Evaluating Probability of Collision (Pc) Uncertainty

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April 2016

Agenda

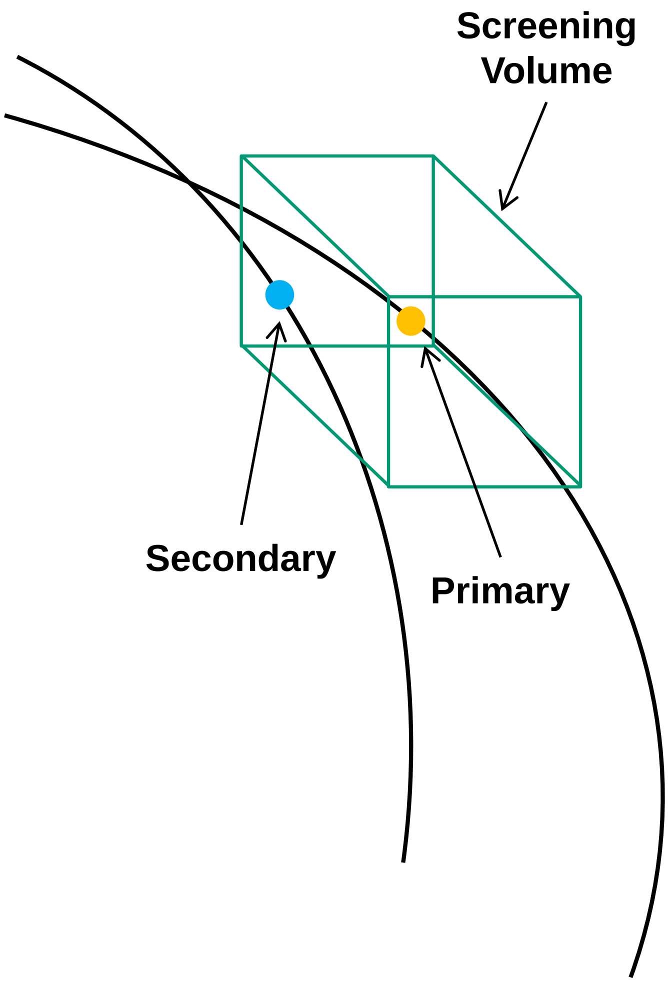
- **Conjunction Assessment Basics**
- **Probability of Collision (Pc) calculation outline**
- **Pc uncertainty overview**
- **Pc uncertainty component: covariance uncertainty**
  - Covariance realism assessment
  - Covariance realism PDF generation
- **Pc uncertainty component: hard-body radius uncertainty**
  - Primary objects using projected-area sampling
  - Secondary objects using radar cross-section values
- **Pc uncertainty component: natural variation in Pc calculation**
- **Example output**
- **Conclusions and future work**

How are Satellite Collision Risks Determined/Mitigated?

- **Certain spacecraft are determined to be "defended assets"**
  - Will be evaluated for collision risk with other objects
- **For seven days into the future, the expected positions of the defended asset and the rest of the objects in the space catalogue are determined**
- **"Keep-out volume" box drawn around the defended asset at each time-step**
  - Typically 5km x 5km x 25km in size, with the longer dimension along the orbit path
- **Any satellite that penetrates the keep-out volume during the 7-day analysis is considered a possible "conjunctor"**
- **Particulars of the close approach analyzed to determine actual conjunction risk**

"Fly By" Ephemeris Comparison

- Ephemerides generated for primary and secondaries that are possible threats
- Screening volume box (or ellipsoid) constructed about primary
- Box "flown" along the primary's ephemeris
- Any penetrations of box constitute possible conjunctions
- For these conjunctions, Conjunction Data Message (CDM) generated
  - State estimates and covariances at TCA
  - Relative encounter information
  - OD information
- CDM data used to calculate probability of collision (Pc)



Calculating Probability of Collision (Pc): 3D Situation at Time of Closest Approach (TCA)

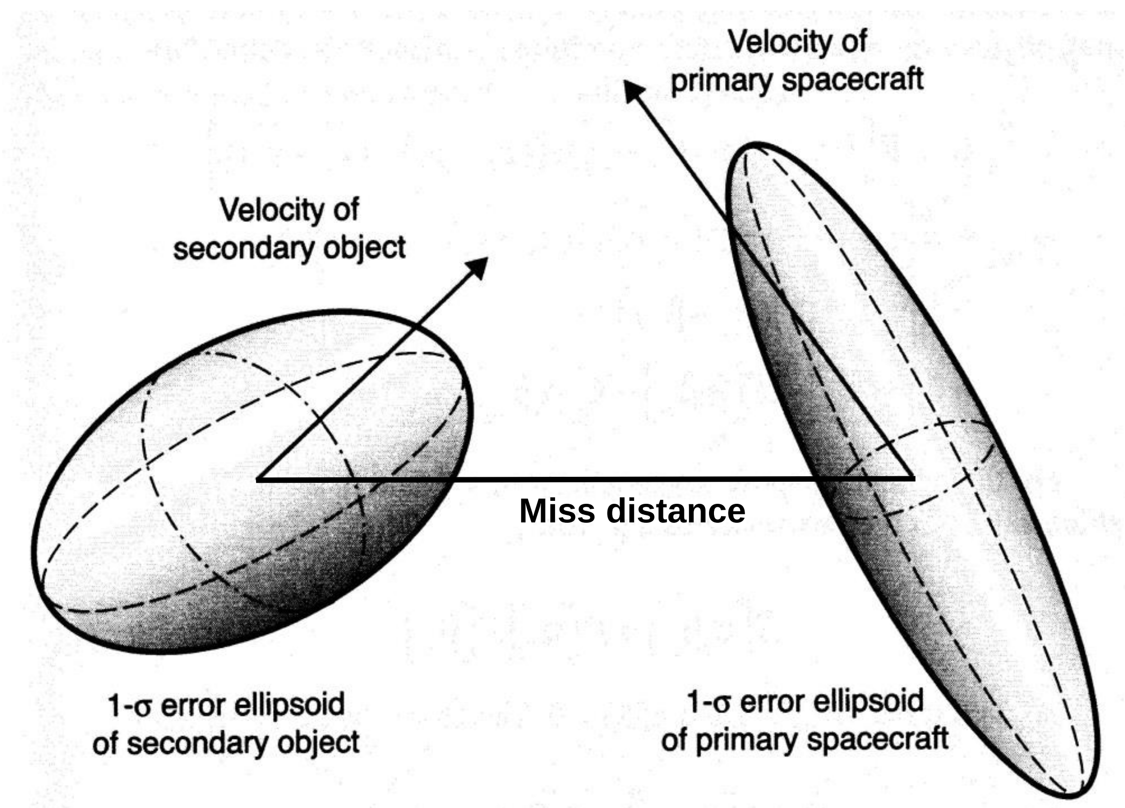


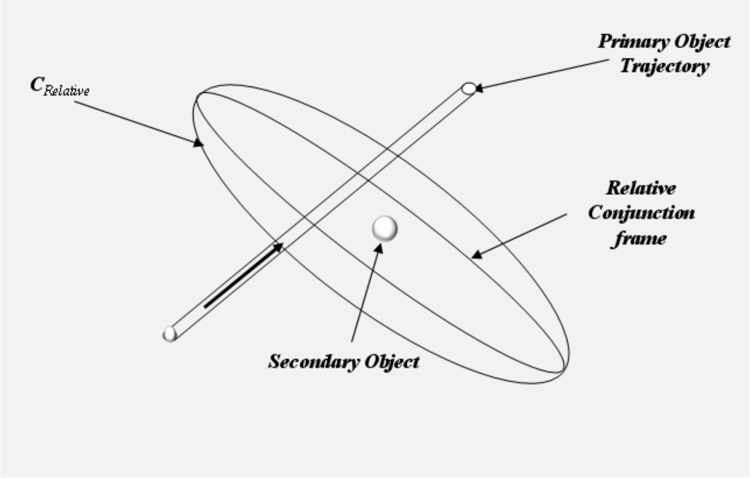
Figure taken from Chan (2008)

Calculating Pc: 2-D Approximation (1 of 3) Combining Error Volumes

- **Assumptions**
  - Error volumes (position random variables about the mean) are uncorrelated
- **Result**
  - All of the relative position error can be centered at one of the two satellite positions
    - Secondary satellite is typically used
  - Relative position error can be expressed as the additive combination of the two satellite position covariances (proof given in Chan 2008)
    - $C_a + C_b = C_c$
  - Must be transformed into a common coordinate system, combined, and then transformed back

