A Comparison of PSD Enveloping Methods for Nonstationary Vibration

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Introduction

There is a need to derive a power spectral density (PSD) envelope for nonstationary acceleration time histories, including launch vehicle data, so that components can be designed and tested accordingly.

Three methods are considered in this paper using an actual flight accelerometer record.

The first method divides the accelerometer data into segments which are idealized as "piecewise stationary" in terms of their respective PSDs. A maximum envelope is then drawn for the superposition of segment PSDs. This method initially requires no assumptions about the response characteristics of the test item, but vibration response spectra may used for peak clipping as shown in the example.

The following two methods apply the time history as a base input to a single-degree-of-freedom system with variable natural frequency and amplification factors. The response of each system is then calculated. Upper and lower estimates of the amplification factor can be used to cover uncertainty.

The first of this pair is the energy response spectrum (ERS), which gives energy/mass vs. natural frequency, as calculated from the relative response parameters.

The final method is the fatigue damage spectrum (FDS), which gives a Miners-type relative fatigue damage index vs. natural frequency based on the response and an assumed fatigue exponent, or upper and lower estimates of the exponent.

The enveloping for each of the response spectra methods is then justified using a comparison of candidate PSD spectra with the measured time history spectra. The PSD envelope can be optimized by choosing the one with the least overall level which still envelops the accelerometer data spectra, or which minimizes the response spectra error.

This paper presents the results of the three methods for an actual flight accelerometer record. Guidelines are given for the application of each method to nonstationary data. The method can be extended to other scenarios, including transportation vibration.

Sample Flight Data

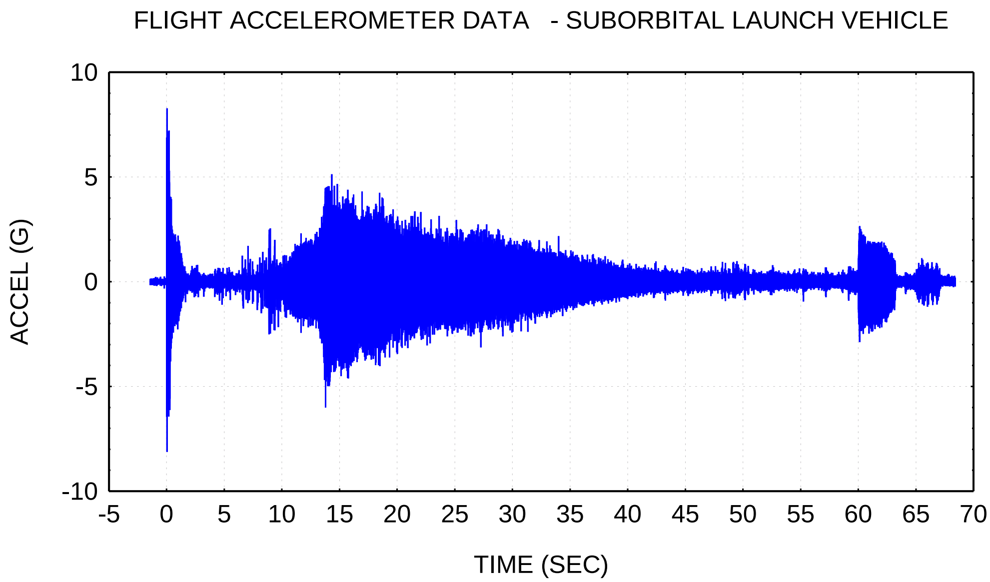

Figure 1.

Derive an optimized PSD with 60-second duration to envelope the nonstationary, base input acceleration time history in Figure 1, via each of the three methods. This PSD is needed for a hypothetical avionics component which is to be mounted near the measurement location on a future flight. The component must be designed and tested accordingly.

An option is to cover the liftoff event from 0 to 2 seconds as a separate shock or transient vibration event, but the entire record will be used as a test of the methods.

Furthermore, a ramp-plateau-ramp envelope PSD will be derived in each of the three cases.

Piecewise Stationary Method

There are numerous ways to implement the piecewise method.

For this example, the data is divided into consecutive 2.5-second segments. Each segment is idealized as being stationary. A narrowband PSD is then calculated for each segment. A maximum envelope PSD is taken for the superposition of segment PSDs. The envelope is converted to 1/12 octave format, as shown in Figure 2.

