frac{\gamma_e\mathsf{M}}{1-\mathsf{M}^2}\left[\mathsf{M} + \frac{iw}{\frac{d\ln(\alpha/\beta)^{1/2}}{d\xi} - iw + \frac{ir}{2w}}\right] \tag{44}$$

Since, for complex periodic time variation of π and $(\rho v)'$, their absolute ratio and their phase are the same as those of their time derivatives, the right-hand side of equation (44) in polar form provides the desired answer. We have yet, however, to express the answer in terms of readily interpreted physical quantities.

Formulation in Terms of Strouhal and Mach Numbers

The following two results can be established after considerable tedious algebraic manipulation:

$$\frac{d\ln(\alpha/\beta)^{1/2}}{d\xi} = -\frac{1}{\zeta\beta}\left[1 - \frac{\gamma^*-1}{2}\mathsf{M}^2\right] \tag{45}$$

$$r(\xi) = \frac{\mathsf{M}^2}{(\zeta\beta)^2}\left[\frac{5\gamma^*-3}{2} + \frac{(\gamma^*-1)}{2}\frac{(3\gamma^*-5)}{2}\mathsf{M}^2\right] \tag{46}$$

where

$$\gamma^* - 1 = \left(\frac{\partial}{\partial\ln\rho}\right)_s \ln\left(\frac{\partial p}{\partial\rho}\right)_s \tag{47}$$

These results are valid for the arbitrary single-phase fluid, except that in obtaining equation (46) the derivative of γ^* at constant entropy has been neglected. For the monatomic gas, $r(\xi)$ reduces to an especially simple expression.

We now proceed to eliminate $\zeta\beta$ in the foregoing expressions in favor of the Strouhal number. This elimination is accomplished as follows. The Strouhal number at a given point in the flow is conventionally defined as $\omega D/v$. To account for the curvature of the flow area, we here modify this definition slightly, taking $\mathsf{S} = [\omega(4A/\pi)^{1/2}]/v$. To relate ω to w, we note that

$$\omega t = w\tau = w\frac{a_0 t}{R} \tag{48}$$

Then, with Ω denoting the solid angle of the cone, we obtain easily enough

$$\mathsf{S} = \frac{a_0 w\left(\frac{4A}{\pi}\right)^{1/2}}{Rv} = \frac{a_0\left(\frac{4}{\pi}\right)^{1/2} w}{v\left(\frac{1}{\Omega}\right)^{1/2}}\left(\frac{A}{A^*}\right)^{1/2} \tag{49}$$

$$= \frac{\beta w}{\mathsf{M}}\left(\frac{4\Omega}{\pi}\right)^{1/2}\zeta \tag{50}$$

As a result

$$w = \frac{\mathsf{MS}^*}{\beta\zeta} \tag{51}$$

where

$$\mathsf{S}^* = \frac{\omega\left(\frac{A}{\Omega}\right)^{1/2}}{v} = \frac{\omega r}{v} \tag{52}$$

When expressions (45), (46) and (51) are used in equation (44), the following final formula for the nozzle performance is obtained:

$$\boxed{\frac{\frac{\partial\pi}{\partial\tau}}{\frac{\partial\frac{(\rho v)'}{\rho v}}{\partial\tau}} = -\frac{\gamma_e\mathsf{M}^2}{1-\mathsf{M}^2}\left[1 - \frac{i\mathsf{S}^*}{1 - \frac{\gamma^*-1}{2}\mathsf{M}^2 + i\mathsf{M}\left\{\mathsf{S}^* - \frac{A(\mathsf{M})}{\mathsf{S}^*}\right\}}\right]} \tag{53}$$

Here $A(\mathsf{M})$ is an abbreviation for

$$A(\mathsf{M}) = \frac{5\gamma^*-3}{2} + \frac{(\gamma^*-1)}{2}\frac{(3\gamma^*-5)}{2}\mathsf{M}^2 \tag{54}$$

