

**Tidal Friction in the Earth-Moon System
and Laplace Planes:
Darwin Redux**

by

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Abstract

The dynamical evolution of the Earth-Moon system due to tidal friction is treated here. George H. Darwin used Laplace planes (also called proper planes) in his study of tidal evolution. The Laplace plane approach is adapted here to the formalisms of W. M. Kaula and P. Goldreich. Like Darwin, the approach assumes a three-body problem: Earth, Moon, and Sun, where the Moon and Sun are point-masses. The tidal potential is written in terms of the Laplace plane angles. The resulting secular equations of motion can be easily integrated numerically assuming the Moon is in a circular orbit about the Earth and the Earth is in a circular orbit about the Sun. For Earth-Moon distances greater than ~ 10 Earth radii, the Earth's approximate tidal response can be characterized with a single parameter, which is a ratio: a Love number times the sine of a lag angle divided by another such product. For low parameter values it can be shown that Darwin's low-viscosity molten Earth, M. Ross's and G. Schubert's model of an Earth near melting, and Goldreich's equal tidal lag angles must all give similar histories. For higher parameter values, as perhaps has been the case at times with the ocean tides, the Earth's obliquity may have decreased slightly instead of increased once the Moon's orbit evolved further than 50 Earth radii from the Earth, with possible implications for climate. This is contrast to the other tidal friction models mentioned, which have the obliquity always increasing with time. As for the Moon, its orbit is presently tilted to its Laplace plane by 5.2 degrees. The equations do not allow the Moon to evolve out of its Laplace plane by tidal friction alone, so that if it was originally in its Laplace plane, the tilt arose with the addition of other mechanisms, such as resonance passages.

1. Introduction

This paper treats the tidal evolution of the Earth-Moon system from the earliest times to the present day, a topic that has been the subject of many previous papers (e.g., Darwin, 1880; MacDonald, 1964; Kaula, 1964; Goldreich, 1966; Mignard, 1981; Webb, 1982; Hansen, 1982; Ross and Schubert, 1989; Touma and Wisdom, 1994; Kagan and Maslova, 1994; Touma and Wisdom, 1998; Ward and Canup, 2000). The rationale for treating this subject once again is to update Darwin's (1880) approach to tidal friction by using modern formalisms and investigate possible histories. See also Ferraz-Mello et al. (2008).

The approach developed here is a marriage of Darwin (1880), Kaula (1964), and Goldreich (1966). Darwin used Laplace planes in his masterly treatment of tidal friction. Kaula developed the remarkable formalism for expressing the tidal perturbations of the Moon's orbit. Goldreich used Kaula's equations in his elegant vector approach. All of these elements are combined here. (For careful considerations of Kaula's formalism, see Efroimsky and Lamey, 2007; Efroimsky and Williams, 2009; and Efroimsky and Marakov, 2013, 2014.)

A three-body problem is assumed here: the Earth, Moon, and Sun. The other planets, which cause small oscillations in the Earth obliquity (e.g., Ward, 1974; Touma and Wisdom, 1994) are ignored. The Moon and Sun are point-masses. The Earth's orbit about the Sun is assumed to be unaffected by tidal friction; changing the Earth's angular momentum has little effect on the angular momentum of its solar orbit. As in Darwin (1880), all of the Laplace plane angles are taken to be small.

The tidal potential is expressed in terms of the Laplace plane angles. The equations governing the lunar orbit and the Earth's spin state are found from the tidal potential. The equations are easily numerically integrated assuming the Moon is in a circular orbit about the Earth, and the Earth is in a circular orbit about the Sun. The equations are applied to Darwin's (1880) original model of the Earth as a viscous liquid, particularly with a viscosity which gives small lag angles proportional to tidal constituent frequency; and to Ross and Schubert's (1989) model of an Earth near melting, with tidal lag angles proportional to the fourth root of the frequency. Both give tidal histories for

the Earth-Moon system that are little different from those of Goldreich (1966) and Touma and Wisdom (1994). It is shown that for lunar distances greater than 10 Earth radii, the equations can be written containing a parameter Δ_{12} , where Δ_{12} is a ratio: a Love number times a lag angle divided by another such product. For values of $\Delta_{12} \leq 1$, as holds for the models just mentioned, all these histories must be quite similar, with the Earth's obliquity increasing nearly linearly with Earth-Moon distance.

A simple model of the ocean tides is also examined here. In this case the value of Δ_{12} depends on past configurations of the ocean basins. If Δ_{12} was ≥ 1.6 once the Moon's orbit evolved further than 50 Earth radii from the Earth, then the Earth's obliquity may have decreased slightly instead of increased, in contrast to the other models. An obliquity decrease would have implications for the Earth's past climate.

As for the Moon, its orbit is currently inclined to its Laplace plane by 5.2° . The equations here do not allow the Moon to evolve out of its Laplace plane, which is also true of the previous tidal friction treatments, so that the orbit probably became tilted through some process other than just tidal friction, such the resonances suggested by Touma and Wisdom (1998) and Ward and Canup (2000).

Only tidal friction is considered here. Other effects, such as climate friction (Rubincam 1990, 1995; Ito et al. 1995; Levrard and Laskar, 2003), (also called obliquity-oblateness feedback; Bills 1994) can increase the Earth's obliquity, while core-mantle coupling can decrease it (e.g., Aoki, 1969; Néron de Surgy and Laskar, 1997; Touma and Wisdom, 2001; Correia, 2006). Both climate friction and core-mantle coupling are ignored, as are the resonances in the early Earth-Moon system which occurred when the Moon was less than 6 Earth radii from the Earth (Touma and Wisdom, 1998; Ward and Canup, 2000). Integrations stop below 7 Earth radii, except for a brief consideration of what happens at 3.8 Earth radii. Moreover, only second degree spherical harmonics in the tidal potential are considered here. Atmospheric tides are ignored.

2. Laplace planes

The Moon orbits the Earth; let \mathbf{b} be the unit vector normal to the Moon's orbital plane. Also, let \mathbf{s} be the unit vector along the Earth's spin axis, and \mathbf{c} be the unit vector

normal to the ecliptic. Let the orbital angular momentum of the Moon be $h\mathbf{b}$, and the spin angular momentum of the Earth be $H\mathbf{s}$, so that h is the magnitude of the Moon's orbital angular momentum, and H is the magnitude of the Earth's spin angular momentum. The equations governing the evolution of the Earth-Moon system are given by:

$$\frac{d(h\mathbf{b})}{dt} = -L(\mathbf{s} \cdot \mathbf{b})(\mathbf{s} \times \mathbf{b}) + 2K_2(\mathbf{r}_s \cdot \mathbf{b})(\mathbf{r}_s \times \mathbf{b}) + \mathbf{T}_M \quad (1)$$

$$\frac{d(H\mathbf{s})}{dt} = +L(\mathbf{s} \cdot \mathbf{b})(\mathbf{s} \times \mathbf{b}) + 2K_1(\mathbf{r}_s \cdot \mathbf{s})(\mathbf{r}_s \times \mathbf{s}) + \mathbf{T}_E \quad (2)$$

The equations and notation largely follow Goldreich (1966). One notation change is that \mathbf{s} is used for the unit vector in the direction of the Earth's spin. Goldreich uses \mathbf{a} , but \mathbf{s} is used here to avoid confusion with the Moon's semimajor axis a . Throughout this paper boldface lower case letters always denote unit vectors. Also, \mathbf{r}_s is the unit vector in the direction from the Earth to the Sun.

In (1) and (2) t is time and $\mathbf{T}_M = \mathbf{T}_{MM} + \mathbf{T}_{SM}$ is the tidal torque acting on the Moon's orbit due to the tides raised on the Earth. The body raising the tides is given by the first subscript, and the body being acted upon by the tides by the second subscript. Thus \mathbf{T}_{MM} is the torque on the Moon's orbit from the Moon's own tides, while \mathbf{T}_{SM} is the torque on the lunar orbit from the solar tides. Likewise, $\mathbf{T}_E = \mathbf{T}_{ME} + \mathbf{T}_{SE}$ is the total tidal torque on the Earth's spin angular momentum from lunar and solar tides.

The first term on the right side of (1) is due to the Earth's equatorial bulge acting on the Moon's orbit, where

$$L = \frac{3}{2} J_2 G M_E M_M \left(\frac{R_E^2}{a^3} \right) \quad (3)$$

with G being the universal constant of gravitation, M_E the mass of the Earth, R_E the radius of the Earth, J_2 the second degree term for the equatorial bulge in the spherical harmonic

expansion of the Earth's gravitational field (currently $J_2 \approx 10^{-3}$; e.g., Stacey (1992, pp. 135-136))

, M_M the Moon's mass, and a the semimajor axis of the Moon's orbit. The second term in (1) is due to the Sun, where

$$K_2 = \frac{3}{4} GM_S M_M \left(\frac{a^2}{a_S^3} \right) \quad (4)$$

and $a_S = 1$ AU is the semimajor axis of the Earth's orbit and M_S is the Sun's mass. The first two terms on the right side of (2) give the effect of the Moon and Sun on the Earth's equatorial bulge, where

$$K_1 = \frac{3}{2} J_2 GM_S M_E \left(\frac{R_E^2}{a_S^3} \right) . \quad (5)$$

Equations (3)-(5) also use Goldreich's notation.

Equations (1) and (2) will first be solved by assuming $\mathbf{T}_M = \mathbf{T}_E = 0$, so that no tidal torques are operative. The reason for doing this is to elicit the Laplace planes, which will be used later. Without the tidal torques and averaged over time (1) and (2) become (Goldreich, 1966):

$$\frac{d(h\mathbf{b})}{dt} = -L(\mathbf{s} \cdot \mathbf{b})(\mathbf{s} \times \mathbf{b}) + K_2(\mathbf{b} \cdot \mathbf{c})(\mathbf{b} \times \mathbf{c}) = h\mathbf{Q}_M \quad (6)$$

$$\frac{d(H\mathbf{s})}{dt} = +L(\mathbf{s} \cdot \mathbf{b})(\mathbf{s} \times \mathbf{b}) + K_1(\mathbf{s} \cdot \mathbf{c})(\mathbf{s} \times \mathbf{c}) = H\mathbf{Q}_E \quad (7)$$

where $h\mathbf{Q}_M$ and $H\mathbf{Q}_E$ are all the cross-product terms in the above equations, and \mathbf{c} is the unit vector normal to the ecliptic.

Laplace (1966) gave an approximate solution to (6) and (7) in terms of Laplace planes, which are also called proper planes (Darwin, 1880; Allan and Cook, 1964, p. 108). A more modern derivation of Laplace planes is given in Appendix A. The results are the following. Let \mathbf{x} , \mathbf{y} , and \mathbf{z} be unit vectors along their respective axes in the (x, y, z) inertial coordinate system shown in Fig. 1. Let the unit vector normal to the Moon's orbital plane be

$$\mathbf{b} = b_x \mathbf{x} + b_y \mathbf{y} + b_z \mathbf{z} \quad (8)$$

in the (x, y, z) system. Let the (x_{LM}, y_{LM}, z_{LM}) coordinate system be the Laplace plane system for the Moon, with unit vectors \mathbf{x}_{LM} , \mathbf{y}_{LM} , and \mathbf{z}_{LM} along the respective axes. The z_{LM} axis is tilted with respect to the z -axis by an angle θ_M . The x_{LM} axis lies in the x - y plane with ϕ_M being the angle between the x - and x_{LM} - axes. In the (x_{LM}, y_{LM}, z_{LM}) system

$$\mathbf{b} = (\sin J_M \sin \Omega_M) \mathbf{x}_{LM} - (\sin J_M \cos \Omega_M) \mathbf{y}_{LM} + (\cos J_M) \mathbf{z}_{LM} \quad (9)$$

where J_M is the angle between the z_{LM} -axis and \mathbf{b} , and Ω_M is the nodal angle of the orbit in the Moon's Laplace plane.

Likewise, the unit vector along the Earth's spin axis is

$$\mathbf{s} = s_x \mathbf{x} + s_y \mathbf{y} + s_z \mathbf{z} . \quad (10)$$

in the (x, y, z) system (see Fig. 2). The Earth's Laplace plane system is (x_{LE}, y_{LE}, z_{LE}) , with the corresponding unit vectors \mathbf{x}_{LE} , \mathbf{y}_{LE} , and \mathbf{z}_{LE} . The z_{LE} -axis is tilted with respect to the z -axis by the angle θ_E . The x_{LE} axis lies in the x - y plane with ϕ_E being the angle between the x - and x_{LE} - axes. In the (x_{LE}, y_{LE}, z_{LE}) system

$$\mathbf{s} = (\sin J_E \sin \Omega_E) \mathbf{x}_{LE} - (\sin J_E \cos \Omega_E) \mathbf{y}_{LE} + (\cos J_E) \mathbf{z}_{LE} . \quad (11)$$

In this case J_E is the angle between the z_{LE} -axis and \mathbf{s} , while Ω_E is the Earth's nodal angle in the Earth's Laplace plane. The unit vector normal to the ecliptic will be denoted by

$$\mathbf{c} = c_x \mathbf{x} + c_y \mathbf{y} + c_z \mathbf{z} . \quad (12)$$

Here the ecliptic is taken to be the x - y plane, so that $\mathbf{c} = \mathbf{z}$.

Equations (6) and (7) have the following approximate solutions (Appendix A). Assume J_M , J_E , θ_M , and θ_E are small and all are >0 . The z_{LM} -, z_{LE} -, and z -axes all lie in a vertical plane, such that $\phi_M = \phi_E = \phi$. Moreover, θ_E and θ_M are each constant, but ϕ precesses approximately uniformly with time at a speed of

$$\dot{\phi}_0 \approx [-L \sin 2(\theta_E - \theta_M) + (2K_1)] / (2H \sin \theta_E) \quad (13)$$

in the negative direction. Further, \mathbf{b} precesses around z_{LM} with constant J_M , with its node Ω_M decreasing at an approximate uniform rate of

$$\dot{\Omega}_0 \approx -[L(\sin J_M + \sin J_E) + K_2 \sin J_M] / (h \sin J_M) - \dot{\phi}_0 . \quad (14)$$

Also, \mathbf{s} precesses around z_{LE} with constant J_E , with its node Ω_E moving with the same rate as Ω_M , but with $\Omega_E = \Omega_M + \pi$. In other words, Ω_M and Ω_E are always 180° out of phase. Finally, $\sin \theta_M$ and $\sin \theta_E$ are related to each other by

$$\sin \theta_M = \alpha \sin \theta_E \quad (15)$$

and $\sin J_M$ and $\sin J_E$ are related to each other by

$$\sin J_E = \beta \sin J_M \quad (16)$$

where by Appendix A

$$2\alpha = 1 + \frac{K_1}{L} - \left(\frac{H}{h} \right) \left(1 + \frac{K_2}{L} \right)$$

$$+ \left[\left(1 + \frac{K_1}{L} \right)^2 + \left(\frac{H}{h} \right)^2 \left(1 + \frac{K_2}{L} \right)^2 - 2 \left(\frac{H}{h} \right) \left(1 + \frac{K_1}{L} \right) \left(1 + \frac{K_2}{L} \right) + 4 \left(\frac{H}{h} \right) \right]^{1/2} \quad (17)$$

and

$$2\beta = - \left[1 + \frac{K_2}{L} - \left(\frac{h}{H} \right) \left(1 - \frac{K_1}{L} \right) \right] + \left[\left(1 + \frac{K_2}{L} \right)^2 + \left(\frac{h}{H} \right)^2 \left(1 - \frac{K_1}{L} \right)^2 - 2 \left(\frac{h}{H} \right) \left(1 + \frac{K_2}{L} \right) \left(1 - \frac{K_1}{L} \right) + 4 \left(\frac{h}{H} \right) \right]^{1/2} . \quad (18)$$

In the next section equations (17)-(18) will be used to reduce the number of independent variables.

3. Tidal torques

In the absence of tidal torques h , H , a , J_M , J_E , θ_M , and θ_E do not change secularly; but when tidal friction is present all of these quantities slowly evolve. Equations (1)-(2) can now be written

$$\frac{d\mathbf{b}}{dt} = \mathbf{Q}_M - \frac{1}{h} \frac{dh}{dt} \mathbf{b} + \frac{\mathbf{T}_M}{h} \quad (19)$$

$$\frac{d\mathbf{s}}{dt} = \mathbf{Q}_E - \frac{1}{H} \frac{dH}{dt} \mathbf{s} + \frac{\mathbf{T}_E}{H} \quad (20)$$

where

$$\frac{dh}{dt} = \mathbf{T}_M \cdot \mathbf{b} \quad (21)$$

and

$$\frac{dH}{dt} = \mathbf{T}_E \cdot \mathbf{s} \quad (22)$$

The components of \mathbf{b} and \mathbf{s} in the (x, y, z) system are

$$b_x = \sin J_M (\sin \Omega_M \cos \phi_M + \cos \theta_M \cos \Omega_M \sin \phi_M) + \cos J_M \sin \theta_M \sin \phi_M \quad (23)$$

$$b_y = \sin J_M (\sin \Omega_M \sin \phi_M - \cos \theta_M \cos \Omega_M \cos \phi_M) - \cos J_M \sin \theta_M \cos \phi_M \quad (24)$$

$$b_z = -\sin J_M \sin \theta_M \cos \Omega_M + \cos J_M \cos \theta_M \quad (25)$$

and similarly

$$s_x = \sin J_E (\sin \Omega_E \cos \phi_E + \cos \theta_E \cos \Omega_E \sin \phi_E) + \cos J_E \sin \theta_E \sin \phi_E \quad (26)$$

$$s_y = \sin J_E (\sin \Omega_E \sin \phi_E - \cos \theta_E \cos \Omega_E \cos \phi_E) - \cos J_E \sin \theta_E \cos \phi_E \quad (27)$$

$$s_z = -\sin J_E \sin \theta_E \cos \Omega_E + \cos J_E \cos \theta_E \quad (28)$$

Differentiating (23)-(28) with respect to time t , and then taking the dot-product of $\mathbf{x} \cos \phi_M$ and $\mathbf{y} \sin \phi_M$ with (19) and adding them together yields

$$\begin{aligned} & \cos \phi_M \frac{db_x}{dt} + \sin \phi_M \frac{db_y}{dt} \\ &= (\cos J_M \sin \Omega_M) \frac{dJ_M}{dt} + (\sin J_M \cos \Omega_M) \frac{d\Omega_M}{dt} \\ &+ (0) \frac{d\theta_M}{dt} + (\sin J_M \cos \theta_M \cos \Omega_M + \cos J_M \sin \theta_M) \frac{d\phi_M}{dt} \end{aligned} \quad (29)$$

$$= \mathbf{Q}_M \cdot (\mathbf{x} \cos \phi_M + \mathbf{y} \sin \phi_M) + R_{MA}$$

where

$$R_{MA} = +\frac{\cos \phi_M}{h}(\mathbf{T}_M \cdot \mathbf{x}) + \frac{\sin \phi_M}{h}(\mathbf{T}_M \cdot \mathbf{y}) - \frac{\sin J_M \sin \Omega_M}{h}(\mathbf{T}_M \cdot \mathbf{b}) \quad (30)$$

Similarly,

$$\begin{aligned} & \cos \theta_M \left(\sin \phi_M \frac{db_x}{dt} - \cos \phi_M \frac{db_y}{dt} \right) - \sin \theta_M \frac{db_z}{dt} \\ &= (\cos J_M \cos \Omega_M) \frac{dJ_M}{dt} - (\sin J_M \sin \Omega_M) \frac{d\Omega_M}{dt} \\ &+ (\cos J_M) \frac{d\theta_M}{dt} - (\sin J_M \cos \theta_M \sin \Omega_M) \frac{d\phi_M}{dt} \\ &= \cos \theta_M \mathbf{Q}_M \cdot (\mathbf{x} \sin \phi_M - \mathbf{y} \cos \phi_M) - \sin \theta_M \mathbf{Q}_M \cdot \mathbf{z} \end{aligned} \quad (31)$$

where

$$\begin{aligned} R_{MB} &= \frac{\cos \theta_M \sin \phi_M}{h}(\mathbf{T}_M \cdot \mathbf{x}) - \frac{\cos \theta_M \cos \phi_M}{h}(\mathbf{T}_M \cdot \mathbf{y}) \\ &- \frac{\sin \theta_M}{h}(\mathbf{T}_M \cdot \mathbf{z}) - \frac{\sin J_M \cos \Omega_M}{h}(\mathbf{T}_M \cdot \mathbf{b}) . \end{aligned} \quad (32)$$