The 1/12 octave curve is then enveloped by the Simplified PSD as shown in Figure 2. Simplification is advisable because the frequencies of the spectral peaks may potentially have flight-to-flight variation. The enveloping is performed using vibration response spectra per Reference 1. This method allows for peak clipping. A single amplification factor of Q=10 was used for the enveloping, but a pair of Q values could be used for added rigor.

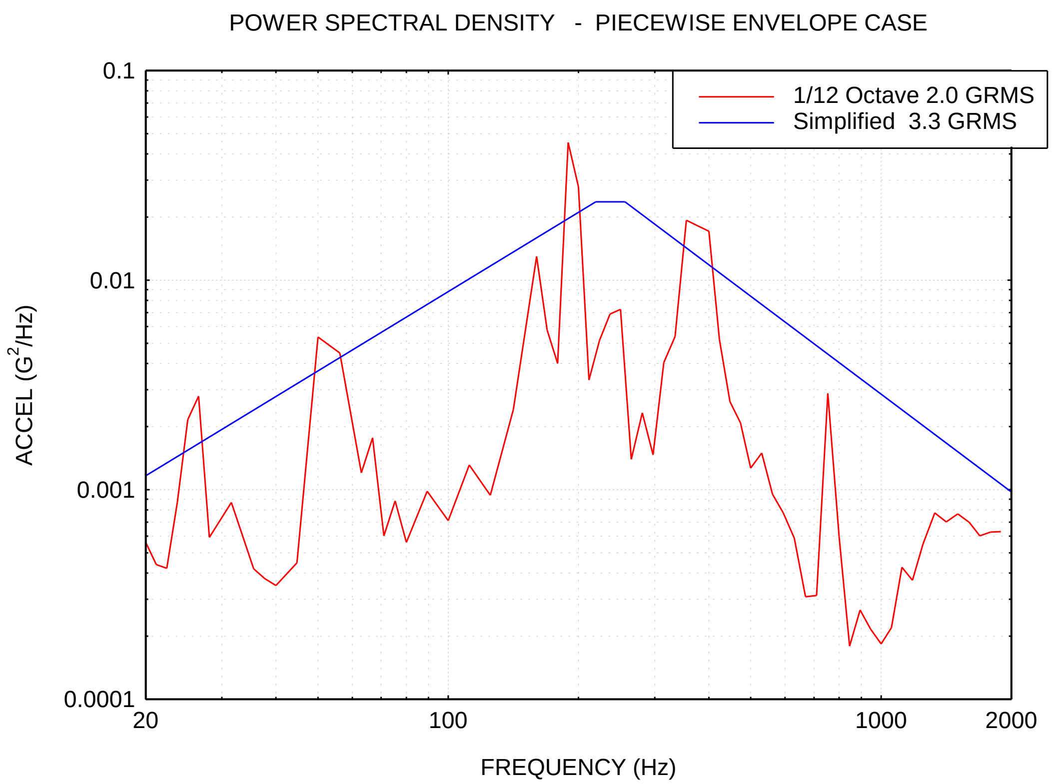

Figure 2.

Energy Response Spectrum Method

The ERS method is taken from Reference 2. The concept seems to have originated in earthquake engineering and has since been a topic of research by Sandia National Labs. The equations for this method are shown in Appendices A & B. There are three energy components: kinetic, dissipated, and absorbed. The energy terms are calculated from the relative response terms of the spring-mass system with variable natural frequency and amplification factor.

The ERS could hypothetically be applied by taking any of these components or some combination thereof. The approach in this example is to take the summation of each of the three, which yields the total input energy. Two amplification factors are used with Q=10 & 30.