Discussion of Results

Although entirely accurate only in the limits of low Mach number and high Strouhal number, equation (53) does, nevertheless, yield the correct quasi-static relations as $\mathsf{S}^* \to 0$; i.e.,

$$\frac{\frac{\partial\pi}{\partial\tau}}{\frac{\partial\frac{(\rho v)'}{\rho v}}{\partial\tau}} = -\gamma_e\frac{\mathsf{M}^2}{1-\mathsf{M}^2} \tag{55}$$

We observe, among other things, that the upstream influence of pressure fluctuations vanishes as the throat Mach number approaches unity.

Now if we assume that

$$\frac{p'}{p} = Pe^{i\omega t} \tag{56}$$

$$\frac{(\rho v)'}{\rho v} = Qe^{i\omega t} \tag{57}$$

Then

$$\frac{P}{Q} = \frac{\gamma_e\mathsf{M}^2}{1-\mathsf{M}^2}Ne^{i\psi} \tag{58}$$

In Figs. 2 and 3, approximate values of the amplitude factor N and of the phase angle ψ, as found from equation (53), are shown for various Mach numbers in the case of a perfect gas with a specific heat ratio of 1.4.

From equation (53) it can be seen that the phase angle for $\mathsf{M} = 0$ starts out at 180 deg for $\mathsf{S}^* = 0$ and approaches 90 deg for $\mathsf{S}^* \to \infty$. At the same time the amplitude factor for $\mathsf{M} = 0$ in-

