JOURNAL OF GEOPHYSICAL RESEARCH VOLUME 66, NO. 2 FEBRUARY 1961

The Theory of Artificial Satellites in Terms of the Orbital True Longitude

PETER MUSEN

Theoretical Division, Goddard Space Flight Center
National Aeronautics and Space Administration
Washington, D. C.

Abstract. The author's previous theory of the artificial satellite is derived in terms of the disturbed eccentric anomaly. The present development, in terms of the orbital true longitude, is a substantial improvement over the earlier work in that it leads to the faster convergence for large eccentricities and to a smaller number of terms in the series representing the perturbations. Moreover, each approximation of the radius vector and of the parameters determining the position of the orbit plane is obtained not in the form of a truncated infinite series but in the form of trigonometric polynomials in two arguments. These arguments are the mean true anomaly and the mean argument of the latitude. The present theory, like the previous one, permits the computation of perturbations of any desired order. Thus, any future information about earth's gravitational field can easily be included.

Introduction. In a numerical theory of artificial satellites developed by the author [*Musen*, 1959] the eccentric anomaly is used as the independent variable. Thus the major limitations of the earlier theory are its slow convergence and its unsuitability for orbits of large eccentricity. Other features of the method given in the previous article are the introduction of the disturbing function with the separated elliptic and nonelliptic anomalies, the appearance of the divisor $\sqrt{1-e_0^2}$ in the equation determining the motion of the perigee, and a comparatively slow convergence of the series for the reciprocal of the radius vector. The divisor $\sqrt{1-e_0^2}$ can become troublesome in the case of elongated orbits. Moreover, the series for the reciprocal of the radius vector converges slowly, even for moderate values of the eccentricity. This slow convergence is reflected in the slow convergence of all following intermediate results. The present theory removes these difficulties to a substantial degree by basing the development on the use of the orbital true longitude v as reckoned in the osculating orbit plane from the departure point. With this fundamental modification the troublesome divisor $\sqrt{1-e_0^2}$ does not appear, and all consecutive approximations are obtained in the form of trigonometric polynomials and not in the form of truncated infinite series. In addition, there is no necessity to introduce the modified disturbing function Ω^* with the two types of eccentric anomalies.

The present method uses the idea of *Hansen* [1838] of the separation of all the perturbations into the perturbations in the orbit plane and the perturbations of the orbit plane, but the basic W function is slightly different from the classical form. The two basic arguments are the mean true anomaly $cv - \pi_0$ and the mean argument of the latitude $gv - \theta_0$.

All notations are the same as in the previous article:

i = the inclination of the orbital plane to the earth's equator.
θ = the longitude of the ascending node.
σ = the angular distance of the node from the departure point.
$a, e, n = a^{-3/2}$, the osculating semimajor axis, the eccentricity, and the mean motion, respectively.
f = the osculating true anomaly.

Disturbing function. The disturbing function is taken in the same form as before:

$$\Omega = k_2u^3(1-3\psi^2) + k_3u^4(3\psi - 5\psi^3) + k_4u^5(3 - 30\psi^2 + 35\psi^4) \tag{1}$$

where

$$u = 1/r \tag{2}$$

and

$$\psi = \sin i \sin (v - \sigma) \tag{3}$$

$$\sigma = (1 - g)v + \theta_0 - N - K \qquad (4)$$

$$\theta = (1 - h')v + \theta_0 - N + K \qquad (5)$$

$$\begin{aligned} \lambda_1 &= \sin \tfrac{1}{2}i \cos N, \quad \lambda_3 = \cos \tfrac{1}{2}i \sin K \\ \lambda_2 &= \sin \tfrac{1}{2}i \sin N, \quad \lambda_4 = \cos \tfrac{1}{2}i \cos K \end{aligned} \qquad (6)$$