The analogous equations for the Earth are by (20)

$$\begin{aligned} & \cos \phi_S \frac{ds_x}{dt} + \sin \phi_E \frac{ds_y}{dt} \\ &= (\cos J_E \sin \Omega_E) \frac{dJ_E}{dt} + (\sin J_E \cos \Omega_E) \frac{d\Omega_E}{dt} \\ &+ (0) \frac{d\theta_E}{dt} + (\sin J_E \cos \theta_E \cos \Omega_E + \cos J_E \sin \theta_E) \frac{d\phi_E}{dt} , \end{aligned} \quad (33)$$

$$= \mathbf{Q}_E \cdot (\mathbf{x} \cos \phi_E + \mathbf{y} \sin \phi_E) + R_{EA}$$

where

$$R_{EA} = \frac{\cos \phi_E}{H} (\mathbf{T}_E \cdot \mathbf{x}) + \frac{\sin \phi_E}{H} (\mathbf{T}_E \cdot \mathbf{y}) - \frac{\sin J_E \sin \Omega_E}{H} (\mathbf{T}_E \cdot \mathbf{s}) \quad (34)$$

and

$$\begin{aligned} & \cos \theta_E \left(\sin \phi_E \frac{ds_x}{dt} - \cos \phi_E \frac{ds_y}{dt} \right) - \sin \theta_E \frac{ds_z}{dt} \\ &= (\cos J_E \cos \Omega_E) \frac{dJ_E}{dt} - (\sin J_E \sin \Omega_E) \frac{d\Omega_E}{dt} \\ &+ (\cos J_E) \frac{d\theta_E}{dt} - (\sin J_E \cos \theta_E \sin \Omega_E) \frac{d\phi_E}{dt} \end{aligned} \quad (35)$$

$$= \cos \theta_E \mathbf{Q}_E \cdot (\mathbf{x} \sin \phi_E - \mathbf{y} \cos \phi_E) - \sin \theta_E \mathbf{Q}_E \cdot \mathbf{z} + R_{EB}$$

where

$$\begin{aligned} R_{EB} &= \frac{\cos \theta_E \sin \phi_E}{H} (\mathbf{T}_E \cdot \mathbf{x}) - \frac{\cos \theta_E \cos \phi_E}{H} (\mathbf{T}_E \cdot \mathbf{y}) \\ &- \frac{\sin \theta_E}{H} (\mathbf{T}_E \cdot \mathbf{z}) - \frac{\sin J_E \cos \Omega_E}{H} (\mathbf{T}_E \cdot \mathbf{s}) . \end{aligned} \quad (36)$$

Equations (29), (31), (33), and (35) are four equations in eight unknowns. Four additional equations must be specified in order to obtain a unique solution. These additional

equations will be those found for the Laplace planes in the preceding section, namely (17), (18), and

$$\phi_E = \phi_M = \phi, \quad (37)$$

$$\Omega_E = \Omega_M + \pi = \Omega + \pi \quad (38)$$

where $\Omega_M = \Omega$. Eliminating ϕ_E , Ω_E , J_E , and θ_M appearing in the derivatives in (29), (31), (33), and (35) yields four equations in four unknowns:

$$\begin{aligned} & (\cos J_M \sin \Omega) \frac{dJ_M}{dt} + (\sin J_M \cos \Omega) \frac{d\Omega}{dt} \\ & + (0) \frac{\alpha \cos \theta_E}{\cos \theta_M} \frac{d\theta_E}{dt} + (\sin J_M \cos \theta_M \cos \Omega + \cos J_M \sin \theta_M) \frac{d\phi}{dt} \\ & = R_{MA} - (0) \frac{\sin \theta_E}{\cos \theta_M} \frac{d\alpha}{dt} \end{aligned} \quad (39)$$

$$\begin{aligned} & (\cos J_M \cos \Omega) \frac{dJ_M}{dt} - (\sin J_M \sin \Omega) \frac{d\Omega}{dt} \\ & + \left(\frac{\alpha \cos \theta_E \cos J_M}{\cos \theta_M} \right) \frac{d\theta_E}{dt} - (\sin J_M \cos \theta_M \sin \Omega) \frac{d\phi}{dt} \\ & = R_{MB} - \left(\frac{\cos J_M \sin \theta_E}{\cos \theta_M} \right) \frac{d\alpha}{dt} \end{aligned} \quad (40)$$

$$\begin{aligned} & -(\beta \cos J_M \sin \Omega) \frac{dJ_M}{dt} - (\sin J_E \cos \Omega) \frac{d\Omega}{dt} \\ & + (0) \frac{d\theta_E}{dt} + (-\sin J_E \cos \theta_E \cos \Omega + \cos J_E \sin \theta_E) \frac{d\phi}{dt} \end{aligned}$$

$$= R_{EA} + \sin J_M \sin \Omega \frac{d\beta}{dt} \quad (41)$$

$$\begin{aligned} & (-\beta \cos J_E \cos \Omega) \frac{dJ_M}{dt} + (\sin J_E \sin \Omega) \frac{d\Omega}{dt} \\ & + \cos J_E \frac{d\theta_E}{dt} + (\sin J_E \cos \theta_E \sin \Omega) \frac{d\phi}{dt} \\ & = R_{EB} + \sin J_M \cos \Omega \frac{d\beta}{dt} \end{aligned} \quad (42)$$

Assuming small angles, so that $\cos \theta_M \approx \cos \theta_E \approx \cos J_M \approx \cos J_E \approx 1$ in (39)-(42), and solving the set of linear equations gives

$$\begin{aligned} \frac{dJ_M}{dt} & \approx \left(\frac{1}{1 + \alpha\beta} \right) \left(R_{MA} \sin \Omega + R_{MB} \cos \Omega - \sin \theta_E \cos \Omega \frac{d\alpha}{dt} \right) \\ & - \left(\frac{\alpha}{1 + \alpha\beta} \right) \left(R_{EA} \sin \Omega + R_{EB} \cos \Omega + \sin J_M \frac{d\beta}{dt} \right) \end{aligned} \quad (43)$$

and

$$\frac{d\theta_E}{dt} \approx \left(\frac{1}{1 + \alpha\beta} \right) \left(\beta R_{MB} + R_{EB} + \sin J_M \cos \Omega \frac{d\beta}{dt} - \beta \sin \theta_E \frac{d\alpha}{dt} \right) \quad (44)$$

where $d\alpha/dt$ and $d\beta/dt$ are given in Appendix A.

4. The equations for dJ_M/dt and $d\theta_E/dt$

Finding R_{MA} , R_{MB} , R_{EA} , and R_{EB} is quite lengthy. General expressions for the torque dot-products are derived in Appendix B. Specific expressions for the torques are found from the tidal potential. The tidal potential is derived in Appendix C. The tidal potential is

$$\begin{aligned}
V_{m>0} = & \frac{GM^* R_E^5}{a^3 (a^*)^3} \sum_{n=0}^2 \sum_{N=0}^2 \sum_{p=0}^2 \sum_{P=0}^2 \sum_{q=-\infty}^{+\infty} \sum_{Q=-\infty}^{+\infty} G_{2pq}(e^*) G_{2PQ}(e) F_{2np}(J^*) F_{2NP}(\tilde{J}) \\
& \sum_{m=1}^2 \frac{2(2-m)!}{(2+m)!} \sum_{j=-2}^2 \sum_{J=-2}^2 \sum_{\gamma=1}^2 k_{2mnj\gamma pq}^* B_{2m1nj\gamma}(J_E^*, d^*) B_{2m1NJ\gamma}(J_E, d) \\
& \cdot \cos\{(2-2p)\omega^* + (2-2p+q)\bar{M}^* + n\Omega^* + j\Omega_E + \delta_{2mnj\gamma pq}^* \\
& - [(2-2P)\omega + (2-2P+Q)\bar{M} + N\Omega + J\Omega_E]\} \\
& + (-1)^m k_{2mnj\gamma pq}^* B_{2m1nj\gamma}(J_E^*, d^*) B_{2m1NJ(3-\gamma)}(J_E, d) \\
& \cdot \cos\{(2-2p)\omega^* + (2-2p+q)\bar{M}^* + n\Omega^* + j\Omega_E + \delta_{2mnj\gamma pq}^* \\
& + [(2-2P)\omega + (2-2P+Q)\bar{M} + N\Omega + J\Omega_E]\} .
\end{aligned} \tag{45}$$

Here M^* is the mass of the tide-raising body, the $G_{2pq}(e^*)$ are the second degree eccentricity functions and the $F_{2np}(J^*)$ are the second degree inclination functions, while the $(a^*, e^*, J^*, \Omega^*, \omega^*, \bar{M}^*)$ are the Keplerian elements of the tide-raising body: a^* is the semimajor axis, e^* is the orbital eccentricity, J^* is the orbital inclination, Ω^* is the nodal position, ω^* is the argument of perigee, and \bar{M}^* is the mean anomaly, all measured in the tide-raising body's Laplace plane system. Also, $J^* = J_M$ or $J^* = J_S$, depending upon whether the Moon or the Sun is the tide-raising body. The Keplerian elements of the body being acted upon by the tides are given without asterisks. The $B_{2m1nj\gamma}(J_E, d)$ functions are derived from the considerations in Appendix D. Table 1 lists the ones which depend on the zeroth- and first-order in the sines of the angles, which are the only ones needed here. In the following $e = 0$ for the lunar and solar orbits so that only the $Q = q = 0$, $p = P = 1$ terms in $G_{2pq}(e^*)$ and $G_{2PQ}(e)$ are non-zero, with $G_{210}(0) = 1$. In (45) $k_{2mnj\gamma pq}^*$ is the Love number, while $\delta_{2mnj\gamma pq}^*$ is the lag angle associated with each trigonometric argument.

Let $\boldsymbol{\mu}_1$, $\boldsymbol{\mu}_2$, and $\boldsymbol{\mu}_3$ be the respective unit vectors along the axes of the Moon's (x_μ, y_μ, z_μ) system as shown in Fig. 3, where $\boldsymbol{\mu}_3$ is identical with the \mathbf{b} vector and is normal to the orbit, $\boldsymbol{\mu}_1$ lies along the nodal line, and $\boldsymbol{\mu}_2$ makes the system right-handed. Let $T_{M1} = \mathbf{T}_M \cdot \boldsymbol{\mu}_1$, $T_{M2} = \mathbf{T}_M \cdot \boldsymbol{\mu}_2$, and $T_{M3} = \mathbf{T}_M \cdot \boldsymbol{\mu}_3$ be the torque components (T_{M1} , T_{M2} , T_{M3}) in the (x_μ, y_μ, z_μ) system. Likewise by analogy to the Moon's (x_μ, y_μ, z_μ) system, let the Earth's (x_ξ, y_ξ, z_ξ) system have unit vectors $\boldsymbol{\xi}_1$, $\boldsymbol{\xi}_2$, $\boldsymbol{\xi}_3$, with $\boldsymbol{\xi}_3 = \mathbf{s}$, where \mathbf{s} is the unit vector along the spin axis, $\boldsymbol{\xi}_1$ lies along the Earth's nodal line, and $\boldsymbol{\xi}_2$ makes the system right-handed (Fig. 4). The torque on the Earth \mathbf{T}_E will have the components $T_{E1} = \mathbf{T}_E \cdot \boldsymbol{\xi}_1$, $T_{E2} = \mathbf{T}_E \cdot \boldsymbol{\xi}_2$, and $T_{E3} = \mathbf{T}_E \cdot \boldsymbol{\xi}_3$ in the (x_ξ, y_ξ, z_ξ) system.

The torques are found from (45). For instance, the first torque that appears on the right-hand side of (B1) is $T_{M2} = T_{MM2} + T_{SM2}$. For circular orbits the tidal torque T_{MM2} on the Moon's orbit from the tidal potential V_{MM} from the lunar tides is

$$T_{MM2} = M_M \left(\frac{1}{\sin J_M} \frac{\partial V_{MM}}{\partial \Omega_M} - \cot J_M \frac{\partial V_{MM}}{\partial \omega_M} \right)$$

(e.g., Goldreich, 1966, p. 429, after multiplying his expression by a missing factor of M_M). Only the secular part of T_{MM2} is desired; hence the periodic parts must vanish. After taking the derivatives, the step in making the mean anomaly vanish in the trigonometric arguments in V_{MM} is to note that this happens when $p = P$ in the first argument, and $p = 2 - P$ in the second argument. This allows the summation over P to be eliminated, yielding

$$T_{MM2} = \frac{2GM_M^2}{R_E} \left(\frac{R_E}{a} \right)^6 \sum_{n=0}^2 \sum_{N=0}^2 \sum_{p=0}^2 \sum_{m=1}^2 \frac{(2-m)!}{(2+m)!}$$

$$\sum_{j=-2J}^2 \sum_{\gamma=-2\gamma}^2 \sum_{\gamma=1}^2 k_{2mnj\gamma p 0}^M B_{2m1nj\gamma}(J_E^*, d^*) B_{2m1NJ\gamma}(J_E, d) F_{2np}(J_M) F_{2Np}(J_M)$$

$$\cdot \left[\frac{N - (2 - 2p) \cos J_M}{\sin J_M} \right] \sin[(n - N)\Omega_M + (j - J)\Omega_E + \delta_{2mnj\gamma p 0}^M]$$

$$+ (-1)^{m+1} k_{2mnj\gamma p 0}^M B_{2m1nj\gamma}(J_E^*, d^*) B_{2m1NJ(3-\gamma)}(J_E, d) F_{2np}(J_M) F_{2N(2-p)}(J_M)$$

$$\cdot \left[\frac{N + (2 - 2p) \cos J_M}{\sin J_M} \right] \sin[(n + N)\Omega_M + (j + J)\Omega_E + \delta_{2mnj\gamma p 0}^M]$$

Other examples are

$$T_{MS1} = M_s \frac{\partial V_{MS}}{\partial J_M}$$

where

$$\vec{T}_{MS1} \cdot \vec{\xi}_2 = \frac{GM_M M_S}{R_E} \left(\frac{R_E}{a} \right)^3 \left(\frac{R_E}{a_S} \right)^3 \sum_{n=0}^2 \sum_{N=0}^2 \sum_{m=1}^2 \frac{(2-m)!}{(2+m)!}$$

$$\sum_{j=-2}^2 \sum_{J=-2}^2 \sum_{\gamma=1}^2 \sum_{f=0}^1 \sum_{g=-1}^1 k_{2mnj\gamma 10}^M B_{2m1nj\gamma}(J_E^*, d^*) B_{2m1NJ\gamma}(J_E, d) F_{2n1}(J_M) \frac{dF_{2N1}(J_S)}{dJ_S} U_{1g}^{S1, \xi 2}$$

$$\cdot \{-\sin[(-N - f)\Omega_S + n\Omega_M + (j - J - g)\Omega_E + \delta_{2mnj\gamma 10}^M]\}$$

$$+ \sin[(-N + f)\Omega_S + n\Omega_M + (j - J + g)\Omega_E + \delta_{2mnj\gamma 10}^M]\}$$

$$+ (-1)^m k_{2mnj\gamma 10}^M B_{2m1nj\gamma}(J_E^*, d^*) B_{2m1NJ(3-\gamma)}(J_E, d) F_{2n1}(J_M) \frac{dF_{2N1}(J_S)}{dJ_S} U_{1g}^{S1, \xi 2}$$

$$\cdot \{-\sin[(N - f)\Omega_S + n\Omega_M + (j + J - g)\Omega_E + \delta_{2mnj\gamma 10}^M]\}$$

$$+\sin[(N+f)\Omega_S+n\Omega_M+(j+J+g)\Omega_E+\delta_{2mnj\gamma 10}^M]\}$$

and

$$T_{SS3} = M_S \frac{\partial V_{SS}}{\partial \bar{M}_S}$$

$$(\vec{T}_{SS3} \cdot \xi_1) \sin \Omega_E = \frac{GM_S^2}{R_E} \left(\frac{R_E}{a_S} \right)^6 \sum_{n=0}^2 \sum_{N=0}^2 \sum_{p=0}^2 \sum_{m=1}^2 \frac{(2-m)!}{(2+m)!}$$

$$\sum_{j=-2}^2 \sum_{J=-2}^2 \sum_{\gamma=1}^2 \sum_{f=0}^1 \sum_{g=-2}^2 k_{2mnj\gamma p 0}^S B_{2m1nj\gamma}(J_E^*, d^*) B_{2m1NJ\gamma}(J_E, d) F_{2np}(J_S) F_{2Np}(J_S) W_{fg}^{S3, \xi 1}$$

$$\cdot \{(2-2p)\sin[(n-N-f)\Omega_S+(j-J-g)\Omega_E+\delta_{2mnj\gamma p 0}^S]$$

$$+(2-2p)\sin[(n-N+f)\Omega_S+(j-J+g)\Omega_E+\delta_{2mnj\gamma p 0}^S]\}$$

$$+(-1)^{m+1} k_{2mnj\gamma p 0}^S B_{2m1nj\gamma}(J_E^*, d^*) B_{2m1NJ(3-\gamma)}(J_E, d) F_{2np}(J_S) F_{2N(2-p)}(J_S) W_{fg}^{S3, \xi 1}$$

$$\cdot \{(2p-2)\sin[(n+N-f)\Omega_S+(j+J-g)\Omega_E+\delta_{2mnj\gamma p 0}^S]$$

$$+(2p-2)\sin[(n+N+f)\Omega_S+(j+J+g)\Omega_E+\delta_{2mnj\gamma p 0}^S]\}$$

where the $U_{fg}^{S1, \xi 2}$, $W_{fg}^{S3, \xi 1}$, etc. functions are given in Appendix B.

These and similar expressions go into (B1) and (B2). Only the secular terms are desired in these expressions, which means choosing values for n, N, j, J, f , and g which make periodic terms vanish, leaving only sines of the lag angles. In choosing, one must be careful to note two things in these expressions. The first is that Ω_E is set to $\Omega_M + \pi$ as

in (36). The second is that Ω_S is set to π after any differentiation with respect to Ω_S , so the coefficient of Ω_S does not necessarily vanish. Hence π must be dealt with inside the arguments.

Only terms which are first-order in $\sin J_M \approx J_M$, $\sin J_E \approx J_E$, $\sin \theta_M \approx \theta_M$, $\sin \theta_E \approx \theta_E$, and $\sin(\theta_E - \theta_M) \approx \theta_E - \theta_M$ on the right side of (B1) and (B2) are retained here, with all the cosines of these angles being ≈ 1 . Even so, there are dozens of terms which must be tediously worked out. The final equations are

$$\begin{aligned}
\frac{1}{J_M} \frac{dJ_M}{dt} = & \frac{3GM_M^2}{4R_E h} \left(\frac{R_E}{a} \right)^6 \left(\frac{1}{1+\alpha\beta} \right) \left\{ (1+\beta) \left[1 + \alpha \left(\frac{h}{H} \right) \right] (k_{10}^M \sin \delta_{10}^M - k_{11}^M \sin \delta_{11}^M - k_{20}^M \sin \delta_{20}^M) \right. \\
& + 2\alpha \left[(1+\beta+\beta_H) \left(\frac{h}{H} \right) - \beta_h \right] k_{20}^M \sin \delta_{20}^M - \left(\frac{M_S}{M_M} \right) \left(\frac{a}{a_S} \right)^3 \left[\alpha(1+\beta) \left(\frac{h}{H} \right) \right] k_{11}^M \sin \delta_{11}^M \\
& - \left(\frac{M_S}{M_M} \right) \left(\frac{a}{a_S} \right)^3 \left[1 + \left(\frac{h}{H} \right) \right] (1+\alpha) \beta k_{11}^S \sin \delta_{11}^S \\
& \left. + \left(\frac{M_S}{M_M} \right)^2 \left(\frac{h}{H} \right) \left(\frac{a}{a_S} \right)^6 \alpha [+\beta k_{10}^S \sin \delta_{10}^S - \beta k_{11}^S \sin \delta_{11}^S + (2\beta_H + \beta) k_{20}^S \sin \delta_{20}^S] \right\}
\end{aligned} \tag{46}$$

and

$$\begin{aligned}
\frac{1}{\theta_E} \frac{d\theta_E}{dt} = & \frac{3GM_M^2}{4R_E h} \left(\frac{R_E}{a} \right)^6 \left(\frac{1}{1+\alpha\beta} \right) \left\{ (1-\alpha) \left(\beta - \frac{h}{H} \right) (-k_{10}^M \sin \delta_{10}^M + k_{11}^M \sin \delta_{11}^M) \right. \\
& + \left[(1-\alpha) \left(\beta + \frac{h}{H} \right) - 2\alpha_h \beta + 2\alpha_H \beta \left(\frac{h}{H} \right) \right] k_{20}^M \sin \delta_{20}^M - \left(\frac{h}{H} \right) \left(\frac{M_S}{M_M} \right) \left(\frac{a}{a_S} \right)^3 (1-\alpha) k_{11}^M \sin \delta_{11}^M
\end{aligned}$$

$$\begin{aligned}
& + \left(\frac{M_S}{M_M} \right) \left(\frac{a}{a_S} \right)^3 \left(\beta - \frac{h}{H} \right) k_{11}^S \sin \delta_{11}^S \\
& + \left(\frac{h}{H} \right) \left(\frac{M_S}{M_M} \right)^2 \left(\frac{a}{a_S} \right)^6 \left[k_{10}^S \sin \delta_{10}^S - k_{11}^S \sin \delta_{11}^S + (1 + 2\alpha_H \beta) k_{20}^S \sin \delta_{20}^S \right] \Big\}
\end{aligned} \tag{47}$$

In these equations the Love numbers $k_{2mnj\gamma p 0}^*$ and lag angles $\delta_{2mnj\gamma p 0}^*$ can be frequency-dependent; if so, it is further assumed that they are controlled by the two fastest variables in the associated argument

$$(2 - 2p)\omega^* + (2 - 2p)\bar{M}^* + n\Omega^* + j\Omega_E + (-1)^\gamma m\psi^* ,$$

namely mean motion \bar{M}^* and the Earth's rotation rate $\dot{\psi}$, where ψ is the rotation angle of a fixed longitude ("Greenwich") on the Earth (Appendix C and Kaula (1964)). Hence the Love numbers and lag angles are characterized only by subscripts m and p , the idea being that the much slower nodal rates will not change their values much. Thus in k_{20}^M , for example, $m = 2$ and $p = 0$.