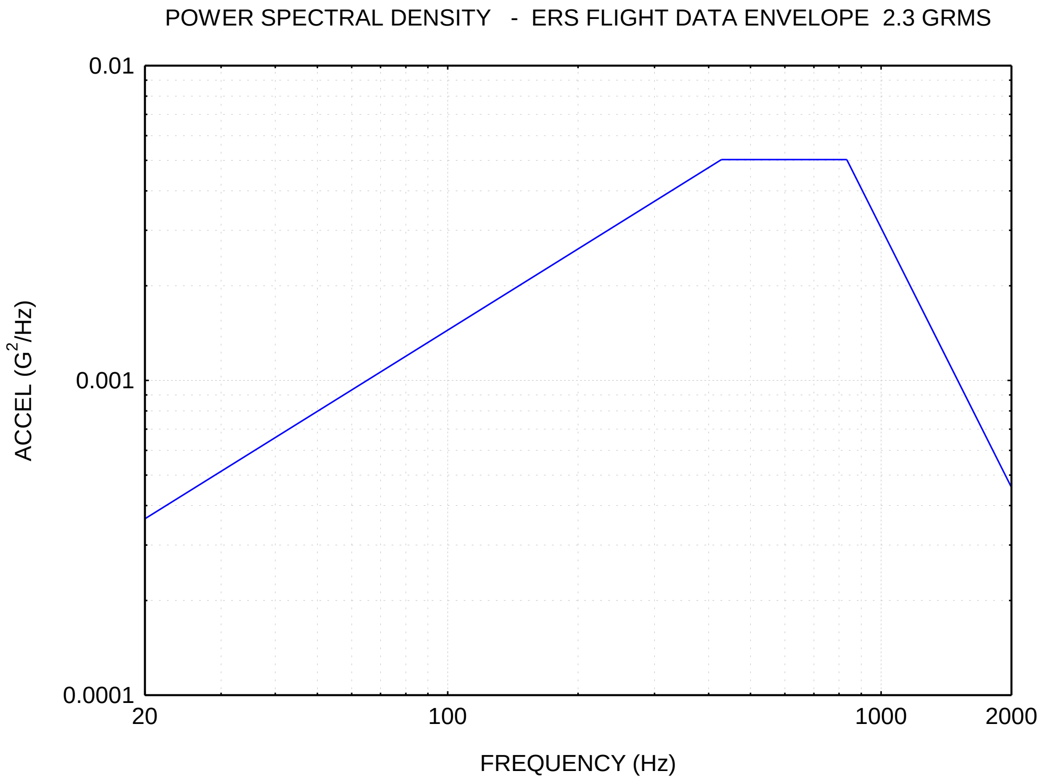

Figure 3.

The method was applied by generated a series of candidate PSD functions. A corresponding time history was then synthesized for each PSD.[1] The ERS was calculated for each time history. The PSD and its ERS were then scaled so that each ERS would be greater than or equal to that of the flight time history at each natural frequency, with the two ERS curve equal at least one frequency. The winning PSD was that which yielded the least overall ERS error for the combined Q cases.

The resulting optimized PSD for the flight data is shown in Figure 3. The ERS justification for this envelope PSD is shown for the two amplification factor cases in Figures 4 & 5. The Q=30 case drives the envelope PSD.

[1] A method for directly calculating an ERS from a PSD is a topic for a future paper.