We deduce

$$\begin{aligned} \psi = {} & 2(\lambda_1\lambda_4 - \lambda_2\lambda_3) \sin (gv - \theta_0) \\ & + 2(\lambda_2\lambda_4 + \lambda_1\lambda_3) \cos (gv - \theta_0) \end{aligned} \qquad (7)$$

and

$$\begin{aligned} \psi^2 = {} & 2(\lambda_3^2 + \lambda_4^2)(\lambda_1^2 + \lambda_2^2) \\ & - 2[(\lambda_1^2 - \lambda_2^2)(\lambda_4^2 - \lambda_3^2) \\ & - 4\lambda_1\lambda_2\lambda_3\lambda_4] \cos (2gv - 2\theta_0) \\ & + 4(\lambda_1\lambda_4 - \lambda_2\lambda_3)(\lambda_2\lambda_4 \\ & + \lambda_1\lambda_3) \sin (2gv - 2\theta_0) \end{aligned} \qquad (8)$$

$$\begin{aligned} \psi^3 = {} & 6\lambda_1^3\lambda_4^3 \sin (gv - \theta_0) \\ & - 2\lambda_1^3\lambda_4^3 \sin (3gv - 3\theta_0) \\ & + 6\lambda_1^2\lambda_4^2(\lambda_2\lambda_4 + \lambda_1\lambda_3) \cos (gv - \theta_0) \\ & - 6\lambda_1^2\lambda_4^2(\lambda_2\lambda_4 + \lambda_1\lambda_3) \cos (3gv - 3\theta_0) \end{aligned} \qquad (9)$$

$$\begin{aligned} \psi^4 = {} & 6\lambda_1^4\lambda_4^4 - 8\lambda_1^4\lambda_4^4 \cos (2gv - 2\theta_0) \\ & + 2\lambda_1^4\lambda_4^4 \cos (4gv - 4\theta_0) \\ & + 16\lambda_1^3\lambda_4^3(\lambda_2\lambda_4 + \lambda_1\lambda_3) \sin (2gv - 2\theta_0) \\ & - 8\lambda_1^3\lambda_4^3(\lambda_2\lambda_4 + \lambda_1\lambda_3) \sin (4gv - 4\theta_0) \end{aligned} \qquad (10)$$

The development of the perturbations can be done either in a completely numerical way, using the method of successive approximations, or in an analytical or semianalytical way. In the method of successive approximations the following system of formulas can be used:

$$\begin{aligned} \frac{\partial \Omega}{\partial u} = {} & 3k_2u^2(1 - 3\psi^2) + 4k_3u^3(3\psi - 5\psi^3) \\ & + 5k_4u^4(3 - 30\psi^2 + 35\psi^4) \end{aligned} \qquad (11)$$

$$\begin{aligned} \frac{1}{u^2}\frac{\partial \Omega}{\partial \psi} = {} & -6k_2u\psi + 3k_3u^2(1 - 5\psi^2) \\ & + 20k_4u^3(-3\psi + 7\psi^3) \end{aligned} \qquad (12)$$

$$\frac{1}{u^2}\frac{\partial \Omega}{\partial v} = \frac{1}{u^2}\frac{\partial \Omega}{\partial \psi}\frac{\partial \psi}{\partial v} \qquad (13)$$

$$\begin{aligned} \frac{1}{2}\frac{\partial \psi}{\partial v} = {} & (\lambda_1\lambda_4 - \lambda_2\lambda_3) \cos (gv - \theta_0) \\ & - (\lambda_2\lambda_4 + \lambda_1\lambda_3) \sin (gv - \theta_0) \end{aligned} \qquad (14)$$

In equation 14 the argument is taken in the form $gv - \theta_0 = v - (1 - g)v - \theta_0$, and the differentiation is performed only with respect to v.