Calculating Pc: 2-D Approximation (2 of 3) Projection to Conjunction Plane

- **Combined covariance centered at position of secondary at TCA**
- **Primary path shown as "soda straw"**
- **If conjunction duration is very short**
  - Motion can be considered to be rectilinear—soda straw is straight
  - Conjunction will take place in 2-d plane normal to the relative velocity vector and containing the secondary position
  - Problem can thus be reduced in dimensionality from 3 to 2
- **Need to project covariance and primary path into "conjunction plane"**



Calculating Pc: 2-D Approximation (3 of 3) Conjunction Plane Construction

- **Combined covariance projected into plane normal to the relative velocity vector and placed at origin**
- **Primary placed on x-axis at (miss distance, 0) and represented by circle of radius equal to sum of both spacecraft circumscribing radii ("hard-body radius" or HBR")**
- **Z-axis perpendicular to x-axis in conjunction plane**

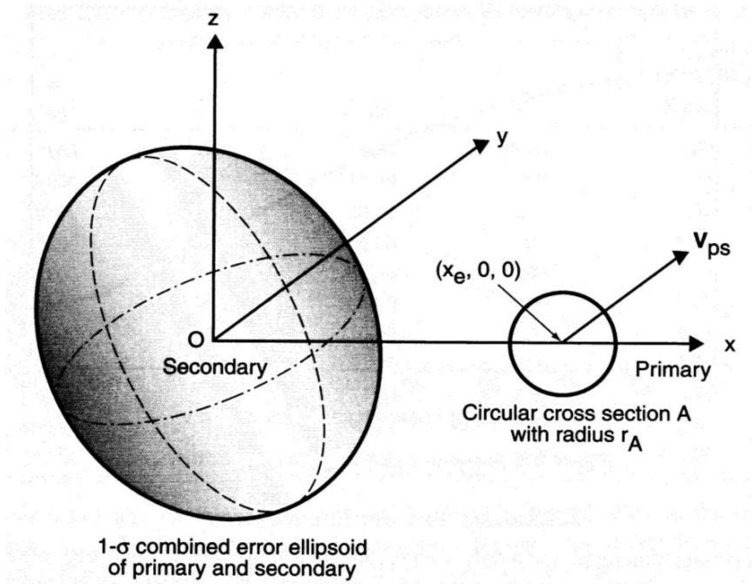
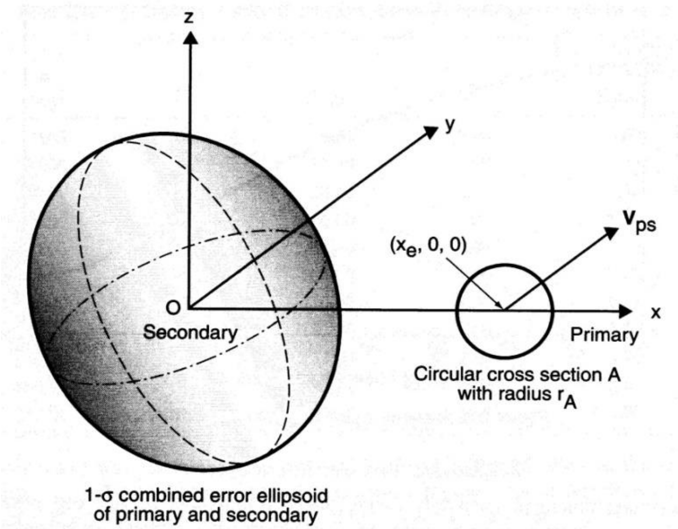


Figure taken from Chan (2008)

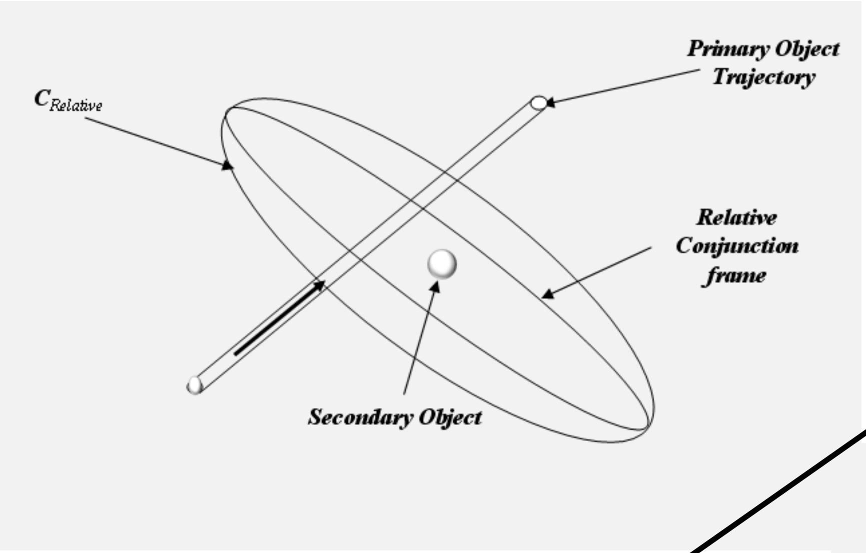
2-D Probability of Collision Computation

- **Rotate axes until they align with principal axes of projected covariance ellipse**
- **Pc is then the portion of the density that falls within the HBR circle**
  - r is [x z] and C* is the projected covariance

$$P_C = \frac{1}{\sqrt{(2\pi)^2 \left|C^*\right|}} \iint_A \exp\left(-\frac{1}{2}\vec{r}^{\,T} C^{*-1} \vec{r}\right) dXdZ$$

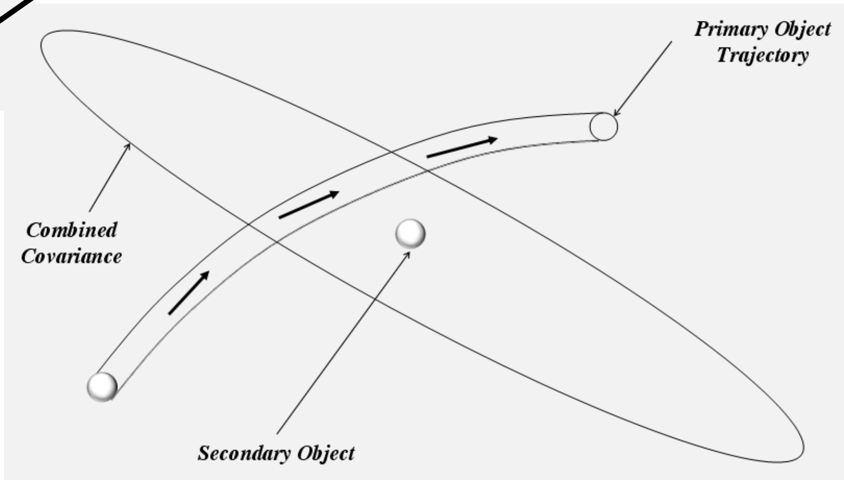


2-D vs. 3-D Conjunction Geometry



2-D Geometry

3-D Geometry

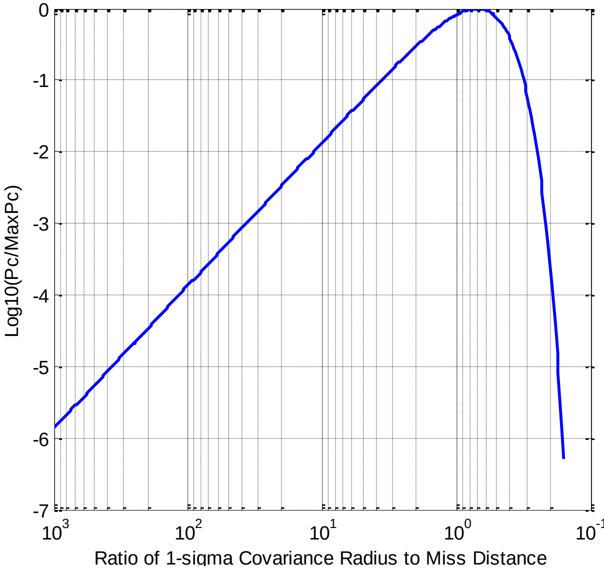


Low Relative Velocity or Long Conjunction Duration Situation

- **2-D approximation not valid**
- **Can attempt 3-D integral**
  - Messy, but Coppola (2012) outlines methodology with Lebedev quadrature
- **Can use Monte Carlo**
  - From TCA
    - Propagate both satellites' states and covariances to nominal TCA
    - Take position (and maybe velocity) perturbations from each covariance to define new states for primary and secondary
    - Find new TCA and record miss distance
    - Tabulate all miss distances; percent that are smaller than HBR is Pc
  - From epoch
    - Similar procedure to above, but perturbations performed at epoch
    - Perturbed states propagated forward to new TCA with full non-linear dynamics