The two variables J_M and θ_E in (46) and (47) decouple from each other: the equation for dJ_M/dt depends only on J_M , and $d\theta_E/dt$ depends only on θ_E . This remarkable fact was discovered by Darwin (1880).

A pitfall to avoid in working out the terms in (46) and (47) has to do with the sign of the lag angle. When the Moon is further than $\sim 3.8R_E$ from the Earth the rate of the argument is negative for $\gamma = 1$ in (45) because the Earth's rotation rate dominates twice the Moon's mean motion and the much slower nodal rates. This means that the lag angles $\delta_{2mnj1p0}^*$ are positive. However, when $\gamma = 2$, the lag angle changes sign. In the above

$$\delta_{2mnj2p0}^* = -\delta_{2mnj1p0}^* .$$

By (21) and (22)

$$\frac{dh}{dt} = \frac{h}{2a} \frac{da}{dt} = T_{MM3} = M_M \frac{\partial V_{MM}}{\partial \bar{M}_M} = \frac{3}{2} \frac{GM_M^2}{R_E} \left(\frac{R_E}{a} \right)^6 k_{20}^M \sin \delta_{20}^M \quad (48)$$

which agrees with Kaula (1964, p. 677). Also,

$$\begin{aligned} \frac{dH}{dt} &= C_E \dot{\psi} = -T_{MM3} - T_{SS3} \\ &= -M_M \frac{\partial V_{MM}}{\partial \bar{M}_M} - M_S \frac{\partial V_{SS}}{\partial \bar{M}_S} = -\frac{3}{2} \frac{GM_M^2}{R_E} \left(\frac{R_E}{a} \right)^6 k_{20}^M \sin \delta_{20}^M - \frac{3}{2} \frac{GM_S^2}{R_E} \left(\frac{R_E}{a_S} \right)^6 k_{20}^S \sin \delta_{20}^S \end{aligned} \quad (49)$$

to the current level of approximation, where $\dot{\psi}$ is the Earth's rotation rate, and C_E is its moment of inertia. These two equations allow the Moon's semimajor axis a and the Earth's rotation rate to be found as a function of time. Also, the Earth's J_2 is to be found from

$$J_2 = J_2^0 \left(\frac{\dot{\psi}}{\dot{\psi}_0} \right)^2 \quad (50)$$

where J_2^0 is the value of J_2 when $\dot{\psi} = \dot{\psi}_0$, so that the Earth's rotational flattening decreases as the rotation rate decreases.

Equations (45) - (50) are the fundamental equations of this paper. They will be used to find J_M , θ_E , a , and $\dot{\psi}$ as functions of time t for circular orbits.

5. Darwin's viscous liquid

Equations (46)-(50) are applied to rheological models of the Earth, the first being Darwin's model. Darwin (1880) chose a constant-density viscous liquid as his rheological model, presumably because no other quantitative model was available. In this case

$$\tan \delta = \frac{19\nu\eta_E}{2g_E\rho_ER_E} = \xi\nu\eta_E$$

where ν is the absolute value of the frequency of a generic tidal constituent, δ is the lag angle, $g_E = GM_E/R_E^2$ is the gravitational acceleration at the Earth's surface, ρ_E is the average density of the Earth, η_E is the Earth's viscosity, and $\xi = 19/(2g_E\rho_ER_E) = 2.8 \times 10^{-11} \text{ kg}^{-1} \text{ m s}^2$. From the above equation $\cos \delta = 1/[1 + (\xi\nu\eta_E)^2]^{1/2}$ and $\sin \delta = \xi\nu\eta_E/[1 + (\xi\nu\eta_E)^2]^{1/2}$. The generic Love number is $k_2 = (3/2)\cos \delta$, so that $k_2 \sin \delta = (3/2)\xi\nu\eta_E/[1 + (\xi\nu\eta_E)^2]$ (see Fig. 5). Here $k_2 \sin \delta \propto \nu\eta_E$ when $\xi\nu\eta_E \ll 1$, so that the tidal lag angle is proportional to tidal frequency, while k_2 hardly changes with frequency. These have been common assumptions in past studies (e.g., Efroimsky and Marakov, 2013). At the other extreme $k_2 \sin \delta \propto (\nu\eta_E)^{-1}$ when $\xi\nu\eta_E \gg 1$.

The choice of $\eta_E \approx 10^{12} \text{ Pa s}$ gives small lag angles and a timescale on the age of the Solar System. The changes in J_M , and θ_E track the canonical results of Goldreich (1966) and Touma and Wisdom (1994) extremely well and are not reproduced here.

Perhaps the only interesting feature of the viscous liquid model is what happens near the resonance when $a \approx 3.8 R_E$, where the frequency $-2n + \dot{\psi}$ of the $m = 1, p = 0$ constituent changes sign. Here for the large viscosity $\eta_E \approx 10^{17} \text{ Pa s}$, angle J_M can dramatically increase from a finite initial value as the Moon moves away from the Earth (Rubincam, 1975). Using (46)-(50) confirms this. However, the problem is that for the dramatic rise in J_M to happen, the Love number has to be extremely low: $k_2 \approx 0.0001$ at the M_2 frequency. Such a low Love number is exceedingly implausible for the Earth regardless of rheology. Moreover, the Moon may have formed further than $3.8 R_E$ from the Earth, as in the giant impact hypothesis (e.g., Benz et al., 1986).

6. Ross-Schubert model

The next rheological model is that of Ross and Schubert (1989), who investigated tidal friction as a two-body problem, considering only the Earth and Moon. Their rheological model is not a theoretical model like a viscous liquid or a Maxwell body, but rather is an empirically-based model. They give the following three equations on their page 9536. For the Love number they give

$$k_2 = \frac{k_0}{1 + \left(\frac{19\mu_E}{2g_E\rho_ER_E} \right)}$$

where

$$\mu_E = \mu_0 \cos (\tau_E/\Xi) . \quad (52)$$

In these equations μ_E is the Earth's shear modulus, τ_E is a bulk temperature for the Earth, while k_0 , μ_0 , and Ξ are constants. After correcting a typographical error in the exponent, their lag angle is given by

$$\delta = \delta_0 \exp (-D/\tau_E)/\nu^\chi \approx \sin \delta . \quad (53)$$

Here ν is once again the absolute value of frequency, and δ_0 , D , and χ are constants. It is to be noted that the Love number k_2 is frequency-independent in their model, so that all frequency-dependence in the product $k_2 \sin \delta$ comes from δ . The functional form of (52) and (53) as well as the associated constants given in Table 2 are based on experiments. It is of interest that $\chi \approx 0.25$ in their model, a value which is quite different from the $\chi = 1$ often assumed in tidal lags, as in Darwin's (1880) low viscosity model (Efroimsky and Lainey, 2007). As for the lag angle, Ross and Schubert note that for the M_2 frequency

when the Moon is near $10R_E$, $\delta \approx 0.1$ radians for rocks on the verge of melting. Moreover, they assume $k_2 = 0.3$ and $\delta \approx 0.004$ radians for the solid part of the Earth today. The choices of δ_0 and D anchor the end points.

As the Earth cools k_2 and δ change. Ross and Schubert do not give a specific equation for temperature τ_E as a function of time, but it is approximately

$$\tau_E = \tau_0 + \tau_1 \exp(-t_{bil}/\tau_2) - \tau_3 t_{bil}$$

where t_{bil} is time in 10^9 y and the constants τ_0 , τ_1 , τ_2 , and τ_3 are given in Table 2. The behavior of k_2 and δ as a function of time is shown in Fig. 6.

The lunar history for the Ross-Schubert model can be integrated using (46)-(50) and the parameters in the right-hand column in Table 2. These parameters are somewhat different from those of Ross and Schubert (left-hand column) but probably lie within the uncertainties of the values. They were chosen to give $k_2 = 1$ in the case where the Earth has no strength, in keeping with the secular Love number k_s being ~ 1 (e.g., Lambeck, 1980, p. 26). The integration starts at $a = 7.3 R_E$ with the Earth's spin rate being 4.1 times its present value, along with $J_M = 7.3^\circ$ and $\theta_E = 12^\circ$. The integration begins past the resonances in the early Earth-Moon system (Touma and Wisdom, 1998; Ward and Canup, 2000). The integration ends after 4.55×10^9 y.

The results are shown in Figs. 7-9. The solid curves give the canonical values of Goldreich (1966) and Touma and Wisdom (1994), while the data points plotted every $5 R_E$ are those of the present integration. The curves and data points track each other well, so that the Ross-Schubert history varies little from the other histories. All of the data points end at $55 R_E$. This is as far as the assumed tidal friction can push the Moon over the age of the Solar System. The Moon's present distance from the Earth is $60.3 R_E$.

Figure 7 shows the length-of-day (LOD) as a function of Earth-Moon distance. Fig. 8 shows J_M , θ_M , and I , with I being the inclination of the Moon's orbital plane with respect to the ecliptic. Here J_M bisects the oscillations in I when the Moon is close to the Earth, while far from the Earth J_M is essentially I because θ_M becomes small, so that the pole of the cone in Fig. 1 approaches the pole of the ecliptic. (Inclination I is always

taken to be positive, which is the reason the lower branch of the curve shows the peculiar “bounce” between $7 R_E$ and $17 R_E$.) Figure 9 shows θ_E , J_E , and the Earth’s obliquity ε . When the Moon is close to the Earth, θ_E bisects the obliquity oscillations caused by the coning motion with amplitude J_E (illustrated in Fig. 2). Far from the Earth the obliquity oscillations die out and θ_E essentially becomes ε because of the small amplitude of J_E , which becomes the nutation angle.

7. The one-parameter approximation

The tidal friction equations can be rewritten in order to understand why the Ross-Schubert model gives a tidal history similar to Darwin’s (1880) low-viscosity Earth and Goldreich’s (1966) equal lag angles. Assume that the Moon is more than $\sim 10 R_E$ from the Earth, so that the frequencies are $\sim \dot{\psi}$ for the $m = 1, p = 0, 1$ lunar and solar tidal constituents. Hence the Love number times the sine of the lag angle are the same and can be written as $k_{11} \sin \delta_{11}$. Likewise the $m = 2, p = 0$ constituents have frequencies $\sim 2 \dot{\psi}$ for the Sun and Moon, and the product of the Love number and lag angle can be written $k_{20} \sin \delta_{20}$. Thus if (46) and (47) are divided by (48), then those equations become

$$\begin{aligned}
 \frac{1}{J_M} \frac{dJ_M}{da} = & \frac{1}{4a} \left(\frac{1}{1+\alpha\beta} \right) \left\{ -(1+\beta) \left[1 + \alpha \left(\frac{h}{H} \right) \right] \right. \\
 & + 2\alpha \left[(1+\beta+\beta_H) \left(\frac{h}{H} \right) - \beta_h \right] - \left(\frac{M_S}{M_M} \right) \left(\frac{a}{a_S} \right)^3 \left[\alpha(1+\beta) \left(\frac{h}{H} \right) \right] \Delta_{12} \\
 & - \left(\frac{M_S}{M_M} \right) \left(\frac{a}{a_S} \right)^3 \left[1 + \left(\frac{h}{H} \right) \right] (1+\alpha)\beta \Delta_{12} \\
 & \left. + \left(\frac{M_S}{M_M} \right)^2 \left(\frac{h}{H} \right) \left(\frac{a}{a_S} \right)^6 \alpha [+ (2\beta_H + \beta)] \right\}
 \end{aligned} \tag{54}$$

$$\begin{aligned}
\frac{1}{\theta_E} \frac{d\theta_E}{da} = \frac{1}{4a} \left(\frac{1}{1+\alpha\beta} \right) & \left\{ + \left[(1-\alpha) \left(\beta + \frac{h}{H} \right) - 2\alpha_h \beta + 2\alpha_H \beta \left(\frac{h}{H} \right) \right] \right. \\
& - \left(\frac{h}{H} \right) \left(\frac{M_S}{M_M} \right) \left(\frac{a}{a_S} \right)^3 (1-\alpha) \Delta_{12} + \left(\frac{M_S}{M_M} \right) \left(\frac{a}{a_S} \right)^3 \left(\beta - \frac{h}{H} \right) \Delta_{12} \\
& \left. + \left(\frac{h}{H} \right) \left(\frac{M_S}{M_M} \right)^2 \left(\frac{a}{a_S} \right)^6 [+(1+2\alpha_H \beta)] \right\}
\end{aligned} \tag{55}$$

where

$$\Delta_{12} = k_{11} \sin \delta_{11} / k_{20} \sin \delta_{20} . \tag{56}$$

Hence the equations governing the evolution of the Earth-Moon system can be characterized with a single parameter Δ_{12} , which is expected to vary with time.

Figure 10 shows θ_E for $0 \leq \Delta_{12} \leq 2$ (grey region), where Δ_{12} is simply a constant for all a . The dashed line is for $\Delta_{12} = 1$, as in Goldreich (1966, p. 434). The lower solid curve is for $\Delta_{12} = 0$. A lower bound of zero is not physical, and only represents the extreme lower limit for solid-Earth tides. Lag angles which depend on linearly on frequency, or frequency to some power < 1 as in the Ross-Schubert model, have $\Delta_{12} \leq 1$ if the associated Love numbers are only weakly frequency-dependent. Thus all such models are trapped between the dashed curve and the lower solid curve in Fig. 10. Since there is not much space between the curves, all models for which $\Delta_{12} \leq 1$ will have quite similar obliquity histories. This is the reason Ross and Schubert's (1989) model does not differ greatly from Goldreich (1966) or Darwin's (1880) low-viscosity Earth in terms of obliquity history.

8. Ocean tides

As indicated above in section 6, solid friction may be responsible for most of the tidal evolution of the Earth-Moon system. This section instead assumes that solid friction is negligible and the tidal evolution is due mainly to the oceans, which may have formed very early in the Earth's history (Wilde et al., 2001). The ocean tides today are the main driver of tidal friction and are in fact anomalously high, in the sense that their operating at the present level would make the Moon come close to the Earth only 1.5×10^9 y ago (e.g., Lambeck, 1980; Bills and Ray, 1999), which is geologically untenable.

With the oceans, each term in the tide-raising potential raises multiple harmonics in the tidal potential (e.g., Lambeck, 1980). This is in contrast to what is assumed for the solid-Earth tides. But only those harmonics whose frequencies are geared the body affected by the tides need be considered to obtain the secular evolution. Therefore (54) and (55) can still be used as a highly simplified model for the ocean tides.

The oceans could give obliquity histories dissimilar to the rheologies for which $\Delta_{12} \leq 1$. The oceans' response to the $m = 1$ harmonic could be quite different from their response to the $m = 2$ harmonic. Thus Δ_{12} would be expected to vary as the ocean basins change shape, depth, and position as the continents drift into various configurations over the course of Earth history. Hence Δ_{12} might be ≥ 1 at times. The upper solid curve in Fig. 10 is for $\Delta_{12} = 2$, so that the region between the dashed curve and the solid upper curve is for $1 \leq \Delta_{12} \leq 2$. Perhaps the most interesting feature of Fig. 10 is that θ_E , which is basically Earth's obliquity ε when the Moon more than halfway to its current distance, can actually decrease when the Moon is more than $\sim 50 R_E$ from the Earth and $\Delta_{12} \geq 1.6$.

What about the tilt of the Moon's orbit to the ecliptic? It turns out that the Moon's J_M , which essentially becomes the orbital inclination I to the ecliptic for distances $> 30 R_E$, is very insensitive to Δ_{12} for $0 \leq \Delta_{12} \leq 2$. Therefore no graph similar to Fig. 10 is shown for it.

9. Nodal and semiannual tides

Equations corresponding to (46) and (47) can be derived for the nodal tide and semiannual tide and are given in Appendix E. They are derived from $V_{m=0}$ given by (C10) (details of the derivations omitted).

The rationale for examining these $m = 0$ tides is that they are long-period, with the nodal tide having a period of $2\pi/\dot{\Omega}$, while the semiannual tide has a period of half a year. The early Earth might respond to these long-period tides more through viscosity than anelasticity, and thus give large lag angles, which might offset the fact that the right sides of (E1)-(E4) are of higher order in the angles than are (46) and (47). However, integration of (E1)-(E4) with the sines of the lag angles being set equal to 1 give only trivial changes in the evolution of J_M and θ_E compared to (46) and (47) and can be neglected.

10. Discussion

Equations (45)-(50) and (54)-(55) are the fundamental equations of this paper, with (46) and (47) being the modern version of Darwin's (1880) equations. The equation for the tidal potential (45) is valid for all orbits regardless of orbital eccentricity. Also, in (45) the variables in the trigonometric arguments tend to change nearly uniformly with time at all Earth-Moon distances when the angles θ_E , J_E , θ_M , and J_M are all small, as is the case for the Earth. This is in contrast to Kaula's (1964) equations, which are formulated in the Earth frame, with the trigonometric arguments changing nearly uniformly with time only when the Moon is close to the Earth.

In contrast to (45), equations (46)-(49) apply only to circular orbits. As Figs. 8 and 9 show, (46)-(50) agree quite well with the integrations by Goldreich (1966) and Touma and Wisdom (1994), indicating that the equations derived here are probably correct. A virtue of (46)-(50) is that they are easy to integrate numerically, although the equations of Goldreich and Touma and Wisdom are not particularly burdensome to integrate with today's computers.

The equations are linear in the sense that each periodic term in the tide-raising potential yields a corresponding term in the tidal potential (45) with the same frequency,

but changed in amplitude and shifted in phase. The equations can be nonlinear in the sense that, for instance, the lag angle is not necessarily proportional to the frequency, but may depend on the frequency to some power, as in the case of the Ross-Schubert model, where the lag angle is proportional to the fourth root of the frequency.

It is not generally recognized that Darwin (1880) realized that not just the tides raised by the Moon secularly affect the Moon, but also the Sun affects the lunar tidal bulge, and the Moon affects the solar tidal bulge. This can be seen in Darwin's equations (250)-(251) in the terms in which his quantities τ and τ' appear together, with τ referring to the Moon and τ' referring to the Sun. These mixed terms are apparent in (46)-(47), in which the mass of the Moon M_M and the mass of the Sun M_S appear together.

Probably the reason the mixed terms escaped modern notice until Goldreich (1966) is the extreme length of Darwin's work, which was necessitated by the lack of mathematical formalisms available to him. For instance, his equations (251)-(251) appear after a dense exposition almost 90 pages into his massive 175 page paper. All in all, Darwin labored mightily with the tools at his command and did a remarkable job.