$$\frac{\partial \Omega}{\partial \lambda_i} = \frac{\partial \Omega}{\partial \psi}\frac{\partial \psi}{\partial \lambda_i} \qquad (i = 1, 2, 3, 4) \qquad (15)$$

$$\begin{aligned} \frac{1}{2}\frac{\partial \psi}{\partial \lambda_1} = {} & +\lambda_4 \sin (gv - \theta_0) \\ & + \lambda_3 \cos (gv - \theta_0) \end{aligned} \qquad (16)$$

$$\begin{aligned} \frac{1}{2}\frac{\partial \psi}{\partial \lambda_2} = {} & -\lambda_3 \sin (gv - \theta_0) \\ & + \lambda_4 \cos (gv - \theta_0) \end{aligned} \qquad (16')$$

$$\begin{aligned} \frac{1}{2}\frac{\partial \psi}{\partial \lambda_3} = {} & -\lambda_2 \sin (gv - \theta_0) \\ & + \lambda_1 \cos (gv - \theta_0) \end{aligned} \qquad (16'')$$

$$\begin{aligned} \frac{1}{2}\frac{\partial \psi}{\partial \lambda_4} = {} & +\lambda_1 \sin (gv - \theta_0) \\ & + \lambda_2 \cos (gv - \theta_0) \end{aligned} \qquad (16''')$$

If we wish to have an analytical or partly analytical development, we obtain

$$\begin{aligned} \Omega = {} & k_2[1 - 6(\lambda_1^2 + \lambda_2^2)(\lambda_3^2 + \lambda_4^2)]u^3 \\ & + 6k_2[(\lambda_1^2 - \lambda_2^2)(\lambda_4^2 - \lambda_3^2) \\ & - 4\lambda_1\lambda_2\lambda_3\lambda_4]u^3 \cos (2gv - 2\theta_0) \\ & - 12k_2(\lambda_1\lambda_4 - \lambda_2\lambda_3)(\lambda_2\lambda_4 \\ & + \lambda_1\lambda_3)u^3 \sin (2gv - 2\theta_0) \\ & + 6k_3(\lambda_2\lambda_4 + \lambda_1\lambda_3) \\ & \cdot(1 - 5\lambda_1^2\lambda_4^2)u^4 \cos (gv - \theta_0) \\ & + 6k_3\lambda_1\lambda_4(1 - 5\lambda_1^2\lambda_4^2)u^4 \\ & \cdot \sin (gv - \theta_0) \\ & + 30k_3\lambda_1^2\lambda_4^2(\lambda_2\lambda_4 + \lambda_1\lambda_3)u^4 \\ & \cdot \cos (3gv - 3\theta_0) \\ & + 10k_3\lambda_1^3\lambda_4^3u^4 \sin (3gv - 3\theta_0) \\ & + k_4(3 - 60\lambda_1^2\lambda_4^2 + 210\lambda_1^4\lambda_4^4)u^5 \\ & + k_4\lambda_1^2\lambda_4^2(60 - 280\lambda_1^2\lambda_4^2)u^5 \\ & \cdot \cos (2gv - 2\theta_0) \\ & - k_4\lambda_1\lambda_4(\lambda_2\lambda_4 + \lambda_1\lambda_3)(120 \\ & - 560\lambda_1^2\lambda_4^2)u^5 \sin (2gv - 2\theta_0) \\ & + 70k_4\lambda_1^4\lambda_4^4u^5 \cos (4gv - 4\theta_0) \\ & - 280k_4\lambda_1^3\lambda_4^3(\lambda_2\lambda_4 \\ & + \lambda_1\lambda_3)u^5 \sin (4gv - 4\theta_0) \end{aligned} \qquad (17)$$

This development of the disturbing function is capable of producing the terms up to the second order in the mean motions of the arguments and the periodic terms of the first order. It can easily be extended if necessary.

Perturbations in the orbit plane. We have

$$u = h^2 + eh^2 \cos (v - \chi) \tag{18}$$

$$h^2 = 1/[a(1 - e^2)] \tag{19}$$

The orbital true longitude χ of the osculating perigee reckoned from the departure point can be represented in the form

$$\chi = (1 - c)v + \pi_0 + \phi \tag{20}$$

where ϕ is a purely trigonometrical part which is of the order of perturbations; we obtain