Conjunction Event Canonical Progression

- **Conjunction typically first discovered 7 days before TCA**
  - Covariances large, so typically Pc below maximum
- **As event tracked and updated, changes to state estimate are relatively small, but covariance shrinks**
  - Because closer to TCA, less uncertainty in projecting positions to TCA
- **Theoretical maximum Pc encountered when 1-sigma covariance size to miss distance ratio is $1/\sqrt{2}$**
  - After this, Pc usually decreases rapidly
- **Behavior shown in graph at right**
  - X-axis is covariance / miss distance
  - Y-axis is $\log_{10}(P_c/\max(P_c))$
  - Order of magnitude change in Pc considered significant, thus log-space more appropriate



Probability of Collision Calculation

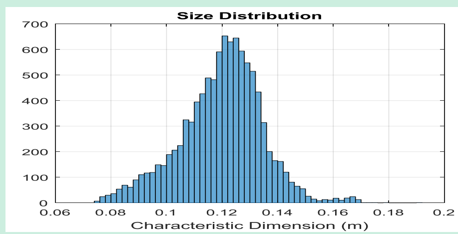
- **Pc is only a nominal solution for the conjunction**
  - Derived from estimates of the mean
    - If underlying distributions not symmetric, then this is not an expression of central tendency
  - Does not include uncertainties on the inputs
    - "Uncertainty of uncertainty volumes" or uncertainty in HBR
- **Thus, while representing the risk, nominal Pc is just a point estimate**
- **Want to know how much variation or uncertainty in the Pc calculated for any given conjunction**
  - Determine uncertainty PDFs for the Pc calculation inputs
  - Through Monte Carlo trials, vary above inputs to the Pc calculation
  - Include a resampling technique to determine natural variation in the calculation
  - Generate a probability density of resultant Pc values
  - Characterize this distribution empirically

Uncertainty in the Probability

- **Generate a Pc distribution, using Monte Carlo (MC) trials of the underlying uncertainties**
  - Determine uncertainty for each of the Pc parameters

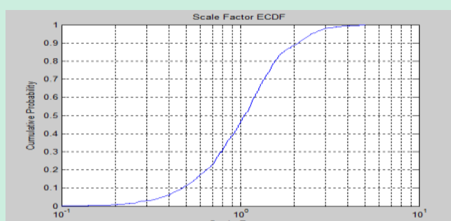
Underlying Uncertainties

HBR Uncertainty



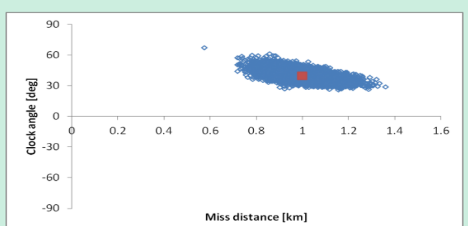
Draw from projected area distribution (primary) and RCS PDF (secondary)

Covariance Uncertainty



Draw from scale factor distributions for both objects

Natural Sampling Variability



Draw from 2D scaled Gaussian covariance

Generate P_c distribution

COVARIANCE REALISM AND SCALE FACTORS

Covariance Realism

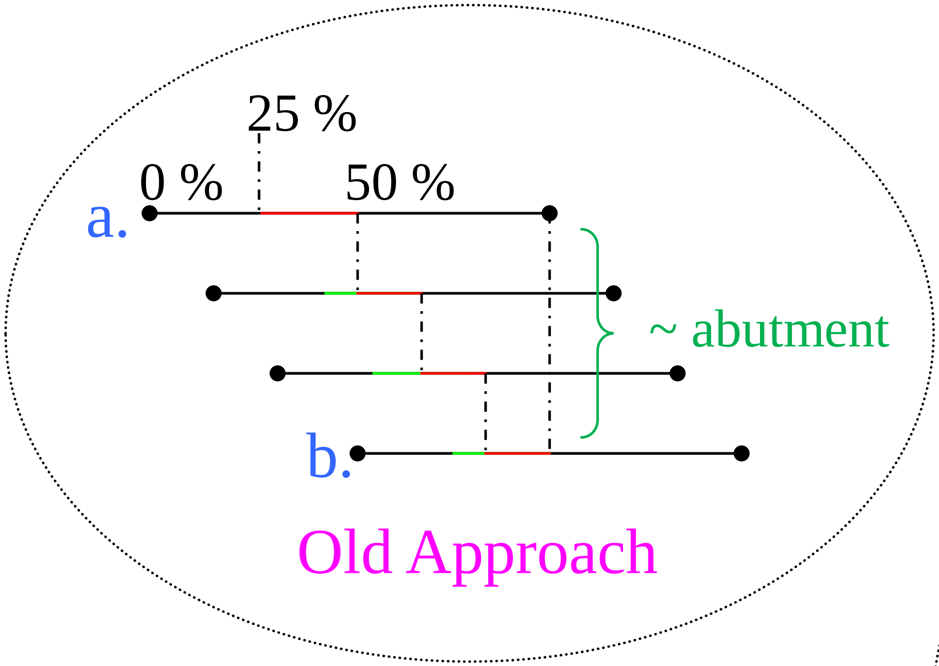
- **Ways a typical covariance can be unrealistic**
  - Much larger or smaller than the “real” error volume
  - Differently oriented from the “real” error volume
  - Representing a different distribution from the “real” error distribution
- **This last item not addressed in present study**
  - Current form of covariance promotes Gaussian assumption
  - *A priori* arguments for presuming component error distributions close to Gaussian
  - *A posteriori* evidence for component errors following a symmetric distribution
  - Study indicates large-Pc events not affected by “bending” covariances*
- **Large covariances not inherently problematic**
  - Rather, quite appropriate if errors themselves are large
- **Covariance realism assessment approach is combined evaluation of size and orientation, presuming error volume is Gaussian ellipsoid**

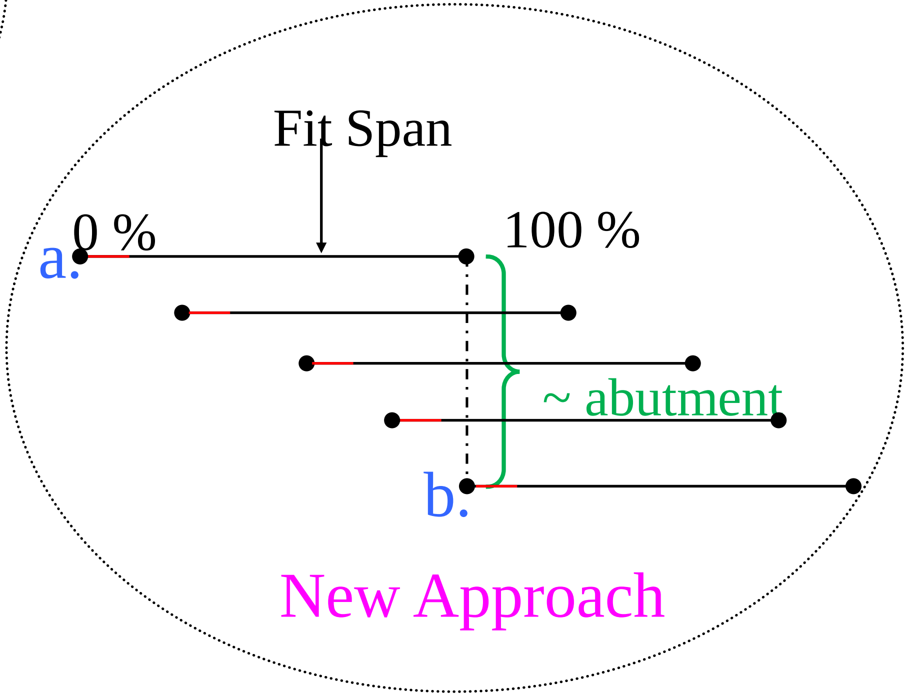
* Ghrist, R.W. and Plakalovic, D. “Impact of Non-Gaussian Error Volumes on Conjunction Assessment Risk Analysis,” 2012.