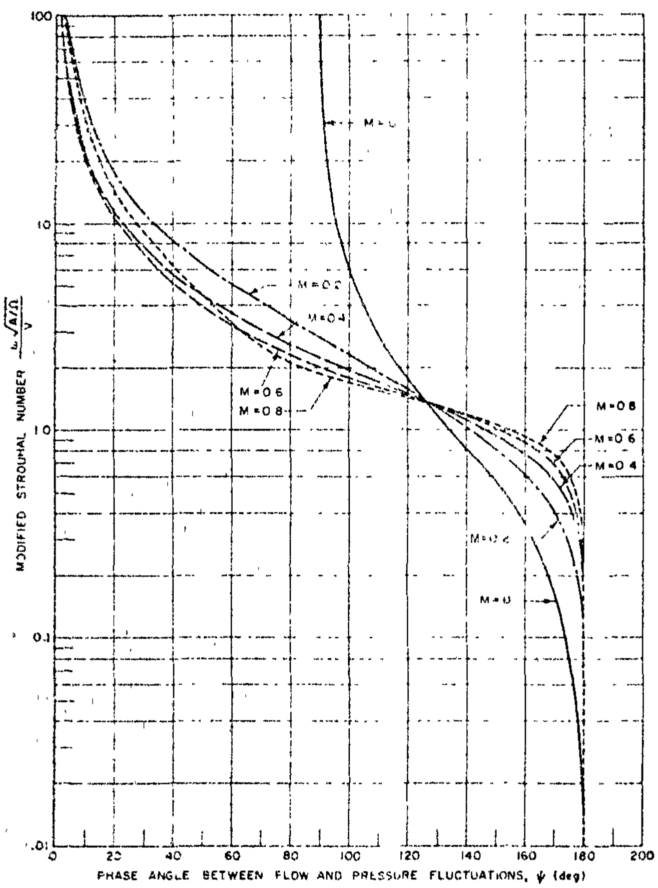

Fig. 2 Modified Strouhal number versus phase angle

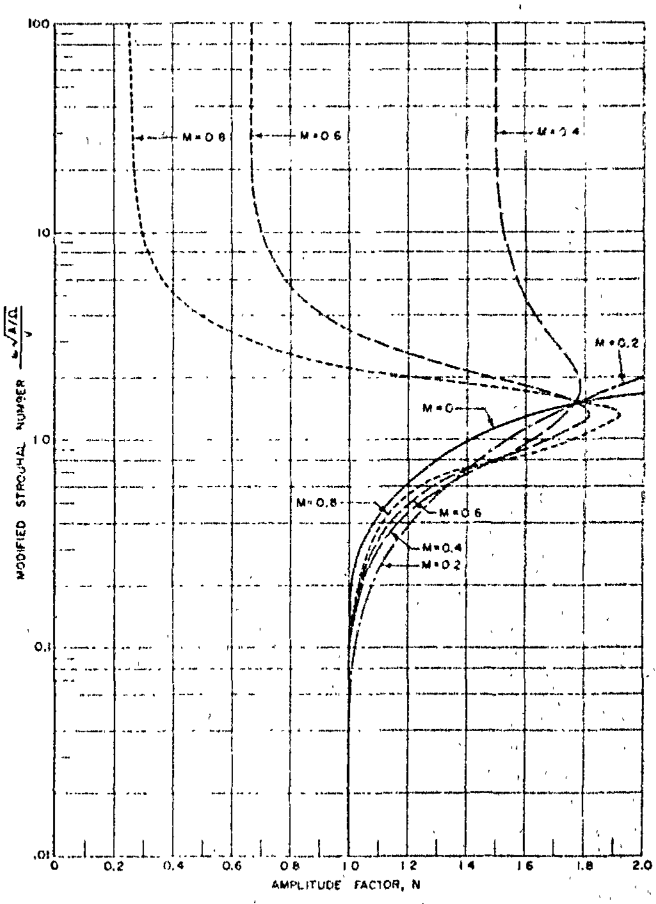

Fig. 3 Modified Strouhal number versus amplitude factor

creases steadily from unity to infinity. It is permissible to think of the $\mathbf{M} = 0$ condition as that for an incompressible fluid, for which every portion of the fluid within the nozzle must oscillate in phase. The foregoing phase and amplitude relations reflect, then, the increasing difficulty for high-frequency pressure oscillations to overcome the inertia of all the fluid in this synchronous column within the nozzle.

When the Mach number at the nozzle is finite, equation (53) shows that

$$Ne^{i\psi} \to -1; \qquad \mathbf{S}^* \to 0 \tag{59}$$

$$Ne^{i\psi} \to -1 + \frac{1}{\mathbf{M}}; \qquad \mathbf{S}^* \to \infty \tag{60}$$

Again, we obtain quasi-static behavior at low Strouhal numbers. The result for $\mathbf{S}^* \to \infty$ can be expressed in complete form as

$$\frac{\dfrac{\partial p'}{\partial \tau}}{\dfrac{\partial(\rho v)'}{\partial \tau}} \to \frac{a}{1 + \mathbf{M}}; \qquad \mathbf{S}^* \to \infty \tag{61}$$

A special case of this last formula is obtained when $\mathbf{M}$ is small. In this case, the pressure fluctuation and mass-flux fluctuation are mutually related in the same manner as they are when a plane wall vibrates adjacent to a semi-infinite medium having a finite sound velocity. The similarity comes about because, at high frequency, the waves within the nozzle are so crowded together that the effects of curvature of the phase fronts are negligible.

The trends just discussed are well illustrated by the curves in Figs. 2 and 3. In addition, we observe that, for Strouhal numbers less than 0.1, deviations in phase and amplitude from quasi-static behavior are small, and, for Strouhal numbers less than 1.0, deviations between incompressible ($M = 0$) and compressible ($\mathbf{M} > 0$) behavior are small.

One of the conclusions of the present analysis is that the angle of nozzle divergence is of some importance. However, for nozzles of reasonable design, the range of total included angle is somewhat limited, a total included angle of 60 deg being representative. For this particular angle, the modified and usual Strouhal numbers are identical.

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APPENDIX

The purpose of this Appendix is to ascertain the order of magnitude of the error incurred through use of the WKB method in the solution of equation (31). The treatment employed will follow completely, for our specific case, the analysis given in Section 3.3 of reference [4].