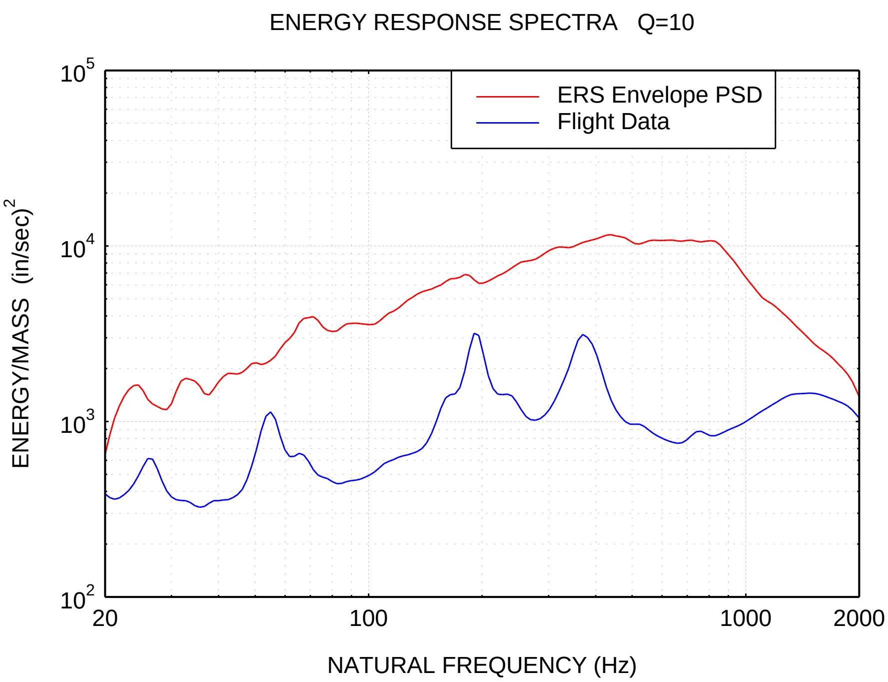

Figure 4.

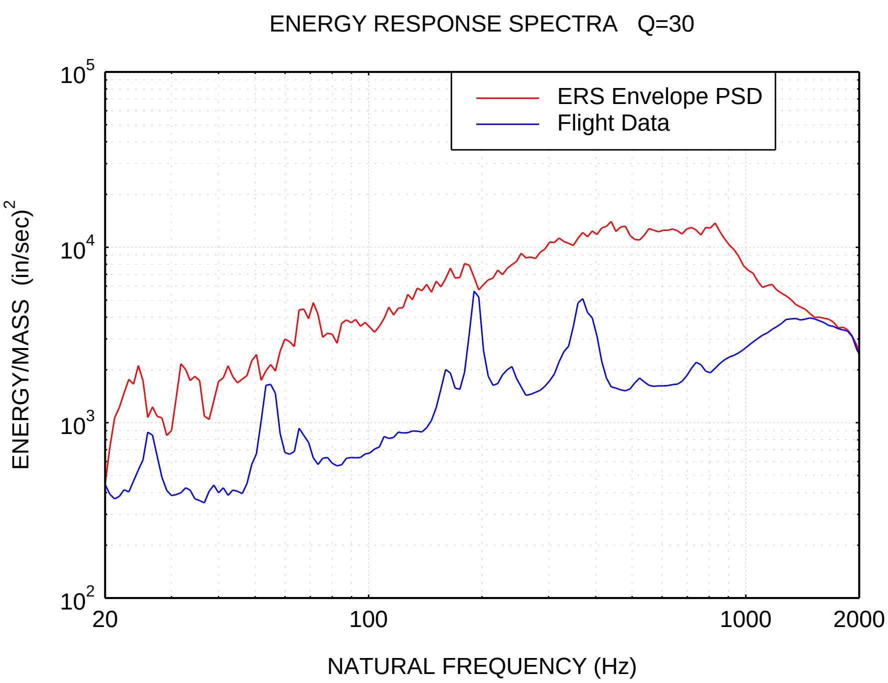

Figure 5.

Fatigue Damage Spectrum Method

The response of the spring-mass oscillators to the flight accelerometer time history can be calculated in the time domain via the method in Reference 3. The rainflow cycles can then be calculated using the method in Reference 4. A relative damage index is calculated using a Miners-type summation for each permutation of natural frequency, amplification factor and fatigue exponent.

The process can be carried out in the frequency domain from the response PSD for each candidate input PSD. The Dirlik method is then used for the FDS calculation, as taken from Reference 5 & 6.

The optimization for this case was done by selecting the candidate PSD which had the least overall levels for displacement, velocity and acceleration. Another possibility would have been to select the one which gave the least error in terms of the FDS comparisons. Both approaches would mostly likely yield similar envelope PSDs, but verification is pending.

Further details for these calculations are given in Appendices C & D, and in Reference 7.

The amplification factors for the example are Q=10 & Q=30. The fatigue exponents are b=4.0 & 6.4. This gives four combinations. The selected fatigue exponents are those common for avionics components.

The resulting optimized PSD is shown in Figure 6.