JSpOC State and Covariance Accuracy Utility

- **Truth ephemeris produced for every satellite**
  - Similar to methodology used for generating precision satellite laser ranging orbits
  - "Stitched together" pieces of ephemeris from a "judiciously chosen" portion of the fit-spans of subsequent batch ODs
    - Methodology to minimize overlap of portions drawing from same observation base
  - Covariance for reference orbit also preserved (epoch covariances from generating ODs)
- **Each produced precision vector for each object compared to its reference orbit at propagation states of interest**
  - Position comparisons at 6, 12, 18, 24, 36, 48, 72, 120, and 168 hrs
  - Propagated position covariance also calculated and retained at each comparison point
- **Raw materials for covariance realism investigations thus available:**
  - State errors
  - Propagated covariance at point of comparison and reference orbit covariance

Reference Orbit Formation Approaches: Previous and Present





Covariance Realism: Normal Deviates and Chi-squared Variables

- **Let q and r be vectors of values that conform to a Gaussian distribution**
  - Commonly called *normal deviates*
- **A normal deviate set can be transformed to a *standard normal deviate* by subtracting the mean and dividing by the standard deviation**
  - This produces the so-called Z-variables

$$Z_q = \frac{q - \mu_q}{\sigma_q}\ , \quad Z_r = \frac{q - \mu_r}{\sigma_r}$$

- **The sum of the squares of a series of standard normal deviates produces a chi-squared distribution, with the number of degrees of freedom equal to the number of series combined**

$$Z_q^2 + Z_r^2 = \chi^2_{2\,dof}$$

Covariance Realism: Normal Deviates in State Estimation

- **In a state estimate, the errors in each component (u, v, and w here) are expected to follow a Gaussian distribution**
  - If all systematic errors have been solved for, only random error should remain
- **These errors can be standardized to the Z-formulation**
  - Mean presumed to be zero (OD should produce unbiased results), so no need for explicit subtraction of mean

$$Z_u = \frac{u}{\sigma_u}, \quad Z_v = \frac{v}{\sigma_v}, \quad Z_w = \frac{w}{\sigma_w}$$

- **Sum of squares of these standardized errors should follow a chi-squared distribution with three degrees of freedom**

$$Z_u^2 + Z_v^2 + Z_w^2 = \chi^2_{3dof}$$

Covariance Realism: State Estimation Example Calculation

- **Let us presume we have a precision ephemeris, state estimate, and covariance about the state estimate**
  - For the present, further presume covariance aligns perfectly with uvw frame (no off-diagonal terms)
- **Error vector ε is position difference between state estimate and precision ephemeris, and covariance consists only of variances along the diagonal**
  - Inverse of covariance matrix is straightforward

$$\varepsilon = \begin{bmatrix} \varepsilon_u \\ \varepsilon_v \\ \varepsilon_w \end{bmatrix}, \quad C = \begin{bmatrix} \sigma_u^2 & 0 & 0 \\ 0 & \sigma_v^2 & 0 \\ 0 & 0 & \sigma_w^2 \end{bmatrix} \qquad C^{-1} = \begin{bmatrix} 1/\sigma_u^2 & 0 & 0 \\ 0 & 1/\sigma_v^2 & 0 \\ 0 & 0 & 1/\sigma_w^2 \end{bmatrix}$$

- **Resultant simple formula for chi-squared variables**

$$\varepsilon C^{-1} \varepsilon^T = \frac{\varepsilon_u^2}{\sigma_u^2} + \frac{\varepsilon_v^2}{\sigma_v^2} + \frac{\varepsilon_u^2}{\sigma_w^2} = \chi^2_{3\,dof}$$

Covariance Realism: Non-Diagonal Covariances

- **Mahalanobis distance formulary naturally accounts for correlation terms**
- **Two-dimensional example:**

$$\varepsilon C^{-1}\varepsilon^{T} = \frac{1}{1-\rho^{2}}\left(\frac{\varepsilon_x^2}{\sigma_x^2}+\frac{\varepsilon_y^2}{\sigma_y^2}-\frac{2\rho\varepsilon_x\varepsilon_y}{\sigma_x\sigma_y}\right)$$

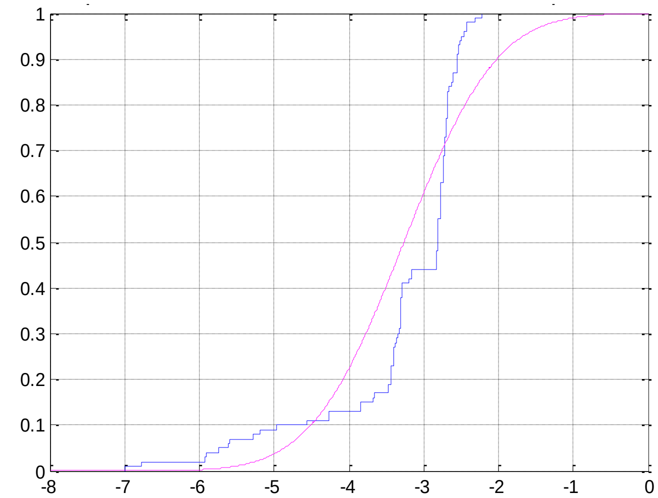
- **Conforms to intuition**
  - As ρ approaches zero, diagonal case recovered

Covariance Realism: Testing for Realism

- **Mahalanobis distance set should conform to 3-DoF χ^2 distribution**
- **Expected value for each calculation is DoF, 3 in this case**
- **Each Mahalanobis point in principle produces a scale factor**
  - mCm sizes covariance such that $\varepsilon C^{-1} \varepsilon^T$ will have a value of 3
  - m^2 thus the proper factor by which to scale the covariance in order to produce the expected value
- **However, not every Mahalanobis calculation expected to equal expected value**
  - Instead, a chi-squared distribution with expected value of 3
- **To set scale factor(s), choose factor that brings entire Mahalanobis distance set into conformity with expected distribution**

Empirical Distribution Function (EDF) GOF: Exquisite Solution

- **Sum of vertical differences between "ideal" and "real" behavior**
  - Hypothetical graph at left
- **Cramér – von Mises formulation the most appropriate for current situation**
  - Equations at right
  - Weighting function (ψ) set to unity a better choice for outlier-infused situations
- ***Q* used to consult tables of *p*-values to determine likelihood of match between ideal and real distribution**
  - Best approach is to be able to use *p*-value, as this has a clear statistical meaning



$$Q = n\int_{-\infty}^{\infty}\left[F_n(x) - F(x)\right]^2 \psi(x)\,dx$$

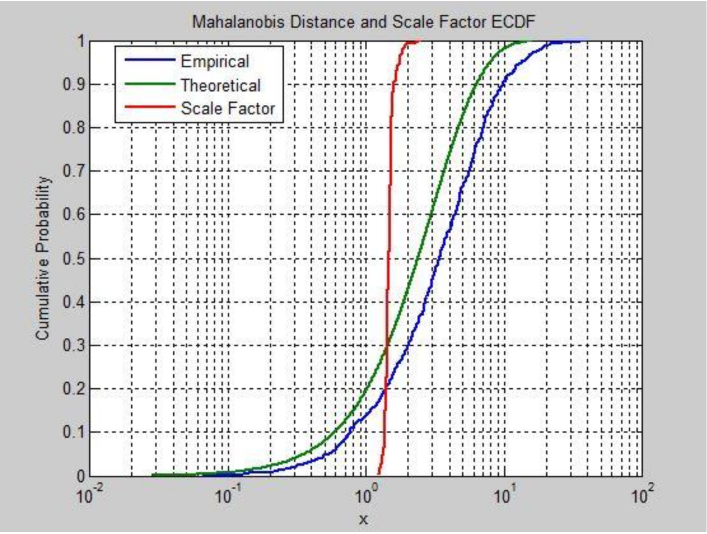
$$\psi(x) = 1$$

- **But what if we want a distribution of scale factors?**

Covariance Realism: Distribution of Scale Factors

- **Rank-ordering of results can give reasonable PDF of scale factors**
  - Presume 100 squared Mahalanobis distance values (M2)
    - Derived from JSpOC covariance realism data
  - Rank order list
  - Align each entry with the 3-DoF χ^2 value that corresponds to that percentile
  - Quotient of two terms is (square of) scale factor that produces the χ^2 value expected for that particular percentile
- **Examples:**