An important feature of (54) and (55) is that the solution to each equation can be written in the form

$$J_M(a) = J_M^0 \exp\left(\int_{a_0}^a F_J da\right)$$

$$\theta_E(a) = \theta_E^0 \exp\left(\int_{a_0}^a F_\theta da\right)$$

where J_M^0 is the value of J_M at starting value a_0 , and likewise for θ_E^0 and θ_E . Here F_J and F_θ are functions which depend on a and the Earth's initial spin state. The angle J_M can grow from some initial finite angle as the Moon evolves past $3.8 R_E$ (Rubincam, 1975); but as stated above, the model for accomplishing this is implausible; and there is no guarantee the Moon was ever that close to the Earth.

If $J_M^0 = 0$, then J_M remains zero regardless of the details of tidal evolution. When the Moon is close to the Earth, J_M is essentially the angle between the Moon's orbital

plane and the Earth's equator. When the Moon is far from the Earth, J_M is basically the angle between the Moon's orbital plane and the ecliptic. If the Moon ever orbited in its Laplace plane ($J_M = 0$) and the orbit evolved outwards through tidal friction alone, then the Moon should be in the ecliptic today; in which case the Earth should see a solar eclipse every month.

However, presently the Moon's orbit is tilted by 5.2° to the ecliptic and solar eclipses occur only when the nodal line points to the Sun, and the Moon happens to be on the nodal line; thus solar eclipses seen from the Earth are fairly rare. The equations developed here and in previous studies do not allow the Moon to leave its Laplace plane if it formed in it. Thus, if these tidal friction histories are taken at face value, then the Moon never orbited in its Laplace plane and would seem to eliminate theories of the Moon's origin, such as forming by accretion close to the Earth in the equatorial plane; fissioning from the Earth and being thrown into an equatorial orbit; and Mars fissioning from the Earth with the Moon forming as a droplet in between the two bodies, as in Lyttleton's (1969) hypothesis. However, the giant impact hypothesis and resonances operating in addition to tidal friction do allow the present tilt (Touma and Wisdom, 1998; Ward and Canup, 2000).

Ross and Schubert (1989) in their two-body treatment found that solid tidal friction alone can account for the tidal evolution of the Earth-Moon system out to $\sim 50 R_E$, implying that most of the evolution comes from the solid Earth and not the oceans. This possibility is confirmed here: using somewhat different parameters from theirs, the Ross-Schubert model can account for evolution out to $55 R_E$, and other parameters can certainly be chosen to take the Moon out to its present distance from the Earth. While tidal friction in the oceans currently plays the largest role in the evolution of the Earth-Moon system, it perhaps played a smaller role early on than previously expected, as proposed by Ross and Schubert.

The oceans may have actually decreased the Earth's obliquity at times instead of increasing it for distances between $50 R_E$ and the present $60.3 R_E$, which is just the range in which the contribution by the solid Earth to tidal friction may have become small, as in the Ross-Schubert model. The large values of Δ_{12} required to make this happen is presumably the reason that Hansen (1982, his Fig. 9) finds an abrupt decrease in one of

his ocean models at an M_2 resonance 1.3×10^9 y ago. Perhaps the denominator in (56) became small. But more gentle decreases over time because $\Delta_{12} > 1.6$ (Fig. 10) may be possible and worth investigating.

The Earth's obliquity oscillates by $\sim \pm 1^\circ$ with a 41,000 y period due to the other planets (e.g., Ward, 1974; Touma and Wisdom, 1994). This small oscillation, which is one of the Milankovitch cycles, is enough to induce ice sheet growth and decay (e.g., Hays et al., 1976; Rubincam, 1995; Bills, 1994). Hence the Earth's climate system is quite sensitive to tilt, so that even a modest obliquity decrease might have implications for the Earth's climate. The problem here is lack of information regarding the ancient oceans. Numerical ocean models with various assumed basin geometries would have to be examined to see whether obliquity decreases are realistic.

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Appendix A

This appendix derives the Laplace planes (also called proper planes; Laplace, 1966; Allan and Cook, 1964). Darwin (1880) used them in his treatment of tidal friction in the Earth-Moon system, as is done here. Boue and Laskar (2006) recently used a Hamiltonian approach to derive them. The more messy but direct approach below is more in the spirit of, but not identical with, Darwin's.

The quantities J_M , J_E , θ_M , and θ_E are all assumed to be greater than zero. The equations to be solved are (29), (31), (33), and (35) with $\phi = \phi_M = \phi_E$, $\Omega = \Omega_M$, $\Omega_E = \Omega_M + \pi$, and $\mathbf{T}_M = \mathbf{T}_E = 0$. It is helpful to note that

b·s

$$= \sin J_M \sin J_E [-\sin^2 \Omega - \cos(\theta_E - \theta_M) \cos^2 \Omega] + \sin J_M \cos J_E [\sin(\theta_E - \theta_M) \cos \Omega] \\ + \sin J_E \cos J_M [\sin(\theta_E - \theta_M) \cos \Omega] + \cos J_M \cos J_E [\cos(\theta_E - \theta_M)] .$$

Using such expressions as $2\cos^2 \Omega = (1 + \cos 2\Omega)$ etc., (29) becomes

$$\begin{aligned}
 & (\cos J_M \sin \Omega) \frac{dJ_M}{dt} + (\sin J_M \cos \Omega) \frac{d\Omega}{dt} \\
 & + (0) \frac{d\theta_M}{dt} + (\sin J_M \cos \theta_M \cos \Omega + \cos J_M \sin \theta_M) \frac{d\phi}{dt} \\
 & = -\frac{L}{h} \sum_{n=0}^4 A_n \cos n\Omega + \frac{K_2}{h} \sum_{n=0}^4 C_n \cos n\Omega
 \end{aligned} \tag{A1}$$

where

$$\sum_{n=0}^4 A_n \cos n\Omega = (\mathbf{b} \cdot \mathbf{s}) [\cos \phi(s_y b_z - s_z b_y) + \sin \phi(s_z b_x - s_x b_z)]$$

$$\sum_{n=0}^2 C_n \cos n\Omega = (\mathbf{b} \cdot \mathbf{c}) [\cos \phi(c_y b_z - c_z b_y) + \sin \phi(c_z b_x - c_x b_z)]$$

with $c_x = c_y = 0$, $c_z = 1$, and

$$\begin{aligned}
 16A_0 = & -8 \cos^2 J_M \cos^2 J_E [\sin 2(\theta_E - \theta_M)] + 4 \sin^2 J_E \cos^2 J_M [\sin 2(\theta_E - \theta_M)] + 4 \sin^2 J_M \\
 & \cos^2 J_E [\sin 2(\theta_E - \theta_M)] + 8 \sin J_M \sin J_E \cos J_M \cos J_E [\sin (\theta_E - \theta_M) + 2 \sin 2(\theta_E - \theta_M)] - 3 \\
 & \sin^2 J_M \sin^2 J_E [\sin 2(\theta_E - \theta_M)]
 \end{aligned}$$

$$\begin{aligned}
 16A_1 = & + 16 \sin J_E \cos^2 J_M \cos J_E [\cos 2(\theta_E - \theta_M)] + 16 \sin J_M \cos J_M \cos^2 J_E [\cos \\
 & 2(\theta_E - \theta_M)] - 4 \sin J_M \sin^2 J_E \cos J_M [\cos (\theta_E - \theta_M) + 3 \cos 2(\theta_E - \theta_M)] - 4 \sin^2 J_M \sin J_E \cos \\
 & J_E [\cos (\theta_E - \theta_M) + 3 \cos 2(\theta_E - \theta_M)]
 \end{aligned}$$

$$4C_0 = -2 \cos^2 J_M \sin 2\theta_M + \sin^2 J_M \sin 2\theta_M$$

$$4C_1 = -4 \sin J_M \cos J_M \cos 2\theta_M .$$

Likewise, (31) becomes

$$\begin{aligned} & (\cos J_M \cos \Omega) \frac{dJ_M}{dt} - (\sin J_M \sin \Omega) \frac{d\Omega}{dt} \\ & + (\cos J_M) \frac{d\theta_M}{dt} - (\sin J_M \cos \theta_M \sin \Omega) \frac{d\phi}{dt} \\ & = -\frac{L}{h} \sum_{n=1}^4 B_n \sin n\Omega + \frac{K_2}{h} \sum_{n=1}^4 D_n \sin n\Omega \end{aligned} \quad (A2)$$

where

$$\begin{aligned} \sum_{n=1}^4 B_n \sin n\Omega &= (\mathbf{b} \cdot \mathbf{s}) \{ \cos \theta_M [\sin \phi (s_y b_z - s_z b_y) \\ & - \cos \phi (s_z b_x - s_x b_z)] - \sin \theta_M (s_x b_y - s_y b_x) \} \\ \sum_{n=1}^2 D_n \sin n\Omega &= (\mathbf{b} \cdot \mathbf{c}) \{ \cos \theta_M [\sin \phi (c_y b_z - c_z b_y) \\ & - \cos \phi (c_z b_x - c_x b_z)] - \sin \theta_M (c_x b_y - c_y b_x) \} \end{aligned}$$

with

$$\begin{aligned} 16B_1 &= -16 \sin J_E \cos^2 J_M \cos J_E [\cos (\theta_E - \theta_M)] - 16 \sin J_M \cos J_M \cos^2 J_E [\cos^2 (\theta_E - \theta_M)] \\ &+ 4 \sin J_M \sin^2 J_E \cos J_M [3 - \sin^2 (\theta_E - \theta_M) + \cos (\theta_E - \theta_M)] + 4 \sin^2 J_M \sin J_E \cos J_E [3 \cos \\ &(\theta_E - \theta_M) + \cos 2(\theta_E - \theta_M)] \end{aligned}$$

$$4D_1 = 4\sin J_M \cos J_M \cos^2 \theta_M .$$

The equations corresponding to (33) and (35) for the Earth are

$$\begin{aligned} & -(\cos J_E \sin \Omega) \frac{dJ_E}{dt} - (\sin J_E \cos \Omega) \frac{d\Omega}{dt} \\ & + (0) \frac{d\theta_E}{dt} + (-\sin J_E \cos \theta_E \cos \Omega + \cos J_E \sin \theta_E) \frac{d\phi}{dt} \\ & = \frac{L}{H} \sum_{n=0}^4 A_n \cos n\Omega + \frac{K_1}{H} \sum_{n=0}^4 F_n \cos n\Omega \end{aligned} \quad (A3)$$

where

$$\sum_{n=0}^2 F_n \cos n\Omega = (\mathbf{s} \cdot \mathbf{c}) [\cos \phi (c_y s_z - c_z s_y) + \sin \phi (c_z s_x - c_x s_z)]$$

with

$$4F_0 = -2\cos^2 J_E \sin 2\theta_E + \sin^2 J_E \sin 2\theta_E$$

$$4F_1 = -4\sin J_E \cos J_E \cos 2\theta_E$$

and

$$\begin{aligned} & -(\cos J_E \cos \Omega) \frac{dJ_E}{dt} + (\sin J_E \sin \Omega) \frac{d\Omega}{dt} \\ & + (\cos J_E) \frac{d\theta_E}{dt} + (\sin J_E \cos \theta_E \sin \Omega) \frac{d\phi}{dt} \end{aligned}$$

$$= \frac{L}{H} \sum_{n=1}^4 E_n \sin n\Omega + \frac{K_1}{H} \sum_{n=1}^4 G_n \sin n\Omega \quad (\text{A4})$$

where

$$\sum_{n=1}^4 E_n \sin n\Omega = (\mathbf{b} \cdot \mathbf{s}) \{ \cos \theta_E [\sin \phi (s_y b_z - s_z b_y) - \cos \phi (s_z b_x - s_x b_z)] - \sin \theta_E (s_x b_y - s_y b_x) \}$$

$$\sum_{n=1}^2 G_n \sin n\Omega = (\mathbf{s} \cdot \mathbf{c}) \{ \cos \theta_E [\sin \phi (c_y s_z - c_z s_y) - \cos \phi (c_z s_x - c_x s_z)] - \sin \theta_E (c_x s_y - c_y s_x) \}$$

with

$$16E_1 = -16 \sin J_E \cos^2 J_M \cos J_E [\cos^2 (\theta_E - \theta_M)] - 16 \sin J_M \cos J_M \cos^2 J_E [\cos (\theta_E - \theta_M)] \\ + 4 \sin J_M \sin^2 J_E \cos J_M [3 \cos (\theta_E - \theta_M) + 4 \cos 2(\theta_E - \theta_M)] + 4 \sin^2 J_M \sin J_E \cos J_E [3 - \sin^2 (\theta_E - \theta_M) + \cos (\theta_E - \theta_M)]$$

$$4G_1 = 4 \sin J_E \cos J_E \cos^2 \theta_E .$$

The rate of change of J_M will be written

$$\frac{dJ_M}{dt} = j_{M1} \sin \Omega + j_{M2} \sin 2\Omega + j_{M3} \sin 3\Omega = \sum_{n=1}^3 j_{Mn} \sin n\Omega \quad (\text{A5})$$

so that the first term on the left side of (A2), for example, becomes

$$\cos J_M \cos \Omega (dJ_M/dt) = \cos J_M [j_{M2} \sin \Omega + (j_{M1}+j_{M3}) \sin 2\Omega + j_{M2} \sin 3\Omega + j_{M3} \sin 4\Omega]/2 .$$

Similarly,

$$\frac{dJ_E}{dt} = \sum_{n=1}^3 j_{En} \sin n\Omega \quad (\text{A6})$$

$$\frac{d\theta_M}{dt} = \sum_{n=1}^3 \theta_{Mn} \sin n\Omega \quad (\text{A7})$$

$$\frac{d\theta_E}{dt} = \sum_{n=1}^3 \theta_{En} \sin n\Omega . \quad (\text{A8})$$

The rate of change of the angles Ω and ϕ will be written

$$\frac{d\Omega}{dt} = \dot{\Omega}_0 + \Omega_1 \cos \Omega + \Omega_2 \cos 2\Omega + \Omega_3 \cos 3\Omega = \dot{\Omega}_0 + \sum_{n=1}^3 \Omega_n \cos n\Omega \quad (\text{A9})$$

$$\frac{d\phi}{dt} = \dot{\phi}_0 + \sum_{n=1}^3 \phi_n \cos n\Omega . \quad (\text{A10})$$

where the first terms reflect the fact that these angles have secular as well as periodic terms for low inclinations (the dot over a quantity means time derivative.) The rationale for writing (A5)-(A8) with sines and (A9)-(A10) with cosines is that it is well-known that there are no secular trends in J_M , J_E , θ_M , and θ_E without tidal torques; mixing sines and cosines would produce such trends.

Equations (A5)-(A10) are to be substituted on the left sides of (A1)-(A4) and all terms on both sides contain $\cos n\Omega$ or $\sin n\Omega$, where $n = 0, 1, 2, \dots$. After making the substitutions, equating the $n = 0$ terms on each side of (A1) gives

$$j_{M1} \cos J_M = -\Omega_1 \sin J_M - \phi_1 \sin J_M \cos \theta_M \\ - 2 \dot{\phi}_0 \cos J_M \sin \theta_M - 2(L/h) A_0 + 2(K_2/h) C_0 \quad (\text{A11})$$

while doing the same with (A3) yields

$$j_{E1} \cos J_E = -\Omega_1 \sin J_E - \phi_1 \sin J_E \cos \theta_E \\ - 2 \dot{\phi}_0 \cos J_E \sin \theta_E + 2(L/H) A_0 + 2(K_1/H) F_0 \quad (\text{A12})$$

From this point forward it will be assumed that $j_{M1} = \theta_{M1} = j_{E1} = \theta_{E1} = \phi_1 = \Omega_1 = 0$. The reason for making this choice is to be rid of terms in $\sin \Omega$ and $\cos \Omega$ in (A5) – (A10), so that J_M and J_E are constant in the summations to $n = 1$. The choice elicits the Laplace plane parameters, as shown next.

From (A11) and (A12) clearly

$$\dot{\phi}_0 = [(L/H) A_0 + (K_1/H) F_0] / (\cos J_E \sin \theta_E) \\ = [-(L/h) A_0 + (K_2/h) C_0] / (\cos J_M \sin \theta_M) \quad (\text{A13})$$

which gives (13) when A_0 and C_0 are substituted in the above equation. On the other hand, eliminating $\dot{\phi}_0$ in (A11) and (A12) using (A13) yields

$$\cos J_E \sin \theta_E [-A_0 + (K_2/L) C_0] = \cos J_M \sin \theta_M [(h/H) A_0 + (hK_1/LH) F_0] \quad .$$

The above equation gives a relationship between J_M , θ_M , J_E , and θ_E . Assuming that J_M and J_E are small in the above equation so that $\sin J_M \approx \sin J_E \approx 0$, $\cos J_M \approx \cos J_E \approx \cos \theta_M \approx \cos \theta_E \approx 1$, and using the expressions for A_0 , C_0 , and F_0 give

$$\begin{aligned}
& (1/2) \sin \theta_E \sin [2(\theta_E - \theta_M)] - (K_2/L) \sin \theta_E \sin \theta_M \\
& = -(h/2H) \sin \theta_M \sin [2(\theta_E - \theta_M)] - (h/H) (K_1/L) \sin \theta_E \sin \theta_M
\end{aligned}$$

to second order in the sines. Using $\sin (2\theta_E - \theta_M) \approx 2\sin \theta_E - 2\sin \theta_M$ allows the above equation to be rewritten

$$\begin{aligned}
& \sin^2 \theta_E - \sin \theta_E \sin \theta_M - (K_2/L) \sin \theta_E \sin \theta_M + (h/H) (\sin \theta_M \sin \theta_E - \sin^2 \theta_M) \\
& + (h/H) (K_1/L) \sin \theta_E \sin \theta_M = 0 .
\end{aligned}$$

Writing $\sin \theta_M = \alpha \sin \theta_E$ as in (15) finally yields the quadratic equation

$$\alpha^2 + \left[-\left(1 + \frac{K_1}{L}\right) + \left(\frac{H}{h}\right) \left(1 + \frac{K_2}{L}\right) \right] \alpha - \frac{H}{h} = 0 \quad (\text{A14})$$

which has the solution given by (17). Differentiating this equation with respect to time t gives

$$\frac{d\alpha}{dt} = \frac{\alpha_h}{h} \frac{dh}{dt} + \frac{\alpha_H}{H} \frac{dH}{dt} \quad (\text{A15})$$

where

$$\alpha_h = \frac{\left(\frac{H}{h}\right) \left\{ -1 + \left[1 + \left(\frac{h}{H}\right) \frac{K_1}{L} - 9 \frac{K_2}{L} \right] \alpha \right\}}{2\alpha - \left(1 + \frac{K_1}{L}\right) + \left(\frac{H}{h}\right) \left(1 + \frac{K_2}{L}\right)} \quad (\text{A16})$$

$$\alpha_H = \frac{\left(\frac{H}{h}\right) \left[1 - \left(1 - \frac{K_2}{L}\right) \alpha \right]}{2\alpha - \left(1 + \frac{K_1}{L}\right) + \left(\frac{H}{h}\right) \left(1 + \frac{K_2}{L}\right)} \quad (\text{A17})$$

after using

$$\frac{d\left(\frac{K_1}{L}\right)}{dt} = 6 \left(\frac{K_1}{L}\right) \left(\frac{1}{h} \frac{dh}{dt}\right) \quad (\text{A18})$$

$$\frac{d\left(\frac{K_2}{L}\right)}{dt} = \left(\frac{K_2}{L}\right) \left(\frac{10}{h} \frac{dh}{dt} - \frac{2}{H} \frac{dH}{dt}\right) \quad (\text{A19})$$

and

$$\frac{d\left(\frac{H}{h}\right)}{dt} = \left(\frac{H}{h}\right) \left(-\frac{1}{h} \frac{dh}{dt} + \frac{1}{H} \frac{dH}{dt}\right) . \quad (\text{A20})$$

The derivation of (A19) assumes that J_2 is proportional to the square of the rotation rate of the Earth as given by (50).