We have the differential equation

$$\eta''(\xi) + \{w^2 - r(\xi)\}\eta(\xi) = 0 \tag{62}$$

where

$$r(\xi) = \frac{\mathsf{M}^2}{(\zeta\beta)^2}\left[\frac{5\gamma^* - 3}{2} + \frac{(\gamma^* - 1)}{2}\frac{(3\gamma^* - 5)}{2}\mathsf{M}^2\right] \tag{63}$$

and

$$\xi = \int \frac{\beta}{1 - \mathsf{M}^2}\, d\zeta \tag{64}$$

Now let

$$\eta(\xi) \equiv e^{-iw\xi}z(\xi) \tag{65}$$

The resulting differential equation for z is, then

$$z''(\xi) - 2iwz'(\xi) - r(\xi)z = 0 \tag{66}$$

A solution of the foregoing differential equation is provided by that $z(\xi)$ which satisfies the following integral equation:

$$z(\xi) = 1 + i\int_\xi^\infty \frac{\{1 - e^{-2iw(t-\xi)}\}}{2w} L(t)z(t)t^{-2}dt \tag{67}$$

$$L(\xi) \equiv -\xi^2 r(\xi); \quad |L| \leq D \tag{68}$$

$z(\xi)$ in equation (67) can be expressed by the infinite series:

$$z(\xi) = \sum_{n=0}^{\infty} z_n(\xi) \tag{69}$$

where $z_0(\xi)$ is defined to be unity, and

$$z_{n+1}(\xi) = i\int_\xi^\infty \frac{\{1 - e^{-i2w(t-\xi)}\}}{2w} L(t)z_n(t)t^{-2}dt \tag{70}$$

As a consequence of equation (67), it can be proven that

$$|z_n(\xi)| \leq \frac{1}{n!}\frac{D^n}{(w\xi)^n} \tag{71}$$

and it can be seen that the series (69) converges for all finite ξ and w.

As a special case of (70), we have

$$z_1(\xi) = \frac{i}{2w}\int_\infty^\xi r(t)dt + \frac{i}{2w}\int_\xi^\infty e^{-i2w(t-\xi)}\, r(t)dt \tag{72}$$

Now $r(\xi)$ is zero at infinity, and we shall presume that it increases monotonically with decreasing ξ. Then, through integration by parts, we can show that

$$\left|\frac{i}{2w}\int_\xi^\infty e^{-i2w(t-\xi)}\, r(t)dt\right| \leq \frac{r}{2w^2} \tag{73}$$

As ξ approaches infinity, it is evident that

$$\mathsf{M} \sim v \sim \frac{1}{r^2}; \qquad \frac{1}{\xi} \sim \frac{1}{\zeta} \sim \frac{1}{r} \sim \mathsf{M}^{1/2} \tag{74}$$

Then

$$r = 0(\mathsf{M}^3); \qquad D = 0(\mathsf{M}^2) \tag{75}$$

$$z(\xi) = 1 + \frac{i}{2w}\int_\infty^\xi r(t)dt + 0\left(\frac{\mathsf{M}^3}{w^2}\right) + 0\left(\frac{\mathsf{M}^5}{w^2}\right) \tag{76}$$

Reference now to equation (34) of the text will show that the WKB approximation for $z(\xi)$ is

$$z(\xi) \cong e^{\frac{i}{2w}\int_\infty^\xi r(t)dt}$$

$$= 1 + \frac{i}{2w}\int_\infty^\xi r(t)dt + \left(\frac{i}{2w}\right)^2\left\{\int_\infty^\xi r(t)dt\right\}^2 + \ldots \tag{77}$$

Since

$$\int^\xi r(t)dt \text{ is } 0(\mathsf{M}^{5/2}),$$

we find by intercomparison of series (76) and (77) that the fractional error of the WKB method as employed in this work is $0(\mathsf{M}^3/w^2)$.

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