$$u = h^2 + h^2 e \cos (cv - \pi_0 - \phi) \tag{21}$$

We introduce also

$$\bar{u} = h_0^2 + h_0^2 e_0 \cos (cv - \pi_0) \tag{22}$$

where

$$h_0^2 = 1/[a_0(1 - e_0^2)]$$

The equation

$$1/\bar{u}^2 \, dv/dz = 1/h_0 \tag{23}$$

which defines the pseudo-time z is equivalent to Kepler's equation of the form

$$E - e_0 \sin E = g_0 + cn_0 z \tag{24}$$

and to

$$1/\bar{u} = a_0(1 - e_0 \cos E) \tag{25}$$

Taking

$$1/u^2 \, dv/dt = 1/h \tag{26}$$

into account we deduce from (23)

$$dz/dt = h_0/h \, u^2/\bar{u}^2 \tag{27}$$

and, for the perturbations of time $\delta z = z - t$, we obtain

$$\frac{d\delta z}{dt} = \frac{h_0}{h} \frac{u^2}{\bar{u}^2} - 1 \tag{28}$$

or the last equation can be put in the form

$$\frac{d\delta z}{dt} = \frac{h_0^2 \bar{W}}{\bar{u}} + \frac{h_0}{h} \left(\frac{\nu}{1 + \nu}\right)^2 \tag{29}$$

similar to the result in Hansen's theory, where

$$1 + \nu = \bar{u}/u = r/\bar{r} \tag{30}$$

But $\bar{W}$ this time is defined by the equation

$$\bar{W} = -\left(1 + \frac{h_0}{h}\right)[1 + e_0 \cos (cv - \pi_0)] + 2 \frac{h}{h_0} [1 + e \cos (cv - \pi_0 - \phi)] \tag{31}$$

or by the equation

$$\bar{W} = -\left(1 + \frac{h_0}{h}\right) \frac{\bar{u}}{h_0^2} + 2 \frac{h}{h_0} [1 + e \cos (cv - \pi_0 - \phi)] \tag{32}$$

This is the fundamental form of Hansen's $\bar{W}$ function if the orbital true longitude is used as the basic variable.

The corresponding W function becomes

$$W = -\left(1 + \frac{h_0}{h}\right)(1 + e_0 \cos \varphi) + 2 \frac{h}{h_0} [1 + e \cos (\varphi - \phi)] \tag{32'}$$

and

$$\bar{W} = W \,|_{\varphi = cv - \pi_0} \tag{32''}$$

We have [*Brown*, 1896]

$$\frac{d}{dt} \{he \cos (\phi - \varphi)\} = \frac{\partial \Omega}{\partial r} \sin (f + \chi - \beta) + \frac{\partial \Omega}{\partial v}\left(\frac{1}{r} + \frac{1}{p}\right) \cos (f + \chi - \beta) + he \frac{d\beta}{dt} \sin (\chi - \beta), \tag{33}$$

where $\beta = (1 - c)v + \pi_0 + \varphi$.

Taking

$$f + \chi = v$$

and (18) and (33) into account we obtain

$$\frac{dW}{dt} = h_0 \frac{\partial \Omega}{\partial v} \left[2 \frac{u}{h_0^2} \cos (cv - \pi_0 - \varphi) - 1 - e_0 \cos \varphi + 2 \frac{h^2}{h_0^2} \{\cos (cv - \pi_0 - \varphi) - 1\} \right] + \frac{2}{h_0} \frac{\partial \Omega}{\partial r} \sin (cv - \pi_0 - \varphi) - 2 \frac{h}{h_0} (1 - c)e \frac{dv}{dt} \sin (\varphi - \phi) \tag{34}$$

Differentiating (32′) with respect to φ, we have

$$-2\frac{h}{h_0} e \sin (\varphi - \phi) = \frac{\partial W}{\partial \varphi} - \left(1 + \frac{h_0}{h}\right) e_0 \sin \varphi$$

and (34) takes the form

$$\frac{dW}{dt} = h_0 \frac{\partial \Omega}{\partial v} \left[2 \frac{u}{h_0^2} \cos (cv - \pi_0 - \varphi) \right.$$
$$- 1 - e_0 \cos \varphi$$
$$\left. + 2 \frac{h^2}{h_0^2} \{\cos (cv - \pi_0 - \varphi) - 1\} \right]$$
$$+ \frac{2}{h_0} \frac{\partial \Omega}{\partial r} \sin (cv - \pi_0 - \varphi)$$
$$+ \left[\frac{\partial W}{\partial \varphi} - \left(1 + \frac{h_0}{h}\right) e_0 \sin \varphi \right] (1 - c) \frac{dv}{dt} \quad (35)$$