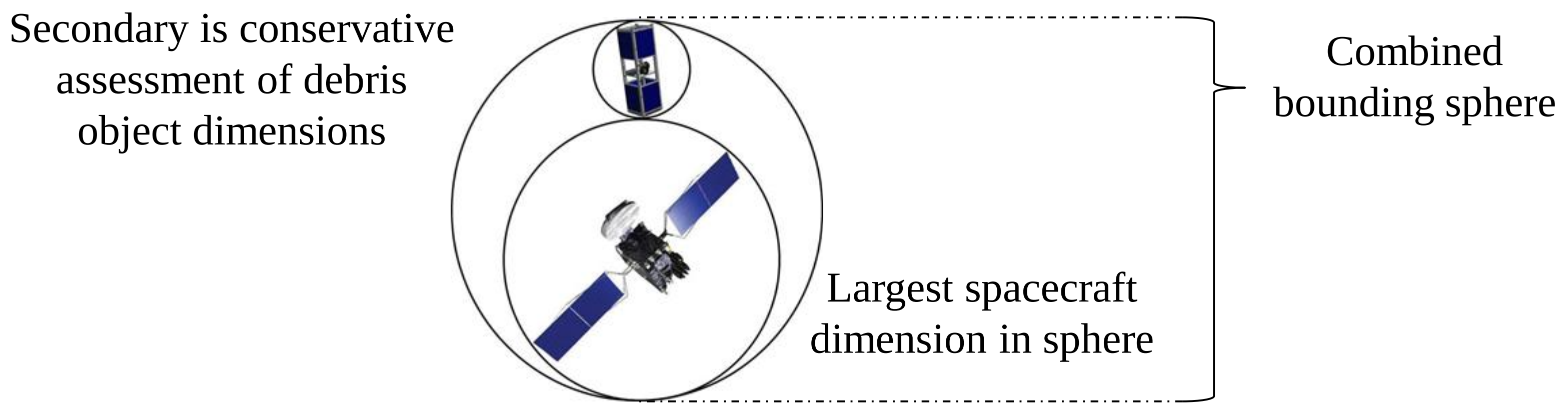
Percentile	x2	Square of M Distance	Quotient
1 (0.01)	0.115	0.183	1.594
2 (0.02)	0.185	0.245	1.326
3 (0.03)	0.245	0.353	1.440
4 (0.04)	0.300	0.418	1.393



HARD-BODY RADIUS

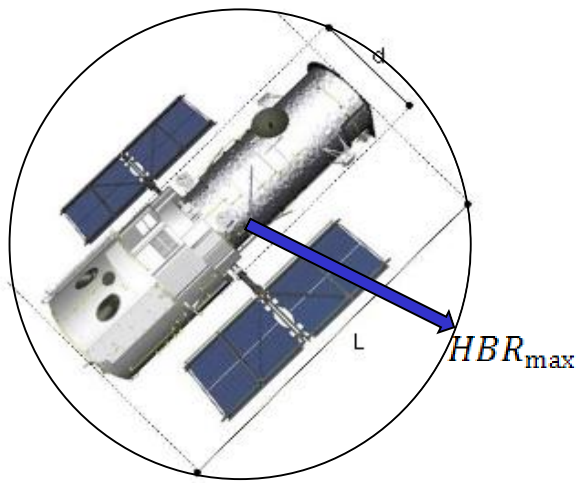
Hard-Body Radius: Introduction

- **HBR is typically found by circumscribing both objects in spheres and combining the objects into one bounding sphere**
  - Size of the secondary is typically not known, so added as a large estimate of debris object dimensions



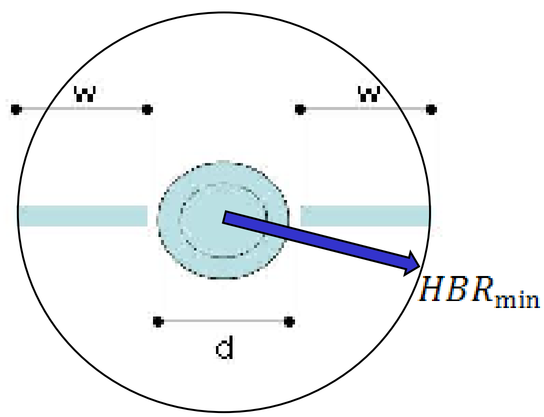
- **HBR uncertainties that follow represent a more realistic estimate of the area in the conjunction plane**
  - The combined uncertainties are much smaller than the bounding sphere

Hard-Body Radius: Min and Max Using Approximation Equations



$$HBR_{\max} = \frac{\sqrt{L^2 + (2w + d)^2}}{2} \cong 8.1\text{m}$$

$$L = 13.2\text{m}$$
$$d = 4.2\text{m}$$
$$w = 2.6\text{m}$$

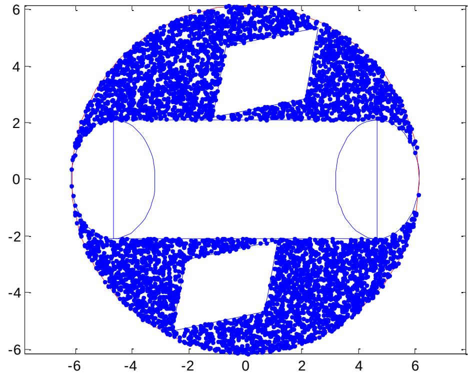


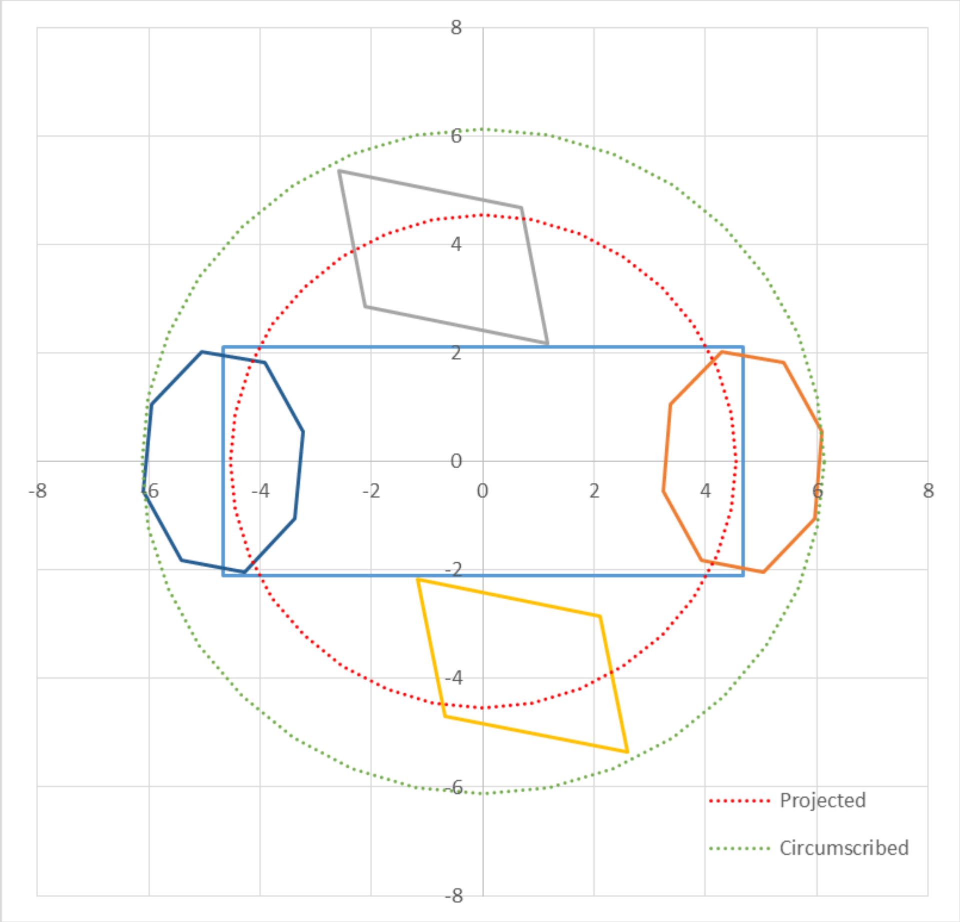
$$HBR_{\min} = \frac{2w + d}{2} \cong 4.7\text{m}$$

Could presume uniform distribution between these values as first-order approximation of PDF, but seems rather arbitrary