For the $n = 1$ terms in (A1) and (A2), after multiplying by 2 one gets

$$\begin{aligned} j_{M2} \cos J_M &= -\sin J_M (2 \dot{\Omega}_0 + \Omega_2) \\ &- \sin J_M \cos \theta_M (2 \dot{\phi}_0 + \phi_2) - 2(L/h) A_1 + 2(K_2/h) C_1 \end{aligned} \quad (\text{A21})$$

and

$$\begin{aligned}
j_{M2} \cos J_M &= \sin J_M (2 \dot{\Omega}_0 - \Omega_2) \\
&+ \sin J_M \cos \theta_M (2 \dot{\phi}_0 - \phi_2) - 2(L/h) B_1 + (K_2/h) D_1 .
\end{aligned} \tag{A22}$$

Subtracting (A22) from (A21) one gets

$$-2 \dot{\Omega}_0 \sin J_M - 2 \dot{\phi}_0 \sin J_M \cos \theta_M = (L/h) (A_1 - B_1) - (K_2/h) (C_1 - D_1) . \tag{A23}$$

Using (A13) in (A23) gives

$$\begin{aligned}
\dot{\Omega}_0 &= [-(L/h)(A_1 - B_1) + (K_2/h)(C_1 - D_1)] / (2 \sin J_M) \\
&- \cos \theta_M [(L/H) A_0 + (K_1/H) F_0] / (\cos J_E \sin \theta_E)
\end{aligned} \tag{A24}$$

which yields (14). The Earth equations corresponding to (A21) and (A22) are

$$\begin{aligned}
-j_{E2} \cos J_E &= \sin J_E (2 \dot{\Omega}_0 + \Omega_2) \\
&+ \sin J_E \cos \theta_E (2 \dot{\phi}_0 + \phi_2) + 2(L/H) A_1 + 2(K_1/H) F_1 \\
-j_{E2} \cos J_E &= -\sin J_E (2 \dot{\Omega}_0 - \Omega_2) \\
&- \sin J_E \cos \theta_E (2 \dot{\phi}_0 - \phi_2) + 2(L/H) E_1 + 2(K_1/H) G_1 .
\end{aligned}$$

Subtracting one equation from the other gives

$$-2 \dot{\Omega}_0 \sin J_E - 2 \dot{\phi}_0 \sin J_E \cos \theta_E = (L/H) (A_1 - E_1) + (K_1/H) (F_1 - G_1) . \tag{A25}$$

Multiplying (A23) by $\sin J_E$ and (A25) by $\sin J_M$ and eliminating Ω_2 and ϕ_2 by subtracting gives

$$\begin{aligned}
& (L/h) (A_1 - B_1) \sin J_E - (K_2/h) (C_1 - D_1) \sin J_E + 2 \dot{\phi}_0 \sin J_M \sin J_E \cos \theta_M \\
& = (L/H) (A_1 - E_1) \sin J_M + (K_1/H) (F_1 - G_1) \sin J_M + 2 \dot{\phi}_0 \sin J_M \sin J_E \cos \theta_E . \quad (\text{A26})
\end{aligned}$$

Now $A_1 - B_1 \approx A_1 - E_1 \approx 2 \sin J_M + 2 \sin J_E$, $C_1 - D_1 \approx -2 \sin J_M$, $F_1 - G_1 \approx -2 \sin J_E$, and $\cos \theta_M \approx \cos \theta_E \approx 1$ to first order in the sines. Using these expressions in (A26), retaining only terms to second order in the sines, and dividing by $\sin^2 J_M$ eventually yields

$$\beta^2 + \left[1 + \frac{K_2}{L} - \left(\frac{h}{H} \right) \left(1 - \frac{K_1}{L} \right) \right] \beta - \frac{h}{H} = 0 \quad (\text{A27})$$

after remembering $\sin J_E = \beta \sin J_M$. The terms with $\dot{\phi}_0$ drop out. The solution to (A27) is given by (18). Differentiating this equation with respect to time gives equations analogous to (A15) and (A17):

$$\frac{d\beta}{dt} = \frac{\beta_h}{h} \frac{dh}{dt} + \frac{\beta_H}{H} \frac{dH}{dt} \quad (\text{A28})$$

where

$$\beta_h = \frac{\left(\frac{h}{H} \right) \left\{ 1 + \left[1 - 7 \frac{K_1}{L} - 10 \left(\frac{H}{h} \right) \frac{K_2}{L} \right] \beta \right\}}{2\beta + 1 + \frac{K_2}{L} - \left(\frac{h}{H} \right) \left(1 - \frac{K_1}{L} \right)} \quad (\text{A29})$$

$$\beta_H = \frac{\left(\frac{h}{H}\right) \left\{ -1 - \left[1 - \frac{K_1}{L} - 2 \left(\frac{H}{h} \right) \frac{K_2}{L} \right] \beta \right\}}{2\beta + 1 + \frac{K_2}{L} - \left(\frac{h}{H} \right) \left(1 - \frac{K_1}{L} \right)} \quad (\text{A30})$$

after using

$$\frac{d\left(\frac{h}{H}\right)}{dt} = \left(\frac{h}{H}\right) \left(\frac{1}{h} \frac{dh}{dt} - \frac{1}{H} \frac{dH}{dt} \right) . \quad (\text{A31})$$

Appendix B

The torques appear in (30), (32), (34), and (36). In the (x, y, z) system $\mathbf{T}_M = T_{Mx}\mathbf{x} + T_{My}\mathbf{y} + T_{Mz}\mathbf{z}$. These components are related to those in the (x_{LM}, y_{LM}, z_{LM}) by

$$\mathbf{T}_M \cdot \mathbf{x} = T_{Mx} = T_{Mx_{LM}} \cos \phi_M - T_{My_{LM}} \cos \theta_M \sin \phi_M + T_{Mz_{LM}} \sin \theta_M \sin \phi_M$$

$$\mathbf{T}_M \cdot \mathbf{y} = T_{My} = T_{Mx_{LM}} \sin \phi_M + T_{My_{LM}} \cos \theta_M \cos \phi_M - T_{Mz_{LM}} \sin \theta_M \cos \phi_M$$

$$\mathbf{T}_M \cdot \mathbf{z} = T_{Mz} = T_{My_{LM}} \sin \theta_M + T_{Mz_{LM}} \cos \theta_M .$$

The torque components (T_{M1}, T_{M2}, T_{M3}) in the (x_μ, y_μ, z_μ) system are related to those in the (x_{LM}, y_{LM}, z_{LM}) system by

$$T_{Mx_{LM}} = T_{M1} \cos \Omega_M - T_{M2} \cos J_M \sin \Omega_M + T_{M3} \sin J_M \sin \Omega_M$$

$$T_{My_{LM}} = T_{M1} \sin \Omega_M + T_{M2} \cos J_M \cos \Omega_M - T_{M3} \sin J_M \cos \Omega_M$$

$$T_{Mz_{LM}} = T_{M2} \sin J_M + T_{M3} \cos J_M .$$

These equations lead to

$$R_{MA} = (T_{M1}/h) \cos \Omega_M - (T_{M2}/h) \cos J_M \sin \Omega_M$$

$$R_{MB} = -(T_{M1}/h) \sin \Omega_M - (T_{M2}/h) \cos J_M \cos \Omega_M$$

$$R_{EA} = (T_{E1}/H) \cos \Omega_E - (T_{E2}/H) \cos J_E \sin \Omega_E$$

$$R_{EB} = -(T_{E1}/H) \sin \Omega_E - (T_{E2}/H) \cos J_E \cos \Omega_E$$

Substituting the above four equations in (43) and (44), using the expressions (A15) for $d\alpha/dt$ and (A28) for $d\beta/dt$ in Appendix A, and remembering (38) yield

$$\begin{aligned} \frac{dJ_M}{dt} \approx & -\left(\frac{1}{(1+\alpha\beta)h}\right) \left\{ T_{M2} + \alpha \left(\frac{h}{H}\right) T_{E2} + \left[\alpha_h T_{M3} + \alpha_H \left(\frac{h}{H}\right) T_{E3} \right] \sin \theta_E \cos \Omega \right. \\ & \left. + \alpha \left[\beta_h T_{M3} + \left(\frac{h}{H}\right) T_{E3} \right] \sin J_M \right\} \end{aligned} \quad (B1)$$

$$\begin{aligned} \frac{d\theta_E}{dt} \approx & +\left(\frac{1}{(1+\alpha\beta)h}\right) \left\{ -\beta [T_{M1} \sin \Omega + T_{M2} \cos \Omega] + \left(\frac{h}{H}\right) [T_{E1} \sin \Omega + T_{E2} \cos \Omega] \right. \\ & \left. + \left[\beta_h T_{M3} + \beta_H \left(\frac{h}{H}\right) T_{E3} \right] \sin J_M \cos \Omega - \beta \left[\alpha_h T_{M3} + \alpha_H \left(\frac{h}{H}\right) T_{E3} \right] \sin \theta_E \right\} \end{aligned} \quad (B2)$$

The task now is to find T_{E1} , T_{E2} , and T_{E3} . The torque on the Moon's orbit is

$$\mathbf{T}_M = \mathbf{T}_{MM1} + \mathbf{T}_{SM1} + \mathbf{T}_{MM2} + \mathbf{T}_{SM2} + \mathbf{T}_{MM3} + \mathbf{T}_{SM3}$$

$$= T_{M1}\boldsymbol{\mu}_1 + T_{M2}\boldsymbol{\mu}_2 + T_{M3}\boldsymbol{\mu}_3$$

where clearly $T_{M1} = T_{MM1} + T_{SM1}$, $T_{M2} = T_{MM2} + T_{SM2}$, and $T_{M3} = T_{MM3} + T_{SM3}$, and once again it is to be remembered that the first subscript refers to the object that raises the tides on the Earth, and the second subscript refers to the body acted upon by those tides.

The torque on the Sun's orbit is

$$\mathbf{T}_S = \mathbf{T}_{SS1} + \mathbf{T}_{MS1} + \mathbf{T}_{SS2} + \mathbf{T}_{MS2} + \mathbf{T}_{SS3} + \mathbf{T}_{MS3}$$

$$= T_{S1}\boldsymbol{\kappa}_1 + T_{S2}\boldsymbol{\kappa}_2 + T_{S3}\boldsymbol{\kappa}_3$$

$$= (T_{SS1} + T_{MS1})\boldsymbol{\kappa}_1 + (T_{SS2} + T_{MS2})\boldsymbol{\kappa}_2 + (T_{SS3} + T_{MS3})\boldsymbol{\kappa}_3 .$$

where $\boldsymbol{\kappa}_1$, $\boldsymbol{\kappa}_2$, and $\boldsymbol{\kappa}_3$ are the unit vectors for the Sun's coordinate system, analogous to $\boldsymbol{\mu}_1$, $\boldsymbol{\mu}_2$, and $\boldsymbol{\mu}_3$ for the Moon.

The torque on the Earth is

$$\mathbf{T}_E = T_{E1}\boldsymbol{\xi}_1 + T_{E2}\boldsymbol{\xi}_2 + T_{E3}\boldsymbol{\xi}_3$$

$$= (T_{ME1} + T_{SE1})\boldsymbol{\xi}_1 + (T_{ME2} + T_{SE2})\boldsymbol{\xi}_2 + (T_{ME3} + T_{SE3})\boldsymbol{\xi}_3$$

where by conservation of angular momentum

$$\begin{aligned} T_{E1} = \mathbf{T}_E \cdot \boldsymbol{\xi}_1 &= -(\mathbf{T}_M + \mathbf{T}_S) \cdot \boldsymbol{\xi}_1 = -[T_{M1}(\boldsymbol{\mu}_1 \cdot \boldsymbol{\xi}_1) + T_{M2}(\boldsymbol{\mu}_2 \cdot \boldsymbol{\xi}_1) + T_{M3}(\boldsymbol{\mu}_3 \cdot \boldsymbol{\xi}_1)] \\ &\quad - [T_{S1}(\boldsymbol{\kappa}_1 \cdot \boldsymbol{\xi}_1) + T_{S2}(\boldsymbol{\kappa}_2 \cdot \boldsymbol{\xi}_1) + T_{S3}(\boldsymbol{\kappa}_3 \cdot \boldsymbol{\xi}_1)] \end{aligned}$$

$$\begin{aligned} &= -[(T_{MM1} + T_{SM1})(\boldsymbol{\mu}_1 \cdot \boldsymbol{\xi}_1) + (T_{MM2} + T_{SM2})(\boldsymbol{\mu}_2 \cdot \boldsymbol{\xi}_1) + (T_{MM3} + T_{SM3})(\boldsymbol{\mu}_3 \cdot \boldsymbol{\xi}_1)] \\ &\quad - [(T_{SS1} + T_{MS1})(\boldsymbol{\kappa}_1 \cdot \boldsymbol{\xi}_1) + (T_{SS2} + T_{MS2})(\boldsymbol{\kappa}_2 \cdot \boldsymbol{\xi}_1) + (T_{SS3} + T_{MS3})(\boldsymbol{\kappa}_3 \cdot \boldsymbol{\xi}_1)] \end{aligned}$$

with similar expressions for T_{E2} and T_{E3} . For the Moon and Earth, the unit vectors (μ_1 , μ_2 , μ_3) and (ξ_1 , ξ_2 , ξ_3) are related to the (x_{LM}, y_{LM}, z_{LM}) and (x_{LE}, y_{LE}, z_{LE}) coordinate systems by

$$\mu_1 = x_{LM} \cos \Omega_M + y_{LM} \sin \Omega_M$$

$$\mu_2 = -x_{LM} \cos J_M \sin \Omega_M + y_{LM} \cos J_M \cos \Omega_M + z_{LM} \sin J_M$$

$$\mu_3 = b = x_{LM} \sin J_M \sin \Omega_M - y_{LM} \sin J_M \cos \Omega_M + z_{LM} \cos J_M$$

and

$$\xi_1 = x_{LE} \cos \Omega_E + y_{LE} \sin \Omega_E$$

$$\xi_2 = -x_{LE} \cos J_E \sin \Omega_E + y_{LE} \cos J_E \cos \Omega_E + z_{LE} \sin J_E$$

$$\xi_3 = s = x_{LE} \sin J_E \sin \Omega_E - y_{LE} \sin J_E \cos \Omega_E + z_{LE} \cos J_E$$

and

$$x_{LE} = x_{LM}$$

$$y_{LE} = y_{LM} \cos d^* + z_{LM} \sin d^*$$

$$z_{LE} = -y_{LM} \sin d^* + z_{LM} \cos d^* .$$

where

$$d^* = \theta_E - \theta_M .$$

The corresponding equations for the Sun are

$$\mathbf{\kappa}_1 = \mathbf{x}_{LS} \cos \Omega_S + \mathbf{y}_{LS} \sin \Omega_S$$

$$\mathbf{\kappa}_2 = -\mathbf{x}_{LS} \cos J_S \sin \Omega_S + \mathbf{y}_{LS} \cos J_S \cos \Omega_S + \mathbf{z}_{LS} \sin J_S$$

$$\mathbf{\kappa}_3 = \mathbf{x}_{LS} \sin J_S \sin \Omega_S - \mathbf{y}_{LS} \sin J_S \cos \Omega_S + \mathbf{z}_{LS} \cos J_S .$$

It is assumed here that the Sun's orbit always lies in the x - y plane of Fig. 1. This is insured by setting $\theta_S = \theta_E$, $J_S = \theta_E$, and $\Omega_S = \pi$, so that $\mathbf{\kappa}_3 = \mathbf{c}$. Hence for the Sun $d^* = \theta_E - \theta_S = \theta_E - \theta_E = 0$. Relaxing these conditions may be a way of treating changes in the orientation of the ecliptic from planetary perturbations.; but this will not be pursued here.

The inner products $(\boldsymbol{\mu}_1 \cdot \boldsymbol{\xi}_1)$, $(\boldsymbol{\mu}_2 \cdot \boldsymbol{\xi}_1)$, ... and $(\mathbf{\kappa}_1 \cdot \boldsymbol{\xi}_1)$, $(\mathbf{\kappa}_2 \cdot \boldsymbol{\xi}_1)$, ... etc., will be written in the form

$$(\boldsymbol{\mu}_1 \cdot \boldsymbol{\xi}_1) = \sum_{f=0}^1 \sum_{g=-1}^1 U_{fg}^{M1, \xi_1} \cos (f\Omega_M + g\Omega_E)$$

with corresponding expressions for $(\boldsymbol{\mu}_2 \cdot \boldsymbol{\xi}_1)$, $(\boldsymbol{\mu}_3 \cdot \boldsymbol{\xi}_1)$, $(\boldsymbol{\mu}_1 \cdot \boldsymbol{\xi}_2)$, etc. Thus

$$\begin{aligned} (\mathbf{T}_{M1}) \cdot \boldsymbol{\xi}_1 &= T_{M1} \sum_{f=0}^1 \sum_{g=-1}^1 U_{fg}^{M1, \xi_1} \cos (f\Omega_M + g\Omega_E) \\ &+ T_{M2} \sum_{f=0}^1 \sum_{g=-1}^1 U_{fg}^{M2, \xi_1} \sin (f\Omega_M + g\Omega_E) \\ &+ T_{M3} \sum_{f=0}^1 \sum_{g=-1}^1 U_{fg}^{M3, \xi_1} \sin (f\Omega_M + g\Omega_E) \end{aligned}$$

with analogous expressions for the dot-products with $\boldsymbol{\xi}_2$ and $\boldsymbol{\xi}_3$. The equation above can be collapsed into the expression

$$\mathbf{T}_M \cdot \boldsymbol{\xi}_1 = \sum_{\sigma=1}^3 \sum_{f=0}^1 \sum_{g=-1}^1 T_{M\sigma} U_{fg}^{M\sigma, \xi_1} \sin [f\Omega_M + g\Omega_E + \delta_{\sigma 1}(\pi/2)]$$

where δ_{ij} is the Kronecker delta ($\delta_{ij} = 1$ if $i = j$ and is zero otherwise). Likewise

$$\mathbf{T}_M \cdot \boldsymbol{\xi}_2 = \sum_{\sigma=1}^3 \sum_{f=0}^1 \sum_{g=-1}^1 T_{M\sigma} U_{fg}^{M\sigma, \xi_2} \cos [f\Omega_M + g\Omega_E - \delta_{\sigma 1}(\pi/2)]$$

and

$$\mathbf{T}_M \cdot \boldsymbol{\xi}_3 = \sum_{\sigma=1}^3 \sum_{f=0}^1 \sum_{g=-1}^1 T_{M\sigma} U_{fg}^{M\sigma, \xi_3} \cos [f\Omega_M + g\Omega_E - \delta_{\sigma 1}(\pi/2)] \quad .$$

Similarly

$$(\mathbf{T}_M \cdot \boldsymbol{\xi}_1) \sin \Omega_E = \sum_{\sigma=1}^3 \sum_{f=0}^1 \sum_{g=-2}^2 T_{M\sigma} W_{fg}^{M\sigma, \xi_1} \cos [f\Omega_M + g\Omega_E - \delta_{\sigma 1}(\pi/2)]$$

$$(\mathbf{T}_M \cdot \boldsymbol{\xi}_2) \cos \Omega_E = \sum_{\sigma=1}^3 \sum_{f=0}^1 \sum_{g=-2}^2 T_{M\sigma} W_{fg}^{M\sigma, \xi_2} \cos [f\Omega_M + g\Omega_E - \delta_{\sigma 1}(\pi/2)]$$

$$(\mathbf{T}_M \cdot \boldsymbol{\xi}_3) \cos \Omega_E = \sum_{\sigma=1}^3 \sum_{f=0}^1 \sum_{g=-2}^2 T_{M\sigma} W_{fg}^{M\sigma, \xi_3} \cos [f\Omega_M + g\Omega_E - \delta_{\sigma 1}(\pi/2)] \quad .$$

The corresponding U - and W - functions for the Sun will be denoted by $U_{fg}^{S\sigma, \xi_1}$, $W_{fg}^{S\sigma, \xi_2}$,

etc.; so that, for example

$$(\mathbf{T}_S \cdot \boldsymbol{\xi}_1) \sin \Omega_E = \sum_{\sigma=1}^3 \sum_{f=0}^1 \sum_{g=-2}^2 T_{S\sigma} W_{fg}^{S\sigma, \xi_1} \cos [f\Omega_S + g\Omega_E - \delta_{\sigma 1}(\pi/2)]$$

and it is to be remembered that Ω_E is related to Ω by (38). The inner products $(\boldsymbol{\mu}_1 \cdot \boldsymbol{\xi}_1)$,

$(\boldsymbol{\mu}_2 \cdot \boldsymbol{\xi}_1)$, ... and $(\boldsymbol{\kappa}_1 \cdot \boldsymbol{\xi}_1)$, $(\boldsymbol{\kappa}_2 \cdot \boldsymbol{\xi}_1)$, ... etc., which make up the U_{fg}^{M1, ξ_1} , U_{fg}^{M2, ξ_1} , ... and U_{fg}^{S1, ξ_1} ,

U_{fg}^{S2, ξ_1} ... functions are easily found from the above equations (Table B1), as are the

$W_{fg}^{M1,\xi1}$, $W_{fg}^{M2,\xi1}$, ... and $W_{fg}^{S1,\xi1}$, $W_{fg}^{S2,\xi1}$... functions which are made up of the inner products multiplied by $\sin \Omega_E$ or $\cos \Omega_E$ (Table B2).