Taking

$$\frac{dt}{dv} = h_0 \cdot \frac{h}{h_0} \frac{1}{u^2} \quad (35')$$

into consideration, we deduce the final form for the derivative of W to be used in this theory

$$\frac{dW}{dv} = h_0^2 \left(\frac{1}{u^2} \frac{\partial \Omega}{\partial v}\right) \left[2 \frac{u}{h_0^2} \cos (cv - \pi_0 - \varphi) \right.$$
$$- 1 - e_0 \cos \varphi$$
$$\left. + 2 \frac{h^2}{h_0^2} \{\cos (cv - \pi_0 - \varphi) - 1\} \right] \frac{h}{h_0}$$
$$- 2 \frac{\partial \Omega}{\partial u} \frac{h}{h_0} \sin (cv - \pi_0 - \varphi)$$
$$+ (1 - c) \left[\frac{\partial W}{\partial \varphi} - \left(1 + \frac{h_0}{h}\right) e_0 \sin \varphi \right] \quad (36)$$

This form of the differential equation for W is much simpler than in the theory based on the use of the eccentric anomaly, and the divisor $\sqrt{1 - e_0^2}$ does not appear.

The integration of (36) leads to the equation of the form

$$W = \Xi + \Upsilon \cos \varphi + \Psi \sin \varphi \quad (37)$$

which is similar to equation 36 of the previous article,

$$\Xi = -1 - \frac{h_0}{h} + 2 \frac{h}{h_0} \quad (38)$$

$$\Upsilon = 2 \frac{h}{h_0} e \cos \phi - \left(1 + \frac{h_0}{h}\right) e_0 \quad (39)$$

$$\Psi = 2 \frac{h}{h_0} e \sin \phi \quad (40)$$

Equations 38 and 39 contain the constants of integration in the left-hand side. These constants are determined in such a way that neither constant nor secular terms are present in $n_0 \delta z$. From (38) we deduce, similarly as before,

$$\frac{h_0}{h} - 1 = -\frac{1}{3} \Xi + \frac{2}{3} \left[\left(\frac{h_0}{h} - 1\right)^2 - \left(\frac{h_0}{h} - 1\right)^3 + \cdots \right] \quad (41)$$

and

$$\frac{h}{h_0} = 1 + \frac{1}{2} \left(\frac{h_0}{h} - 1 + \Xi\right) \quad (42)$$

For the disturbed radius vector we deduce from (31) a new form for u:

$$u = \bar{u} + \frac{1}{2} \left(\frac{h}{h_0} - 1\right) \bar{u} + \frac{1}{2} h_0^2 \bar{W} \frac{h}{h_0} \quad (43)$$

or

$$\nu = -\frac{1}{2} (1 + v) \left(\frac{h}{h_0} - 1 + \frac{\bar{r}}{p_0} \bar{W} \frac{h}{h_0}\right) \quad (43')$$

The system of equations 36 and 41–43 is basic in the solving of the problem either by the numerical process of iteration or analytically. The numerical process is easier to produce the perturbations of any desired order. At each step of the iteration we will have the perturbations of different orders in the form of finite polynomials, as the development (17) shows, and not in the form of truncated trigonometrical series.

If the analytical approach is taken, the results are obtained in the form

$$k_2 T_1 + k_2^2 T_2 + k_2^3 T_3 + \cdots$$

where

$$T_1, T_2, T_3, \cdots$$

are trigonometric polynomials in $cv - \pi_0$ and $gv - \theta_0$. *Brenner* [1960] was the first who recognized the existence of this development.