Hard-Body Radius: Projected Area Approach

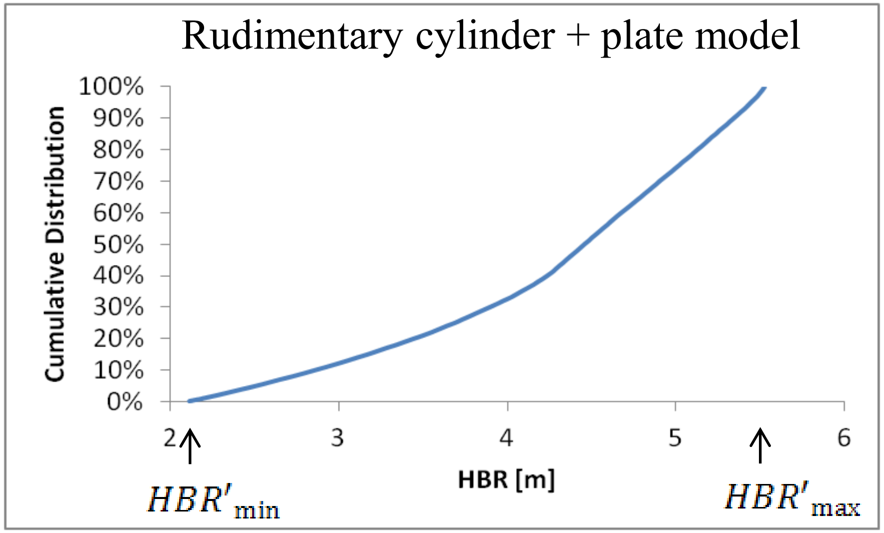
- **Randomized orientation of primary satellite to capture the average area**
  - Ball-and-stick model to be created for each primary asset
  - Includes rotating solar panels
- **Projected radius**
  - Actual hit area of the satellite expressed as a circular radius
  - $r = \sqrt{\frac{A}{\pi}}$





Hard-Body Radius: Projected Area Approach Performance

- **NASA/JSC Orbital Debris Program Office (ODPO) has sophisticated satellite model and full Euler angle rotation software to generate projected area PDFs**
- **Comparison of results for Hubble Space Telescope between ODPO software and ball-and-stick model:**



	Average Area [m²]	Average Effective HBR [m]
Crude HST model (corresponding to chart)	60.3	4.3
Sophisticated HST model (Matney*)	63.7	4.5

Good agreement

* M. Matney, "How to Calculate the Average Cross Sectional Area," Orbital Debris Quarterly Newsletter, Vol. 8, issue 2.

Hard-Body Radius: Projected Area Approach Implementation

- **Assemble ball-and-stick model of primary satellite**
- **Rotate through all Euler angles and project into plane**
- **Create empirical PDF of projected areas**
- **Express as PDF of radii of circles of equivalent area**

- **If satellite orientation is known at TCA, then area can be projected directly into conjunction plane**
  - Can then perform integration by means of a contour integral
  - Lingering problem of how to incorporate area for secondary object

Hard-Body Radius: Secondary Object HBR Uncertainty

- **For intact spacecraft, possible to use published dimensions**
  - For payloads, these are often not precise enough to be useful, and at least some canonical models would have to be imposed
    - Error in all of this great enough that approach is questionable
  - For rocket bodies, published dimensions are probably adequate
    - But many booster types lack published dimensions
- **Most common secondaries are debris objects, for which no size information is available**
- **Thus, forced to estimate size from radar cross-section (RCS) value**
  - Objects do not have single RCS value but PDF of values, depending on radar response and object aspect function
  - PDFs of individual objects' RCS values not available, only averaged values
  - As proxy could use canonical distribution
    - Swerling III distribution is most common for debris, and also most conservative in terms of size*

* Hejduk, M. D. and DePalma, D. "Comprehensive Radar Cross-Section "Target Typing" Investigation for Spacecraft," 2010.

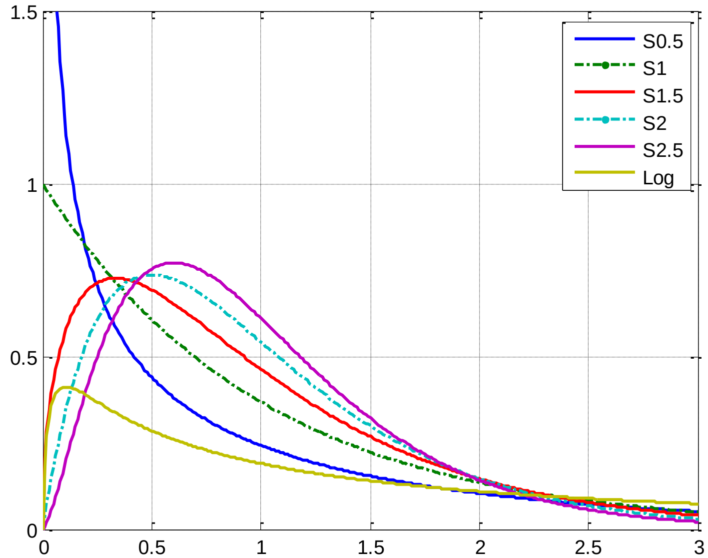
Hard-Body Radius: Swerling Distribution Family

- **Swerling distributions derive from the gamma distribution family**
  - Location parameter (γ) = 0
  - Shape parameter (m) fixed
  - Scale parameter (β) estimated from sample (MLE)
- **Swerling I/II is gamma with *m*=1**
  - Exponential distribution
  - Presumes Rayleigh scattering
- **Swerling III/IV is gamma with *m*=2**
  - Erlang distribution
  - Presumes correlation with object orientation; more correct assumption
- **S-notation is gamma with *m* given**
  - S1.5 = gamma with m=1.5 &c.

$$f(x;\gamma,\beta,m)=\frac{1}{\beta^{m}\Gamma(m)}(x-\gamma)^{m-1}\exp\left(\frac{-(x-\gamma)}{\beta}\right)$$

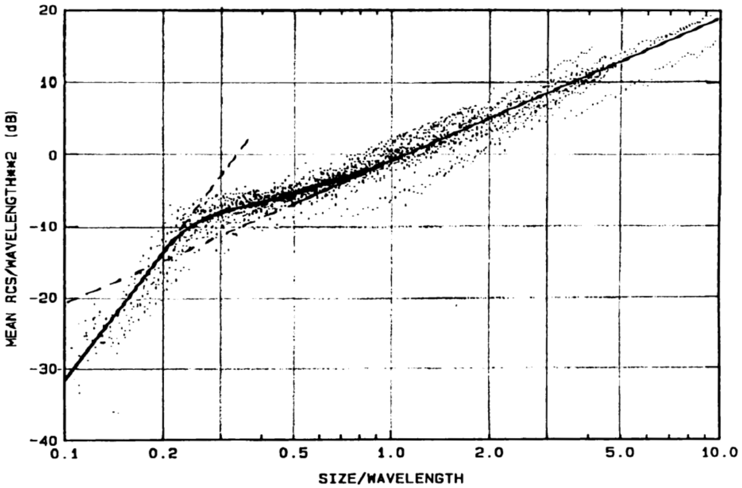
$$f(x;\beta)=\frac{1}{\beta}\exp\left(\frac{-x}{\beta}\right)$$

$$f(x;\beta)=\frac{1}{\beta^{2}}x\exp\left(\frac{-x}{\beta}\right)$$



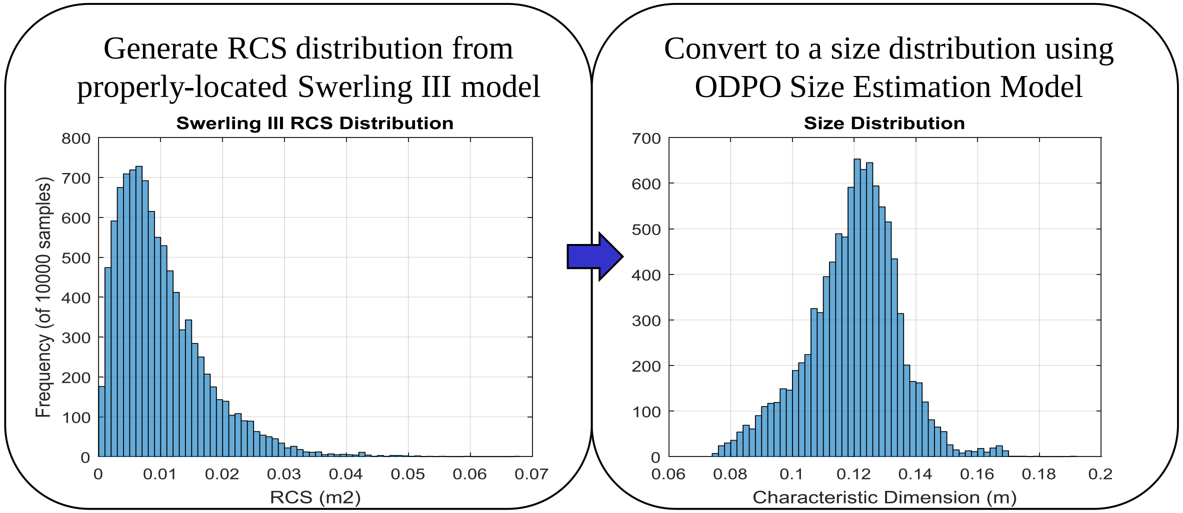
Hard-Body Radius: Radar OSEM Basic Rubric