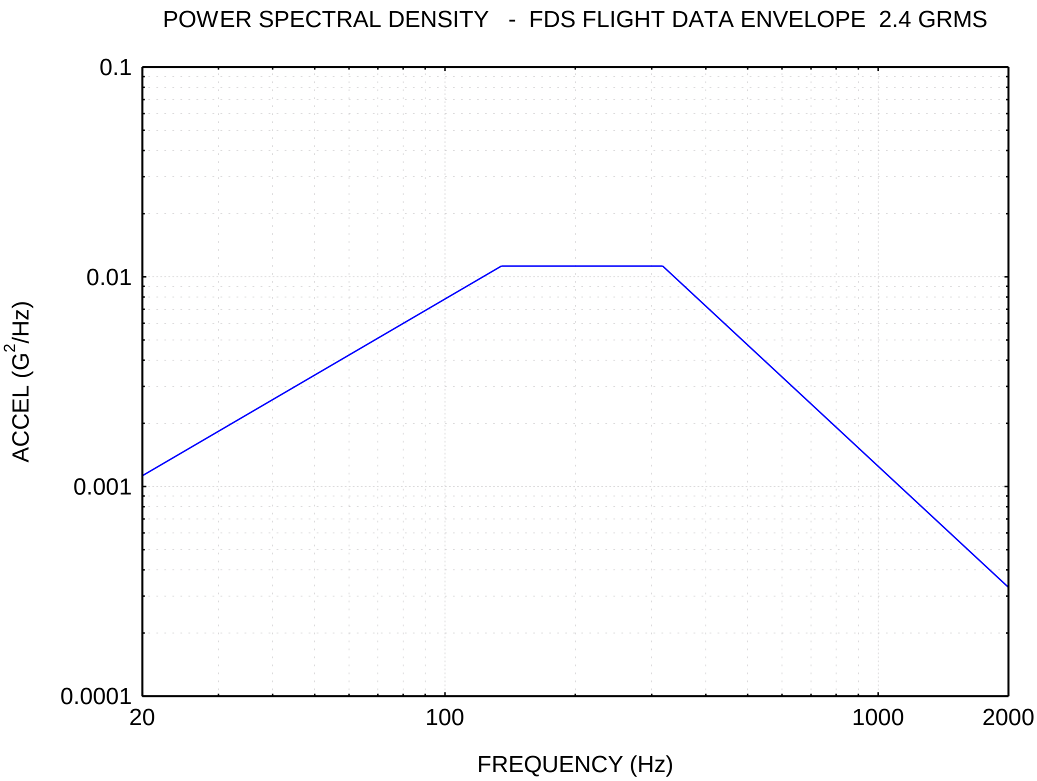

Figure 6.

