Reprinted from
THE PHYSICS OF FLUIDS VOLUME 4, NUMBER 3 MARCH, 1961
Printed in U.S.A.

Existence of Solutions of One-Dimensional Flow of a Real Gas

J. MENKES
Jet Propulsion Laboratory, California Institute of Technology, Pasadena, California
(Received June 20, 1960; revised manuscript received November 30, 1960)

Formal solutions to the equations of motion of the one-dimensional, unsteady flow of a viscous, compressible, heat-conducting gas are presented. The relationship between the existence of these solutions and the problem of hydrodynamic stability is discussed.

INTRODUCTION

THE present investigation aims to demonstrate that the one-dimensional flow of a viscous, compressible, and heat-conducting fluid can be represented formally by the heat-conduction equation in the stream function–time plane. No explicit solutions are obtained; nevertheless the behavior of the fluid can be deduced without actually solving the equations of motion.

In order to simplify the analysis, the stream function is introduced as a streamwise coordinate and the Prandtl number, Pr, is taken to be $\frac{3}{4}$. This coordinate transformation is patterned after the von Mises transformation.[1] The momentum equation is nonlinear in the space-time plane by virtue of its convective acceleration term; in the stream function–time plane the term drops out and the equation assumes the form of the well-known diffusion equation in a homogeneous medium. This equation is, of course, linear, and the classical methods of dealing with this type of equation can be brought to bear on the investigation.

ANALYSIS

The equation expressing the conservation of energy[2] is given by

$$\frac{\mathfrak{D}}{\mathfrak{D}t}(c_pT + \tfrac{1}{2}v^2) = \frac{1}{\rho}\frac{\partial p}{\partial t} + \frac{1}{\rho}\frac{\partial}{\partial x}\left(\lambda\frac{\partial T}{\partial x}\right) + \frac{1}{\rho}\frac{\partial}{\partial x}\left(v\frac{4}{3}\mu\frac{\partial v}{\partial x}\right), \tag{1}$$

where c_p is the specific heat at constant pressure, λ is the thermal conductivity, and v is the streamwise component of velocity. Substituting Pr λ/c_p for μ, and setting Pr $= \frac{3}{4}$ yields

$$\rho\frac{\mathfrak{D}H}{\mathfrak{D}t} - \frac{\partial}{\partial x}\left(\frac{\lambda}{c_p}\frac{\partial H}{\partial x}\right) = \frac{\partial p}{\partial t}, \tag{2}$$

where $H = c_pT + \frac{1}{2}v^2$. The momentum equation for Pr $= \frac{3}{4}$ is simply

$$\rho\frac{\mathfrak{D}v}{\mathfrak{D}t} - \frac{\partial}{\partial x}\left(\frac{\lambda}{c_p}\frac{\partial v}{\partial x}\right) = -\frac{\partial p}{\partial x}. \tag{3}$$

The continuity equation is given by

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v)}{\partial x} = 0. \tag{4}$$

The stream function, defined by

$$\rho = \partial\psi/\partial x, \qquad \rho v = -\partial\psi/\partial t, \tag{5}$$

may be introduced as a streamwise coordinate, yielding

$$\frac{\partial}{\partial t} \to \frac{\partial}{\partial t} - \rho v\frac{\partial}{\partial \psi} \quad \text{and} \quad \frac{\partial}{\partial x} \to \rho\frac{\partial}{\partial \psi}.$$

Application of this transformation to the energy equation gives

$$\frac{\partial H}{\partial t} - k^2\frac{\partial^2 H}{\partial \psi^2} = \frac{1}{\rho}\frac{\partial p}{\partial t} - v\frac{\partial p}{\partial \psi}, \tag{6}$$

while the momentum equation reads

$$\frac{\partial v}{\partial t} - k^2\frac{\partial^2 v}{\partial \psi^2} = -\frac{\partial p}{\partial \psi}, \tag{7}$$

where $k^2 = \lambda\rho/c_p$.