We deduce from (43)

$$\frac{h_0}{h} = \left(1 + \frac{h_0^2}{\bar{u}} \bar{W}\right) \frac{1 + \nu}{1 - \nu} \qquad (44)$$

and taking (26) and (29) into consideration we obtain

$$\frac{d\,\delta z}{dt} = \frac{h_0^2 \bar{W}}{\bar{u}} + \frac{\nu^2}{1 - \nu^2}\left(1 + \frac{h_0^2 \bar{W}}{\bar{u}}\right) \qquad (45)$$

and

$$\frac{dt}{dv} = \frac{h_0}{\bar{u}^2} \cdot \frac{1 - \nu^2}{1 + h_0^2 \bar{W}/\bar{u}} \qquad (46)$$

it follows from the last two equations that

$$\frac{dn_0\ \delta z}{dv} = (1 - e_0^2)^{3/2}\left(\frac{h_0^2}{\bar{u}}\right)^2 \cdot \frac{h_0^2 \bar{W}/\bar{u} + \nu^2}{1 + h_0^2 \bar{W}/\bar{u}} \qquad (47)$$

or, in the form more convenient for the use of the process of iteration and with accuracy compatible with our information about the form of the earth,

$$\frac{dn_0\ \delta z}{dv} = (1 - e_0^2)^{3/2}\left(\frac{\bar{r}}{p_0}\right)^3 \bar{W} + (1 - e_0^2)^{3/2}\left(\frac{\bar{r}}{p_0}\right)^4 \frac{(u - \bar{u})^2}{h_0^4} - \frac{\bar{r}}{p_0} \bar{W} \frac{dn_0\ \delta z}{dv} \qquad (48)$$

We see that in the case of a large eccentricity the slowly convergent series of the form [*Hansen*, 1855]

$$\left(\frac{\bar{r}}{p_0}\right)^n = \frac{(1 + \beta^2)^n}{(1 - \beta^2)^{2n-1}} \cdot \left\{v_0^{(n)} + \sum_{i=1}^{\infty} 2v_i^{(n)} \cos i\,(cv - \pi_0)\right\} \qquad (49)$$

where

$$\beta = \frac{e_0}{1 + \sqrt{1 - e_0^2}}$$

$$v_i^{(n)} = (-1)^i \beta^i \binom{n + i - 1}{i} \cdot F(-n + 1,\ -n + i + 1,\ 1 + i,\ \beta^2)$$

reappear in the solution of our problem, but only in the very last equation, without interfering with all intermediate results.

The term of the form

$$A + B \cos (cv - \pi_0)$$

in (48) will disappear after the proper determination of the constants of integration in W. Only the first few terms in the series for

$$(\bar{r}/p_0)^n \qquad (n = 1, 3, 4) \qquad (50)$$

are important for the separating of this term, because of the form of the integrals. The convergence of the remaining parts of the series for (50) can be speeded up by a summability process.

Perturbations of the orbit plane. Putting, as before,

$$\sigma = (1 - g)v + \theta_0 - N - K \qquad (51)$$

$$\theta = (1 - h')v + \theta_0 - N + K \qquad (52)$$

$$\begin{aligned} \lambda_1 &= \sin \tfrac{1}{2}i \cos N \\ \lambda_2 &= \sin \tfrac{1}{2}i \sin N \\ \lambda_3 &= \cos \tfrac{1}{2}i \sin K \\ \lambda_4 &= \cos \tfrac{1}{2}i \cos K \end{aligned} \qquad (53)$$

and taking into consideration that $\bar{f} + (w)$ of the previous article becomes $gv - \theta_0$, we can transform the previous equations for $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ to the form

$$\frac{d\lambda_1}{dt} = +\left(\frac{h' + g}{2} - 1\right)\lambda_2 \frac{dv}{dt} + \frac{1}{2} rh \frac{\partial \Omega}{\partial Z} \cdot [+\lambda_4 \cos (gv - \theta_0) - \lambda_3 \sin (gv - \theta_0)]$$