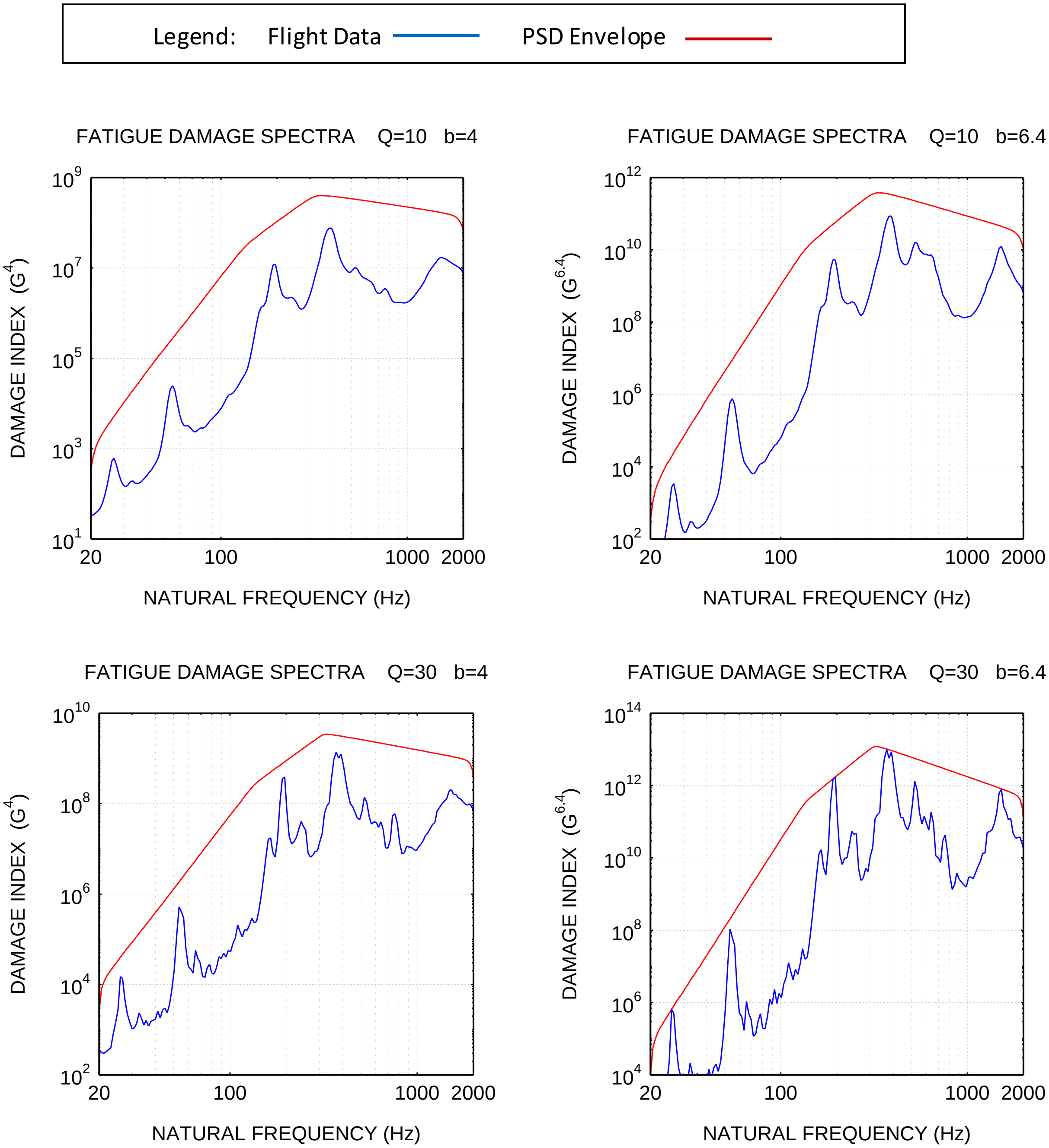

Figure 7.

The corresponding FDS plots are shown in Figure 7. The case with Q=30 and b=6.4 is the driver for the envelope PSD.

Envelope Comparison

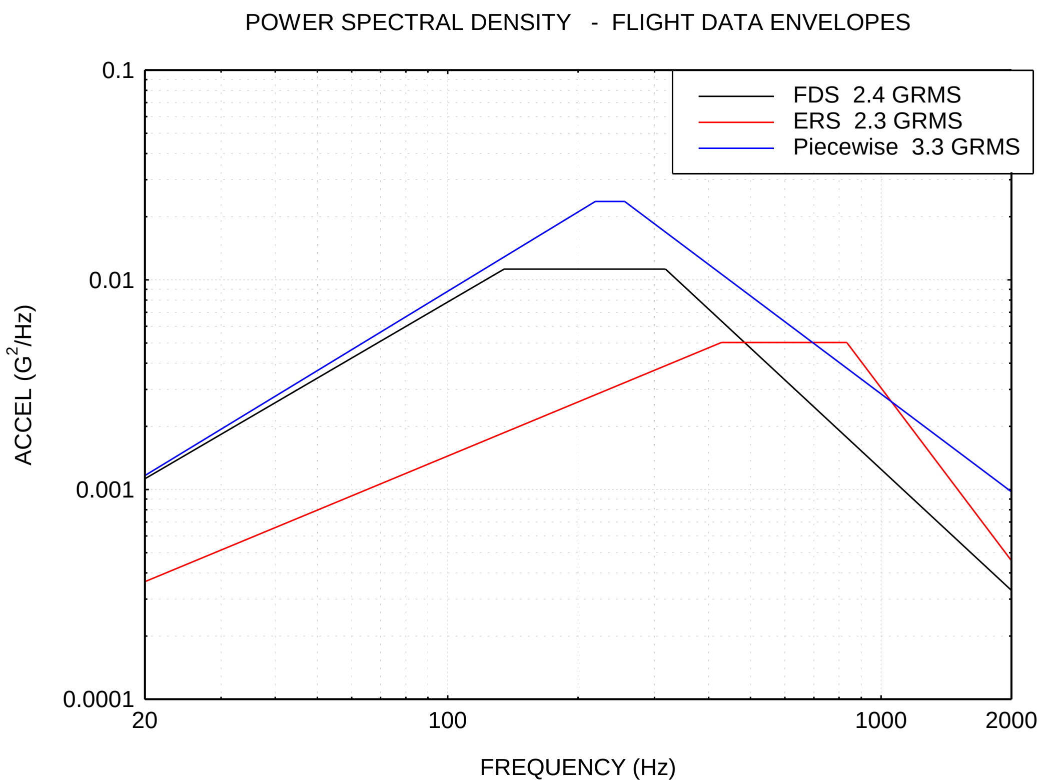

Figure 8.