Appendix C

The torques are found from tidal potentials. The Moon and Sun each incur a tide-raising potential which acts on the Earth. The Earth deforms, creating a tidal potential which reacts back on the Moon and Sun.

The tide-raising potential V^* acting on the Earth due to a body with mass M^* (Moon or Sun) at some point \mathcal{P} is (e.g., Kaula, 1964)

$$V^*(r, \Theta) = \frac{GM^*}{r^*} \sum_{\ell=2}^{+\infty} \left(\frac{r}{r^*} \right)^\ell P_\ell(\cos \Theta) \quad (C1)$$

where G is the universal constant of gravitation, r^* is the distance from the center of the Earth to the center of the tide-raising body, r is the distance from the center of the Earth to \mathcal{P} , and Θ is the angle between the line joining the Earth's center to M^* and the line joining the Earth's center to \mathcal{P} . $P_\ell(\cos \Theta)$ is the Legendre polynomial of degree ℓ . Let θ be the colatitude and λ be the east longitude of \mathcal{P} in an Earth-fixed frame (x_E, y_E, z_E) . The (x_E, y_E, z_E) frame is rigidly attached to the rotating Earth, with the z_E -axis being the rotation axis and the x_E - and y_E -axes lying in the equator, with the x_E -axis passing through a fixed point on the equator (at the longitude of "Greenwich"). Also, let θ^* and λ^* be the colatitude and east longitude of M^* in the Earth-fixed frame; then by the addition theorem (Kaula, 1964), the $\ell=2$ part of the potential of the above equation can be written

$$V^*(r, \theta, \lambda) = \frac{GM^*}{r^*} \left(\frac{r}{r^*} \right)^2 \sum_{i=1}^2 \sum_{m=0}^2 \frac{(2 - \delta_{0m})(2 - m)!}{(2 + m)!} Y_{2mi}(\theta^*, \lambda^*) Y_{2mi}(\theta, \lambda) \quad (C2)$$

where $Y_{\ell m 1}(\theta, \lambda) \equiv P_{\ell m}(\cos \theta) \cos m\lambda$ and $Y_{\ell m 1}(\theta, \lambda) \equiv P_{\ell m}(\cos \theta) \sin m\lambda$ are spherical harmonics of degree ℓ and order m , with $P_{\ell m}(\cos \theta)$ being the associated Legendre polynomial and δ_{0m} being the Kronecker delta.

The following two equations can be extracted from Kaula (2000, pp. 30-37) for the second degree harmonics:

$$\frac{Y_{2n1}(\theta^{*'}, \lambda^{*'})}{(r^{*})^3} = \frac{1}{(a^{*})^3} \sum_{p=0}^2 \sum_{q=-\infty}^{+\infty} G_{2pq}(e^{*}) F_{2np}(J^{*})$$

$$\cdot \left[\frac{\cos}{\sin} \right]_{2-n \text{ odd}}^{2-n \text{ even}} [(2-2p)\omega^{*} + (2-2p+q)\bar{M}^{*} + n\Omega^{*}] \quad (\text{C3})$$

$$\frac{Y_{2n2}(\theta^{*'}, \lambda^{*'})}{(r^{*})^3} = \frac{1}{(a^{*})^3} \sum_{p=0}^2 \sum_{q=-\infty}^{+\infty} G_{2pq}(e^{*}) F_{2np}(J^{*})$$

$$\cdot \left[\frac{\sin}{-\cos} \right]_{2-n \text{ odd}}^{2-n \text{ even}} [(2-2p)\omega^{*} + (2-2p+q)\bar{M}^{*} + n\Omega^{*}] \quad (\text{C4})$$

Combining these equations (C2) and (C3) with the expressions relating the second degree spherical harmonics of one frame with those of another (Appendix D) yields

$$\frac{Y_{2mi}(\theta^{*}, \lambda^{*})}{(r^{*})^3} = \frac{1}{(a^{*})^3} \sum_{n=0}^2 \sum_{p=0}^2 \sum_{q=-\infty}^{+\infty} \sum_{j=-2}^{+2} \sum_{\gamma=1}^2 G_{2pq}(e^{*}) F_{2np}(J^{*}) B_{2minj\gamma}(J_E^{*}, d^{*})$$

$$\cdot \left[\frac{\sin}{\cos} \right]_{m+i \text{ odd}}^{m+i \text{ even}} [(2-2p)\omega^{*} + (2-2p+q)\bar{M}^{*} + n\Omega^{*} + j\Omega_E^{*} + (-1)^{\gamma} m\psi^{*}] \quad (\text{C4})$$

where

$$d^{*} = \theta_E - \theta_M \quad (\text{C5})$$

or

$$d^* = \theta_E - \theta_S \quad (C6)$$

depending upon whether the Moon or the Sun is the tide-raising body. Also, ψ^* is the rotation angle of “Greenwich” and takes care of the Earth rotating on its axis. The $B_{2minj\gamma}(J_E^*, d^*)$ are derived from the rotation matrix given in Appendix D. Below it will be shown only the $i = 1$ values are needed in $B_{2minj\gamma}(J_E^*, d^*)$; these are given in Appendix D to zeroth- and first-order in the angles. Moreover, $J_E^* = J_E$.

The body being acted upon by the tides will be denoted by variables without the asterisk (*), so that the spherical harmonics for that body are

$$\frac{Y_{2mi}(\theta, \lambda)}{r^3} = \frac{1}{a^3} \sum_{N=0}^2 \sum_{P=0}^2 \sum_{Q=-\infty}^{+\infty} \sum_{J=-2\Gamma}^{+2} \sum_{\Gamma=1}^2 G_{2PQ}(e) F_{2NP}(\tilde{J}) B_{2miNJ\Gamma}(J_E, d)$$

$$\cdot \left[\frac{\sin}{\cos} \right]_{m+i \text{ odd}}^{m+i \text{ even}} [(2-2P)\omega + (2-2P+Q)\bar{M} + N\Omega + J\Omega_E + (-1)^\Gamma m\psi] \quad (C7)$$

Here $\psi = \psi^*$ (Kaula, 1964; Efroimsky and Williams, 2009) and \tilde{J} is the inclination of the body being acted upon, so that \tilde{J} is equal to J_M or J_S . (The tilde (\sim) is necessary to distinguish this variable from the index J used in the summation.)

The tide-raising potential V^* distorts the Earth. The distorted Earth in turn produces the tidal potential V . The tidal potential V at a point (r, θ, λ) in space is related to the tide-raising potential V^* by

$$V = [kV^*(R_E, \theta, \lambda)]_{\text{lag}}(R_E^3/r^3) \quad (C8)$$

for a linear response, where $[kV^*(R_E, \theta, \lambda)]_{\text{lag}}$ symbolically denotes the tide-raising potential at the Earth’s surface multiplied by an appropriate Love number k and lagged in time (Kaula, 1964). If the tidal potential V acts on an object (the Moon or Sun) at (r, θ, λ) , which has Keplerian elements $(a, e, I, \Omega, \omega, \bar{M})$, then by (C4)-(C8) the tidal potential becomes

$$\begin{aligned}
V = & \frac{GM^* R_E^5}{a^3 (a^*)^3} \sum_{n=0}^2 \sum_{N=0}^2 \sum_{p=0}^2 \sum_{q=-\infty}^2 \sum_{Q=-\infty}^{+\infty} G_{2pq}(e^*) G_{2PQ}(e) F_{2np}(J^*) F_{2NP}(\tilde{J}) \\
& \sum_{m=0}^2 \frac{(2-\delta_{0m})(2-m)!}{2(2+m)!} \sum_{j=-2J=-2\gamma=1}^2 \sum_{\Gamma=1}^2 \sum_{i=1}^2 k_{2minj\gamma pq}^* B_{2minj\gamma}(J_E^*, d^*) B_{2miNJ\Gamma}(J_E, d) \\
& \cdot \left\{ \cos\{(2-2p)\omega^* + (2-2p+q)\bar{M}^* + n\Omega^* + j\Omega_E + (-1)^\gamma m\psi^* + \delta_{2minj\gamma pq}^* \right. \\
& \left. - [(2-2P)\omega + (2-2P+Q)\bar{M} + N\Omega + J\Omega_E + (-1)^\Gamma m\psi] \right\} \\
& \pm_{m=i}^{m \neq i} [\cos\{(2-2p)\omega^* + (2-2p+q)\bar{M}^* + n\Omega^* + j\Omega_E + (-1)^\gamma m\psi^* + \delta_{2minj\gamma pq}^* \\
& \left. + [(2-2P)\omega + (2-2P+Q)\bar{M} + N\Omega + J\Omega_E + (-1)^\Gamma m\psi] \right\} \quad (C9)
\end{aligned}$$

where $k_{2minj\gamma pq}^*$ is the Love number and $\delta_{2minj\gamma pq}^*$ is the lag angle.

Both the Love number and the lag angle depend on the frequencies of the tide-raising object, with their values determined by whatever rheological model is assumed. The asterisks (*) on $k_{2minj\gamma pq}^*$ and $\delta_{2minj\gamma pq}^*$ are a reminder that they are associated with frequencies of the tide-raising body, and not the body acted upon by V . Originally Kaula (1964) defined the lag angle $\delta_{2minj\gamma pq}^*$ with the sign opposite to that here. Lambeck (1980, p. 118) later reversed the sign convention, so that the lag angle of the major M_2 tide (for which $m = 2$, $p = q = 0$, $\gamma = 1$) has a positive value. This paper follows Lambeck's convention. Also, to save space

$$\mathcal{A}_{2mnj\gamma pq}^* = (2-2p)\omega^* + (2-2p+q)\bar{M}^* + n\Omega^* + j\Omega_E + (-1)^\gamma m\psi^*$$

$$\mathcal{A}_{2mNJ\Gamma PQ} = (2-2P)\omega + (2-2P+Q)\bar{M} + N\Omega + J\Omega_E + (-1)^\Gamma m\psi$$

are used in the equations below.

The expression for V can be considerably simplified. First, it is assumed that the Earth has isotropic properties, so that the subscript i is banished from the subscripts on

the Love numbers and lag angles. Second, the potential is then split into two parts: $V = V_{m=0} + V_{m>0}$, where

$$V_{m=0} = \frac{GM^* R_E^5}{2a^3 (a^*)^3} \sum_{n=0}^2 \sum_{N=0}^2 \sum_{p=0}^2 \sum_{q=-\infty}^{+\infty} \sum_{Q=-\infty}^{+\infty} G_{2pq}(e^*) G_{2PQ}(e) F_{2np}(J^*) F_{2NP}(\tilde{J})$$

$$\sum_{j=-2}^2 \sum_{J=-2}^2 \sum_{\gamma=1}^2 \sum_{\Gamma=1}^2 k_{20nj\gamma pq}^* B_{201nj\gamma}(J_E^*, d^*) B_{201NJ\Gamma}(J_E, d)$$

$$[\cos(\mathcal{A}_{20nj\gamma pq}^* + \delta_{20nj\gamma pq}^* - \mathcal{A}_{20NJ\Gamma PQ}) + \cos(\mathcal{A}_{20nj\gamma pq}^* + \delta_{20nj\gamma pq}^* + \mathcal{A}_{20NJ\Gamma PQ})]$$

and

$$V_{m>0} = \frac{GM^* R_E^5}{a^3 (a^*)^3} \sum_{n=0}^2 \sum_{N=0}^2 \sum_{p=0}^2 \sum_{q=-\infty}^{+\infty} \sum_{Q=-\infty}^{+\infty} G_{2pq}(e^*) G_{2PQ}(e) F_{2np}(J^*) F_{2NP}(\tilde{J})$$

$$\sum_{m=1}^2 \frac{(2-m)!}{(2+m)!} \sum_{j=-2}^2 \sum_{J=-2}^2 \sum_{\gamma=1}^2 \sum_{\Gamma=1}^2 \sum_{i=1}^2 k_{2mnj\gamma pq}^* B_{2minj\gamma}(J_E^*, d^*) B_{2miNJ\Gamma}(J_E, d)$$

$$\cdot [\cos(\mathcal{A}_{2mnj\gamma pq}^* + \delta_{2mnj\gamma pq}^* - \mathcal{A}_{2mNJ\Gamma PQ}) \pm \cos(\mathcal{A}_{2mnj\gamma pq}^* + \delta_{2mnj\gamma pq}^* + \mathcal{A}_{2mNJ\Gamma PQ})]$$

Next, i is summed over in (C11):

$$V_{m>0} = \frac{GM^* R_E^5}{a^3 (a^*)^3} \sum_{n=0}^2 \sum_{N=0}^2 \sum_{p=0}^2 \sum_{q=-\infty}^{+\infty} \sum_{Q=-\infty}^{+\infty} G_{2pq}(e^*) G_{2PQ}(e) F_{2np}(J^*) F_{2NP}(\tilde{J})$$

$$\sum_{m=1}^2 \frac{(2-m)!}{(2+m)!} \sum_{j=-2}^2 \sum_{J=-2}^2 \sum_{\gamma=1}^2 \sum_{\Gamma=1}^2 k_{2mnj\gamma pq}^* [B_{2m1nj\gamma}(J_E^*, d^*) B_{2m1NJ\Gamma}(J_E, d)$$

$$\cdot [\cos(\mathcal{A}_{2mnj\gamma pq}^* + \delta_{2mnj\gamma pq}^* - \mathcal{A}_{2mNJ\Gamma PQ}) + (-1)^m \cos(\mathcal{A}_{2mnj\gamma pq}^* + \delta_{2mnj\gamma pq}^* + \mathcal{A}_{2mNJ\Gamma PQ})]$$

$$+B_{2m2nj\gamma}(J_E^*, d^*)B_{2m2NJ\Gamma}(J_E, d)$$

$$\cdot [\cos(\mathcal{A}_{2mnj\gamma pq}^* + \delta_{2mnj\gamma pq}^* - \mathcal{A}_{2mNJ\Gamma PQ}) + (-1)^{m+1} \cos(\mathcal{A}_{2mnj\gamma pq}^* + \delta_{2mnj\gamma pq}^* + \mathcal{A}_{2mNJ\Gamma PQ})] \quad .$$

It turns out that

$$B_{2m2nj\gamma}(J_E^*, d^*) = (-1)^{m+\gamma-1} B_{2m1nj\gamma}(J_E^*, d^*) \quad \text{for } m > 0$$

so that

$$V_{m>0} = \frac{GM^* R_E^5}{a^3 (a^*)^3} \sum_{n=0}^2 \sum_{N=0}^2 \sum_{p=0}^2 \sum_{Q=-\infty}^{+\infty} \sum_{q=-\infty}^{+\infty} G_{2pq}(e^*) G_{2PQ}(e) F_{2np}(J^*) F_{2NP}(\tilde{J})$$

$$\sum_{m=1}^2 \frac{(2-m)!}{(2+m)!} \sum_{j=-2}^2 \sum_{J=-2}^2 \sum_{\gamma=1}^2 \sum_{\Gamma=1}^2 k_{2mnj\gamma pq}^* [1 + (-1)^{\gamma+\Gamma}] B_{2m1nj\gamma}(J_E^*, d^*) B_{2m1NJ\Gamma}(J_E, d)$$

$$\cdot [\cos(\mathcal{A}_{2mnj\gamma pq}^* + \delta_{2mnj\gamma pq}^* - \mathcal{A}_{2mNJ\Gamma PQ})]$$

$$+ (-1)^m [1 + (-1)^{\gamma+\Gamma-1}] B_{2m1nj\gamma}(J_E^*, d^*) B_{2m1NJ\Gamma}(J_E, d)$$

$$\cdot [\cos(\mathcal{A}_{2mnj\gamma pq}^* + \delta_{2mnj\gamma pq}^* + \mathcal{A}_{2mNJ\Gamma PQ})] \quad .$$

This is can be rewritten

$$V_{m>0} = \frac{GM^* R_E^5}{a^3 (a^*)^3} \sum_{n=0}^2 \sum_{N=0}^2 \sum_{p=0}^2 \sum_{Q=-\infty}^{+\infty} \sum_{q=-\infty}^{+\infty} G_{2pq}(e^*) G_{2PQ}(e) F_{2np}(J^*) F_{2NP}(\tilde{J})$$

$$\sum_{m=1}^2 \frac{(2-m)!}{(2+m)!} \sum_{j=-2}^2 \sum_{J=-2}^2 \sum_{\gamma=1}^2 \sum_{\Gamma=1}^2 k_{2mnj\gamma pq}^* B_{2m1nj\gamma}(J_E^*, d^*) B_{2m1NJ\Gamma}(J_E, d)$$

$$\cdot \{ [1 + (-1)^{\gamma+\Gamma}] [\cos(\mathcal{A}_{2mnj\gamma pq}^* + \delta_{2mnj\gamma pq}^* - \mathcal{A}_{2mNJ\Gamma PQ})]$$

$$+(-1)^m [1 + (-1)^{\gamma+\Gamma-1}] [\cos(\mathcal{A}_{2mnj\gamma pq}^* + \delta_{2mnj\gamma pq}^* + \mathcal{A}_{2mNJ\Gamma PQ})]$$

Next Γ is summed over, yielding (45), which is one of the fundamental equations of this paper. It is to be noted that ψ and ψ^* disappear in the trigonometric arguments in (45) because $\psi = \psi^*$ (Efroimsky and Williams, 2009) and they are subtracted from each other in the nonzero terms.