- **Simulated hyperkinetic destruction of satellite in vacuum chamber**
- **Collected pieces and subjected them to individual analysis**
  - “Observed” each piece with radar in anechoic chamber
  - Articulated full range of aspect angles and full range of radar frequencies
  - Recorded resultant RCS of each aspect/frequency configuration
- **Collected results and plotted in dimensionless format**
  - RCS / λ^2; size / λ
  - Results follow basic theory of Rayleigh, Mie, and optical regions



Hard-Body Radius: Full Process for Secondary Object

- **Begin with average RCS**
- **Produce RCS PDF using Swerling III distribution**
  - Scale parameter estimated by mean RCS divided by shape parameter
- **Send distribution through ODPO size estimator to generate size PDF**
  - Certified only for objects smaller than 20cm, but this is most debris



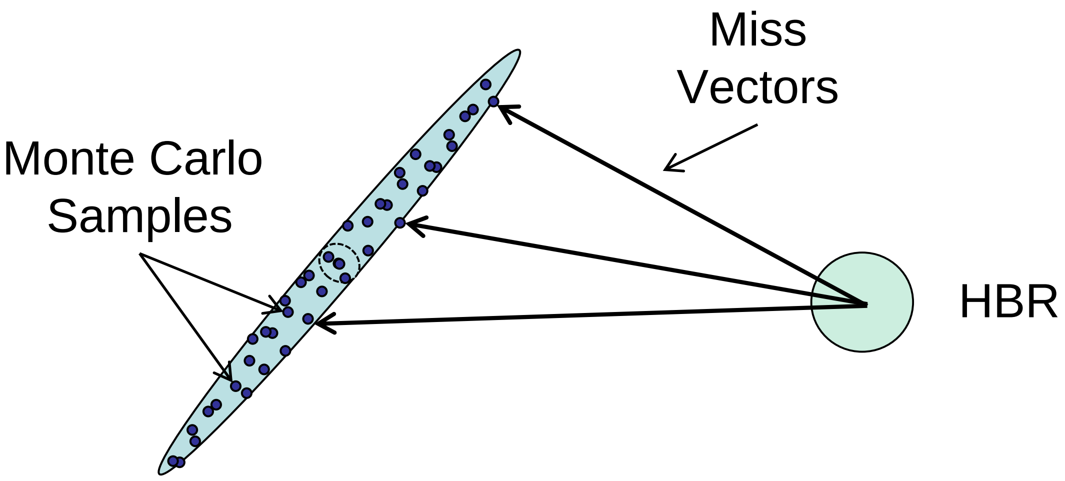
PC CALCULATION RESAMPLING

Pc Calculation Resampling

- **Resampling/bootstrap methods often used to generate confidence intervals when calculation final distribution unknown**
- **Early attempts at this with Pc used resampling with invariant covariances**
  - Take position draw on primary and secondary covariance at TCA
  - Find new TCA; this defines new nominal miss vector
  - Recompute Pc with this new miss vector and unaltered covariances
  - Problem: covariance is clearly correlated with conjunction geometry
    - Cannot produce new miss distance from covariance-based sampling and then recompute Pc using those same covariances
- **Need approach that considers miss distance / covariance linkage**

Pc Calculation Resampling Proposed Approach

- **J.H. Frisbee proposed a resampling technique that would also address the correlation problem**
  - Choose samples from the combined covariance to generate *m* miss vectors
  - Take mean of *m* miss vectors—this is new nominal miss
  - Take sample covariance of *m* miss vectors—this is new combined covariance
  - Compute Pc using this mean miss distance and sample combined covariance
  - Repeat procedure *n* times—this produces bootstrap dataset



* J. Frisbee, "International Space Station Collision Probability Analysis," OFD-03-48300-010, 2003.

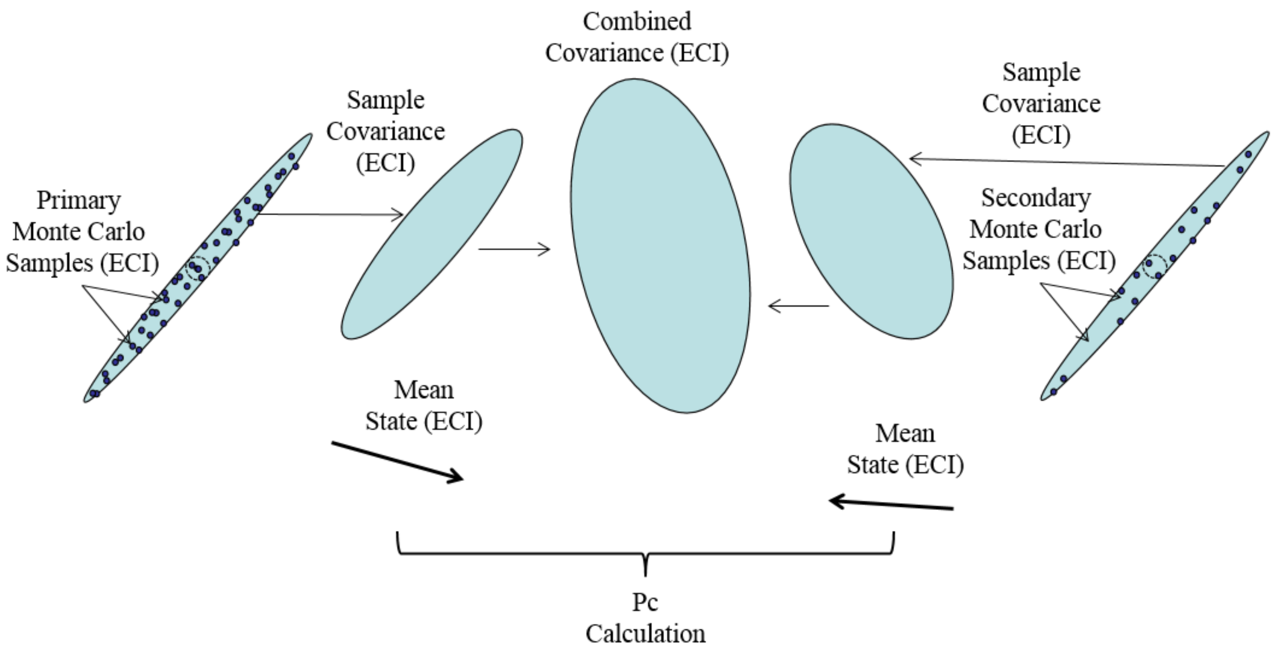
Resampling Approach Issues

- **In this framework, covariances are considered representatives of parent distributions, here further characterized by resampling**
- **Issue: what should be the value of *m*?**
  - In bootstrapping, want the bootstrap sample size to equal the single-sample size that would have been used (or was used) to estimate the parameter
  - Thus, want the number of samples (DoF) of the bootstrap resampling (*m*) to equal the DoF that produced the covariance in the first place
    - That is, the DoF of the generating OD

Tracking Levels and Degrees of Freedom

- **DoF is usually calculated as the number of data points minus the number of estimated parameters**
  - JSpOC ODs calculated with SSN obs (usually have range, azimuth, and elevation—three observables)
  - Obs provided in “tracks”—group of obs taken during one tracking session
- **Thus, tabulation issues arise**
  - Each ob provides 3 DoF, minus the estimated parameters
  - However, rather little information content in interior obs of a track
    - JSpOC “track weighting” confirms this—all tracks weighted the same in the OD, regardless of length
  - Better tabulation to count each track as equivalent of one state estimate
    - Longish track about enough data to execute a single state estimate, to first order
    - Total estimated parameters in OD would thus be only one—one state estimated
  - DoF calculation is thus “# of tracks – 1”
    - Would need to be amended for DS, where obs report only two parameters, and needs more work in general