The PSD envelope comparison is shown in Figure 8.

The Piecewise PSD is the most conservative in terms of overall level, as expected. The FDS-derived PSD shows the justifiable reduction within the scope of the assumed amplification factor and fatigue exponent combinations.

The ERS-derived PSD is the least of the three in terms of overall level.

Conclusion

Selecting the enveloping method for a given nonstationary time history is a matter of engineering judgment.

The piecewise method is the traditional approach and requires the fewest assumptions, but it may be overly conservative for many cases.

The FDS method requires the most assumptions, but appears to be most sound because fatigue is the most likely failure mode for random vibration environments.

The ERS method needs justification that accumulated energy/mass correlates with damage in aerospace components.[2]

The FDS method seems the most sensible of the three, but further research is needed.

Post Script

The Matlab scripts for performing the calculations in this paper are given at:

https://vibrationdata.wordpress.com/2013/05/29/vibrationdata-matlab-signal-analysis-package/

References

1. T. Irvine, Enveloping Data via the Vibration Response Spectrum, Vibrationdata, 1999.
2. T. Irvine, Energy Response Spectrum, Vibrationdata, 2015.
3. David O. Smallwood, An Improved Recursive Formula for Calculating Shock Response Spectra, Shock and Vibration Bulletin, No. 51, May 1981.
4. ASTM E 1049-85 (2005) Rainflow Counting Method, 1987.
5. Halfpenny & Kim, Rainflow Cycle Counting and Acoustic Fatigue Analysis Techniques for Random Loading, RASD 2010 Conference, Southampton, UK.
6. Halfpenny, A frequency domain approach for fatigue life estimation from Finite Element Analysis, nCode International Ltd., Sheffield, UK.
7. T. Irvine, Optimized PSD Envelope for Nonstationary Vibration, Revision B, Vibrationdata, 2014

[2] Note that peak energy is typically used in earthquake engineering in a roundabout way such that the resulting spectrum is really a pseudo velocity spectrum.

APPENDIX A

Equation of Motion

Consider a single degree-of-freedom system subjected to base excitation.

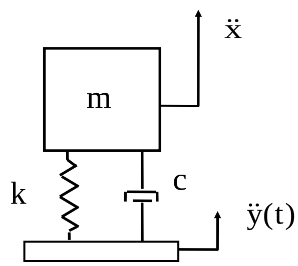

Figure A-1.

where

m	is the mass
c	is the viscous damping coefficient
k	is the stiffness
x	is the absolute displacement of the mass
y	is the base input displacement

The double-dot denotes acceleration.

The free-body diagram for the system is

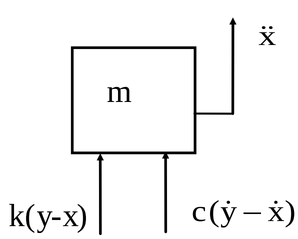

Figure A-2.

Summation of forces in the vertical direction,

$$\sum F = m\ddot{x} \tag{A-1}$$

$$m\ddot{x} = f(t) + c(\dot{y} - \dot{x}) + k(y - x) \tag{A-2}$$

Define a relative displacement

$$z = x - y \tag{A-3}$$

Substituting the relative displacement terms into equation (A-2) yields

$$m(\ddot{z} + \ddot{y}) = -c\dot{z} - kz \tag{A-4}$$

$$m\ddot{z} + c\dot{z} + kz = -m\ddot{y} \tag{A-5}$$

Dividing through by mass yields,

$$\ddot{z} + (c/m)\dot{z} + (k/m)z = -\ddot{y} \tag{A-6}$$

By convention,

$$(c/m) = 2\xi\omega_n \tag{A-7}$$

$$(k/m) = \omega_n{}^2 \tag{A-8}$$

where ω_n is the natural frequency in (radians/sec), and ξ is the damping ratio.