$$\frac{d\lambda_2}{dt} = -\left(\frac{h' + g}{2} - 1\right)\lambda_1 \frac{dv}{dt} + \frac{1}{2} rh \frac{\partial \Omega}{\partial Z} \cdot [-\lambda_3 \cos (gv - \theta_0) - \lambda_4 \sin (gv - \theta_0)]$$

$$\frac{d\lambda_3}{dt} = +\frac{h' - g}{2} \lambda_4 \frac{dv}{dt} + \frac{1}{2} rh \frac{\partial \Omega}{\partial Z} \cdot [+\lambda_2 \cos (gv - \theta_0) + \lambda_1 \sin (gv - \theta_0)]$$

$$\frac{d\lambda_4}{dt} = -\frac{h' - g}{2} \lambda_3 \frac{dv}{dt} + \frac{1}{2} rh \frac{\partial \Omega}{\partial Z} \cdot [-\lambda_1 \cos (gv - \theta_0) + \lambda_2 \sin (gv - \theta_0)]$$

Taking (16) and (16′)–(16‴), also

$$\frac{\partial \Omega}{\partial Z} = \frac{1}{r} \frac{\partial \Omega}{\partial \psi} \cos i$$

into account, we deduce

$$
\begin{aligned}
\frac{d\lambda_1}{dt} &= +\left(\frac{h' + g}{2} - 1\right)\lambda_2 \frac{dv}{dt} + \frac{h_0}{4}\cdot\frac{h}{h_0}\frac{\partial\Omega}{\partial\lambda_2}\cos i \\
\frac{d\lambda_2}{dt} &= -\left(\frac{h' + g}{2} - 1\right)\lambda_1 \frac{dv}{dt} - \frac{h_0}{4}\cdot\frac{h}{h_0}\frac{\partial\Omega}{\partial\lambda_1}\cos i \\
\frac{d\lambda_3}{dt} &= +\frac{h' - g}{2}\lambda_4 \frac{dv}{dt} + \frac{h_0}{4}\cdot\frac{h}{h_0}\frac{\partial\Omega}{\partial\lambda_4}\cos i \\
\frac{d\lambda_4}{dt} &= -\frac{h' - g}{2}\lambda_3 \frac{dv}{dt} - \frac{h_0}{4}\cdot\frac{h}{h_0}\frac{\partial\Omega}{\partial\lambda_3}\cos i
\end{aligned}
\tag{54}
$$

Multiplying (54) by (35) we obtain

$$
\begin{aligned}
\frac{d\lambda_1}{dv} &= +\left(\frac{h' + g}{2} - 1\right)\lambda_2 + \frac{h_0^2}{4}\cdot\frac{h^2}{h_0^2}\frac{1}{u^2}\frac{\partial\Omega}{\partial\lambda_2}\cos i \\
\frac{d\lambda_2}{dv} &= -\left(\frac{h' + g}{2} - 1\right)\lambda_1 - \frac{h_0^2}{4}\cdot\frac{h^2}{h_0^2}\frac{1}{u^2}\frac{\partial\Omega}{\partial\lambda_1}\cos i \\
\frac{d\lambda_3}{dv} &= +\frac{h' - g}{2}\lambda_4 + \frac{h_0^2}{4}\cdot\frac{h^2}{h_0^2}\frac{1}{u^2}\frac{\partial\Omega}{\partial\lambda_4}\cos i \\
\frac{d\lambda_4}{dv} &= -\frac{h' - g}{2}\lambda_3 - \frac{h_0^2}{4}\cdot\frac{h^2}{h_0^2}\frac{1}{u^2}\frac{\partial\Omega}{\partial\lambda_3}\cos i.
\end{aligned}
\tag{55}
$$