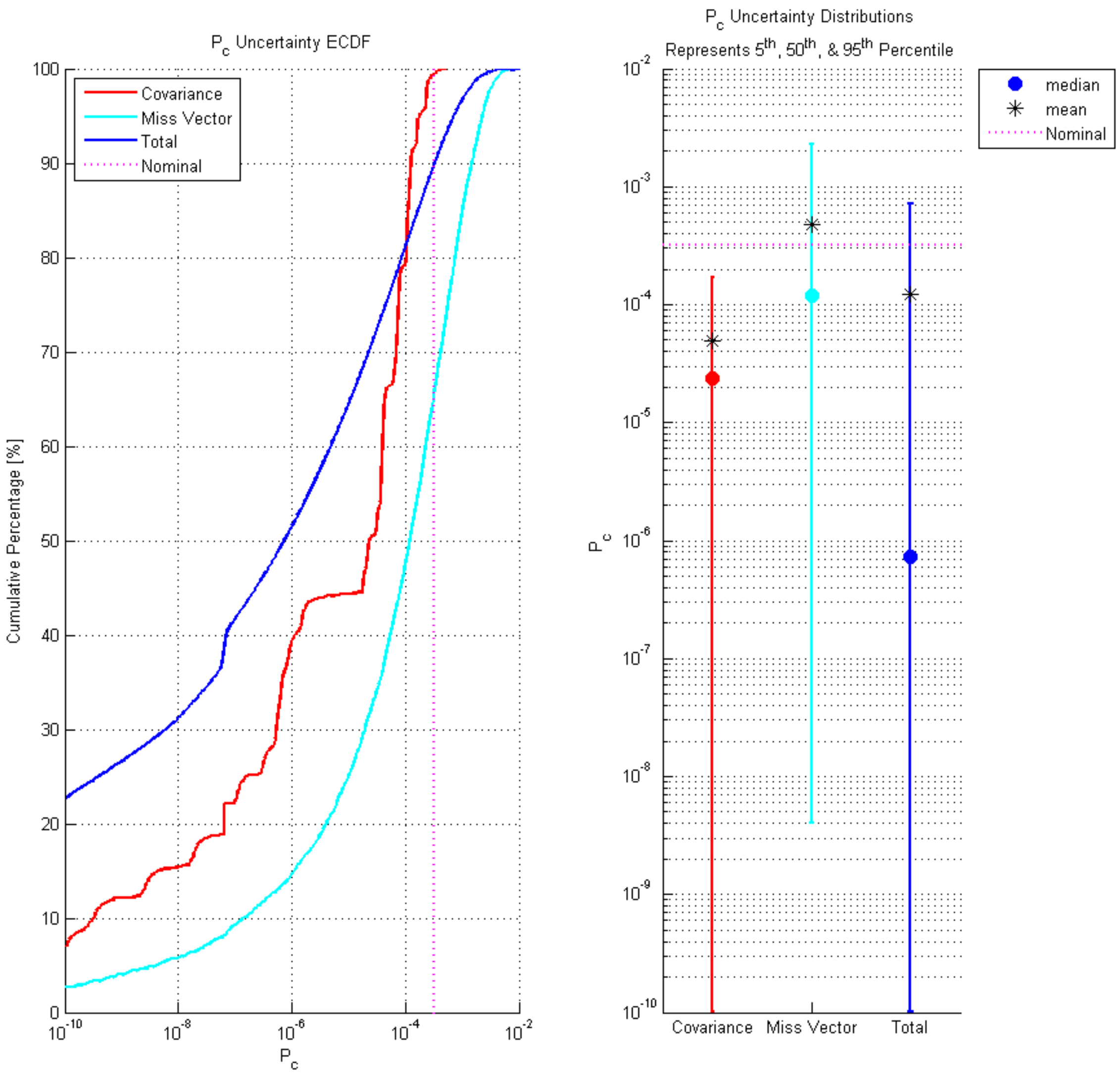
Resampling Approach Schematic



- **Repeated thousands of times to calculate distribution of Pc values**
- **Benefits**
  - Correlation of the miss vector and the covariance
  - Maintains an equivalent sampling level to the original OD
    - Naturally responds to variations in tracking density

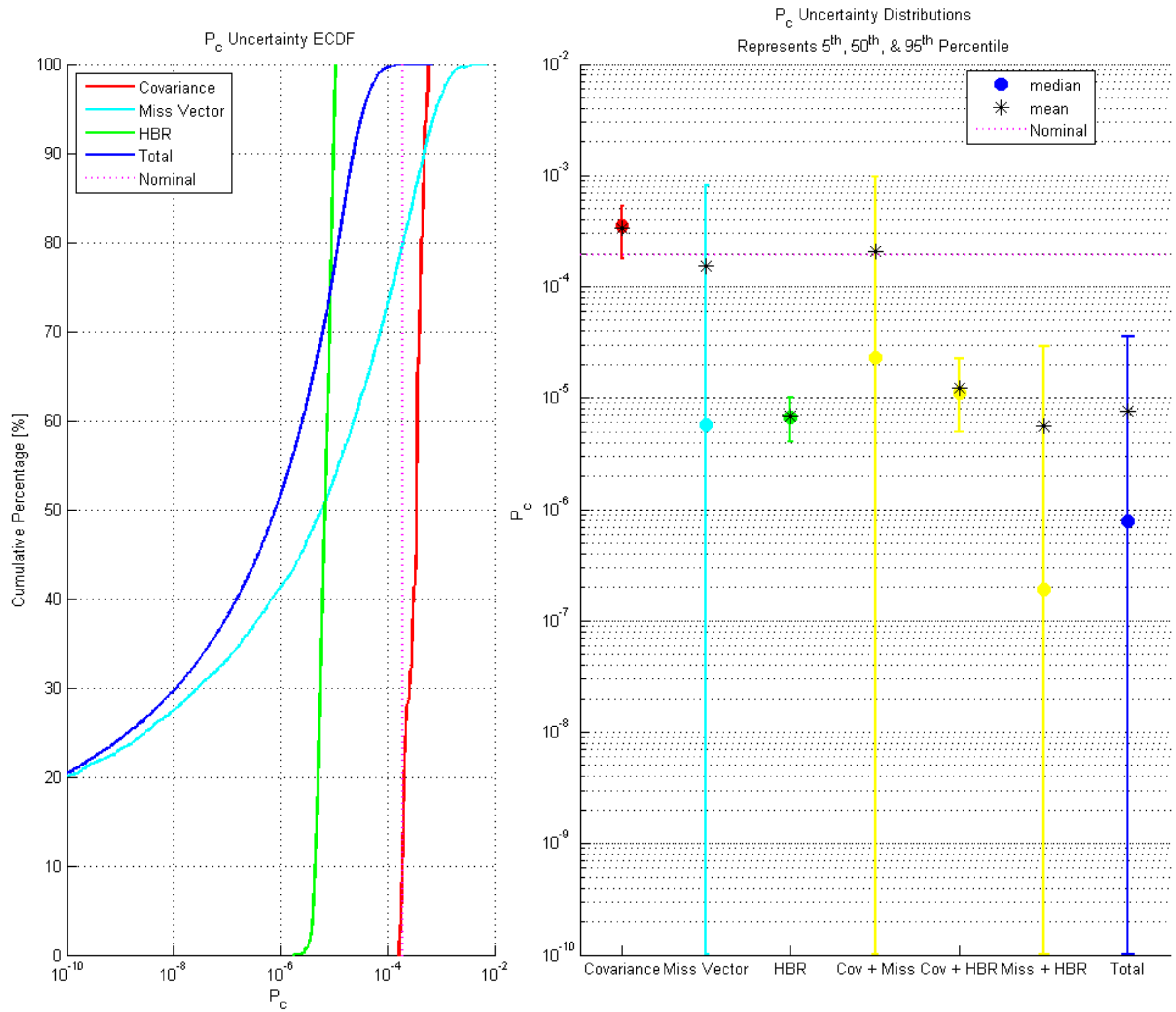
PROCESS RESULTS

Example #1



P_c Uncertainty ECDF
Covariance
Miss Vector
Total
Nominal
Cumulative Percentage [%]
P_c
P_c Uncertainty Distributions
Represents 5th, 50th, & 95th Percentile
median
mean
Nominal
Covariance
Miss Vector
Total

Example #2



P_c Uncertainty ECDF
Covariance
Miss Vector
HBR
Total
Nominal
Cumulative Percentage [%]
100
90
80
70
60
50
40
30
20
10
0
10^{-10}
10^{-8}
10^{-6}
10^{-4}
10^{-2}
P_c
P_c Uncertainty Distributions
Represents 5^{th}, 50^{th}, & 95^{th} Percentile
median
mean
Nominal
P_c
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
10^{-8}
10^{-9}
10^{-10}
Covariance
Miss Vector
HBR
Cov + Miss
Cov + HBR
Miss + HBR
Total

Conclusions and Future Work

- **Proposed method**
  - Characterizes the PDF that can represent the Pc from a particular conjunction, given the uncertainties in covariances, HBR and natural variation in the Pc calculation
  - Gives a sense of the dynamic range of the Pc and allow maneuver decisions to be based on percentile points of this range rather than the nominal value alone
  - Provides a mechanism for obtaining a better expression of the calculation's central tendency (here the median)
- **Future Work**
  - Refine DoF calculation and generate expansion for angles-only cases
  - Survey results from runs of large datasets
    - Stability studies of simplifying assumptions for faster processing
  - Examine potential as a Pc forecaster