Appendix D

This appendix finds the relationship between the second degree spherical harmonics in some system (x_E, y_E, z_E) to those of some other system (x', y', z') . For any general coordinate system (x, y, z)

$$r^2 = x^2 + y^2 + z^2$$

$$r^2 Y_{201}(\theta, \lambda) = z^2 - (x^2 + y^2)/2$$

$$r^2 Y_{202}(\theta, \lambda) \equiv 0$$

$$r^2 Y_{211}(\theta, \lambda) = 3xz$$

$$r^2 Y_{212}(\theta, \lambda) = 3yz$$

$$r^2 Y_{221}(\theta, \lambda) = 3x^2 - 3y^2$$

$$r^2 Y_{222}(\theta, \lambda) = 6xy$$

The following relations are also helpful:

$$\frac{x^2}{r^2} = \frac{1}{3} - \frac{1}{3} Y_{201}(\theta, \lambda) + \frac{1}{6} Y_{221}(\theta, \lambda),$$

$$\frac{y^2}{r^2} = \frac{1}{3} - \frac{1}{3} Y_{201}(\theta, \lambda) - \frac{1}{6} Y_{221}(\theta, \lambda),$$

$$\frac{z^2}{r^2} = \frac{1}{3} + \frac{2}{3} Y_{201}(\theta, \lambda),$$

Let the x, y, z coordinates be related to those of the (x', y', z') system by $\mathbf{X} = \mathbf{C}\mathbf{X}'$, where \mathbf{C} is the rotation matrix; then

$$\begin{aligned} Y_{201}(\theta, \lambda) = & [(4c_{33}^2 - 2c_{31}^2 - 2c_{32}^2)/4] Y_{201}(\theta', \lambda') \\ & + (c_{31}c_{33}) Y_{211}(\theta', \lambda') \\ & + (c_{32}c_{33}) Y_{212}(\theta', \lambda') \\ & + [(c_{31}^2 - c_{32}^2)/4] Y_{221}(\theta', \lambda') \end{aligned}$$

$$\begin{aligned} Y_{211}(\theta, \lambda) = & (2c_{13}c_{33} - c_{11}c_{31} - c_{12}c_{32}) Y_{201}(\theta', \lambda') \\ & + (c_{13}c_{31} + c_{11}c_{33}) Y_{211}(\theta', \lambda') \\ & + (c_{13}c_{32} + c_{12}c_{33}) Y_{212}(\theta', \lambda') \\ & + [(c_{11}c_{31} - c_{12}c_{32})/2] Y_{221}(\theta', \lambda') \\ & + [(c_{12}c_{31} + c_{11}c_{32})/2] Y_{222}(\theta', \lambda') \end{aligned}$$

$$\begin{aligned} Y_{212}(\theta, \lambda) = & (2c_{23}c_{33} - c_{21}c_{31} - c_{22}c_{32}) Y_{201}(\theta', \lambda') \\ & + (c_{21}c_{33} + c_{23}c_{31}) Y_{211}(\theta', \lambda') \\ & + (c_{23}c_{32} + c_{22}c_{33}) Y_{212}(\theta', \lambda') \\ & + [(c_{21}c_{31} - c_{22}c_{32})/2] Y_{221}(\theta', \lambda') \\ & + [(c_{22}c_{31} + c_{21}c_{32})/2] Y_{222}(\theta', \lambda') \end{aligned}$$

$$\begin{aligned}
Y_{221}(\theta, \lambda) = & (2c_{13}^2 - 2c_{23}^2 - c_{11}^2 + c_{21}^2 - c_{12}^2 + c_{22}^2) Y_{201}(\theta', \lambda') \\
& + 2(c_{11}c_{13} - c_{21}c_{23}) Y_{211}(\theta', \lambda') \\
& + 2(c_{12}c_{13} - c_{22}c_{23}) Y_{212}(\theta', \lambda') \\
& + [(c_{11}^2 - c_{21}^2 - c_{12}^2 + c_{22}^2)/2] Y_{221}(\theta', \lambda') \\
& + (c_{11}c_{12} - c_{21}c_{22}) Y_{222}(\theta', \lambda')
\end{aligned}$$

$$\begin{aligned}
Y_{222}(\theta, \lambda) = & (4c_{13}c_{23} - 2c_{11}c_{21} - 2c_{12}c_{22}) Y_{201}(\theta', \lambda') \\
& + 2(c_{13}c_{21} + c_{11}c_{23}) Y_{211}(\theta', \lambda') \\
& + 2(c_{13}c_{22} + c_{12}c_{23}) Y_{212}(\theta', \lambda') \\
& + (c_{11}c_{21} - c_{12}c_{22}) Y_{221}(\theta', \lambda') \\
& + (c_{12}c_{21} + c_{11}c_{22}) Y_{222}(\theta', \lambda')
\end{aligned}$$

Choosing the tide-raising body's Laplace plane system as the primed system and writing in matrix form $\mathbf{x}_E = (x_E, y_E, z_E)^T$ and $\mathbf{x}^* = (x^*, y^*, z^*)^T$, the relationship between the two systems is

$$\mathbf{x}_E = \mathbf{C}\mathbf{x}^* \text{ where } \mathbf{C} = \mathbf{A}\mathbf{B}.$$

The elements a_{ij} of the rotation matrix are given by

$$\begin{aligned}
a_{11} &= \cos \Omega_E \cos \psi - \cos J_E \sin \Omega_E \sin \psi \\
a_{12} &= \sin \Omega_E \cos \psi + \cos J_E \cos \Omega_E \sin \psi \\
a_{13} &= \sin J_E \sin \psi \\
a_{21} &= -\cos \Omega_E \sin \psi - \cos J_E \sin \Omega_E \cos \psi \\
a_{22} &= -\sin \Omega_E \sin \psi + \cos J_E \cos \Omega_E \cos \psi \\
a_{23} &= \sin J_E \cos \psi \\
a_{31} &= \sin J_E \sin \Omega_E \\
a_{32} &= -\sin J_E \cos \Omega_E \\
a_{33} &= \cos J_E
\end{aligned}$$

where ψ is the rotation angle from “Greenwich”, and $b_{11}=1$, $b_{22}=b_{33}=\cos d^*$, $b_{23}=\sin d^*=-b_{32}$ where $d^*=\theta_E-\theta_M$ or $d^*=\theta_E-\theta_S$, depending on which body is the tide-raising object. All the other $b_{ij}=0$. The **B** matrix transforms from the Laplace plane frame of the tide-raising body into the Laplace plane frame of the Earth. The **A** matrix transforms from the Earth’s Laplace plane frame into the (x_E, y_E, z_E) rigidly fixed in the Earth. The transformations give rise to the $B_{2m1nj\gamma}(J_E, d)$ functions; those used here are listed in Table 1.

Appendix E

The equations for dJ_M/dt and $d\theta_E/dt$ for the nodal and semiannual tides are given below. The right sides are left in terms of the sines rather than just the approximation $\sin J_M \approx J_M$, etc. For the nodal tides,

$$\begin{aligned}
 \frac{dJ_M}{dt} = & \frac{9GM_M^2}{16R_E h} \left(\frac{R_E}{a} \right)^6 \left(\frac{1}{1+\alpha\beta} \right) \\
 & \cdot \left\{ - \left[1 + \alpha \left(\frac{h}{H} \right) \right] (\sin^2 2d^* \sin J_M) k_{node}^M \sin \delta_{node}^M \right. \\
 & + \left(\frac{M_S}{M_M} \right) \left(\frac{a}{a_S} \right)^3 \left[1 + \alpha \left(\frac{h}{H} \right) \right] (\sin 2J_E \sin 2d^* \sin \theta_E) k_{node}^S \sin \delta_{node}^S \\
 & - \left(\frac{M_S}{M_M} \right) \left(\frac{a}{a_S} \right)^3 \left(\frac{h}{H} \right) \alpha (2 \sin J_M + \sin 2J_E) \sin 2d^* \sin \theta_E k_{node}^M \sin \delta_{node}^M \\
 & \left. - \left(\frac{M_S}{M_M} \right)^2 \left(\frac{a}{a_S} \right)^6 \left(\frac{h}{H} \right) \alpha (2 \sin 2J_E \sin^2 \theta_E) k_{node}^S \sin \delta_{node}^S \right\} \quad (E1)
 \end{aligned}$$

$$\begin{aligned}
\frac{d\theta_E}{dt} = & \frac{9GM_M^2}{16R_E h} \left(\frac{R_E}{a} \right)^6 \left(\frac{1}{1+\alpha\beta} \right) \\
& \cdot \left\{ \left[-\beta + \left(\frac{h}{H} \right) \right] [+2\sin^2 J_M + \sin 2J_E \sin J_M + (1/2)\sin^2 2J_E] \sin 2d k_{node}^M \sin \delta_{node}^M \right. \\
& + \left(\frac{M_S}{M_M} \right) \left(\frac{a}{a_S} \right)^3 \left[-\beta + \left(\frac{h}{H} \right) \right] (+2\sin J_M + 2\sin 2J_E) \sin 2J_E \sin \theta_E k_{node}^S \sin \delta_{node}^S \\
& + \left(\frac{M_S}{M_M} \right) \left(\frac{a}{a_S} \right)^3 \left(\frac{h}{H} \right) [(-3/2)\sin J_M + (1/8)\sin 2J_E] \sin 2J_E \sin 2d^* k_{node}^M \sin \delta_{node}^M \\
& \left. + \left(\frac{M_S}{M_M} \right)^2 \left(\frac{a}{a_S} \right)^6 \left(\frac{h}{H} \right) (\sin^2 2J_E \sin \theta_E) k_{node}^S \sin \delta_{node}^S \right\} , \tag{E2}
\end{aligned}$$

where the period of the nodal tide is $2\pi/\dot{\Omega}$. For the semidiurnal tide

$$\frac{dJ_M}{dt} = -\frac{9GM_S^2}{32R_E H} \left(\frac{R_E}{a_S} \right)^6 \left(\frac{\alpha}{1+\alpha\beta} \right) [+(1/2)\sin^2 \theta_E + \sin^2 J_E] \sin 2J_E k_{semi}^S \sin \delta_{semi}^S \tag{E3}$$

and

$$\frac{d\theta_E}{dt} = -\frac{9GM_S^2}{32R_E H} \left(\frac{R_E}{a_S} \right)^6 \left(\frac{1}{1+\alpha\beta} \right) (+\sin^2 2J_E) \sin \theta_E k_{semi}^S \sin \delta_{semi}^S \tag{E4}$$

for which the period is half a year.

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TablesTable 1. The $B_{2m1nj\gamma}(J_E, d)$ functions to zeroth and first order in the sines.

m	n	j	γ		
0	0	+0	1	$+(1/8) (3 \cos^2 J_E - 1) (1 + 3 \cos 2d^*)$	0
0	1	-1	1	$+(1/4) \sin 2J_E (\cos d + \cos 2d^*)$	1
0	1	+0	1	$+(1/4) (3 \cos^2 J_E - 1) \sin 2d^*$	1
1	0	-1	1	$-(3/8) (\cos J_E + \cos 2J_E) \sin 2d^*$	1
1	0	+0	1	$-(3/16) \sin 2J_E (1 + 3 \cos 2d^*)$	1
1	0	+0	2	$+(3/16) \sin 2J_E (1 + 3 \cos 2d^*)$	1
1	0	+1	2	$+(3/8) (\cos J_E + \cos 2J_E) \sin 2d^*$	1
1	1	-1	1	$+(1/4) (\cos J_E + \cos 2J_E) (\cos d^* + \cos 2d^*)$	0
1	2	-2	1	$-(1/64) (2 \sin J_E + \sin 2J_E) (3 + 4 \cos d^* + \cos 2d^*)$	1
1	2	-1	1	$-(1/16) (\cos J_E + \cos 2J_E) (2 \sin d + \sin 2d)$	1
2	1	-2	1	$+(1/8) (1 + \cos J_E)^2 (2 \sin d^* + \sin 2d^*)$	1
2	1	-1	1	$+(1/4) (2 \sin J_E + \sin 2J_E) (\cos d^* + \cos 2d^*)$	1
2	2	-2	1	$+(1/32) (1 + \cos J_E)^2 (3 + 4 \cos d^* + \cos 2d^*)$	0

Table 2. Constants in the Ross-Schubert model.

Symbol	Ross & Schubert	Present paper
k_0	1.5	1.0
μ_0	9.7×10^{10} Pa	1.505×10^{11} Pa
Ξ	1679 K	1679 K
δ_0	25	12.4663
D	17937	17258.75
χ	0.25	0.25
τ_0	— ^a	2000
τ_1	— ^a	565
τ_2	— ^a	0.2
τ_3	— ^a	81.319
a_0	$10 R_E$	$7.27 R_E$

^a No equation for τ_E was given.

Table B1. U functions for the Moon.

f	g	$2U_{fg}^{M1,\xi1}$	$2U_{fg}^{M1,\xi2}$	$2U_{fg}^{M1,\xi3}$
0	-1	+0	+0	+0
0	+0	+0	+0	+0
0	+1	+0	+0	+0
1	-1	$+(1 + \cos d^*)$	$+(1 + \cos d^*) \cos J_E$	$-(1 + \cos d) \sin J_E$
1	+0	+0	$-2 \sin d^* \sin J_E$	$-2 \sin d^* \cos J_E$
1	+1	$+(1 - \cos d^*)$	$-(1 - \cos d^*) \cos J_E$	$+(1 - \cos d^*) \sin J_E$
		$2U_{fg}^{M2,\xi1}$	$2U_{fg}^{M2,\xi2}$	$2U_{fg}^{M2,\xi3}$
0	-1	+0	+0	+0
0	+0	+0	$+2 \cos d^* \sin J_M \sin J_E$	$+2 \cos d^* \sin J_M \cos J_E$
0	+1	$+2 \sin d^* \sin J_M$	$+2 \sin d^* \sin J_M \cos J_E$	$-2 \sin d^* \sin J_M \sin J_E$
1	-1	$-(1 + \cos d^*) \cos J_M$	$+(1 + \cos d^*) \cos J_M \cos J_E$	$-(1 + \cos d^*) \cos J_M \sin J_E$
1	+0	+0	$-2 \sin d^* \cos J_M \sin J_E$	$-2 \sin d^* \cos J_M \cos J_E$
1	+1	$-(1 - \cos d^*) \sin J_M$	$-(1 - \cos d^*) \cos J_M \cos J_E$	$+(1 - \cos d^*) \cos J_M \sin J_E$
		$2U_{fg}^{M3,\xi1}$	$2U_{fg}^{M3,\xi2}$	$2U_{fg}^{M3,\xi3}$
0	-1	+0	+0	+0
0	+0	+0	$+2 \cos d^* \cos J_M \sin J_E$	$+2 \cos d^* \cos J_M \cos J_E$
0	+1	$+2 \sin d^* \cos J_M$	$+2 \sin d^* \cos J_M \cos J_E$	$-2 \sin d^* \cos J_M \sin J_E$
1	-1	$+(1 + \cos d^*) \sin J_M$	$-(1 + \cos d^*) \sin J_M \cos J_E$	$+(1 + \cos d^*) \sin J_M \sin J_E$
1	+0	+0	$+2 \sin d^* \sin J_M \sin J_E$	$+2 \sin d^* \sin J_M \cos J_E$
1	+1	$+(1 - \cos d^*) \sin J_M$	$+(1 - \cos d^*) \sin J_M \cos J_E$	$-(1 - \cos d^*) \sin J_M \sin J_E$

Table B2. W functions for the Moon.

f	g	$4W_{fg}^{M1,\xi1}$	$4W_{fg}^{M1,\xi2}$	$4W_{fg}^{M1,\xi3}$
0	+0	+0	+0	+0
0	+1	+0	+0	+0
0	+2	+0	+0	+0
1	-2	$-(1 + \cos d^*)$	$+(1 + \cos d^*) \cos J_E$	$-(1 + \cos d^*) \sin J_E$
1	-1	+0	$-2 \sin d^* \sin J_E$	$-2 \sin d^* \cos J_E$
1	+0	$+2 \cos d^*$	$+2 \cos d^* \cos J_E$	$-2 \cos d^* \sin J_E$
1	+1	+0	$-2 \sin d^* \sin J_E$	$-2 \sin d^* \cos J_E$
1	+2	$+(1 - \cos d^*)$	$-(1 - \cos d^*) \cos J_E$	$+(1 - \cos d^*) \sin J_E$
		$4W_{fg}^{M2,\xi1}$	$4W_{fg}^{M2,\xi2}$	$4W_{fg}^{M2,\xi3}$
0	+0	$+2 \sin d^* \sin J_M$	$+2 \sin d^* \sin J_M \cos J_E$	$-2 \sin d^* \sin J_M \sin J_E$
0	+1	+0	$+2 \cos d^* \sin J_M \sin J_E$	$+4 \cos d^* \sin J_M \cos J_E$
0	+2	$-2 \sin d^* \sin J_M$	$+2 \sin d^* \sin J_M \cos J_E$	$-2 \sin d^* \sin J_M \sin J_E$
1	-2	$-(1 + \cos d^*) \cos J_M$	$+(1 + \cos d^*) \cos J_M \cos J_E$	$-(1 + \cos d^*) \cos J_M \sin J_E$
1	-1	+0	$-2 \sin d^* \cos J_M \sin J_E$	$-2 \sin d^* \cos J_M \cos J_E$
1	+0	$+2 \cos d^* \cos J_M$	$+2 \cos d^* \cos J_M \cos J_E$	$-2 \cos d^* \cos J_M \sin J_E$
1	+1	+0	$-2 \sin d^* \cos J_M \sin J_E$	$-2 \sin d^* \cos J_M \cos J_E$
1	+2	$+(1 - \cos d^*) \cos J_M$	$-(1 - \cos d^*) \cos J_M \cos J_E$	$+(1 - \cos d^*) \cos J_M \sin J_E$
		$4W_{fg}^{M3,\xi1}$	$4W_{fg}^{M3,\xi2}$	$4W_{fg}^{M3,\xi3}$
0	+0	$+2 \sin d^* \cos J_M$	$+2 \sin d^* \cos J_M \cos J_E$	$-2 \sin d^* \cos J_M \sin J_E$
0	+1	+0	$+2 \cos d^* \cos J_M \sin J_E$	$+4 \cos d^* \cos J_M \cos J_E$
0	+2	$-2 \sin d^* \cos J_M$	$+2 \sin d^* \cos J_M \cos J_E$	$-2 \sin d^* \cos J_M \sin J_E$
1	-2	$+(1 + \cos d^*) \sin J_M$	$-(1 + \cos d^*) \sin J_M \cos J_E$	$+(1 + \cos d^*) \sin J_M \sin J_E$
1	-1	+0	$+2 \sin d^* \sin J_M \sin J_E$	$+2 \sin d^* \sin J_M \cos J_E$
1	+0	$-2 \cos d^* \sin J_M$	$-2 \cos d^* \sin J_M \cos J_E$	$+2 \cos d^* \sin J_M \sin J_E$
1	+1	+0	$+2 \sin d^* \sin J_M \sin J_E$	$+2 \sin d^* \sin J_M \cos J_E$
1	+2	$-(1 - \cos d^*) \sin J_M$	$+(1 - \cos d^*) \sin J_M \cos J_E$	$-(1 - \cos d^*) \sin J_M \sin J_E$

Figures

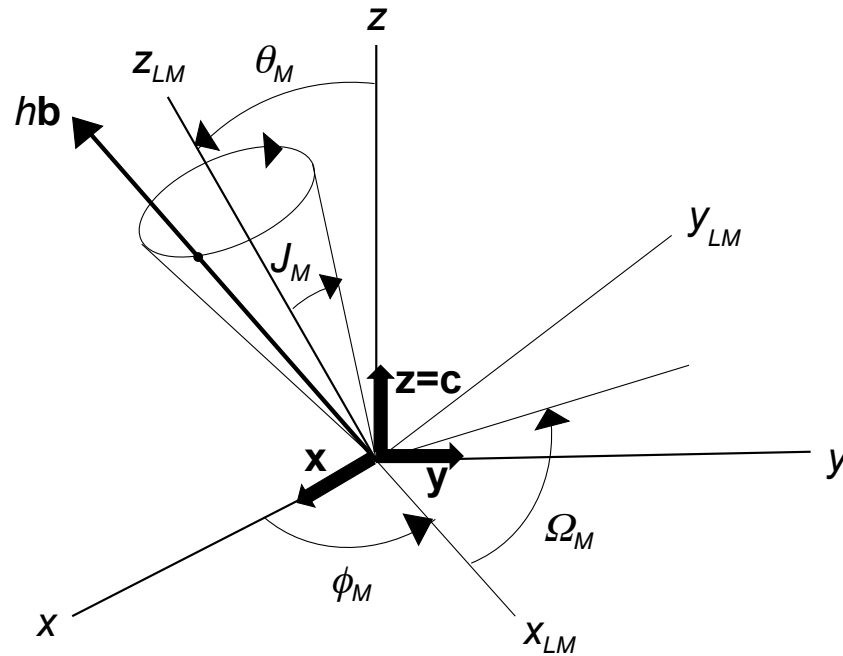


Fig. 1. The Laplace plane of the Moon. The x, y, z system is inertial with the plane of the ecliptic lying in the x - y plane. \mathbf{x} , \mathbf{y} , and \mathbf{z} are unit vectors along the respective axes, with $\mathbf{z} = \mathbf{c}$, where \mathbf{c} is the unit vector normal to the ecliptic. The Moon's Laplace plane lies in the x_{LM} - y_{LM} plane with the z_{LM} axis being normal to the Laplace plane. The z_{LM} axis is tilted by an angle θ_M to the z axis. The Laplace plane intersects the x - y plane along the x_{LM} axis, with the x_{LM} axis making an angle ϕ_M with the x axis. The unit vector \mathbf{b} is normal to the plane of the Moon's orbit; the magnitude of the orbital angular momentum is h . The Moon's orbit precesses around the z_{LM} axis, with J_M being the angle between the z_{LM} axis and \mathbf{b} . The intersection of the Moon's orbital plane with the Laplace plane makes an angle Ω_M with the x_{LM} axis. Both ϕ_M and Ω_M precess nearly uniformly in the negative sense.