It is to be noted that the momentum equation (which is coupled to the energy equation only through the pressure gradient term) is now linear while the energy equation remains nonlinear. Our plan is to treat the pressure gradient term in the momentum equation as a *known* function of ψ and t, and to treat Eq. (7) as if it were a nonhomogeneous heat conduction equation. We shall be able to demonstrate that without actually knowing the dependence of p on ψ and t, and by imposing only certain physically plausible restrictions, we can establish whether or not the equations of motion possess solutions that are bounded for all times.

The linear nature of Eq. (7), together with the hypothesis concerning the pressure gradient, per-

[1] R. von Mises, *Mathematical Theory of Compressible Fluid Flow,* (Academic Press, Inc., New York, 1958), pp. 157–158.
[2] J. O. Hirschfelder, and C. F. Curtiss, J. Chem. Phys. 17, 1076 (1949).

mits the use of the principle of superposition. Thus we may first solve the homogeneous equation with inhomogeneous boundary conditions and then solve the inhomogeneous equation with homogeneous boundary conditions. The complete solution is a linear combination of the two. Calling the solution to the homogeneous equation v_1, we obtain by the method of Green's function

$$v_1 = \frac{1}{2k(\pi t)^{\frac{1}{2}}} \int_{-\infty}^{\infty} f(\psi') \exp\left[-\frac{(\psi - \psi')^2}{4k^2 t}\right] d\psi', \quad (8)$$

where $v_1 = f(\psi)$, at $t = 0$. We are, of course, interested in the behavior of the solution as $t \to \infty$; thus we have to take the limit of the integral. Since the integration is independent of t, and, moreover, since the integrand is uniformly continuous with respect to t, it is permissible to interchange the order of integrating and of taking the limit, provided the improper integral (8) converges. A sufficient condition for the convergence of the integral is the convergence of

$$\int_{-\infty}^{\infty} f(\psi')\, d\psi'. \quad (9)$$

In other words, any initial velocity distribution, compatible with the boundary conditions, that satisfies the convergence condition is a permissible one. In turn, any permissible initial distribution satisfying the boundary conditions and the convergence criterion will die out as $t^{-\frac{1}{2}}$; thus

$$\lim_{t\to\infty} v_1 = 0. \quad (10)$$

The particular solution to the inhomogeneous equation $v_2(\psi, t)$ is obtained with the aid of the Laplace transform. We take the Laplace transform of Eq. (7) with respect to ψ, calling the transform variable s and the transformed functions by their respective capital letters, V_2 and P. We thus obtain for the transformed momentum equation

$$\frac{dV_2}{dt} + k^2 s^2 V_2 = sP(s, t), \quad (11)$$

the solution to which can be written down immediately

$$V_2 = \exp(-k^2 s^2 t) \int_0^t \exp(k^2 s^2 \tau) sP(s, \tau)\, d\tau. \quad (12)$$

Taking cognizance of the fact that we are dealing with a real as opposed to an ideal gas, we now assert on a physical basis that whatever the pressure may be—and for that matter, its image function $P(s, t)$—it must be bounded. The rationale for this is simply that a viscous and heat conducting gas cannot support an infinite pressure. Setting the maximum value of the pressure equal to M^2 permits us to perform the integration and obtain

$$V_2 \leq \frac{M^2 s[1 - \exp(-k^2 s^2 t)]}{k^2 s^2}. \quad (13)$$

Before performing the inversion integration to return to the physical plane, we let $t \to \infty$, and thus obtain

$$\lim_{t\to\infty} v_2 \leq (M/k)^2. \quad (14)$$

This inequality asserts that v_2 is bounded. Our apparent success in dealing with the momentum equation by showing the existence of a bounded solution obviates the need to consider the energy equation, which can be dealt with in precisely the same fashion.

DISCUSSION

We have demonstrated that the unsteady equations of motion possess only bounded solutions. A bounded solution is also a stable one if after the initial conditions are changed slightly the new solution will approach the original one asymptotically. Since the initial conditions are involved only in the solution to the homogeneous equation, which we have shown to be zero provided that $\int_{-\infty}^{\infty} f(\psi') d\psi'$ converges, it follows that the solution is stable.

ACKNOWLEDGMENT

This work was sponsored by the National Aeronautics and Space Administration.