The form (55) of the differential equations for $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ is new, and it is convenient for the application of the process of numerical iteration as well as for the analytical development. This system (55) is incomparably more elegant than the system (31) of the previous article. The quantities h' and g are determined in such a way that no constant terms appear on right-hand sides of (55). The position vector of the satellite can be written in the form

$$
\mathbf{r} = \frac{1}{u} A_3[\theta]\cdot A_1[i]\cdot A_3[-\sigma]\cdot\begin{bmatrix} +\cos v \\ +\sin v \\ 0 \end{bmatrix} \tag{56}
$$

where

$$
A_3[\alpha] = \begin{bmatrix} +\cos\alpha & -\sin\alpha & 0 \\ +\sin\alpha & +\cos\alpha & 0 \\ 0 & 0 & +1 \end{bmatrix} \tag{57}
$$

$$
A_1[\alpha] = \begin{bmatrix} +1 & 0 & 0 \\ 0 & +\cos\alpha & -\sin\alpha \\ 0 & +\sin\alpha & +\cos\alpha \end{bmatrix} \tag{58}
$$

We put

$$
(\theta) = (1 - h')v + \theta_0 \tag{59}
$$

then

$$
\theta = (\theta) - N + K \tag{60}
$$

We also have

$$
\sigma = (1 - g) + \theta_0 - N - K \tag{61}
$$

Taking (59)–(61) into account, we deduce

$$
\mathbf{r} = \frac{1}{u} A_3[(\theta)]\cdot A_3[K - N]\cdot A_1[i] \cdot A_3[K + N]\begin{bmatrix} \cos(gv - \theta_0) \\ \sin(gv - \theta_0) \\ 0 \end{bmatrix}
$$

or

$$
\mathbf{r} = \frac{1}{u} A_3[(\theta)]\cdot\Lambda\cdot\begin{bmatrix} \cos(gv - \theta_0) \\ \sin(gv - \theta_0) \\ 0 \end{bmatrix}
$$

where

$$
\Lambda = \begin{bmatrix} +\lambda_1^2 - \lambda_2^2 - \lambda_3^2 + \lambda_4^2, & -2(\lambda_3\lambda_4 + \lambda_1\lambda_2), & +2(\lambda_1\lambda_3 - \lambda_2\lambda_4) \\ +2(\lambda_3\lambda_4 - \lambda_1\lambda_2), & -\lambda_1^2 + \lambda_2^2 - \lambda_3^2 + \lambda_4^2, & -2(\lambda_1\lambda_4 + \lambda_2\lambda_3) \\ +2(\lambda_1\lambda_3 + \lambda_2\lambda_4), & +2(\lambda_1\lambda_4 - \lambda_2\lambda_3), & -\lambda_1^2 - \lambda_2^2 + \lambda_3^2 + \lambda_4^2 \end{bmatrix}
$$

$A_3[(\theta)]$ is the matrix connected with the regression of the node, and Λ gives small oscillations of the orbit plane around the mean position of the node.

Conclusion. The theory of artificial satellites in terms of the orbital true longitude as given in this article has a simpler mathematical formulation than the theory based on the use of the eccentric anomaly. All formulas are more compact and have a wider field of validity.

This theory is also more convenient from the standpoint of numerical computation. The programming will be simpler than for the previous theory, requiring less space in the memory, thus making the program more flexible and providing room for future refinements as our knowldege about the external gravitational field of the earth becomes more accurate.

Acknowledgment. I wish to express my gratitude to my colleague Mr. Philip Meyers for his valuable assistance in checking the development.

References

Brenner, J. L., The motion of an equatorial satellite of an oblate planet, *Tech. Rept., Stanford Research Inst., SU-3163*, p. 1, 1960.

Brown, E. W., *An Introductory Treatise on the Lunar Theory*, Cambridge University Press, 1896.

Hansen, P. A., Fundamenta nova investigationis orbitae verae quam Luna perlustrat, pp. 1–331, Gotha, 1838.

Hansen, P. A., Entwickelungen des Products einer Potenz des Radius Vectors, etc., *Abhandl. Königl. Ges. Wiss.*, *2*, 183, 1855.

Musen, P., Application of Hansen's theory to the motion of an artificial satellite in the gravitational field of the earth, *J. Geophys. Research*, *64*, 2271–2279, 1959.

(Manuscript received November 21, 1960.)