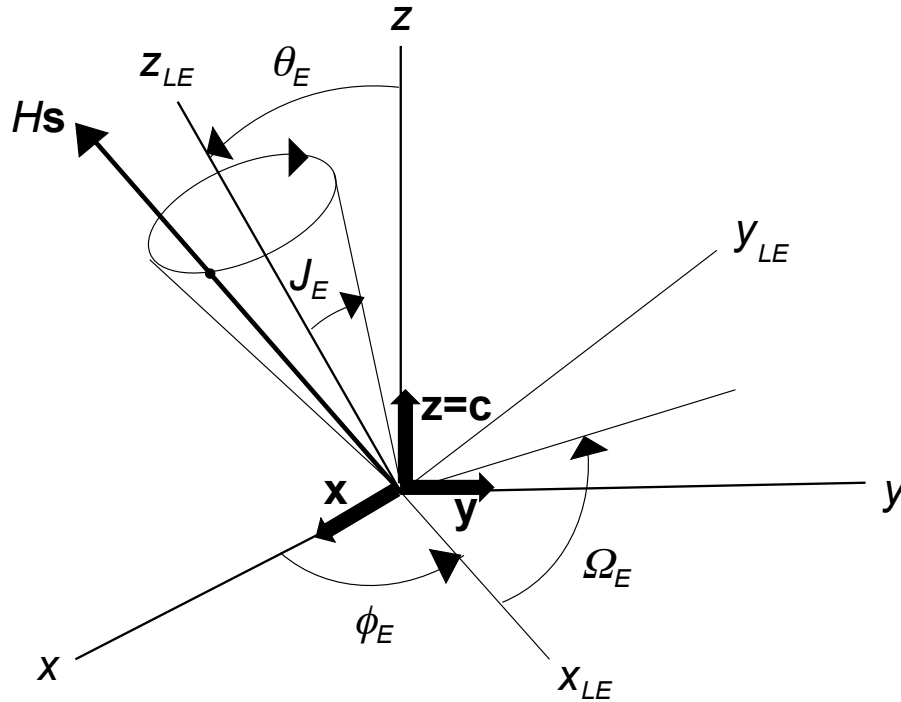


Fig. 2. The Laplace plane of the Earth. The x, y, z system is inertial with the plane of the ecliptic lying in the x - y plane. \mathbf{x} , \mathbf{y} , and \mathbf{z} are unit vectors along the respective axes, with $\mathbf{z} = \mathbf{c}$, where \mathbf{c} is the unit vector normal to the ecliptic. The Earth's Laplace plane lies in the x_{LE} - y_{LE} plane with the z_{LE} axis normal to the Laplace plane. The z_{LE} axis is tilted by an angle θ_E to the z axis. The Laplace plane intersects the x - y plane along the x_{LE} axis, with the x_{LE} axis making an angle ϕ_E with the x axis. The unit vector \mathbf{s} is along the Earth's spin vector; the magnitude of the rotational angular momentum is H . The Earth precesses around the z_{LE} axis, with J_E being the angle between the z_{LE} axis and \mathbf{s} . The intersection of the Earth's equatorial plane with the Laplace plane makes an angle Ω_E with the x_{LE} axis. Both ϕ_E and Ω_E precess nearly uniformly in the negative sense.

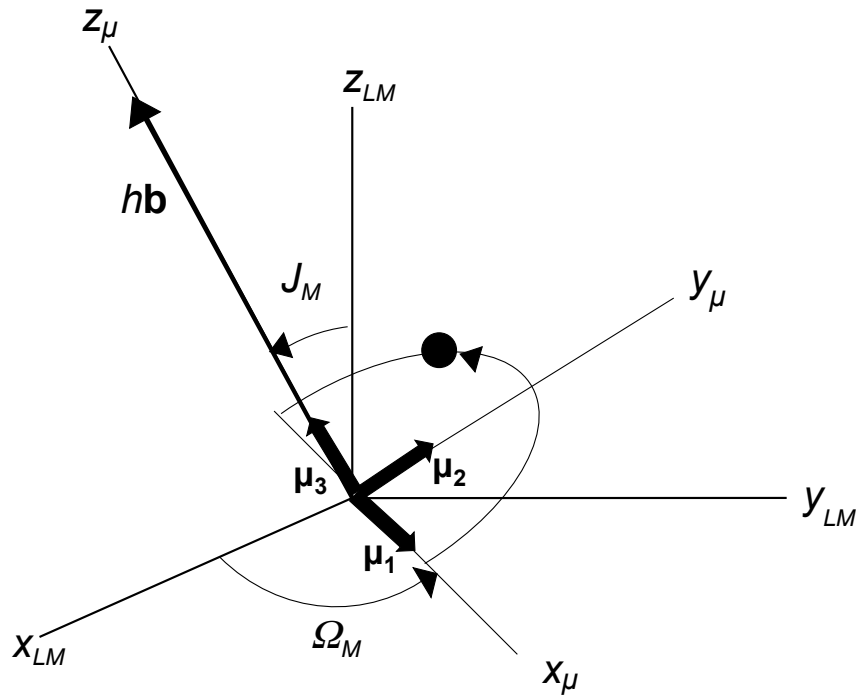


Fig. 3. The relationship between the Moon's x_μ, y_μ, z_μ system and the x_{LM}, y_{LM}, z_{LM} system. The Moon is shown as the large black dot. The Moon's orbit lies in the x_μ - y_μ plane. The unit vectors μ_1, μ_2, μ_3 lie along the respective axis of the Moon's x_μ, y_μ, z_μ system. The unit vector b normal to the Moon's orbital plane makes an angle J_M with the z_{LM} axis. The x_μ axis makes an angle Ω_M with the x_{LM} axis.

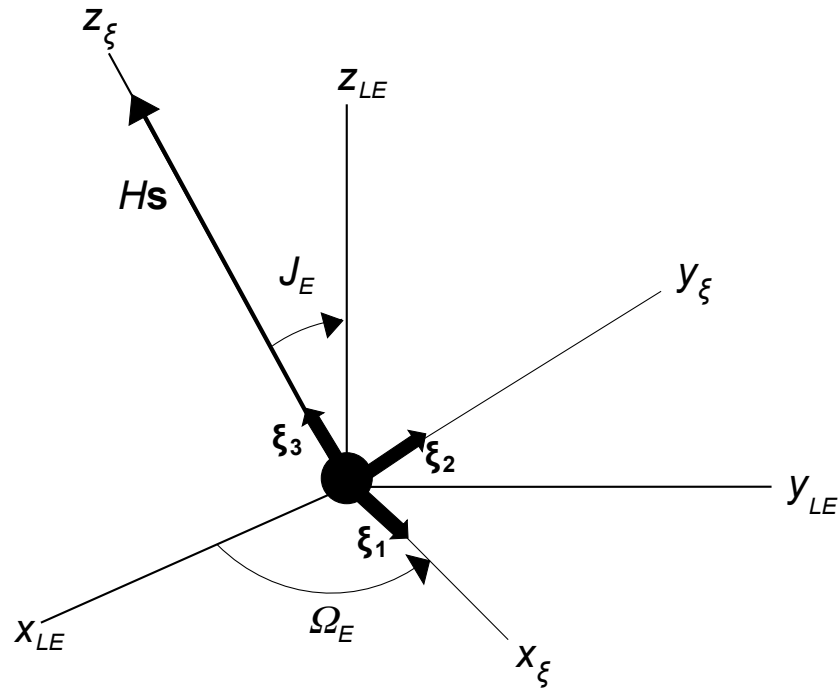


Fig. 4. The relationship between the Earth's x_{ξ} , y_{ξ} , z_{ξ} system and the x_{LE} , y_{LE} , z_{LE} system. The Earth is shown as the large black dot at the origin. The Earth's equatorial plane lies in the x_{ξ} - y_{ξ} plane. The unit vectors ξ_1 , ξ_2 , ξ_3 lie along the respective axis of the Earth's x_{ξ} , y_{ξ} , z_{ξ} system. The unit vector s along the Earth's spin axis makes an angle J_E with the z_{LE} axis. The x_{ξ} axis makes an angle Ω_E with the x_{LE} axis.

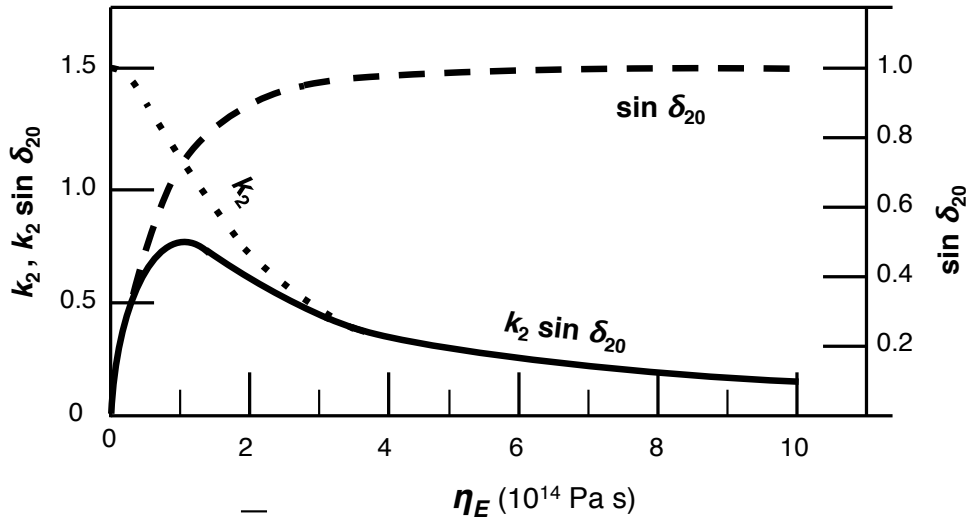


Fig. 5. Love numbers and lag angles as a function of viscosity η_E for a viscous liquid for an angular frequency with a 4 hour period. The value of the sine of the lag angle δ_{20} is read off the right-hand scale, while the values of the Love number k_2 and the product $k_2 \sin \delta_{20}$ are both read off the left-hand scale.

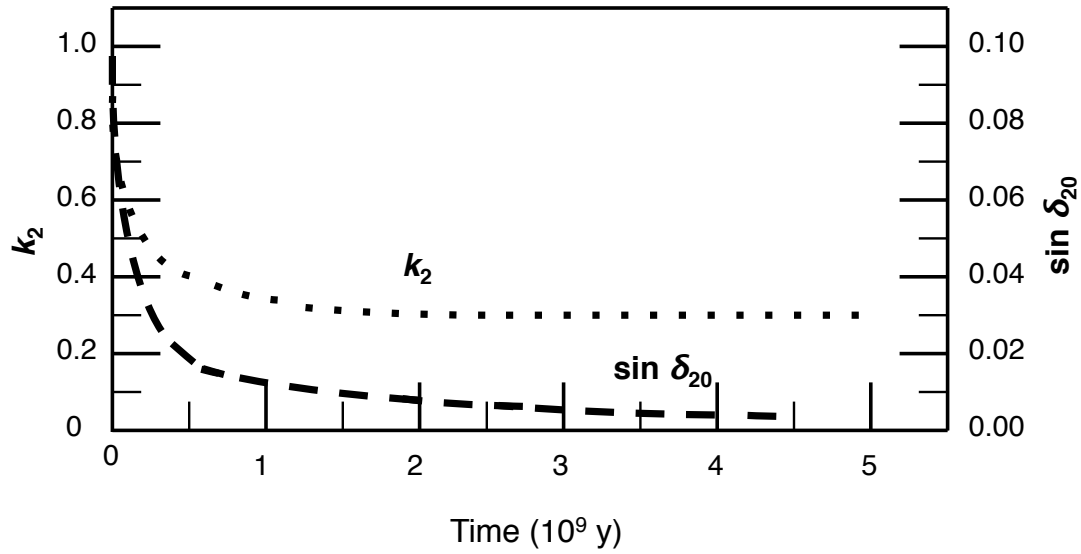


Fig. 6. Love number and lag angles in the Ross-Schubert model as a function of time. The sine of the lag angle δ_{20} is read off the right-hand scale, while the value of the Love number k_2 is read off the left-hand scale. The initial Love number is 0.85.

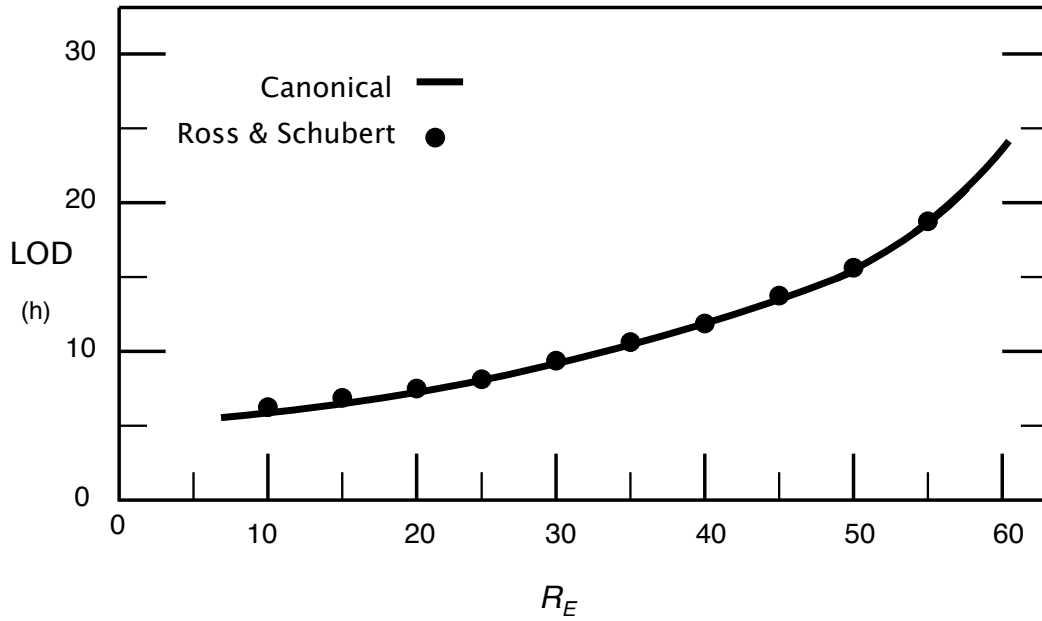


Fig. 7. The Earth’s length-of-day (LOD) as a function of the Moon’s distance from the center of the Earth. The distance is in terms of Earth radii, where R_E is the Earth’s radius. The usual LOD found from the small lag angle approximation is the curve labeled “Canonical”, while the Ross-Schubert model with parameters listed on the right side of Table 2 are shown as black dots every 5 R_E .

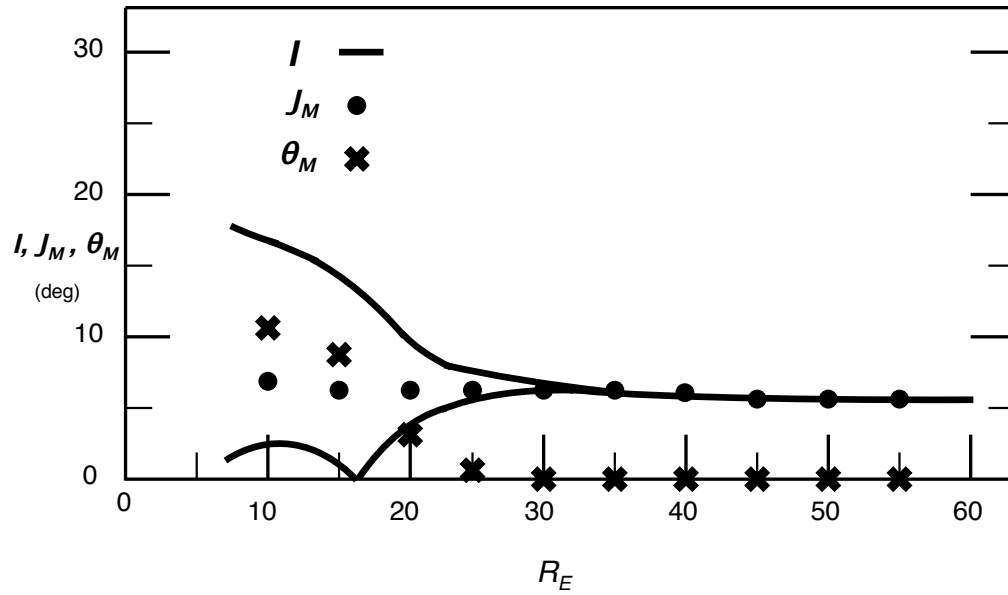


Fig. 8. Angles for the Moon as a function of the Moon’s distance from the center of the Earth. The inclination of the Moon’s orbit I to the ecliptic for the small lag angle approximation is shown as the black curve. The inclination oscillates between the upper and lower branches of the curve. The “bounce” in the lower branch between $7 R_E$ and $17 R_E$ is due to I always being taken to be positive. The oscillations become small for distances $> \sim 30 R_E$. The angle J_M for the Ross-Schubert model using the parameters adopted here are shown as dots every $5 R_E$, while the angle θ_M is shown as \times ’s.

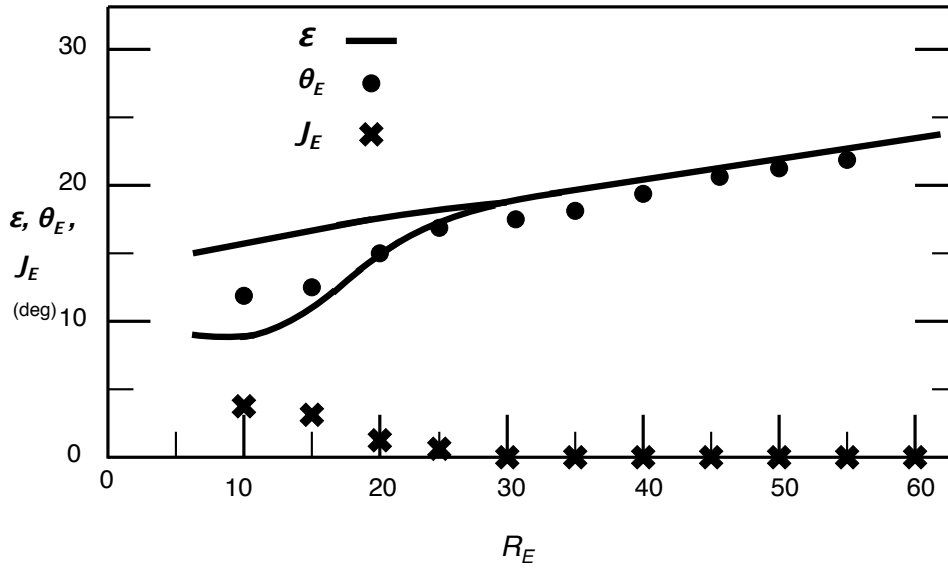


Fig. 9. Angles for the Earth as a function of the Moon's distance from the center of the Earth. The Earth's obliquity ε for the small lag angle approximation is shown as the black curve. The obliquity oscillates between the upper and lower branches of the curve. The oscillations become small for distances $> \sim 30 R_E$, so that $\theta_E \approx \varepsilon$. The angle θ_E for the Ross-Schubert model using the parameters adopted here are shown as dots every $5R_E$, while the angle J_E is shown as \times 's.

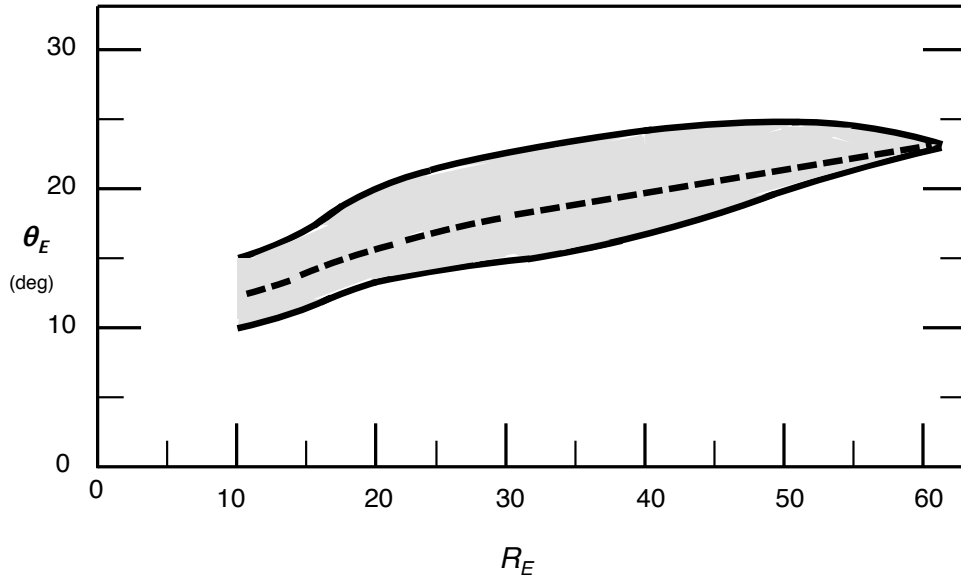


Fig. 10. Envelope for θ_E for parameterized ocean tide models as a function of the Moon's distance from the center of the Earth. θ_E is essentially obliquity ε for Earth-Moon distances $> 30 R_E$. The dotted line gives θ_E for $\Delta_{12} = (k_{10} \sin \delta_{10} / k_{20} \sin \delta_{20}) = (k_{11} \sin \delta_{11} / k_{20} \sin \delta_{20}) = 1$. The lower curve is for $\Delta_{12} = 0$. The upper curve is for $\Delta_{12} = 2$. The gray region in between is for $0 \leq \Delta_{12} \leq 2$. Note that for the upper curve θ_E reaches a peak around $50 R_E$ and then decreases as the Earth-Moon distance increases.