Substituting the convention terms into equation (A-6) yields

$$\ddot{z} + 2\xi\omega_n\dot{z} + \omega_n{}^2 z = -\ddot{y} \tag{A-9}$$

The equation of motion can solved by the method in Reference 3 for the case of an arbitrary base input.

Furthermore, allow the natural frequency to be an independent variable. The task is to calculate the “damage” at each natural frequency of interest for a fixed Q value.

$$Q = 1/(2\xi) \tag{A-10}$$

APPENDIX B

Energy Equations

The input energy per mass E_I is defined by integrating the equation of motion over the relative displacement.

$$E_I = \int_0^z \ddot{z}(t)\,dz + 2\xi\omega_n \int_0^z \dot{z}(t)\,dz + \omega_n{}^2 \int_0^z z(t)\,dz \qquad \text{(B-1)}$$

The kinetic energy per mass E_K is

$$E_K = \int_0^z \ddot{z}(t)\,dz = \int_0^t \ddot{z}(t)\,\dot{z}(t)\,dt \qquad \text{(B-2)}$$

The dissipated energy per mass E_D is

$$E_D = 2\xi\omega_n \int_0^z \dot{z}(t)\,dz = 2\xi\omega_n \int_0^t \dot{z}(t)^2\,dt \qquad \text{(B-3)}$$

The absorbed energy per mass E_A is

$$E_A = \omega_n{}^2 \int_0^z z(t)\,dz = \omega_n{}^2 \int_0^t z(t)\dot{z}(t)\,dt \qquad \text{(B-4)}$$

Note that the energy terms are functions of the relative response amplitudes.

APPENDIX C

A relative damage index D can be calculated using

$$D = \sum_{i=1}^{m} A_i^b \, n_i \tag{C-1}$$

where

A_i is the *response* amplitude from the rainflow analysis

n_i is the corresponding number of cycles

b is the fatigue exponent

Note that the amplitude convention for this paper is (peak-valley)/2.

APPENDIX D

Dirlik Method

The *n*th spectral moment m_n for a PSD is

$$m_n = \int_0^\infty f^n G(f)\,df \tag{D-1}$$

Where

f	is frequency
$G(f)$	is the one-sided PSD

The expected peak rate $E[P]$ is

$$E[P] = \sqrt{m_4/m_2} \tag{D-2}$$

The Dirlik histogram formula $N(S)$ for stress cycles ranges is

$$N(S) = E[P] \cdot T \cdot p(S) \tag{D-3}$$

where

T	is the duration
S	is the stress cycle range (peak-to-peak)

The function $p(S)$ is

$$p(S) = \frac{\dfrac{D_1}{Q}\exp\left(-\dfrac{Z}{Q}\right) + \dfrac{D_2 Z}{R^2}\exp\left(\dfrac{-Z^2}{2R^2}\right) + D_3 Z \exp\left(\dfrac{-Z^2}{2}\right)}{2\sqrt{m_0}} \tag{D-4}$$

The coefficients and variables are

$$D_1 = \frac{2\left(x_m - \gamma^2\right)}{1+\gamma^2} \tag{D-5}$$

$$D_2 = \frac{1-\gamma-D_1+D_1^2}{1-R} \tag{D-6}$$

$$D_3 = 1 - D_1 - D_2 \tag{D-7}$$

$$Z = \frac{S}{2\sqrt{m_0}} \tag{D-8}$$

$$Q = \frac{1.25\left(\gamma - D_3 - D_2 R\right)}{D_1} \tag{D-9}$$

$$R = \frac{\gamma - x_m - D_1^2}{1-\gamma-D_1+D_1^2} \tag{D-10}$$

$$\gamma = \frac{m_2}{\sqrt{m_0\, m_4}} \tag{D-11}$$

$$x_m = \frac{m_1}{m_0}\sqrt{\frac{m_2}{m_4}} \tag{D-12}$$

A cumulative histogram of the peaks can then be calculated from equation (D-3).

The stress range of individual cycles can then be obtained by interpolating the cumulative histogram.